Chapter 9

Forms, module morphisms and Gram matrices

9.1 Forms, module morphisms and Gram matrices (Draft)

Here we consider some useful forms on modules for algebras that are isomorphic to their opposites.

An algebra isomorphic to its opposite via an involution has an involutive antiautomorphism. (Let $op : A \to A^{op}$ be the identity map on A as a set; and hence an algebra antihomomorphism. Since the image of op is A as a set, we can apply op again: $op \circ op = 1$. Let $i : A \to A^{op}$ be an algebra isomorphism. Then $(op \circ i) : A \to A$ is an antiautomorphism.)

Overview:
The basic idea is this... Firstly, *any* A-module morphism $\psi : M \to N$ gives us information about M and N. The kernel is a submodule of M for example. If there is an algebra antiautomorphism then module morphisms are in bijection with suitable contravariant forms — thus the latter become a useful source of morphisms. Next, for certain special classes of modules, only morphisms of special kinds are possible in principle; explicit morphisms are then particularly revealing.

... the idea has been used historically to study the symmetric group (see e.g. [66]). See also [52]. Outside the classical group representation theory setting, it has proved very useful for 'diagram algebras' (see [91, 92], [**?**] for some early examples).
An outline of the section is this... We start with a brief review of basics designed to make the section as self-contained as possible. Then...

9.1.1 Some basic preliminaries recalled: ordinary duality

(**9.1.1**) Recall the convention (e.g. from Ch.7) that we write the action of ring or algebra A on left A-module M 'on the left':

$$\begin{aligned} A \times M &\to M && (9.1)\\ (a, m) &\mapsto am && (9.2) \end{aligned}$$

so that $b(am) = (ba)m$; and the action of $a \in A$ on a right A-module on the right: ma.
(If we write ma for a left action we get $(ma)b = m(ba)$ which just looks odd.)

de:dualM (**9.1.2**) Recall that if A is an R-algebra and $M = {}_AM$ a left A-module then the *dual right module* is

$$M^* \; = \; M^*_A \; := \; \mathrm{Hom}_R({}_AM, R)$$

Thus elements of M^* are maps

$$\mu : M \;\; \to \;\; R \tag{9.3}$$

$$m \;\; \mapsto \;\; \mu(m) \tag{9.4}$$

It is a *right* A-module by the action of $a \in A$ on any μ as above, to give μa as follows:

$$\mu a : m \mapsto \mu(am)$$

We check the right action property by comparing $(\mu a)b$ with $\mu(ab)$:

$$(\mu a)b : m \mapsto \mu a(bm) = \mu(a(bm))$$

$$\mu(ab) : m \mapsto \mu((ab)m)$$

(**9.1.3**) Does this $*$ lift to a functor from $A-$mod to mod$-A$? Consider the possible image of a map in $A-$mod (or indeed a sequence in $A-$mod):

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[d, "\hom_R(-,R)"'] \\ (M')^* & M^* \arrow[l, "?"] & (M'')^* \arrow[l, "?"] \end{tikzcd}$$

Given a map $f \in \hom_R(M', M)$, then for each $a \in \hom_R(M, R)$ we can form $f^*(a) \in \hom_R(M', R)$ by $f^*(a) = a \circ f$. That is

$$\begin{tikzcd} M' \arrow[rr, "f^*(a)"] \arrow[dr, "f"'] & & R \\ & M \arrow[ur, "a"'] & \end{tikzcd}$$

Thus $f^* \in \hom_R(M^*, (M')^*)$ and we have

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[d, "\hom_R(-,R)"'] \\ (M')^* & M^* \arrow[l, "f^*"] & (M'')^* \arrow[l, "g^*"] \end{tikzcd}$$

In other words $\hom_R(-, R) : A-\mathrm{mod} \;\to\; \mathrm{mod}-A$ defined in this way is a *contravariant* functor. Indeed this $H- = \hom_R(-, R)$ is a left-exact contravariant functor, meaning that an exact sequence $M' \to M \to M'' \to 0$ passes to an exact sequence $0 \to HM'' \to HM \to HM'$.

(**9.1.4**) Exercise. (I) Suppose an A-module M is simple. What can we say about M^*?
(II) Suppose we have a Jordan–Holder series (see e.g. §7.3.2) for an A-module M. What can we say about M^*?

(**9.1.5**) If in particular A is a finite dimensional algebra over a *field* R then every finitely-generated A-module M is a finite-dimensional R-vector space, and M^* has the same dimension, and we have the following (see e.g. [3, §23]).

(I) M and M^{**} are isomorphic A-modules.

(II) M is (semi)simple if and only if M^* is (semi)simple.

(III) A sequence $0 \to M' \to M \to M'' \to 0$ is (split) exact if and only if the dual sequence is (split) exact.

(IV) $\mathrm{Soc}\, M^* \cong (M/\mathrm{Rad}\ M)^*$.

9.1.2 Contravariant duality

de:opM0 (**9.1.6**) Let R be a commutative ring and let A be any R-algebra. We may regard $M = {}_AM$ as a right A^{op}-module $M = M_{A^{op}}$ by defining a right action as follows:

$$ma := am$$

(Check: recall that for $a, b \in A^{op}$, with multiplication denoted $*$ say, then $a * b = ba$ as computed in A; thus $(ma)b = b(ma) = b(am) = (ba)m = m(a * b)$). Similarly each right module gives a left module.

A left A-module homomorphism $M' \xrightarrow{f} M$ becomes a right A^{op}-module homomorphism, so this construction $\Phi : A-\mathrm{mod} \to \mathrm{mod}-A^{op}$ (say) is a (covariant) functor. We also use Φ for the right-to-left version.

avariant duality (**9.1.7**) A group algebra over a commutative ring R is isomorphic to its opposite (defined as for opposite ring) since $g \mapsto g^{-1}$ defines a group antiautomorphism (an isomorphism $G \cong G^{op}$); and this extends to RG via: $\sum_i r_i g_i \mapsto \sum_i r_i g_i^{-1}$.

There may be other isomorphisms. For example, a suitable group of matrices may be mapped to its opposite by $g \mapsto g^{tr}$ (transpose matrix). Here, when considering any algebra isomorphic to its opposite, we will generally fix a given involutive antiautomorphism.

de:opM (**9.1.8**) Let R be a commutative ring and let A be any R-algebra with an involutive antiautomorphism (generally denoted $g \mapsto g^t$, or $g \mapsto g^\tau$). We may regard $M = {}_AM$ as a right A-module by

$$ma := a^t m$$

(Check: $(ma)b = b^t(ma) = b^t a^t m = (ab)^t m = m(ab)$); and similarly each right module gives a left module.

(**9.1.9**) It follows from (9.1.2) and (9.1.8) that for each $M \in A-\mathrm{mod}$ there is another left module M^o obtained from the dual right module M^* by applying Φ:

$$M^o := \Phi(M^*)$$

(i.e. via the opposite isomorphism, regarding M^* as a left module for the opposite). This construction has the property that $R-\mathrm{mod}$ is invariant under taking to its dual combined with taking all $M \mapsto M^o$. (I.e. if defined, $(\mathrm{Head}\ M)^o \cong \mathrm{Soc}\ M^o$, and so on.)

We will call the map $M \mapsto M^o$ *contravariant duality* (see e.g. [52]). We have $(M^o)^o = M$.

(**9.1.10**) Exercise. Let G be a finite group and R a field. The 'contragredient' of a projective RG-module is projective (claim (10.29) in Curtis–Reiner [33]). Prove this. Give a counter-example for general finite-dimensional R-algebra A with t.
(For the symmetric group, and indeed any finite G, we will see in (9.1.27) that the regular module is contravariant self-dual for any R. Thus the collection of indecomposable summands over R a field must be fixed under duality, which verifies the claim in this case. However the regular module is not always self-dual for an algebra A with t (we shall have an example from the Temperley–Lieb algebras shortly).)

(**9.1.11**) Exercise. (Optional) Why are duals of lattices done differently in Curtis–Reiner [33] p.89 cf. p.245?

(**9.1.12**) Exercise. (Optional) Claim: Suppose A is in fact a finite group algebra over R. Let $x \in A$ be mapped to x^o by the opposite isomorphism (and regard x^o as an element of A). Then $M = Ax$ implies $M^o = Ax^o$.
Prove this, or provide a counter-example.

9.1.3 A Schur Lemma for 'standard' modules

Here a standard module M is one with a simple head L, with L having composition multiplicity 1 in M.
Some questions to keep in mind:
Given an algebra A with a t as above, when is a simple A-module $L = L^o$?

prepre (**9.1.13**) Proposition. *Suppose that R is a field, and A is an R-algebra with a given involutive antiautomorphism. If left A-module M has a unique maximal proper submodule (call it M_o) and hence simple head $L = M/M_o$, and this composition factor L has multiplicity one in M, and $L \cong L^o$, then* $\dim Hom_A(M, M^o) = 1$, *and* $\psi \in Hom_A(M, M^o)$ *has rank* $\dim(L)$.

Proof: NB, every simple factor in M^o is extended by L below it. There *is* a map $\psi \in \mathrm{Hom}_A(M, M^o)$ — that which kills the unique maximal proper submodule M_o and so makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M_o \arrow[r] & M \arrow[r] \arrow[d, "\psi"'] & L \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 & (M_o)^o \arrow[l] & M^o \arrow[l] & L^o \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

(or any scalar multiple thereof). No reduction is possible in the kernel, since this would require factors appearing in the image below L, which M^o does not have, as already noted. No enlargement of the image is possible since this, correspondingly, requires factors above L in M. □
(NB, the converse does not hold in general.)

9.1.4 Bilinear forms

(**9.1.14**) Bilinear form: For R a field (or commutative ring), a bilinear form on $M, N \in R-\mathrm{mod}$ is an R-bilinear function

$$<,>: M \times N \to R$$

(cf. Perlis [105], Maclane-Birkoff [79, §X.1]).

(**9.1.15**) Suppose M, N free R-modules. Let $B_M = \{b_1, b_2, \ldots, b_k\}$ be an ordered basis of M and $B_N = \{c_1, c_2, \ldots, c_l\}$ of N. Then a matrix $B(M,N)$ of form

$$(B(M,N))_{ij} \; = < b_i, c_j >$$

determines $<,>$.

(**9.1.16**) In an obvious notation,

$$B(M,N) = \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \begin{pmatrix} \; |c_1\rangle, \; |c_2\rangle, \cdots, \; |c_l\rangle \; \end{pmatrix}$$

By linearity, the effect of basis changes $B'_M = B_M U$, $B'_N = B_N W$ (B_M arranged as a row vector $(b_1, b_2, ...)$; U, W unimodular matrices, as in §7.2.4) is

$$B'(M,N) = U^t \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \begin{pmatrix} \; |c_1\rangle, \; |c_2\rangle, \cdots, \; |c_l\rangle \; \end{pmatrix} W \; = \; U^t B(M,N) W$$

Recall (e.g. from §7.2.4) that if R is a PID then among the possible such basis changes would be a pair that bring $B'(M,N)$ into Smith normal form.
Remark. The reason for keeping the ground ring R at the level of generality of commutative ring (despite the fact that our eventual objects of study are typically algebras over fields) will become apparent shortly.

pa:form-map (**9.1.17**) Keep $M, N \in R-\text{mod}$ as before. Recall that the dual $N^* = \text{Hom}_R(N,R)$ has the structure of R-module (since R is commutative, 'left' and 'right' modules are the same here). Note that form $<,>$ defines an R-module homomorphism

$$\psi : M \to N^*$$

by

$$\psi(m)(n) \; = < m, n > .$$

From this perspective we can think of $B(M,N)$ (or $B'(M,N)$) as characterising the image of M in N^*.

If R is a field, then the rank of matrix $B(M,N)$ is independent of the specific choice of bases (to see this note that it is unchanged on replacing b_1 by a linear combination with nonzero component of b_1). Accordingly

$$\text{rank} \;\; <,> \;\; := \;\; \text{rank}\ (B(M,N)).$$

(**9.1.18**) For I an ideal of R define $L^I_{<,>}$ as the subset of M such that $< m, n > \in \; I$ for every $m \in L^I_{<,>}$ and $n \in N$. By linearity $L^I_{<,>}$ is a submodule.
The Left radical $L_{<,>} = L^0_{<,>}$ of $<,>$ is the submodule M' of M such that $< m, n > = \; 0$ for every $m \in M'$ and $n \in N$.

If R is a field then

$$\mathrm{rank}\ <,> \;=\; \dim M - \dim L_{<,>}.$$

(**9.1.19**) Note that $\psi(b_j)$ is the map such that $(\psi(b_j))(c_i) = \ < b_j, c_i >$. Our basis of N^* is $\{f_i\}$ such that $f_i c_j = \delta_{i,j}$ therefore

$$\psi(b_j) = \sum_i < b_j, c_i > f_i$$

(check: we have $(\sum_i \alpha_i f_i)c_j = \alpha_j$ so $(\sum_i < b_j, c_i > f_i)c_k \;=\; < b_j, c_k >$ as required).

In an example, this says that, expressing the basis of M as $(1,0,0),(0,1,0),(0,0,1)$ (say); and of N^* as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can express ψ acting on M by $B(M,N)$ acting on the right:

$$(1,0,0)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix} = (B_{11}, B_{12}) = B_{11}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T + B_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$$

or by $B(M,N)^T$ acting on the left. In this formulation the inner product can be written, say,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (B_{11})$$

9.1.5 Contravariant forms on A-modules

In (9.1.17) we gave a correspondence between bilinear forms and maps from modules to dual-modules over a commutative ring R. Next we lift this to modules over an R-algebra A (NB, suitably changing dual right modules to left modules for a *non*-commutative algebra A requires an antiautomorphism).

This will not work for an arbitrary form $<,>: M \to N$, or corresponding $\psi : M \to N^*$, since we have the A-module morphism condition $\psi(am) = a\psi(m)$ for all $a \in A, m \in M$ to satisfy. We can think of this directly or over specific bases for M and N, where a acts as a matrix $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$ respectively and we must check $B\,\rho_{B_M}(a) \;=\; \rho_{B_N} B$ for all $a \in A$ — i.e. that $B = B(M,N)$ 'intertwines' representations $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$. It turns out that this condition can be expressed quite neatly.

(**9.1.20**) Contravariant form.
Let A be an R–algebra with involutive antiautomorphism t as above. For $M, N \in A-\mathrm{mod}$, an R–bilinear form $<,>: M \times N \to R$ is *contravariant* if $< am, n > = < m, a^t n >$ for all $a \in A$, $m \in M$, $n \in N$.

(**9.1.21**) Note that there is a bilinear form $<,>$ for every choice of matrix $B(N,M)$. The requirement of satisfying the constraints of *contravariant form* on A-modules, however, will in general be very restrictive on possible choices of $B(M,N)$.

(**9.1.22**) Example. Let R be a commutative ring and G a finite group. Let $M = R\{m_1, m_2, ..., m_l\}$ be an RG-module that is free as an R-module, with the given basis. Define a bilinear form on M (i.e. on the pair $(M,N) = (M,M)$) by setting $B(M,M)$ to the identity matrix, that is, by

$$< m_i, m_j > = \ \delta_{m_i, m_j}.$$

We have $gm_i = m_j \iff m_i = g^{-1}m_j$. Thus *if G acts on M by permuting basis elements* we have $< gm_i, m_j > = < m_i, g^{-1}m_j >$, and hence $< am_i, m_j > = < m_i, a^t m_j >$ for $a \in RG$. Thus this $<,>$ is a contravariant form.

In particular if $M = RG$ is the left regular module then G indeed acts by permuting basis elements, so this $<,>$ is contravariant in this case. (We will give a more explicit example in (9.1.27).)

(**9.1.23**) Exercise. (Optional) Construct an example as above but where G does not act on M by permutation, and where the $<,>$ above is indeed not contravariant.

contraformperp (**9.1.24**) Proposition. *Let A be an R-algebra with involutive antiautomorphism t, and $M, N \in A-mod$. If $<,>: M \times N \to R$ is a contravariant form and I an ideal in R then the subset $S = L^I_{<,>} \subseteq M$ (such that $< s, N > \in I$ for all $s \in S$) is an A-submodule of M.*

Proof. For $a \in A$, $s \in S$, $n \in N$ we have $< as, n > = < s, a^t n > \in I$ since $a^t n \in N$, so $as \in S$. □

(**9.1.25**) Remark. Note that if $I' \subset I$ then $L^{I'}_{<,>} \subseteq L^I_{<,>}$. Of course if R is a field then the only possibility is $I = 0$.
Note that if we start with R a commutative ring and compute $S = L^0_{<,>}$; then base change to some $A^k = k \otimes_R A$ and write S^k for the corresponding submodule computed here, then S^k may be bigger than $k \otimes_R S$.

contraformx (**9.1.26**) Proposition. *Let R be a commutative ring and $A = RG$ for some group G (or else an R-algebra with involutive antiautomorphism).*
(I) To each contravariant form $<,>: M \times N \to R$ we may associate an element

$$\psi \in \hom_A(M, N^o)$$

given by $\psi(m)(n) = <m, n>$.
(II) This association defines a bijective correspondence between such forms and morphisms.
(III) If R is a field and $M = N$ satisfies the assumptions in Proposition (9.1.13) then there is a unique form up to scalars, and the form is non–singular iff the associated ψ is an isomorphism (in particular it is non–singular if $M = N$ is simple). (cf. [52, §2.7].)

Proof. (I) We first need to show that ψ defined in this way is a homomorphism of left A–modules, i.e. that $a\psi(m) = \psi(am)$. Putting aside the way A acts on it for a moment we have $N^o = N^* = \mathrm{Hom}_R(N, R)$, so $\psi(m) \in \mathrm{Hom}_R(N, R)$. By construction we have that $\psi(am) \in \mathrm{Hom}_R(N, R)$ is given by:

$$\psi(am)(n) \; = < am, n > = < m, a^t n > = \; \psi(m)(a^t n)$$

Meanwhile for $a\psi(m)(n)$ the action of a on the left is achieved by the action of a^t on the right of N^*, which we recall is given by $(\phi a^t)(n) = \phi(a^t n)$ for any $\phi \in N^*$. Thus $(a \circ \psi(m))(n)) = (\psi(m)a^t)(n) = \psi(m)(a^t n)$ as required.

(II) Note that for given $\psi \in \hom_A(M, N^o)$ we can define a form by $<m, n>_\psi \; = \; \psi(m)(n)$.

(III) Finally observe (cf. proposition 7.5.12, noting that the difference between N^o and N^* is not relevant, since the algebra action will not be used) that the rank of the image under ψ is rank $<,>$. □

9.1.6 Examples: contravariant forms

ex:S_3 form (**9.1.27**) Example. The symmetric group S_3 acts on the set of sequences $T = \{211, 121, 112\}$ by place permutation:

$$g_1 211 = 121, \qquad\qquad g_1 112 = 112$$

and so on (g_i denotes elementary transposition $(i\; i{+}1) \in S_n$). This is a left-action:

$$g_2(g_1 211) = g_2 121 = 112 = (g_2 g_1) 211$$

These sequences thus form a basis for a left $\mathbb{Z}S_3$-module, $M = \mathbb{Z}T$. (Or similarly over any ground ring.)

The bilinear form on M given on T by

$$< t, t' > \;=\; \delta_{t,t'}$$

is contravariant. We can see this as follows. If $gt = t'$ then $g^{-1}t' = t$ so $< gt, t' > = < t, g^{-1}t' >$.

Note that the rank of a form depends, in general, on the ground field. However in our case there is clearly no such dependence. Since this form is of full rank it defines an isomorphism between M and M^o. (Of course M does not satisfy the conditions of proposition 9.1.13.)

(**9.1.28**) We can restrict our form above to a form on a submodule S. For example, consider the element of $\mathbb{Z}S_3$ given by $V_{13} = 1 - (13)$ and the submodule of M generated by

$$e_{112} \;:=\; V_{13} 112 \;= 112 - 211$$

This is spanned by e_{112} and $e_{121} = 121 - 211$. (This submodule will turn out to satisfy the conditions of proposition 9.1.13.) The restricted form has Gram matrix

$$\begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |e_{112}\rangle, & |e_{121}\rangle \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(in the obvious notation), for which

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad (9.5)$$

eq:unimexa

is an equivalent, giving Gram determinant 3. So the Gram matrix has full rank over $\mathbb{Q}$, but not so over certain other fields. As we shall see, over a field F this submodule S has a simple head which may or may not be the whole thing, depending on F. If it is not the whole thing then the submodule is not a direct summand of the original module M (since this is contravariant self-dual).

We note that the unimodular transformations applied in (9.5) give

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |e_{112}\rangle, & |e_{121}\rangle \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_{112} + e_{121}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |2e_{112} - e_{121}\rangle, & |-e_{112} + e_{121}\rangle \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

(**9.1.29**) Fixing a field F, let S' be the set of vectors in M orthogonal to the above submodule S with respect to the original form ($112 + 121 + 211 \in S'$ for example, over any field). Note that S'

is another submodule. Note that, depending on F, this new submodule is either non-intersecting of S, in which case the restricted form is of full rank, or intersects S in a submodule of dimension given by the discrepancy between the full rank and the actual rank of the form on S. (In our case $S \ni 2e_{112} - e_{121} = 2.112 - 121 - 211 \equiv -(112 + 121 + 211)$ over a field of characteristic 3, so the new module, over such a field, intersects S.) We will see that the quotient of S by $S \cap S'$ is simple (indeed absolutely irreducible). This is because of the following key result.
Evidently $V_{13}121 = 0$, so for any $m \in M$ we have $V_{13}m \in Fe_{112}$.
Suppose m is in some submodule T of M, so either $V_{13}m = 0$ or $V_{13}m \neq 0$. In the latter case $e_{112} \in T$ so $S \hookrightarrow T$. In the former case

$$0 =< V_{13}m, 112 >=< m, V_{13}112 >=< m, e_{112} >$$

so $T \hookrightarrow S'$. Now suppose in particular that T is a submodule of S in M. Then either it is the whole of S, or it is also in S', and hence in $S \cap S'$. This shows that $S/S \cap S'$ is irreducible.
This is the same as to say that its Gram matrix is non-singular over F.

(**9.1.30**) There are several further exampled later. See (10.4.1), (**??**) ...

(**9.1.31**) JOBS: FIX AND FINISH THE ABOVE!