Part IV

Partition

Chapter 15

On representations of the partition algebra

ch:pa

For each commutative ring k and $\delta \in k$ the partition algebras over k may be constructed as the end-algebras of a certain k-linear category [92]. This category is useful in representation theory, so we construct it first.

15.1 Partition categories

We can use a 'diagram calculus' to describe 'morphisms' in the partition category. We do this in §15.1.1 *et seq*, but it is convenient to start more formally. Recall the following notations from §3.2.3 (see also §1.3, §2.2).

For S a set, P_S is the set of partitions of S, and $\mathsf{P}_{n,m} := \mathsf{P}_{\underline{n}\cup\underline{m}'}$, and $\mathsf{P}_n := \mathsf{P}_{n,n}$. Also E_S is the set of equivalence relations on S (and we may apply the 'standard bijection' $\mathsf{E}_S \leftrightarrow \mathsf{P}_S$ without further comment).

For a, b equivalence relations, we define ab as the transitive closure of the relation $a \cup b$:

$$ab \; := \; \overline{a \cup b}$$

If $b \in \mathsf{P}_{l,m}$ then $b' \in \mathsf{P}_{\underline{l}'\cup\underline{m}''}$ is obtained by adding a (further) prime to every element. If $c \in \mathsf{P}_{\underline{n}\cup\underline{l}'\cup\underline{m}''}$ then $r(c) \in \mathsf{P}_{\underline{n}\cup\underline{m}''}$ is obtained by restriction. If $c \in \mathsf{P}_{\underline{l}\cup\underline{m}''}$ then $u(c) \in \mathsf{P}_{\underline{l}\cup\underline{m}'}$ is obtained by removing a prime from double-primed elements.

We then define a product $\mathsf{P}_{n,l} \times \mathsf{P}_{l,m} \to \mathsf{P}_{n,m}$ by

$$a \circ b = u(r(ab'))$$

(**15.1.1**) Theorem. *The triple* $\mathsf{P} = (\mathbb{N}_0, \mathsf{P}_{n,m}, \circ)$ *is a category.*

Proof. We will treat this as a special case of the following Theorem 15.1.4. □

(**15.1.2**) If $p \in \mathsf{P}_S$ and T a set, then $\#^T(p)$ denotes the (cardinal) number of parts of p which contain only elements of T.

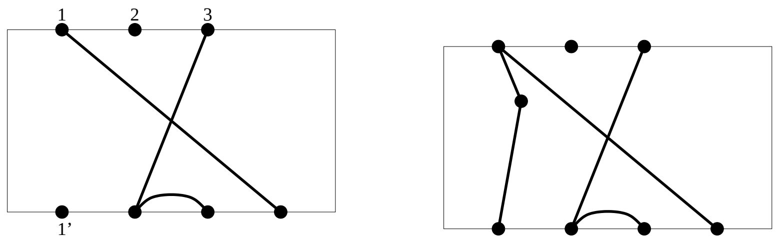


Figure 15.1: Partition diagram examples fig:part diag gr

(**15.1.3**) Now consider a categorical triple $\mathsf{P}^o = (\mathbb{N}_0, \mathsf{P}_{n,m} \times \mathbb{N}_0, *)$ where $*$ is defined as follows. Consider $a * b$ for $a = (a_1, a_2)$, $b = (b_1, b_2)$, with $a_1 \in \mathsf{P}_{n,l}$ and $b_1 \in \mathsf{P}_{l,m}$. Then

$$(a_1, a_2) * (b_1, b_2) \;=\; (a_1 \circ b_1,\; a_2 + b_2 + \#^{\underline{l}'}(a_1 b_1'))$$

We call the $\#(-)$ in this setting the *vacuum number*.

th:p cat (**15.1.4**) Theorem. *The triple* P^o *is a category.*

Proof. The main issue is associativity. Diagrams provide a convenient way to give the argument, so we discuss these next. (For a direct approach see [91].) We complete the proof in (15.1.7).

15.1.1 Partition diagrams and proof of theorem

ss:pdiagramat

(**15.1.5**) A shorthand for elements of $\mathsf{P}_{n,m}$ is as follows. Draw a rectangular frame, and draw n vertices on the northern edge (labeled $1, 2, ..., n$ in the natural order) and m vertices on the southern edge. Any graph on these vertices, with edges drawn embedded in the rectangle (as in figure 15.1), describes a partition: two vertices are in the same part if there is an edge (or chain of edges) between them.

Note that the map from the set of such graphs to the set of partitions of the vertices is surjective. Of course it is not injective in general. The graph, G say, maps in the obvious way to the set $b(G)$ of symmetric binary relations on its vertices (there is a pair $(x, y) \in b(G)$ iff there is a direct edge (x, y) in G); and one then forms the reflexive, transitive closure of $b(G)$.

If d is such an embedded graph we shall write $[d]$ for the corresponding partition.

Indeed a graph G on any collection of vertices together with a 'structure' map λ_G from $\underline{n} \cup \underline{m}'$ to the vertex set of G defines an element of $\mathsf{P}_{n,m}$ similarly (see e.g. [100]). We have in mind something slightly less general however —

Consider the frame construction as above with (possibly) some extra vertices in the interior of the rectangle. It will be convenient to *exclude* graphs with interior components disconnected from the exterior components as representatives of partitions, although $[d]$ is still well-defined for such graphs.

(**15.1.6**) In P^o, the composition $a * b$, for the object triple (n, l, m) say, can then be envisaged as follows. Consider a d such that $[d] = a$ and a d' such that $[d'] = b$. We call these *juxtaposable* if they can be juxtaposed in such a way that the l vertices on the l-vertex edges meet up (and become 'interior' vertices of the resultant graph). It will be evident that juxtaposable representatives exist, so take in particular any such pair d, d'. Now form their juxtaposition. Call this new graph dd'.

Note that dd' has three primary rows of vertices (together with any interior vertices of d and d'): a top row of n; a 'middle' interior row of l; and a bottom row of m. It thus defines, in particular, a partition $p_-(dd')$ of $\underline{n} \sqcup \underline{l} \sqcup \underline{m}$, and hence in a natural way an element $p(dd')$ of $\mathsf{P}_{\underline{n} \cup \underline{l}' \cup \underline{m}''}$. Specifically

$$p_{n,l,m}(dd') = ab'$$

To see this, pick an x in $\underline{n} \cup \underline{l}' \cup \underline{m}''$ and note that $x \sim^{ab'} y$ if (and only if) there is a path from x to y in dd'.

Comparing the constructions for $[-]$ and dd' with the definition of $\circ$ we see (1) that

$$[dd'] = [d] \circ [d']$$

and (2) that there may be some components of the graph disconnected from the top and bottom edges; and the number of these is the vacuum number.

`pf:p cat` (**15.1.7**) *proof of Theorem 15.1.4:* Note that

$$(dd')d'' = d(d'd'')$$

for any suitably juxtaposable triple of graphs. Now note that the computation of $a * (b * c)$ and of $(a * b) * c$ can be done by drawing the same diagram $dd'd''$ (where $[d''] = c$). This verifies associativity of $*$.

□

15.1.2 More partition categories

(**15.1.8**) Let k be a commutative ring. We may formally extend P^o to a k-linear category. For each $\delta \in k$ the relation

$$(a_1, b_1 + 1) \ \sim \ \delta(a_1, b_1)$$

defines a congruence on the k-linear category, and hence a quotient category P^δ_k.

$$\mathsf{P}^\delta_k = (\mathbb{N}_0, k\mathsf{P}_{n,m}, \cdot)$$

Note that the end-sets, $\hom_{\mathsf{P}^\delta_k}(n, n)$, in the category P^δ_k are k-algebras. For any given k, the n-th case has basis $\mathsf{P}_{n,n}$ and is the partition algebra $P_n = P_n(\delta)$ over k.

(**15.1.9**) In our construction we forced the top and bottom vertex sets of $P_n(\delta)$ to be disjoint. However this is not necessary. We define algebra $P_{n+}(\delta)$ similarly, except that the basis is the subset of elements of $\mathsf{P}_{n+1,n+1}$ is which $n+1$ and $n+1'$ are identified (or equivalently are always in the same part). Thus P_{n+} has a basis of partitions of a set of $2n+1$ objects.

(**15.1.10**) There are a large number of other interesting generalisations and subcategories of the partition category. For example, the product closes on the span of partitions of at most two parts. Mazorchuk calls the corresponding algebra the *rook Brauer algebra*, so have the 'rook Brauer category'. We shall write $RB_{n,m}$ for the corresponding subset of $\mathsf{P}_{n,m}$, and so on.

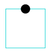
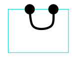
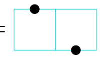
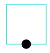
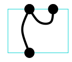
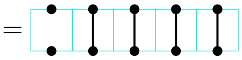
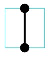
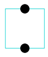
$\mathsf{v} = \{\{1\}\} =$ 	$\mathsf{U} = \{\{1,2\}\} =$ 	$\mathsf{u} = \mathsf{v} \otimes \mathsf{v}^\star =$ 
$\mathsf{v}^\star = \{\{1'\}\} =$ 	$\Gamma = \{\{1,2,1'\}\} =$ 	$\mathsf{u}_1 := \mathsf{u} \otimes 1 \otimes 1 \otimes \ldots \otimes 1 =$ 
$1 = \{\{1,1'\}\} =$ 	$\sigma = \{\{1,2'\},\{2,1'\}\}$	$\mathsf{u}_2 := 1 \otimes \mathsf{u} \otimes 1 \otimes \ldots \otimes 1$
$\mathsf{u} = \{\{1\},\{1'\}\} =$ 	$\square = \{\{1,2,1',2'\}\}$	$\mathsf{e} := \mathsf{U} \otimes \mathsf{U}^\star$

Table 15.1: Set partitions: examples and notations tab:part1-15

15.1.3 Examples and useful notation for set partitions

This section is reproduced from §1.3.

dedsidelbp-15 (**15.1.11**) See Table 15.1 for examples and notations.

Given a partition p of some subset of $N(n,m) = \underline{n} \cup \underline{m}'$, take $p^\star$ to be the image under toggling the prime.

de:pa tensor (**15.1.12**) Define partition $p_1 \otimes p_2$ by side-by-side concatenation of diagrams (and hence renumbering the p_2 factor as appropriate). See Table 15.1 for examples.

de:pnotations-15 (**15.1.13**) We say a part in $p \in \mathsf{P}_{n,m}$ is *propagating* if it contains both primed and unprimed elements. Write $\mathsf{P}_{n,l,m}$ for the subset of $\mathsf{P}_{n,m}$ with l propagating parts; and $\mathsf{P}^l_{n,m}$ for the subset of $\mathsf{P}_{n,m}$ with at most l propagating parts. Thus

$$\mathsf{P}^l_{n,m} = \bigsqcup_{l=0}^{l} \mathsf{P}_{n,l,m} \qquad \text{and} \qquad \mathsf{P}_{n,m} = \bigsqcup_{l=0}^{n} \mathsf{P}_{n,l,m}.$$

E.g. $\mathsf{P}_{2,2,2} = \{1 \otimes 1, \sigma\}$, $\mathsf{P}_{2,1,1} = \{\mathsf{v} \otimes 1, 1 \otimes \mathsf{v}, \Gamma\}$, $\mathsf{P}_{2,0,0} = \{\mathsf{v} \otimes \mathsf{v}, \mathsf{U}\}$ and

$$\mathsf{P}_{2,1,2} = \mathsf{P}_{2,1,1}\mathsf{P}_{1,1,2} = \{\mathsf{u} \otimes 1, 1 \otimes \mathsf{u}, \mathsf{v} \otimes 1 \otimes \mathsf{v}^\star, \mathsf{v}^\star \otimes 1 \otimes \mathsf{v}, \Gamma\Gamma^\star, \ldots\}.$$

Note that $\mathsf{P}_{n,n,n}$ spans a multiplicative subgroup:

$$\mathsf{P}_{n,n,n} \cong S_n \tag{15.1}$$ eq:PnSnsub-15

(**15.1.14**) Define $L : \mathsf{P}_{n,l,m} \to S_l$ by deleting all but the (top and bottom row) leftmost elements in each propagating part, and renumbering consecutively. Define

$$\mathsf{P}^L_{n,l,m} = \{p \in \mathsf{P}_{n,l,m} \mid L(p) = 1 \in S_l\}$$

(**15.1.15**) We have $P_0 \cong k$, $P_1 = k\{1, \mathsf{u}\}$ and

$$P_2 = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \mathsf{P}_{2,0,2}) = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \{\mathsf{U} \otimes \mathsf{U}^\star, (\mathsf{v} \otimes \mathsf{v}) \otimes \mathsf{U}^\star, (\mathsf{v} \otimes \mathsf{v})^\star \otimes \mathsf{U}, \mathsf{u} \otimes \mathsf{u}\}).$$

We have $\mathsf{u}^2 = \delta\mathsf{u}$ and $\mathsf{v}^\star\mathsf{v} = \delta\emptyset$ and $\mathsf{v}\mathsf{v}^\star = \mathsf{u}$.

15.2 Properties of partition categories

ss:partition9

(**15.2.1**) We make

$$\mathsf{P} := \cup_{n,m} \mathsf{P}_{n,m}$$

a monoid $(\mathsf{P}, \otimes)$ by lateral composition (as in (15.1.12)).

(**15.2.2**) Note that there is a unique element in $\mathsf{P}_{1,0}$. Let us define v, v^* as the unique elements of $\mathsf{P}_{1,0}$ and $\mathsf{P}_{0,1}$ respectively. Write

$$\mathsf{u} := \mathsf{v} \otimes \mathsf{v}^* \quad \in \mathsf{P}_{1,1}.$$

(**15.2.3**) Write 1_n for the identity element in P_n. If we write $\mathsf{u} \in \mathsf{P}_{n,n}$ we shall mean $\mathsf{u} \otimes 1_{n-1}$. We shall extend this notation in the obvious way to other elements.

(**15.2.4**) There is a sequence of unital algebra injections

$$P_n \subset P_{n+} \subset P_{n+1}$$

— the first is by $p \mapsto p \otimes 1_1$; the second is inclusion.

de:hashp

(**15.2.5**) Define

$$\# : \mathsf{P}_{n,m} \to \mathbb{N}_0$$

by $p \mapsto \#$ propagating parts (as in (15.1.13)).

15.2.1 Initial filtration: Propagating ideals

(**15.2.6**) Define $\mathsf{P}^l_{n,m} := \mathsf{P}_{n,l} \circ \mathsf{P}_{l,m} \quad \subset \mathsf{P}_{n,m}$ as in (15.1.13). Note that this is the subset of partitions with at most l propagating parts. Define

$$\mathsf{P}^{=l}_{n,m} := \mathsf{P}_{n,l,m} = \mathsf{P}^l_{n,m} \setminus \mathsf{P}^{l-1}_{n,m}$$

Automatically then, we have the following.

de:Pfilt1

(**15.2.7**) Proposition. For any k, $\delta \in k$,

$$k\mathsf{P}_{n,n} \supset k\mathsf{P}^{n-1}_{n,n} \supset k\mathsf{P}^{n-2}_{n,n} \supset ... \supset k\mathsf{P}^0_{n,n} \tag{15.2}$$

eq:Ppure1

is a chain of two-sided ideals in P_n. The l-th ideal, $k\mathsf{P}^{n-l}_{n,n}$, is generated by $\mathsf{u}^{\otimes l} \otimes 1_{n-l}$:

$$k\mathsf{P}^{n-l}_{n,n} = P_n(\mathsf{u}^{\otimes l} \otimes 1_{n-l})\, P_n$$

or indeed by any partition with $n-l$ propagating parts.
The l-th section, counting from the *right*, has basis $\mathsf{P}^{=l}_{n,n}$. Let us write $\mathsf{P}^{l/}_{n,n}$ for this section. □

(**15.2.8**) Write

$$P^{l+1/}_n = P^{/l}_n := P_n / k\mathsf{P}^l_{n,n}$$

for the quotient algebra. (Note the significance of position of the slash.)

pr:Pbimnm

(**15.2.9**) Proposition. A bimodule $\mathsf{P}^{l/}_{n,m}$ may be defined similarly to $\mathsf{P}^{l/}_{n,n}$.

Proof. This is a section in an analogous sequence to (15.2). □

(**15.2.10**) In particular $\mathsf{P}^{l/}_{n,l}$ has the nice property that its basis $\mathsf{P}^{=l}_{n,l}$ consists of all diagrams in $\mathsf{P}^{l}_{n,l}$ such that each vertex on the bottom edge is in a distinct propagating part.

Note that $(\mathsf{P}^{=n}_{n,n}, \circ)$ gives a copy of the symmetric group S_n.

lem:flipinvheck (**15.2.11**) Consider $\mathsf{P}^{=l}_{l,n} \circ \mathsf{P}^{=l}_{n,l}$ for a moment. Evidently some of these diagrams lie in $\mathsf{P}^{=l}_{l,l}$ and some in $\mathsf{P}^{<l}_{l,l}$. Note however the following.

LEMMA. The particular products of form $p \circ p^\star$ in $\mathsf{P}^{=l}_{l,n} \circ \mathsf{P}^{=l}_{n,l}$ all equal 1_l in $\mathsf{P}^{=l}_{l,l}$.

Proof. The propagating parts in p take the form $\{i, j_1', j_2', ...\}$ for $i = 1, 2, ..., l$; and for each such part there is a corresponding part $\{j_1, j_2, ..., i'\}$ in $p^\star$. Thus the composition gives a part $\{i, i'\}$ for each i. □

(**15.2.12**) Note that there is a unique element in $\mathsf{P}^{=1}_{1,1}$, denoted 1_1. There are two elements in $\mathsf{P}^{=2}_{2,2}$, one of which is $1_2 := 1_1 \otimes 1_1$. Write (12) for the other element.

(**15.2.13**) We may consider the parts of $p \in \mathsf{P}_{n,m}$ that meet the top set of vertices to be totally ordered by the natural order of their lowest numbered elements (from the top set). We may define a corresponding order for parts that meet the bottom set of vertices. We say that p is *non-permuting* if the subset of propagating parts has the same order from the top and from the bottom.

For $l \leq n$ let $\mathsf{P}^{||}_{l,n}$ denote the subset of $\mathsf{P}^{=l}_{l,n}$ of non-permuting partitions (and analogously for $l \geq n$). That is, $\mathsf{P}^{||}_{l,n} = \mathsf{P}^{L}_{l,l,n}$.

de:Psect1 (**15.2.14**) Now consider the section $\mathsf{P}^{l/}_{n,n}$ as a left-module. We have the left-module isomorphism

$$\mathsf{P}^{l/}_{n,n} \;\cong\; \bigoplus_{w \in \mathsf{P}^{||}_{l,n}} \mathsf{P}^{l/}_{n,l} w$$

Every summand is isomorphic to $\mathsf{P}^{l/}_{n,l}$.

(**15.2.15**) PROPOSITION. *Regarded as a set of classes in the obvious way, $\mathsf{P}^{=l}_{n,l}$ is a basis for $\mathsf{P}^{l/}_{n,l}$.* □

15.2.2 Polar decomposition

pa:usefac3 (**15.2.16**) The set $\mathsf{P}^{=l}_{n,l}$ has a useful factorisation:

$$\mathsf{P}^{=l}_{n,l} = \mathsf{P}^{||}_{n,l} \circ \mathsf{P}^{=l}_{l,l} \tag{15.3}$$ eq:psbas

where $\mathsf{P}^{=l}_{l,l} \;\cong\; S_l$.

(**15.2.17**) Indeed for $l < n, m$

$$\mathsf{P}^{=l}_{n,m} = \; \mathsf{P}^{||}_{n,l} \circ \mathsf{P}^{=l}_{l,l} \circ \mathsf{P}^{||}_{l,m} \tag{15.4}$$ eq:psbas2

(**15.2.18**) In fact (given our particular definition of non-crossing) the factorisation $p = l_p \circ c_p \circ r_p$ of $p \in \mathsf{P}_{n,m}$ as above is unique.

We call this the *polar decomposition* of p.

...

15.2.3 I-Functors

(**15.2.19**) Lemma. Consider the left P_n- right P_l bimodule $\mathsf{P}^{l/}_{n,l} = k\mathsf{P}^{=l}_{n,l}$ (Prop.15.2.9). The restriction of the P_l action to S_l makes it a P_n-kS_l-bimodule. It is free as a right kS_l-module.

Proof. Freeness follows from (15.2.16). □

(**15.2.20**) Proposition. The functor

$$I^n_l : kS_l - \text{mod} \;\rightarrow\; P_n - \text{mod}$$

$$M \;\mapsto\; \mathsf{P}^{l/}_{n,l} \otimes_{kS_l} M$$

is exact.

15.3 Partition algebra $\dot{\Delta}$ and Δ-modules

(**15.3.1**) For $\lambda \vdash l$ let $\mathcal{S}_\lambda$ denote the corresponding kS_l Specht module.

15.3.1 Δ-module constructions

(**15.3.2**) For $|\lambda| \leq n$ define a P_n-module:

$$\ddot{\Delta}_n(\lambda) \;=\; I^n_l \mathcal{S}_\lambda$$

pr:PDbasis (**15.3.3**) Proposition. *For each basis b_λ of $\mathcal{S}_\lambda$ then*

$$b_{\ddot{\Delta}_n(\lambda)} \;:=\; \{a \otimes_{kS_l} b \mid (a,b) \in \mathsf{P}^{||}_{n,l} \times b_\lambda\}$$

is a basis for $\ddot{\Delta}_n(\lambda)$.

Proof. It is easy to see that $b_{\ddot{\Delta}_n(\lambda)}$ is spanning ... □

de:PDdef-1 (**15.3.4**) For each $\lambda \vdash l$ let us choose an element $f_\lambda \in kS_l$ such that

$$\mathcal{S}_\lambda = kS_l f_\lambda$$

is the corresponding Specht module (cf. e.g. James [**?**]).

de:PDdef1 (**15.3.5**) Given f_λ as in (15.3.4), define a subset of $\mathsf{P}^{l/}_{n,l}$

$$\dot{\Delta}_n(\lambda) \;:=\; \mathsf{P}^{l/}_{n,l} f_\lambda$$

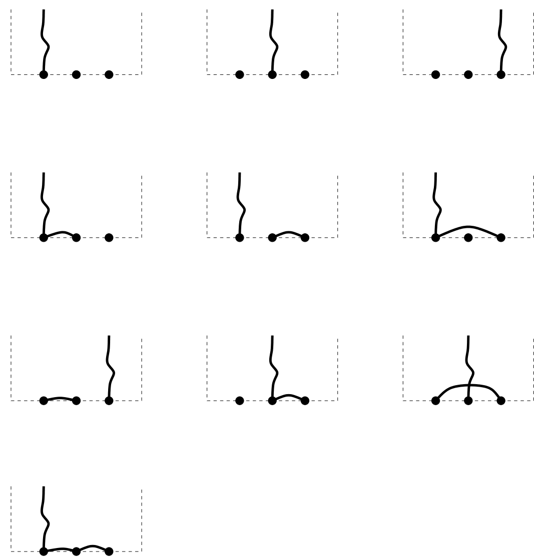
This is a sub-P_n-module of $\mathsf{P}^{l/}_{n,l}$.

(**15.3.6**) Including $f_\lambda \in kS_l$ in P_l in the obvious way allows us to draw a picture for $\dot{\Delta}_n(\lambda)$.
Exercise: CLARIFY THIS!

pr:PDbas1 (**15.3.7**) Proposition. *For each basis b_λ of $\mathcal{S}_\lambda$ there is a basis $\mathsf{P}^{||}_{n,l} \times b_\lambda$ of $\dot{\Delta}_n(\lambda)$.*

Proof. Note that the module is spanned by elements of form abf_λ where $a \in \mathsf{P}^{||}_{n,l}$ and $b \in S_l$ (consider (15.3)). □

(**15.3.8**) An example of a basis for a Δ-module is as follows. For $n = 3$, $\lambda = (1)$:



By 15.3.7 the basis elements for $\dot{\Delta}_n(\lambda)$ with $|\lambda| = l$ say, are elements of $\mathsf{P}^{||}_{n,l}$ with S_l Specht module basis elements 'attached'. A convenient way to attach is to label the legs on the l end with the appropriate Young sequences (the sequences associated to the basis of Young tableau in **??**). In our example there is only one trivial Young sequence and we omit it.

Suppose we fix a basis b_λ of the Specht module and associate its elements to Young sequences. Then write $\mathsf{P}^{b_\lambda}_n$ for the corresponding basis of $\dot{\Delta}_n(\lambda)$.

(**15.3.9**) Proposition.

$$\dot{\Delta}(\lambda) \cong \ddot{\Delta}_n(\lambda)$$

Proof. Consider the bases in (15.3.3) and (**??**). □

15.3.2 Δ-filtration

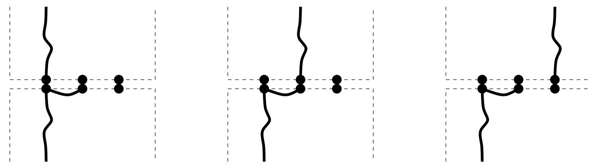
(**15.3.10**) Note from (15.2.7), (15.2.14) and (15.3.5) that if k is such that kS_l has a Specht filtration (e.g. if $k \supseteq \mathbb{Q}$) then the regular P_n module has a filtration by Δ-modules, and indeed a filtration in which all the modules labelled with partitions of a given degree are consecutive. Indeed, if $k = \mathbb{C}$ (or at least contains $\mathbb{Q}$ so that kS_l is semisimple) they do not extend each other (so can be arranged, among themselves, in any order in the filtration).

15.3.3 On simple modules, labelling and Brauer reciprocity

For now this section is here to match the B_n one: §17.3.4.

15.3.4 Contravariant form on $\Delta_n(\lambda)$

(**15.3.11**) If $k \supseteq \mathbb{Q}$ then each of these Δ-modules $\Delta_n(\lambda)$ can be considered as a primitive-idempotently generated left-ideal in a quotient algebra. Thus by (**??**) there is a contravariant form defined on these modules. The form (a, b) is computed by juxtaposing b with $a^\star$. For example



These three yield δ, δ, and 0 respectively.

In general there is some choice in the form coming from the choice of basis/form on the $S_{|\lambda|}$ Specht module associated to λ. By default we use the form described in §10.1.7 and in (11.4.5) *et seq*, and the diagonal version as in (10.7).

(**15.3.12**) Examples: Fig.15.2 and Fig.15.3. Note that a concrete form comes in Fig.15.3 on fixing a form for the S_3 Specht module part, such as that in (11.4.5) *et seq.*

(**15.3.13**) In studying the form it will be convenient to introduce some notation:

Let b^o denote the set of isolated (non-propagating) connected parts in $b \in \mathsf{P}_n^{b_\lambda}$. Write $a^o \leq b^o$ if a^o, b^o have the same underlying set and a^o is a refinement of b^o, that is if every edge in a^o (in the obvious sense) is also in b^o.
We also define a partial order on the basis by $a < b$ if $|a^o| < |b^o|$.

We arrange the basis $\mathsf{P}_n^{b_\lambda}$ into blocks in which the diagram part is fixed and only the b_λ part is varying.

(**15.3.14**) Lemma. Note that the blocks on the block diagonal of the gram matrix are then diagonal if the seed basis b_λ can be chosen orthogonal (since the flip acts as inversion on the 'symmetric subgroup' — cf. Lem.15.2.11). □

(**15.3.15**) For given b_λ define $G_n(\lambda)$ as the gram matrix of the form with respect to the basis $\mathsf{P}_n^{b_\lambda}$, totally ordered in a way consistent with the above. That is:

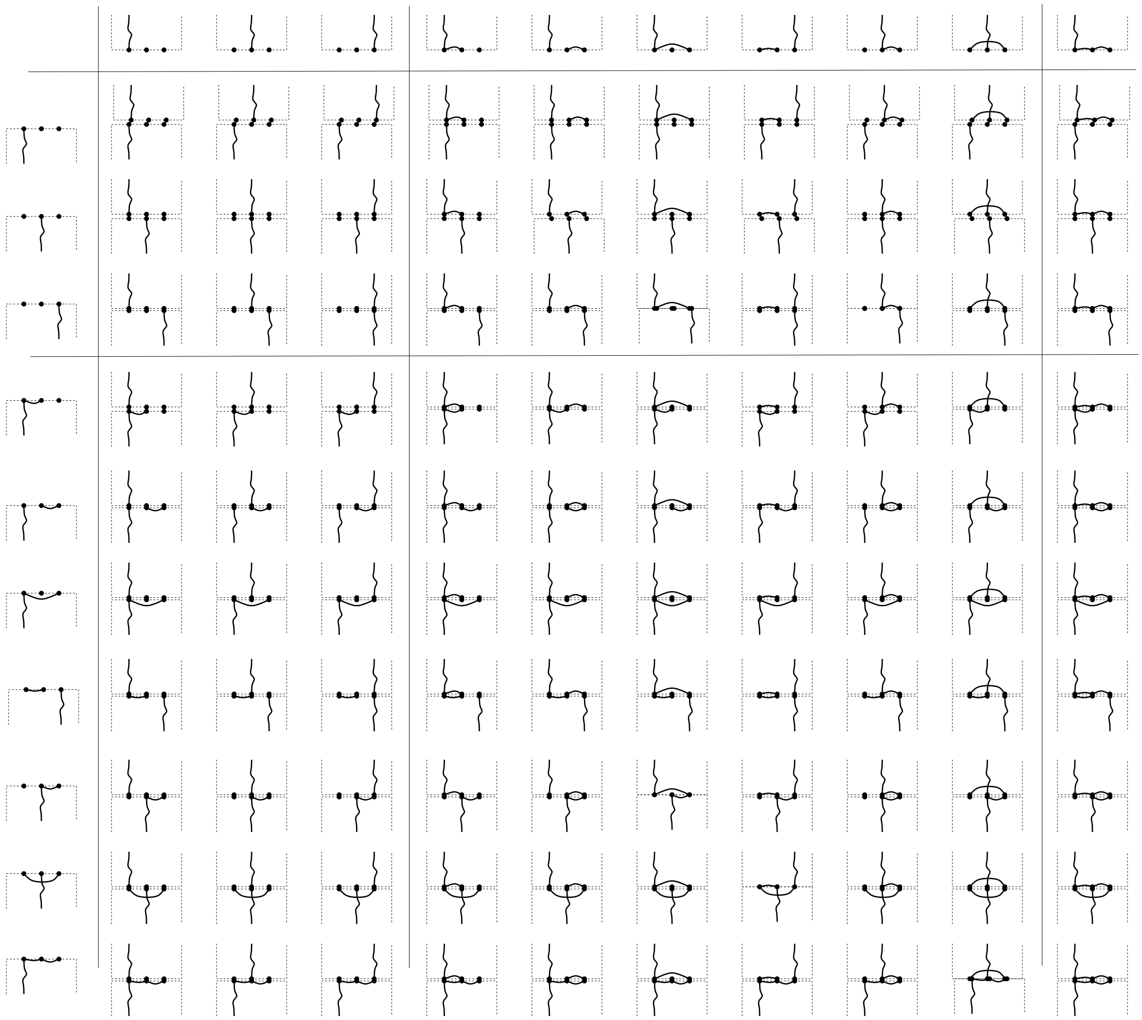
$$G_n(\lambda) \;=\; ((a, b))_{a,b \in \mathsf{P}_n^{b_\lambda}}$$

(see e.g. Fig.15.2 and 15.3).

(**15.3.16**) Proposition. This form is non-degenerate for generic δ (i.e. over a ring $k[\delta]$ where k is a field of char.0).

Proof. Remark: This is Prop.12 in [91] (where the outline proof is, however, somewhat terse). See also [99] and [**?**] for example.

To verify the proposition now consider the following points.
(1) Each product of basis elements (a, b) gives either (i) some power of δ together with a rank-l permutation diagram (such as the identity diagram — see Fig.15.3 for examples) with Young sequences at both ends or (ii) 0 by the quotient.
(2) The maximum power of δ in the row $(-, b)$ is realised by (b, b). Then the power here is $\delta^{|b^o|}$.



$$G_3((1)) \;=\; \left(\begin{array}{ccc|cccccc|c}
\delta^2 & 0 & 0 & \delta & \delta & \delta & 0 & 0 & 0 & 1 \\
0 & \delta^2 & 0 & \delta & 0 & 0 & 0 & \delta & \delta & 1 \\
0 & 0 & \delta^2 & 0 & 0 & \delta & \delta & \delta & 0 & 1 \\
\hline
\delta & \delta & 0 & \delta & 1 & 1 & 0 & 1 & 1 & 1 \\
\delta & 0 & 0 & 1 & \delta & 1 & 1 & 0 & 1 & 1 \\
\delta & 0 & \delta & 1 & 1 & \delta & 1 & 1 & 0 & 1 \\
0 & 0 & \delta & 0 & 1 & 1 & \delta & 1 & 1 & 1 \\
0 & \delta & \delta & 1 & 0 & 1 & 1 & \delta & 1 & 1 \\
0 & \delta & 0 & 1 & 1 & 0 & 1 & 1 & \delta & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

Figure 15.2: Example gram matrix calculation for $\Delta_3((1))$. `fig:pb31xl`

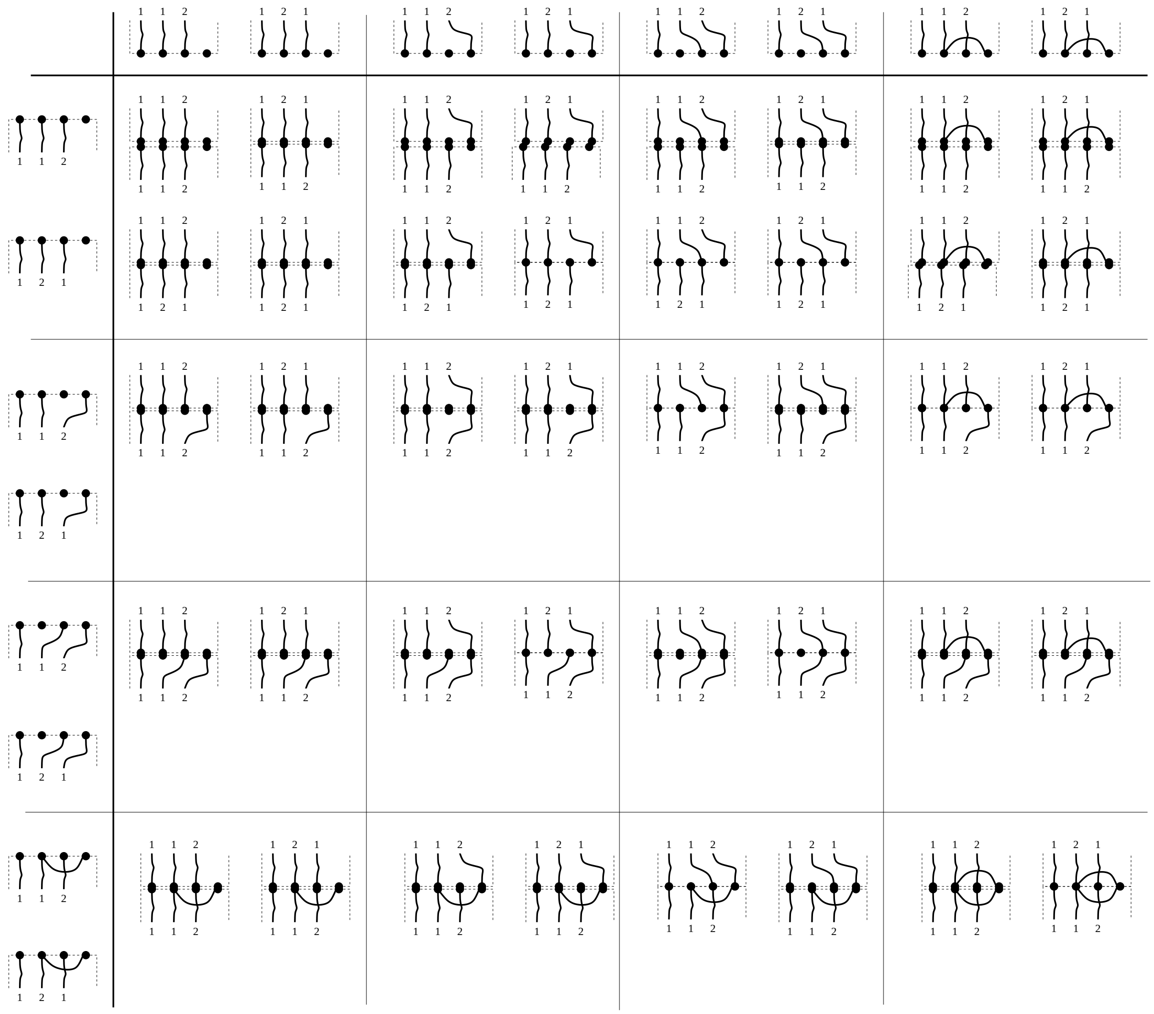


Figure 15.3: Partial gram matrix for $\Delta_4((2,1))$ with entries blocked according to the symmetric group $(2,1)$ part. Note that each block-diagonal block is a power of δ times the corresponding symmetric group Specht gram matrix. However off-diagonal blocks do not necessarily take this form (see the figure for both zero and nonzero examples).

fig:pgram43

Fixing b, this power is also realised in general by other (a,b), but a necessary condition is that (i) the underlying sets of a^o and b^o agree; and (ii) $a^o \leq b^o$.

To see (i) note that otherwise a propagating line meets b^o in composition and the number of isolated connected components goes down.

To then see (ii) note that if $a^o \not\leq b^o$ then by definition there is an edge in a^o not in b^o — but then the number of connected components in the composite goes down.

(3) Next we claim that if the underlying sets of a^o and b^o agree then composition is 0 by the quotient unless the propagating parts all agree (i.e. unless we are on the block diagonal).

To see this note that if the propagating parts do not all agree then an edge in a propagating part in either a or b is not present in the other, in which case it combines two propagating parts in composition.

(4) Since k is char.0 we may take the basis b_λ to be orthogonal and normalizable.

Note that the product of diagonal terms at the top of the partial order strictly dominates any other Leibniz product from these rows. On discarding all these rows and corresponding columns, the product of diagonal terms strictly dominates any other Leibniz product from the rows in the next layer in the partial order. And so on. The proposition now follows from the Laplace expansion. □

15.4 Globalisation functors

de:Boopp

(**15.4.1**) Let $\square_1 \in \mathsf{P}_n$ be $\square \otimes 1 \otimes 1 \otimes ...$, and define $\square_i$ analogously. Note the following identities: $\mathsf{u}_1\square_1\mathsf{u}_1 = \mathsf{u}_1$, $\square_1\mathsf{u}_1\square_1 = \square_1$.

Define $y = \mathsf{v}^\star\Gamma$, so $y_1 = \square_1\mathsf{u}_1$. We have

$$y_1 y_1 = y_1$$

Note that $\square_1, \mathsf{u}_1$ and y_1 all have $n-1$ propagating lines. Note that the ideal generated by any $z \in \mathsf{P}_{n,l,n}$ contains $\mathsf{P}^l_{n,m}$.

de:Poopp

(**15.4.2**) Note that provided δ is invertible then the vector space isomorphism

$$P_n \;\cong\; \mathsf{u}_1 P_{n+1} \mathsf{u}_1$$

given by $p \mapsto \mathsf{u} \otimes p$, is an algebra isomorphism.

In case δ invertible then, the functor

$$G_{\mathsf{u}} \;:=\; P_{n+1}\mathsf{u}_1 \otimes_{P_n} - \tag{15.5}$$

eq:Gu1

fully embeds P_n-mod in P_{n+1}-mod. (We will give examples of applications in (15.4.16).)

By the same token, for $n > 0$, $\mathsf{P}_n \cong \mathsf{u}_1\square_1\mathsf{P}_{n+1}\mathsf{u}_1\square_1$ (given by $p \mapsto (\mathsf{u} \otimes p)\square_1$) and

$$P_n \cong y_1 P_{n+1} y_1$$

(given by $p \mapsto \square_1(\mathsf{u} \otimes p)$) are algebra isomorphisms for any δ.

15.4.1 Long G functors and fair categories: $\otimes_{C_n}$ versus category composition

Look at [?] (or maybe Green—Martin's construction [?]) for 'propagating' object orders on k-linear categories.

We are interested in partition categories as giving examples of modular towers (??) [53]. In particular we are interested in the effect on representations (particularly Δ-modules) of various kinds of inflation. Given a k-linear category C, the basic inflation in C is

$$C(x,y) \otimes_{C_y} - \;:\; C_y - \mathrm{mod} \;\; \to \;\; C_x - \mathrm{mod}$$

This raises the question of what we can say about $C(x,y)$ as a C_y-module. For example, When is it projective? (Note that this construction is defined for arbitrary k, so these modules have 'positive Δ-characters'.)

Recall the notation $C^y(x,z) := C(x,y)C(y,z) \subseteq C(x,z)$. (These $C^y(x,z)$ are also bimodules, so we can investigate them as C_z-modules too.)

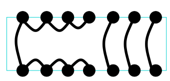
de:kgood (**15.4.3**) We say a k-linear category C is *k-good* at object x if $C(x,y)$ is projective as a left $C(x,x)$-module for all y.

(**15.4.4**) We say k-linear category C is *fair* if $C(x,y) \otimes_{C(y,y)} C(y,z) \cong C(x,z)$ whenever $C(x,y)C(y,z) = C(x,z)$ (in the category composition).

More generally we could ask, when is $C(x,y) \otimes_{C(y,y)} C(y,z) \cong C^y(x,z)$?

Partition category case

(**15.4.5**) Next we consider the example of the partition categories. Define partitions

H_m =  $= \{\{1,2,...,m,1',2',...,m'\}\} \otimes 1_{n-m} \quad \in P_n$

$$J_m \;=\; \{\{1\},\{1'\},\{2\},\{2'\},...,\{m\},\{m'\}\} \otimes 1_{n-m}$$

Note that H_m is an idempotent in P_n.

le:projhom (**15.4.6**) Lemma. *For $l > 0$ the map $P(m+l,l) \cong P(m+l,m+l)H_{m+1}$ given by*

$$p \mapsto ((\mathsf{v}^\star)^{\otimes m} \otimes p)H_{m+1}$$

is an isomorphism of left P_{m+l}-modules.

Proof. Firstly this is a set isomorphism. The inverse is to 'forget' the first m primed vertices, which are all connected and connected to $(m+1)'$ in every basis element on the RHS. From this it will also be clear that these vertices play no active role in the algebra action from the left, leaving the action identical on each side. □

(**15.4.7**) ("We see from the above that P is never k-good, and this is not the definition that we are looking for!!!" — Why?) What about fairness?

le:factorset (**15.4.8**) Lemma. *For category P with any k we have $P(n,m) = P(n,l)P(l,m)$ so long as $l \geq m$ or $l \geq n$.* □

le:tensorvcat (**15.4.9**) LEMMA. *Suppose $n \geq m \geq l$ and consider $P(n,m) \otimes_{P_m} P(m,l)$. The multiplication in the category P (as in (15.4.8)) provides a multiplication map $\mu : a \otimes b \mapsto ab$, that defines a bimodule isomorphism*

$$P(n,m) \otimes_{P_m} P(m,l) \xrightarrow{\sim} P(n,l)$$

unless δ is a non-unit, $l = 0$ and $m = 1$.

Proof. Recall that multiplication is balanced (8.4.4), so μ is well-defined by the universal property of tensor products (8.9.2). It is surjective by (15.4.8). We need to show that it does not have a kernel.

By (15.4.6) we have that ${}_{P_m}P(m,l)$ is projective for $l > 0$ (if $\delta \neq 0$ then $l = 0$ may be included similarly, using J_m as (pre)idempotent in Lemma (15.4.6)). Now use the property of the multiplication map (8.9.2).

For $\delta = 0$ and $l = 0$ neither H_{m+1} nor the alternate idempotent are defined. The natural spanning set is $\{a \otimes b \mid a \in \mathsf{P}(n,m),\ b \in \mathsf{P}(m,0)\}$. We need to show that we end up with a basis in bijection with $\mathsf{P}_{n,0}$.

Write ω_m for the all-singleton partition in $\mathsf{P}(m,0)$. In $\mathsf{P}(m,m)$ write h^i_m for the partition with part $\{i, 1', 2', ..., m'\}$ and other parts singletons. Note

$$\omega_m = h^i_m \omega_m$$

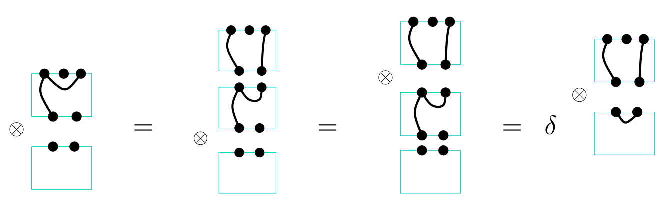
One can see that a smaller spanning set is elements of form $a \otimes \omega_m$, where a has at least one propagating part. Next note that any such $a \otimes \omega_m$ equals $a' \otimes \omega_m$ where a' has all but one propagating part 'cut': consider $a \otimes \omega_m = a \otimes h^i_m \omega_m = a h^i_m \otimes \omega_m$. Indeed the propagating part can be chosen to be the part containing 1. If it does not then we can first add this as a second propagating part:

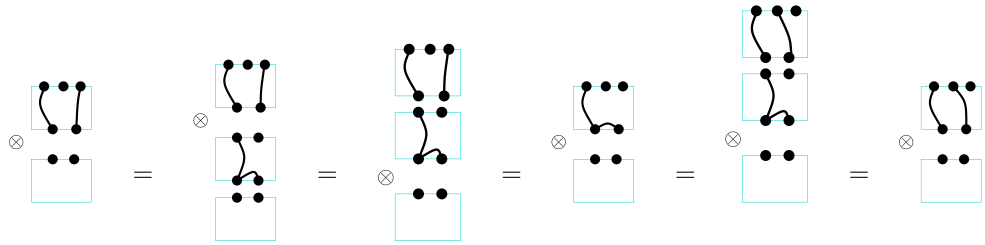
$$a \otimes \omega_m = a^x h^m_m \otimes \omega_m = a^x \otimes \omega_m$$

where a^x is obtained from a by removing all primed elements but m' from the original propagating part and adding $\{1', 2', ..., m-1'\}$ to the part containing 1. Then cut the original by applying h^1_m. NB this requires two propagating parts in the intermediate step, so $m \geq 2$.

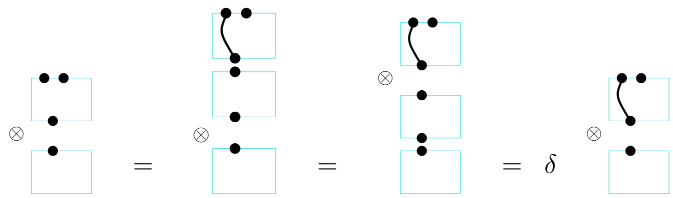
Note that we now have a spanning set in bijection with $\mathsf{P}(n,0)$ (and which passes to $\mathsf{P}(n,0)$ under μ). Specifically the elements are $a \otimes \omega_m$ where a is obtained from $a_- \in \mathsf{P}(n,0)$ by appending $\{1', 2', ..., m'\}$ to the part containing 1. $\square$

Example: Consider the case $m = 2$. Linear dependences between elements from the natural spanning set can arise, for example, like this:





To see that the case $m = 1$ must be excluded when $\delta = 0$ consider



There are 5 elements in the spanning set, and 2 in a basis for the nominal target (isomorphic to $P(2, 0)$). The above shows that we can remove the one on the left. By symmetry there is a chain from this ending in the LR image of the one on the right, so if $\delta \neq 0$ then all three of these are linearly dependent. However if $\delta = 0$ then the one on the right and its LR image cannot be made dependent in this way. The remaining two span a 1d subspace similarly to the above. Thus if $\delta = 0$ we have 3 independent elements. □

(**15.4.10**) Exercise: Generalise this to other diagram categories. Cf. e.g. (17.4.2) and (17.4.5) for the Brauer algebra case.

(**15.4.11**) Aside: Let us come back to the question of projectivity of $P(n, m)$ as a left P_n-module. Obviously every module is projective when P_n is semisimple, so we should look at non-semisimple examples (although there are plenty of these that are projective, as Lemma **??** shows).

Example: When $\delta = 0$ the left P_1-module $P(1, 2)$ is not projective over $\mathbb{C}$. To see this note that $P(1, 2)$ has dimension 5, but there is only one indecomposable projective and this has dimension 2.

Exercise: say more about ${}_{P_1}P(1, 2)$.

(**15.4.12**) CLAIM: Fix field k and $\delta \in k$. The partition category $P = \mathsf{P}_k^\delta$ is fair if $\delta \neq 0$ and almost fair otherwise.

Proof. First note Lem.15.4.8. Given the flip symmetry (and some elementary cases) it is enough to consider $n > l > m$.

...

15.4.2 G-functors and Δ-modules

We have several kinds of 'standard' modules with useful properties. We construct them and show how they are related.

(**15.4.13**) We have several functors between P_n-module categories. For example:

$$G_n^m = P(m,n) \otimes_{P_n} - : \; P_n - \mathrm{mod} \to P_m - \mathrm{mod}$$

and in particular $G_n = G_n^{n+1}$ and $F_n = F_n^{n-1}$;
and also G_{u} from (**??**).

By (**??**) we have that $G_n \cong G_{\mathsf{u}}$ provided that ...

(**15.4.14**) Consider the 'long inflation' from the $\lambda \vdash l$ Specht module $\mathcal{S}_\lambda = \dot{\Delta}_l(\lambda)$, that is:

$$\Delta_n(\lambda) \;:=\; G_l^n \Delta_l(\lambda).$$

Is it clear that this is defined integrally? What is a basis? (What are the problems with determining if a construction 'works' integrally? A construction might make sense over any base, but not be free over every base... Examples?)

(**15.4.15**) Proposition. *For $\lambda \vdash l$ and $n > l$*

$$G_l^n \Delta_l(\lambda) \cong \dot{\Delta}_n(\lambda)$$

Proof. By (**??**) we can work with $\dot{\Delta}_n(\lambda) = I_l^n \mathcal{S}_\lambda$. So we are comparing $G_l^n I_l^l -$ with $I_l^n -$ here.

We first argue that we can essentially identify $I_l^l \mathcal{S}_\lambda$ with $\mathcal{S}_\lambda = k b_\lambda$ as k-modules.

Unpacking we have $G_l^n I_l^l - = k\mathsf{P}(n,l) \otimes_{P_l} I_l^l -$. Compare the bases: For $I_l^n \mathcal{S}_\lambda$ we have the basis

$$b_\Delta = \{a \otimes_{kS_l} b \mid a \in \mathsf{P}^l(n,l); \; b \in b_\lambda\}$$

by (**??**). For $G_l^n I_l^l \mathcal{S}_\lambda$ we have the subset given by the image of the set $\mathsf{P}(n,l) \times b_\lambda$ in $k\mathsf{P}(n,l) \otimes_{P_l} I_l^l \mathcal{S}_\lambda$. It will be clear that this subset is spanning.

If $l = 0$ this set is a basis and is in natural bijection with b_Δ.

More generally, the set $\mathsf{P}(n,l) \times b_\lambda = \mathsf{P}^{=l}(n,l) \times b_\lambda \cup \mathsf{P}^{<l}(n,l) \times b_\lambda$. Let us consider the image of $\mathsf{P}^{<l}(n,l) \times b_\lambda$. By (**??**) a partition in $\mathsf{P}^{<l}(n,l)$ with $l > 0$ can be expressed in the form $du_1 d'$ where $d \in \mathsf{P}(n,l)$ and $u_1, d' \in \mathsf{P}(l,l)$.

TRUE?!

But then $du_1 d' \otimes_{P_l} b_\lambda = d \otimes_{P_l} u_1 d' b_\lambda = 0$

...

pr:GD1 (**15.4.16**) Proposition. *Provided δ invertible (or $n > 2$) we have*

$$G\Delta_n(\lambda) = \Delta_{n+1}(\lambda)$$

Proof. First recall from (**??**) that $G_l^{l+2} \cong G.G$ ...

15.5 Enumerating set partitions and diagrams

In this section we discuss enumerations of the sets $\mathsf{P}_{\underline{n}}$ of set partitions (noting that these sets are the bases of our various partition algebras). One idea, for example, is to parallel the Robinson–Schensted correspondence [**?**], regarded as an enumeration of S_n (noting that the Young graph facilitates an enumeration of Young tableau). There are various ways to do this. We describe one that is natural from a representation theory perspective.

(**15.5.1**) First note that a partition p of S can be described by giving the restriction to some subset S', denoted $p|_{S'}$; the restriction to the complement of S' in S; and the details of the connections between the parts thus described. These connection details must give the list of parts in $p|_{S'}$ that are connected; the equinumerate list of parts for the complement; and a bijection between these lists. If a canonical list order is fixed (for any such list), then the bijection may be represented by an element of the symmetric group. Via the RS correspondence this element may be represented as an ordered pair of Young tableaux. We may think of giving the first tableau to S' and the second to the complement. In this way, p is split into two 'halves'.

(**15.5.2**) A *half-partition* is an ordered partition of a set partition into two parts, called *non-propagating* and propagating respectively.

(**15.5.3**) Consider the graph G shown in figure 15.4 (figure taken from Marsh–Martin[85]).

(**15.5.4**) A vertex in G is labelled by a pair consisting of a natural number n (or $n+$) and a Young diagram. The vertex labelled (n, λ) consists in the set of ordered pairs where the first element is a half-partition of $\underline{n}$ with $|\lambda|$ propagating parts; and the second part is a Young tableau of shape λ. If the label is $(n+, \lambda)$ then one takes instead a half-partition of $\underline{n+1}$, but requires that $n+1$ itself lies in a propagating part.

(**15.5.5**) We shall use shortly the following construction (again see [85]). Let S_ν be an S_{l-1}-Specht module, with a basis of standard tableau of shape ν, and $\mathbb{C}S_l \otimes_{\mathbb{C}S_{l-1}} S_\nu$ the induced module. This has a basis of elements of form $\pi_k \otimes T$ ($k \in \{0, 1, 2, ..., l-1\}$) where, in cycle notation, $\pi_k := (l\ l-1\ ...\ l-k)$ is a coset representative for $S_{l-1} \subset S_l$ and T is a tableau from the standard tableau basis above. The induced module also has a basis of standard tableau of various shapes corresponding to its Specht content (i.e. shapes of form $\nu + e_i$). Thus one may choose a bijection between these bases. Such a bijection gives a correspondence between the subset of standard tableau of shape $\lambda = \nu + e_i$ and some subset of the elements of form $\pi_k \otimes T$.

(**15.5.6**) We now introduce maps describing how to construct each vertex set of G from those in the layer above in G. (Firstly we shall define some maps, and then later we shall show that they do the job.)

There is a map from (n, λ) to $(n+, \lambda)$ adding a propagating part $\{n+1\}$ and keeping the same tableau. (Note that no permutation of the part involving $n+1$ is possible if $n+1$ is common to both halves.)

There is a map from (n, λ) to $(n+, \lambda - e_i)$ [1] as follows. Let (h, T') be the pair to be mapped. Then T' has shape λ and our bijection in (**??**) above takes this to some element $\pi_k \otimes T$. In this case add $n+1$ to the k-th propagating part of h, and replace T' with T.

There is a map from $(n+, \lambda)$ to $(n+1, \lambda)$ that makes the part containing $n+1$ non-propagating, and leaves the tableau unchanged.

[1] WHICH IS FIDDLY — CHECK IT!

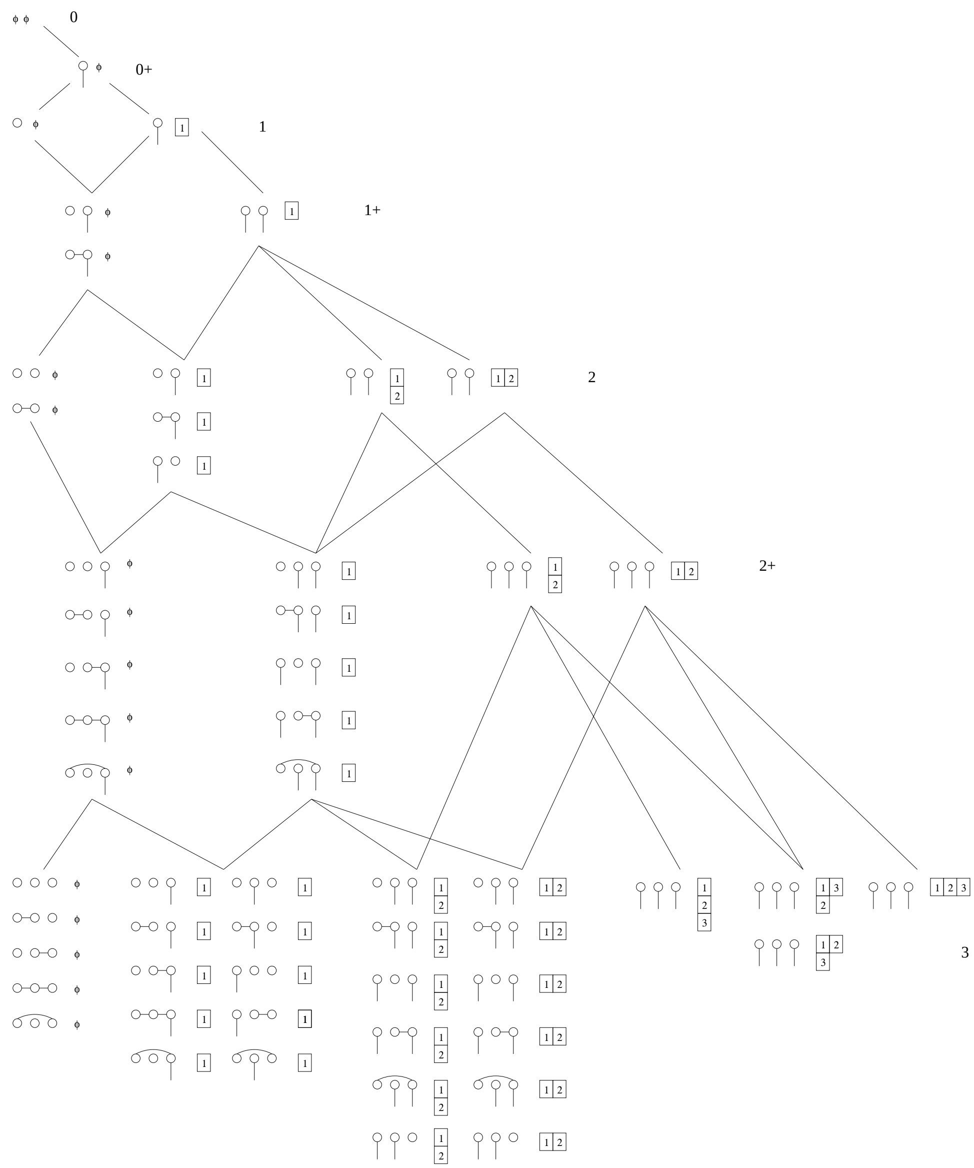


Figure 15.4: The array of 'half-partitions'. fig:cat set part

There is a map from $(n+, \lambda)$ to $(n+1, \lambda + e_i)$ that leaves the half-partition unchanged and inserts $|\lambda| + 1$ into the new box in the Young diagram, leaving the rest unchanged.

(**15.5.7**) Theorem. *This is everything.*

Proof. ...

15.6 Representation theory

This section is based on [92]

15.7 Representation theory via Schur algebras

This section is designed as a companion to Martin–Woodcock [100]. This is an approach to partition algebra representation theory using generalised Schur algebras, motivated by certain n-stability properties of tensor product rules for symmetric group representations. We start by recalling some notations used in [100].

15.7.1 Local notations

Here Λ is the set of all compositions:

$$\Lambda \;=\; \{\lambda : \mathbb{N} \to \mathbb{N}_0 \mid supp(\lambda)\ finite\}$$

$$\Lambda_0 \;=\; \{\lambda : \mathbb{N}_0 \to \mathbb{N}_0 \mid supp(\lambda)\ finite\}$$

(caveat: this is not the notation we use elsewhere). The elements of Λ and Λ_0 are called *weights*. If Γ is any set of weights and $Q \in \mathbb{R}$ then $\Gamma(Q)$ is the subset of weights of degree Q.

The *dominant weights* Λ^+ (and Λ_0^+) are the set of not-strictly descending weights. There is a natural action of the symmetric group $S_{\mathbb{N}}$ (resp. $S_{\mathbb{N}_0}$) on Λ, for which Λ^+ is a fundamental region.

For $\lambda \in \Lambda$ and $Q \in \mathbb{N}_0$ (with $Q \geq |\lambda|$) define $\lambda^{(Q)} \in \Lambda_0(Q)$ by

$$\lambda^{(Q)} := (Q - |\lambda|, \lambda_1, \lambda_2, ...)$$

(**15.7.1**) Define

$$\mathcal{A} \;=\; \{g \in \mathbb{Q}[v] \mid g(\mathbb{Z}) \subset \mathbb{Z}\}$$

Note that this is a subring — the ring of 'numerical polynomials'. For example,

$$\binom{v}{i} = \frac{v(v-1)...(v-i)}{i!} \;\in \mathcal{A}$$

Indeed

$$\mathcal{A} \;=\; \bigoplus_{i \geq 0} \mathbb{Z}\binom{v}{i} \;=\; \mathbb{Z} \,\oplus\, \mathbb{Z}v \,\oplus \mathbb{Z}\frac{v(v-1)}{2} \oplus ...$$

15.7.2 The Schur algebras

Here $n \in \mathbb{N}$ and K is an infinite field. Following Green [53] we define $I(n,r) = \hom(\underline{r}, \underline{n})$; an action of S_r on the right on $I(n,r)$ by place permutation; and an equivalence relation on $I(n,r)^2$ by $ij \sim kl$ if there is a $w \in S_r$ such that $k = iw$ and $l = jw$. We write $I(n,r)^{2'}$ for a fixed but arbitrary transversal of $I(n,r)^2/\sim$.

Function $c_{ij} : GL_n(K) \to K$ takes $g \in GL_n(K)$ to its i,j-entry. The set of all functions $f : GL_n(K) \to K$ is an algebra via $(f + f')(g) = f(g) + f'(g)$ and $(ff')(g) = f(g)f'(g)$. Following Green [53] we define $A_K(n)$ as the subalgebra generated by the c_{ij}s. The subspace of elements expressible as homogeneous polynomials of degree r in the c_{ij}s is denoted $A_K(n,r)$. In fact $A_K(n,r)$ has a coalgebra structure.

(**15.7.2**) We then define the *Schur algebra*

$$S_K(n,r) = \mathrm{Hom}_K(A_K(n,r), K)$$

This has K-basis $\{\zeta_{ij} \mid i,j \in I(n,r)^{2'}\}$. The multiplication is

$$\zeta_{ij}\zeta_{kl} \;=\; \sum_{p,q} Z(i,j,k,l,p,q)\zeta_{pq} \qquad (15.6) \quad \boxed{\texttt{bidet mult}}$$

where ...

Put

$$E_k \;=\; \oplus_{i \geq 0} k e_i$$

(**15.7.3**) The *global Schur algebra of degree Q* is

$$S_k(Q) \;=\; \mathrm{End}^{fin}_{kS_Q}(E_k^{\otimes Q})$$

There are sub-k-modules of $E_k^{\otimes Q}$ of form

$$M_k(\lambda) \;=\; k\{e_i \mid wt_0(i) = \lambda\,\}$$

so that

$$E_k^{\otimes Q} \;=\; \bigoplus_{\lambda \in \Lambda_0(Q)} M_k(\lambda)$$

is a decomposition of kS_Q-modules.

Let $\xi_\lambda \in S_k(Q)$ be the idempotent projecting onto $M_k(\lambda)$. For $n \in \mathbb{N}$ let

$$\xi = \sum_{\lambda \,:\, |\lambda| = Q;\; \lambda_i = 0 \;for\; i \geq n} \xi_\lambda .$$

Then

$$S_k(n,Q) \;=\; \xi S_k(Q) \xi \qquad (15.7) \quad \boxed{\texttt{glob schur}}$$

(**15.7.4**) If $n \geq Q$ then (15.7) defines a Morita equivalence of $S_k(Q)$ with $S_k(n,Q)$.

(**15.7.5**) For $i, j \in I(\mathbb{N}_0, Q)$ write $(i, j) \sim (k, l)$ if the pairs are conjugate under the right S_Q-action on $I(\mathbb{N}_0, Q)^2$. Let $\xi_{ij} \in S_k(Q)$ be

$$\xi_{ij} : e_m \mapsto \sum_{(i,j)\sim(l,m)} e_l$$

Multiplication of these elements is essentially the same as for the ζ_{ij} in (15.6).

(**15.7.6**) We now take a kind of inverse limit of large Q.
Let $\mathcal{T}_{\mathcal{A}}$ be the free $\mathcal{A}$-module with basis $\{\xi_{ij} \mid (i, j) \in I(\mathbb{N}_0, \mathbb{N})^2/\sim\}$.

(**15.7.7**) Proposition. *There are unique elements* $\hat{Z}(i, j, l, m, p, q)$ *such that*

$$\xi_{ij}\xi_{lm} \; = \; \sum_{(p,q)} \hat{Z}(i, j, l, m, p, q)\xi_{pq}$$

(where the sum is over a transversal of $I(\mathbb{N}_0, \mathbb{N})^2/\sim$*)) makes* $\mathcal{T}_{\mathcal{A}}$ *an associative* $\mathcal{A}$*-algebra without identity.*

(**15.7.8**) Example. An $i \in I(\mathbb{N}_0, \mathbb{N})$ is an infinite list of integers, almost all zero. In writing them we may omit trailing zeros. Thus 11=11000, 11111, 101, 0011 are all examples. We write elements ξ_{ij} as bracketed pairs in this notation, with i over j, such as

$$\xi_{ij} \; = \; \begin{bmatrix} 11000 \\ 11111 \end{bmatrix} \; = \; \begin{bmatrix} 10100 \\ 11111 \end{bmatrix}$$

Then for example

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix}\begin{bmatrix} 11111 \\ 11000 \end{bmatrix} \; = \; \ldots$$

may be computed by considering a 'general Q' case of the finite problem. This has a given pq on the right only if there is an s such that $(11000, 11111) \sim (p, s)$ and $(s, q) \sim (11111, 11000)$ (we continue to omit trailing zeros even in the general-finite case). Note that any such s must have five 1s, but there are potentially many possible distributions, depending on p, q. We may fix $p = 11000$ in the transversal. There are then various possibilities for q.

Clearly there are solutions when $p = q = 11000$. This requires that s has five 1s, with the first two in the first two positions, so there are (as it were) $(v-2)(v-3)(v-4)/6!$ possibilities.

Another possibility for q is then $q = 101$. Here s must start 111, but the remaining two 1s can go anywhere: $(v-3)(v-4)/2$ possibilities.

The last possibility in the transversal is $q = 0011$. Here s must start 1111, but the remaining 1 can go anywhere: $(v-4)$ possibilities.

Altogether we have

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix}\begin{bmatrix} 11111 \\ 11000 \end{bmatrix} = \binom{v-2}{3}\begin{bmatrix} 11 \\ 11 \end{bmatrix} + \binom{v-3}{2}\begin{bmatrix} 110 \\ 101 \end{bmatrix} + \binom{v-4}{1}\begin{bmatrix} 1100 \\ 0011 \end{bmatrix}$$

(**15.7.9**) For k a commutative ring and $Q \in \mathbb{Z}$ we write $k^{(Q)}$ for k made into an $\mathcal{A}$-algebra via evaluation of polynomials at Q.

When $Q \in \mathbb{N}$ there is an isomorphism between suitable finite pieces of $\mathcal{T}_{k(Q)}$ and $S_k(Q)$:

$$\left(\sum_{\lambda \in \Lambda[Q/2]} \xi_\lambda \right) \mathcal{T}_{k(Q)} \left(\sum_{\lambda \in \Lambda[Q/2]} \xi_\lambda \right) \;\cong\; S_k(\Gamma, Q)$$

where

$$\Gamma \;=$$

15.7.3 The global partition algebra as a localisation

The idea is to identify the Potts module U_k (for fixed Q), viewed as a right S_Q-module, with a summand of the defining module $E_k^{\otimes Q}$ of $S_k(Q)$ and then to "take limits".

15.7.4 Representation theory

(**15.7.10**) Here F is a commutative ring that we shall specify shortly. For any such F, and $Q \in \mathbb{Z}$, we write $F^{(Q)}$ for F made into a $\mathcal{A}$-algebra or $\mathbb{Z}[v]$-algebra (say) by evaluating polynomials at $v = Q$ (see [100, (3.2),(3.8)]).

Now let $R \in \mathbb{N}_0$ and $F = \mathbb{F}_p^{(R)}$ for some characteristic $p > 0$ (precise choice of which will eventually not matter). Our first objective is to say something about the modules of the global Schur algebra $\mathcal{T}_k$, where k is an $\mathcal{A}$-algebra which is a field of char.0 in which element v maps to $R \in \mathbb{N}_0$. Under suitable circumstances, simple modules for $\mathcal{T}_F$ are obtained by reduction mod.p of those for $\mathcal{T}_k$. We can thus study $\mathcal{T}_k$ (at the level of characters, say) by studying $\mathcal{T}_F$. But $\mathcal{T}_F$ in turn can be studied by studying a suitable collection of ordinary Schur algebras, and hence via the representation theory of the general linear groups.

For $\nu \in \Lambda_0^+$ with support at most in positions 0 through n (note that for each ν this just sets a lower bound for n), let $\Delta_F(\nu)$ denote the Weyl module for the F-group scheme GL_{n+1} (rows and columns of matrices indexed from 0). Let $\Delta_F^l(\nu)$ denote the l-th term in the Jantzen filtration [69, II.8] of $\Delta_F(\nu)$. Write $e_0, e_1, ..., e_n$ for the standard ordered basis in the weight lattice $\mathbb{Z}^{n+1}$.

(**15.7.11**) Set

$$Q = R + p.$$

Now fix n (some $n >> 0$, say) and set $\rho = \rho_n = (n, n-1, ..., 0)$. If $\nu = \lambda^{(Q)}$ then the Jantzen sum formula [69, II.8.19] gives:

$$\sum_{l>0} ch\ \Delta_F^l(\nu) \;=\; \sum_{\substack{1 \le j \le n \\ \langle e_0 - e_j, \nu + \rho_n \rangle > p}} \chi(\nu(j)) \tag{15.8}$$

eq:Jantzen sum

where $\chi(\mu) = ch\ \Delta_F(\mu)$ if μ dominant and $\chi((ij).\mu) = -\chi(\mu)$; and

$$\begin{aligned} \nu(j) \;&=\; (0j).\nu + p(e_0 - e_j) \\ &=\; (\lambda_j - j, \lambda_1, ..., \lambda_{j-1}, Q - |\lambda| + j, \lambda_{j+1}, ...) + (p, 0, ..., 0, -p, 0, ...) \end{aligned}$$

Here

$$w.\lambda \;:=\; w(\lambda+\rho_n)-\rho_n$$

(**15.7.12**) Remarks. (1) The dot action is used here so that the nominal index scheme for modules is the natural scheme for GL. One could work with ρ-shifted weights from the start, whereupon the dot action would be replaced by ordinary reflections.
(2) The $+(p,0,...,-p,0,...)$ in $\nu(j)$ makes it the image of ν in an affine wall. However, $\lambda^{(Q)}=\lambda^{(R+p)}$ has a p in the 0-th term, so $(0j).\nu$ has a p in the j-th term, which is then just moved back to the first term by $+(p,0,...,-p,0,...)$.

(**15.7.13**) Let us examine the sum on the RHS in (15.8). We have

$$\langle e_0-e_j,\nu+\rho\rangle \;=\; (Q-|\lambda|+n)-(\lambda_j+n-j)$$

so there is a j-term in the sum on the RHS in (15.8) iff

$$R+j>|\lambda|+\lambda_j$$

If there is no such j then $\Delta_F(\nu)$ is simple. If there is such a j, let i be the least such. Then $\nu(i)$ fails to be dominant iff

$$R+i-1=|\lambda|+\lambda_{i-1}$$

If this holds, then not only is $\nu(i)$ non-dominant, but it lies on the $(i-1,i)$-reflection wall:

$$\nu(i) \;=(...,\lambda_{i-1},\lambda_{i-1}+1,...)$$

so $\chi(\nu(i))=0$. It follows that $\chi(\nu(j))=0$ for all $j>i$ too, so again $\Delta_F(\nu)$ is simple.

On the other hand if $\nu(i)$ is dominant then, noting that $R+i>|\lambda|+\lambda_i$ implies $R+j>|\lambda|+\lambda_j$ for all $j>i$, we have

$$\sum_{l>0} ch\;\Delta_F^l(\nu) \;=\; ch\;\Delta_F(\nu(i)) \;+\; \sum_{j>i}\chi(\nu(j))$$

Is $\nu(i+1)$, say, dominant? We can bypass this question.

The sum formula for $\Delta_F(\nu(i))$ in this case involves:

$$\langle e_0-e_i,\nu(i)+\rho\rangle \;=\; (\lambda_i-i+p)\;-\;(R+i-|\lambda|)+i \;<\; p$$

$$\langle e_0-e_{i+1},\nu(i)+\rho\rangle \;=\; (\lambda_i-i+p)\;-\;(\lambda_{i+1}-(i+1)) \;>\; p$$

so j gives a contribution iff $j>i$:

$$\sum_{l>0} ch\;\Delta_F^l(\nu(i)) \;=\; \sum_{j>i}\chi((\nu(i))(j))$$

In this case

$$(\nu(i))(j) \;=\; (\lambda_j-j+p,...,Q+i-|\lambda|,...,\lambda_i-i+j,...)$$

(displaying positions $1,i,j$). In fact

$$(\nu(i))(j) \;=\; (ij).\nu(j)$$

Thus

$$\sum_{l>0} ch\ \Delta^l_F(\nu) \ = \ ch\ \Delta_F(\nu(i)) \ - \ \sum_{l>0} ch\ \Delta^l_F(\nu(i)) \ = \ ch\ L_F(\nu(i)) \ - \ \sum_{l>1} ch\ \Delta^l_F(\nu(i))$$

Since the LHS is a non-negative sum of simple characters the nominally negative part must vanish, and we have

$$\Delta^1_F(\nu) = L_F(\nu(i))$$

(**15.7.14**) JOB. Recast all this in the P-natural ρ-shift setting.

(**15.7.15**) EXAMPLES. The simplest examples is $\lambda = \emptyset$, $R = 0$. In principle we need to choose n and p. We note (a) that this can always be done; and (b) that the choice plays no subsequent role. Indeed the primeness of p is a vestige of the Schur algebra 'finesse' that allows us to use the Jantzen sum formula. With this in mind, we shall take n large and shift so that the first p-affine reflection wall parallel to $(0i)$ (each i) is drawn as if the 'non-affine' wall. This corresponds, combinatorially, to setting $p = 0$ — we must then remember that the $(0i)$ wall drawn is not at the boundary of the 'dominant region'. We will also apply (an n-independent version of) the ρ-shift derived above to weights at the outset, so we can replace the dot action of the 'Weyl group' by the ordinary action. After the ρ-shift we have the embedding:

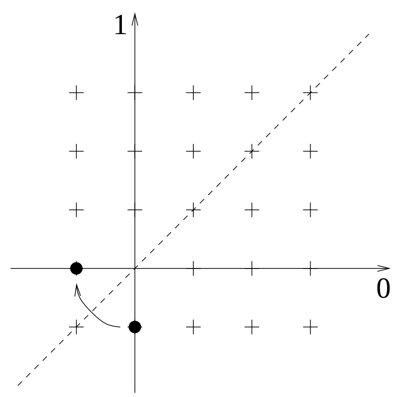
$$\Lambda \to \mathbb{Z}^{\mathbb{N}_0}$$

$$\mathfrak{e}_R : \lambda \mapsto (R - |\lambda|, \lambda_1, \lambda_2, ...) + (0, -1, -2, -3, ...)$$

so

$$\emptyset^{(0)} \ \mapsto \ (0, -1, -2, -3, ...)$$

We can represent this diagramatically by projecting onto some i, j-subspace, such as the $0, 1$-subspace:



The figure also shows

$$(01)(0, -1, -2, -3, ...) \ = \ (-1, 0, -2, -3, ...)$$

Note that since we are working with the ρ-shifted weight we use the simple reflection, not the dot action. Note that $\mathfrak{e}_R$ is invertible:

$$\mathfrak{e}_o^{-1}(-1, 0, -2, -3, ...) = \ (1)$$

Let us consider the images of some other weights:

λ	$\mathfrak{e}_0(\lambda)$	$\mathfrak{e}_1(\lambda)$	$\mathfrak{e}_2(\lambda)$
0	$(0,-1,-2,-3,...)$	$(1,-1,-2,-3,...)$	$(2,-1,-2,-3,...)$
(1)	$(-1,0,-2,-3,...)$	$(0,0,-2,-3,...)$	$(1,0,-2,-3,...)$
(2)	$(-2,1,-2,-3,...)$	$(-1,1,-2,-3,...)$	$(0,1,-2,-3,...)$
(1^2)	$(-2,0,-1,-3,...)$	$(-1,0,-1,-3,...)$	$(0,0,-1,-3,...)$

Note that some of these weights do not look 'dominant', but this is because we have omitted $+p$ from the 0-th term. Note that $\mathfrak{e}_0(2)$ lies on the (02)-wall. Recall that this is an affine wall in the GL setting (with large p):

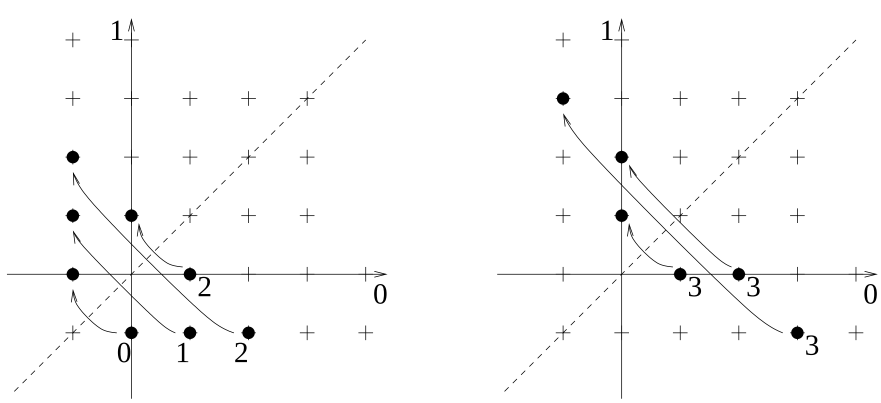
$$(2) \mapsto (p-2,1,-2,-3,...)$$

so does not imply that $\Delta_F(2)$ has vanishing character. However, it follows that all images under reflections of form $(0i)$ lie on an (ij)-wall. This implies that $\Delta_F(2)$ has vanishing radical. Note indeed that every λ lies on a wall — the (first affine) $(0\,|\lambda|)$-wall, unless it takes the form $\lambda = (1^m)$ for some m.

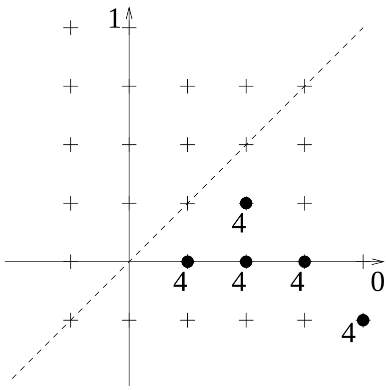
Similarly both $\mathfrak{e}_1(1)$ and $\mathfrak{e}_1(1^2)$ give simple modules.

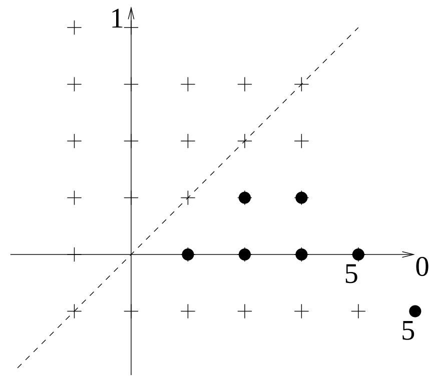
(**15.7.16**) Lemma. *(I) each block consists either of a singleton, or of a chain of weights.*
(II) for given R each chain block begins with a weight which is a partition of R with the first row removed.

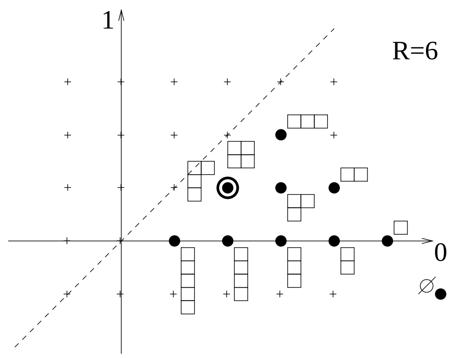
The first two elements in each chain for $R = 0,1,2,3$ are here:



The first elements in each chain for $R = 4, 5, 6$ are here:







In the latter figure ($R = 6$) we recall the original labels for these elements. Note that position in the 0,1-projection is no longer sufficient to distinguish all the weights depicted here.

15.7.5 Alcove geometric charaterisation

If we consider the $S_{\mathbb{N}}$ parabolic in $S_{\mathbb{N}_0}$ then the weights $\cup_R \mathfrak{e}_R(\Lambda)$ (a disjoint union???) all lie in the 'dominant' fundamental chamber. If we maintain our convention of considering $p = 0$ then dominance with respect to the 0-th position is not imposed.

(1) The weights that label the various chain blocks are the fully dominant weights — the weights in the fundamental alcove (in the sense of the Coxeter group/parabolic above).

(2) The other weights in the chain blocks are images of these weights in the various $(0, j)$-walls (which I guess are not walls intersecting facets of the fundamental alcove), which can also be realised via simple reflection chains of the form $(i\ i+1)$...(23)(12)(01).

(3) The weights in the fundamental chamber that lie on a wall (i.e. on a $0, j$-wall) are singletons. Because of the $\mathfrak{e}_R$ embedding we use for $\mathcal{T}_k$ and the partition algebra (cf. that used for the Brauer algebra, say), there are many of these. If we make n large compared to R then almost every block is a singleton.

15.7.6 More

What about walks and so on? Can we embed the walks on the multiplicity-free Bratelli diagram in the same setup?

(1) can we embed the index set for the odd-partition algebras in the same setup?

15.8 Notes and references

The partition algebra first appears in the context of the partition vector formalism for Potts models in computational Statistical Mechanics [88, 89] ('partition vector' refers to a vector of partition functions, not to set partitions). In this setting it is a quotient of a case of the so-called graph Temperley–Lieb algebras. It appears as a focus for study in its own right in the 1992 Yale preprints YCTP-P33-92 and YCTP-P34-92 [91, 96].

15.8.1 Notes on the Yale papers on the partition algebra

A set of subsets of a set M *covers* M if its union is M.

First we focus on the partition category in the form introduced in [91, §7]. There the set P_M of partitions of a set M is denoted S_M. Further, for q a covering set of subsets, $\mathcal{Q}(q)$ is defined as the transitive closure.

(**15.8.1**) Fix a field k. For sets $N \subseteq M$ define

$$In_N : S_M \to k(Q)S_N$$

by $In_N(p) = Q^{f_N(p)} p|_N$, where $p|_N$ is the restriction of p to N and $f_N(p)$ is the number of parts of p not intersecting N.

(**15.8.2**) With $N \subset M \cup M'$ define $Ag : S_M \times S_{M'} \to S_{M\cup M'}$ by $Ag(A, B) = \mathcal{Q}(A \cup B)$; and composition $\mathcal{P}_N$ by commutativity of

$$\begin{tikzcd} S_M \times S_{M'} \arrow[rr, "\mathcal{P}_N"] \arrow[dr, "Ag"'] & & k(Q)S_N \\ & S_{M\cup M'} \arrow[ur, "In"'] & \end{tikzcd}$$

(**15.8.3**) It will be clear how to use this composition to define a composition on, say, $S_{\underline{m}\cup\underline{n'}} \times S_{\underline{n}\cup\underline{l'}} \to k(Q)S_{\underline{m}\cup\underline{l'}}$ (by mapping $\underline{n} \cup \underline{l'} \to \underline{n'} \cup \underline{l''}$ and then using $\mathcal{P}_{\underline{n'}}$ directly). This then extends $k(Q)$-linearly to define composition in the partition category. .

15.8.2 Aside on notation

Suppose $M = \{1, 2, ..., n, 1', 2', ..., n'\}$ and let $m = 2n = |M|$. Where convenient and unambiguous write $S_m = S_M$.

Define an equivalence relation on S_m by $A \sim B$ if they are the same up to a perm of the propagating 'lines'.

Part V

Brauer