Notes in representation theory
(Rough Draft)

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Part I

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Foreword

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0.0.1 Definition summary

ss:defsum

There follows a list of definitions in the form
ALGEBRAIC SYSTEM $A = (A \text{ a set}, \ n-\text{ary operations})$, axioms.
(The selection of a special element $u \in A$, say, counts as a 0-ary operation.)
Extended examples are postponed to the relevant sections.

SEMIGROUP	$S = (S, \Box)$, $\Box$ a closed associative binary operation on S.
MONOID	$M = (M, \Box, u)$, $(M, \Box)$ a semigroup, $u \in M$ a *unit element* (i.e. $au = a = ua \ \ \forall a \in M$). Example: $(\mathbb{N}_0, +, 0)$.
GROUP	$G = (G, ., u)$, G a monoid, $\forall a \in G \exists a'$ such that $aa' = u = a'a$.
ABELIAN GROUP	$G = (G, +, 0)$, G a group, $a + b = b + a$.
RING	$R = (R, +, ., 1, 0)$, $(R, +, 0)$ an abelian group, $(R, ., 1)$ a monoid, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.
DIVISION RING	D, D a ring, every non–zero element has a multiplicative inverse.
LOCAL RING	A, A a ring, sum of two nonunits is a nonunit (a a nonunit means there does not exist b such that $ab = ba = 1$). [1]
DOMAIN	K, K a ring, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$.
INTEGRAL DOMAIN	K, K a ring, . commutative, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$. (I.e. an integral domain is a commutative domain.)
PRINCIPAL IDEAL DOMAIN	K, K an integral domain, every ideal $J \subseteq K$ is principal (i.e. $\exists$ $a \in K$ such that $J = aK$).
FIELD	F, F an integral domain, every $a \neq 0$ has a multiplicative inverse.

Our other core definitions are, for S a semigroup, R a ring as above:

S–IDEAL J: $J \subset S$ and $rj, jr \in J$ for all $r \in S$, $j \in J$.
R–IDEAL J: $J \subset R$ and $rj, jr \in J$ for all $r \in R$, $j \in J$.

(LEFT) R–MODULE M: M an abelian group with map $R \times M \to M$ (we write rx for the image of (r, x)) such that $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$ ($r \in R$, $x, y \in M$). Right modules defined similarly, but with $(rs)x = s(rx)$.

(LEFT) R–MODULE HOMOMORPHISM : Ψ from left R-module M to N is a map $\Psi : M \to N$ such that $\Psi(x + y) = \Psi(x) + \Psi(y)$, $\Psi(rx) = r\Psi(x)$ for $x, y \in M$ and $r \in R$.

(**0.0.1**) EXERCISE. $\mathbb{Z}$ is a ring. Form examples of as many of the other structures as possible from this one. (And some non-examples.)

In the following table k is a field and $\mathbb{H}$ is the ring of real quaternions (see §4.2).

	$DivR$	LR	ID	PID
$\mathbb{Z}$	$\times$	$\times$	$\surd$	$\surd$
$\mathbb{Z}[x]$	$\times$	$\times$	$\surd$	$\times$
$k[x]$	$\times$	$\times$	$\surd$	$\surd$
$k[x, y]$	$\times$	$\times$	$\surd$	$\times$
$\mathbb{H}$	$\surd$	$\surd$	$\times$	$\times$

ss:glossary

0.0.2 Glossary

		alternatives and references
$M_N(R)$	ring of $N \times N$ matrices over ring R	[6]
$GL(N)$	general linear group on $\mathbb{C}^N$	GL_N, $GL(N, \mathbb{C})$
$SL(N)$	special (det=1) linear group on $\mathbb{C}^N$	SL_N, $SL(N, \mathbb{C})$
$O(N)$	orthogonal ($g^T g = 1$) group on $\mathbb{R}^N$	$O(N, \mathbb{R})$
$O(N, \mathbb{C})$	orthogonal ($g^T g = 1$) group on $\mathbb{C}^N$	$O(N, \mathbb{C})$
$SO(N, \mathbb{C})$	special orthogonal group on $\mathbb{C}^N$	$SO(N, \mathbb{C})$
$U(N)$	unitary ($g^\dagger g = 1$) group on $\mathbb{C}^N$	U_N
$SU(N)$	special unitary group on $\mathbb{C}^N$	SU_N
Λ	set of integer partitions	$\mathcal{P}$ [96, I.1]
Λ_n	set of integer partitions of n	$\mathcal{P}_n$ [96, I.1]
P_S	partitions of a set S	
J_S	pair partitions of a set S	
$\mathtt{P}(S)$	power set (lattice) of a set S	
$\mathtt{P}_n(S)$	subset of $\mathtt{P}(S)$ of sets of order n	
$U_{S,T}$	the set of relations $\mathtt{P}(S \times T)$	
E_S	set of equivalence relations on set S	
$\underline{n}$	$\{1, 2, ..., n\}$	[57, §2]
$\underline{l}^{\underline{n}}$	set of functions $f : \underline{n} \to \underline{l}$	$I(l, n)$ [57, §2]
Σ_n, S_n, $\mathfrak{S}_n$	symmetric group $\subset (\underline{n}^{\underline{n}}, \circ)$	S_n [96, I.7], $G(n)$ [57, §2]
$\Lambda(l, n)$	S_n orbits of $\underline{l}^{\underline{n}}$ / compositions of n into l parts	[57, §3.1]
$\Lambda^+(l, n)$	S_l orbits of $\Lambda(l, n)$ / partitions of n into l parts	[57, §3.1]
Set	category of sets and set maps	
$\mathsf{C}_{\mathbb{N}}$	skeleton category in **Set**	§4.1.1
$B_n(\delta)$	Brauer algebra	§**??** [14]
$P_n(\delta)$	partition algebra	
P	partition category	§16.7.1
$T_n(\delta)$	Temperley–Lieb algebra	
T_n	full transformation semigroup	§4.1.1
T_n'	presented monoid $T_n' \cong T_n$	§4.1.1
T	diagonal category in $\mathsf{C}_{\mathbb{N}}$	
$Z(\mathbb{H})$	polyhedral complex defined by set of hyperplanes $\mathbb{H}$	
$\Gamma(G, S)$	Cayley graph of group G over subset S	
$G(W, S)$	directed Cayley graph of Coxeter system (W, S)	
$W_{\mathbb{H}}$	reflection group generated by set of hyperplanes $\mathbb{H}$	
$D(\mathbb{H})$	dual graph of complex defined by hyperplanes $\mathbb{H}$	
$\mathcal{C}_{\mathbb{H}}$	set of chambers of defined by $\mathbb{H}$	
$\mathcal{C}_W$	set of chambers of reflection group W	
$\mathcal{A}_{\mathbb{H}}$	set of alcoves of defined by $\mathbb{H}$	
$\mathbb{H}_a$	subset of hyperplanes, walls of chamber $a \in \mathcal{C}_{\mathbb{H}}$	

Chapter 1

Introduction

ch:basic

Contents

Chapters 1 - 3 give a brief introduction to representation theory, and a review of some of the basic algebra required in later Chapters. A more thorough grounding may be achieved by reading the works listed in §2.10: *Notes and References*.

Section 1.1 (upon which later chapters do not depend) attempts to provide a sketch overview of topics in the representation theory of finite dimensional algebras. In order to bootstrap this process, we use some terms without prior definition. We assume you know what a vector space is, and what a ring is (else see Section 3.1.1). For the rest, either you know them already, or you must intuit their meaning and wait for precise definitions until after the overview.

1.1 Representation theory preamble

s:ov

1.1.1 Matrices

ss:matrices1

Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices over a ring $R = (R, +, ., 1, 0)$. The additive identity in $M_{m,n}(R)$ can be denoted $0_{m,n}$. There is also a scalar multiplication, which makes $M_{m,n}(R)$ an R-module. Indeed a free R-module - see later - with basis the elementary matrices ϵ_{ij} given by $(\epsilon_{ij})_{ij} = 1$ and $(\epsilon_{ij})_{kl} = 0$ otherwise.

For each $l, m, n \in \mathbb{N}$ we have matrix multiplication

$$\mu : M_{l,m}(R) \times M_{m,n}(R) \to M_{l,n}(R)$$

This is associative, and makes the set of all such matrices a *category* (a certain very useful kind of partial semigroup) with object class $\mathbb{N}$ - as we will discuss in §1.7 and later. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

Here R can be any ring, and computational and structural complexity may arise both from R and from the matrix structure. It is often convenient to separate these challenges, resticting R to be a commutative ring or even a field. In this direction algebraically closed fields, fields of characteristic zero, and fields containing the real field all correspond to well-motivated restrictions, as we shall see. The confluence point of all these is the complex field (see for example my Topology UG Lecture Notes). The complex field has the involutive complex-conjugation operation here denoted $z \mapsto z^*$. We may extend this entry-wise to matrices, thus $A \mapsto A^*$. We may then define an involutive map $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $A \mapsto A^\dagger = (A^t)^*$, where A^t is the matrix transpose - called Hermitian conjugate.

A complex square matrix is *unitary* if $A^\dagger A = 1$; and Hermitian if $A^\dagger = A$. In applications one is often interested in the properties of linear operators acting on a space (as models of physical processes, for example), and in this setting such general properties of matrices are significant, as we shall see.

Define a block diagonal composition (matrix direct sum)

$$\begin{aligned} \oplus : M_m(R) \times M_n(R) &\to M_{m+n}(R) \\ (A, A') &\mapsto A \oplus A' = \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & A' \end{pmatrix} \end{aligned}$$

(sometimes we write $\oplus^\cdot$ for matrix/exterior $\oplus$ for disambiguation).

Define Kronecker product

$$\otimes : M_{a,b}(R) \times M_{m,n}(R) \to M_{am,bn}(R) \tag{1.1}$$

eq:kronecker12

$$(A, B) \mapsto \begin{pmatrix} a_{11}B & a_{12}B & ... \\ a_{21}B & a_{22}B & ... \\ \vdots & & \end{pmatrix} \tag{1.2}$$

In general $A \otimes B \neq B \otimes A$, but (if R is commutative then) for each pair A, B there exists a pair of permutation matrices S, T such that $S(A \otimes B) = (B \otimes A)T$ (if A, B square then $T = S$ — the *intertwiner* of $A \otimes B$ and $B \otimes A$).

Square matrice rings and direct sums (subrings of larger matrix rings) form prototypical algebras, with arithmetic structures that serve as prototypes for a number of algebraic ideas. Consider matrix 'algebra' $A = M_n(K)$ - or indeed any direct sum of such:

$$A = M_{n_1}(K) \oplus M_{n_2}(K) \oplus ... \oplus M_{n_l}(K) \subset M_{\sum_i n_i}(K)$$

- where for now a K-algebra is just a ring with a K-linear structure like that of $M_n(K)$, and for now we suppose K is an algebraically closed field. Then

$$Trace : M_n(K) \to K \tag{1.3}$$

is linear and obeys $Trace(ab) = Trace(ba)$; and furthermore $Trace(Ab) = 0$ implies $b = 0$ (this is an arithmetic exercise — do this exercise! and consider some subalgebras where Trace is defined but the map is degenerate after all!).

Define $f_T : A \times A \to K$ by

$$f_T(a, b) = Trace(ab) \tag{1.4}$$

Observe that this function/form:
1. is bilinear,
2. obeys $f_T(a, b) = f_T(b, a)$ — we say such a form is 'symmetric',
3. obeys $f_T(A, b) = \{0\}$ implies $b = 0$ — see say such a form is 'non-degenerate' (see also later), and
4. obeys $f_T(a, cb) = f_T(ac, b)$ — we say such a form is 'associative'.

Consider the basis of A of elementary matrices ϵ^l_{ij}, where l indicates the summand. Then

$$\epsilon^l_{ij}\epsilon^{l'}_{i'j'} = \delta_{l,l'}\delta_{ji'}\epsilon^l_{ij'} \tag{1.5}$$

so

$$Trace(\epsilon^l_{ij}\epsilon^{l'}_{i'j'}) = \delta_{l,l'}\delta_{ji'}\delta_{ij'} \tag{1.6}$$

Define $\overline{\epsilon}^l_{ij} = \epsilon^l_{ji}$ — transpose. Consider two ordered sets of elements of A:

$$\{a_r\}_{\underline{\mathsf{n}}} = \{\epsilon^l_{ij}\} \; ; \qquad\qquad \{b_r\}_{\underline{\mathsf{n}}} = \{\overline{\epsilon}^l_{ij}\} \tag{1.7}$$ eq:arbr

ordered in book order. Observe that n is the dimension of A.

Observe that ϵ^l_{ii} is an idempotent, as indeed is every conjugate of it by an invertible element of A. Indeed ϵ^l_{ii} and ϵ^l_{jj} are conjugate (but there are in general many more such conjugate idempotents besides). On the other hand, the elements

$$\epsilon^l = \sum_{i=1}^{n_l} \epsilon^l_{ii} \tag{1.8}$$ eq:idel1

are central and thus *invariant* under conjugation. Note

$$\epsilon^l\epsilon^{l'} = \delta_{l,l'}\epsilon^l \tag{1.9}$$ eq:eleldel

Observe that the two bases of A in (1.7) obey

$$f_T(a_r, b_{r'}) = Trace(a_r b_{r'}) = Trace(\epsilon^l_{ij}\epsilon^{l'}_{j'i'}) = \delta_{l,l'}\delta_{jj'}\delta_{ii'} = \delta_{r,r'}.$$

Thus (we say that) these are a pair of f_T-dual bases — bases that are dual with respect to the SNDAB (symmetric non-degenerate associative bilinear) form f_T.

(To continue our arithmetic journey, any pair of bases dual with respect to any SNDAB form will do. More examples later. For now, we can see above that every semisimple algebra *has* a SNDAB form and a corresponding dual pair of bases. The form is not unique and, fixing the form, the dual pair is not unique. But both can be used to make invariants, as we see next.)

Now suppose $\mu : A \to K$ is linear, i.e. $\mu \in \mathcal{A}^* := Hom_K(A, K)$. Define

$$d^\mu = \sum_r \mu(a_r)b_r \;\in\; A \tag{1.10}$$ eq:dmu1

It looks like this depends on all our construction choices above (of f_t and the bases). But let B be any SNDAB form, $\{a_r\}, \{b_r\}$ be any B-dual bases, and $a \in A$ — which we can write as $a = \sum_r \alpha_r a_r \in A$ ($\alpha_r \in K$). We have

$$B(d^\mu, a) \overset{defn.}{=} B(\sum_r \mu(a_r) b_r, a) \overset{linear}{=} \sum_s \alpha_s B(\sum_r \mu(a_r) b_r, a_s) \overset{linear}{=} \sum_s \alpha_s \sum_r \mu(a_r) B(b_r, a_s)$$

$$\overset{dual}{=} \sum_r \alpha_r \mu(a_r) \overset{linear}{=} \mu(a) \tag{1.11}$$ eq:Bdmua

so we deduce that d^μ does not depend on any of the choices - we can use any SNDAB form and any dual bases to get the same element.

What is more, if $\mu : A \to K$ is a character, i.e. $\mu(ab) = \mu(ba)$, then

$$B(xd^\mu, y) \overset{sym.}{=} B(y, xd^\mu) \overset{assoc.}{=} B(yx, d^\mu) \overset{(1.11)}{=} \mu(yx) \overset{char}{=} \mu(xy)$$

$$\overset{(1.11)}{=} B(xy, d^\mu) \overset{sym.}{=} B(d^\mu, xy) \overset{assoc.}{=} B(d^\mu x, y)$$

By linearity this implies $B(xd^\mu - d^\mu x, y) = 0$ for all y, so by ND we have $xd^\mu = d^\mu x$ for all $x \in A$. We say d^μ is *central* in A.

Consider the left-module $A\epsilon^i_{11}$, with dimension denoted dim_i. Let μ_i be the character induced on the module $A\epsilon^i_{11}$. This is the same as on $A\epsilon^i_{jj}$ for any j by a suitable conjugation. Let

$$d_i = d^{\mu_i}$$

We have $d_i \epsilon^j = 0$ for $i \neq j$, since $\epsilon_i A \epsilon_j = 0$. Thus

$$d_i = d_i \epsilon^i \ \in A\epsilon^i$$

(and note that this does not vanish). Note that the centre of the simple matrix algebra $A\epsilon^i$ is one-dimensional. Thus $d_i d_i = \kappa_i d_i$ for some κ. We have

$$\mu_i(1) = dim(A\epsilon^i_{11}) = dim_i = \mu_i(\epsilon^i) = \frac{1}{\kappa_i}\mu_i(d_i)$$

thus $\kappa_i = \mu_i(d_i)/dim_i$ and

$$\epsilon^i = \frac{dim_i}{\mu_i(d_i)} d_i$$

1.1.2 Aside: binary operations, magmas and associativity

Most of the algebraic structures we consider here satisfy an associativity condition (or something similarly strong). Here we say a few words about the more general case, for context. See §3.2.4 for some exercises.

(**1.1.1**) A set with a closed binary operation is sometimes called a *magma*.

We may define the *free magma* M_S generated by a set S as follows. First of all the elements $S \subset M_S$ (elements of *length* 1). Given a pair of elements x, y then the free magma product is the ordered pair (x, y). Thus in particular $S \times S \subset M_S$ (elements of length 2). But then obviously we also get $((x, y), z)$ and $((x, y), (y, z))$ and so on. For $n > 0$ define sets $S^{!n}$ iteratively as follows: $S^{!1} = S$; $S^{!2} = S \times S$; then

$$S^{!n} = \bigcup_{a+b=n} S^{!a} \times S^{!b}$$

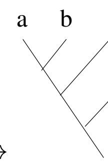
We have $M_s = \bigcup_n S^{!n}$.

(**1.1.2**) Proposition. *The product $a * b = (a, b)$ closes on M_S.*

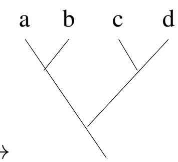
(**1.1.3**) Magma $M_S = (M_S, *)$ is free in the sense that no conditions have been imposed on the product. It is also free in the sense that if $f : S \to G$ is any map to a magma, then this extends uniquely to a magma map $f' : M_S \to G$.

(**1.1.4**) Note that an element of $S^{!n} \subset M_S$ corresponds to a word w in S of length n together with a planar binary tree with n leaves. One labels the leaves by w in the natural way, then reads off the order of composition from the tree. Examples:

$((ab)c)d \quad \mapsto$



$(ab)(cd) \quad \mapsto$



(**1.1.5**) We say a few words about imposing a congruence on M_S corresponding to associativity. Let us define a relation on M_S by $a(bc) \sim (ab)c$. (We assume this notation engenders no ambiguity.) Consider the pentagon of congruences induced by this relation on $((ab)c)d$:

$$\begin{tikzcd}
& ((ab)c)d \arrow[ddl, "ab"'] \arrow[dr, "g"] & \\
& & (a(bc))d \arrow[dd, "bc"] \\
(ab)(cd) \arrow[ddr, "cd"'] & & \\
& & a((bc))d) \arrow[dl, "g"] \\
& a(b(cd)) == a(b(cd)) &
\end{tikzcd}$$

Each step is a congruence implied by an element of the relation. (The edge label g means it is via a relation $\sim$ exactly as originally written. The label ab means that ab is an atom in the congruence.) The composites are congruences by transitive closure. Note that the two routes to the bottom result not only in congruent elements but in *identical* elements.

1.1.3 Aside: Some notations for monoids and groups

(See §3.2 for a more extended discussion of set theory notations. See §3.2.4 for exercises on binary operations.)

de:freemonoid (**1.1.6**) Given a set S, then a *word* in S is a finite sequence from 'alphabet' S, i.e. a map from $\underline{n}$ to S for some $n \in \mathbb{N}_0$. E.g. for $S = \{a, b, c\}$ then write $w = abc$ for the word $abc : \underline{3} \to S$ given by $abc(1) = a$ and so on.

The *free monoid* S^* is the set of words in the alphabet S, together with the operation of juxtaposition: $a * b = ab$. (Note associativity.) That is, for $w : \underline{n} \to S$ (written, for example, as $w = w_1 w_2 ... w_n$, with $w_i = w(i)$) and $v : \underline{m} \to S$ we have $w * v : \underline{n+m} \to S$ given by

$$(w * v)(i) = \begin{cases} w(i) & i \leq n \\ v(i-n) & i > n \end{cases}$$

i.e. $w * v = w_1 w_2 ... w_n v_1 v_2 ... v_m$.

pr:f1 (**1.1.7**) If M is a monoid with generating subset S' in bijection with set S (bijection $s \leftrightarrow s'$, say) then there is a map $f : S^* \to M$ given by $f(s) = s'$.

(**1.1.8**) Let ρ be a relation on set S, a monoid. Then ρ is *compatible* with monoid S if $(s,t),(u,v) \in \rho$ implies $(su, tv) \in \rho$.

We write $\rho\#$ for the intersection of all compatible equivalence relations ('congruences') on S containing relation ρ.

(**1.1.9**) If ρ is an equivalence relation on set S then S/ρ denotes the set of classes of S under ρ.

(**1.1.10**) If ρ is a congruence on semigroup S then S/ρ has a semigroup structure by:

$$\rho(a) * \rho(b) = \rho(a * b)$$

(Exercise: check well-definedness and associativity.)

(**1.1.11**) For set S finite we can define a monoid by *presentation*. This is the monoid $S^*/\sim$, where the presentation $\sim$ is a relation on S.

...

(**1.1.12**) For more on semigroups see for example Howie [67].

(**1.1.13**) A monoid M is *regular* if $m \in mMm$ for all $m \in M$.

Fix a monoid M. The equivalence relation $\mathcal{J}$ on M is given by $a\mathcal{J}b$ if $MaM = MbM$. Note that the classes are partially ordered by inclusion.

de:solvableg (**1.1.14**) A group G is *solvable* if there is a chain of subgroups $...G_i \subset G_{i+1}...$ such that $G_i \trianglelefteq G_{i+1}$ (normal subgroup) and G_{i+1}/G_i is abelian.

(**1.1.15**) Example. $(\mathbb{Z}, +)$ and S_3 are solvable; S_5 is not.

(**1.1.16**) A group G is *simple* if it has no proper normal subgroups.

(**1.1.17**) Example. The alternating group A_n is simple for $n > 4$; S_n is not simple for $n > 2$.

1.2 Group representations

de:rep (**1.2.1**) A matrix representation of a group G over a commutative ring R is a map

$$\rho : G \to M_n(R) \tag{1.12}$$ try345

such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

In other words a matrix representation is a map from the group to a different system, which nonetheless respects the extra structure (of multiplication) in some way. The study of representations — models of the group and its structure — is a way to study the group itself.

(**1.2.2**) The map ρ above is an example of the notion of representation that generalises greatly. A mild generalisation is the representation theory of R-algebras that we shall discuss, but one could go further. Physics consists in various attempts to model or represent the observable world. In a model, Physical entities are abstracted, and their behaviour has an image in the behaviour of the model. We say we understand something when we have a model or representation of it mapping to something we understand (better), which does not wash out too much of the detailed behaviour.

de:repIII (**1.2.3**) Representation theory itself seeks to classify and construct representations (of groups, or other systems). Let us try to be more explicit about this.

(I) Suppose ρ is as above, and let S be an arbitrary invertible element of $M_n(R)$. Then one immediately verifies that

$$\rho_S : G \;\to\; M_n(R) \tag{1.13}$$ aaas

$$g \;\mapsto\; S\rho(g)S^{-1} \tag{1.14}$$

is again a representation.

(II) If ρ' is another representation (by $m \times m$ matrices, say) then

$$\rho \oplus \rho' : G \;\to\; M_{m+n}(R) \tag{1.15}$$ dsum

$$g \;\mapsto\; \rho(g) \oplus \rho'(g) \tag{1.16}$$

is yet another representation.

(III) For a finite group G let $\{g_i \; : \; i = 1, ..., |G|\}$ be an ordering of the group elements. Each element g acts on G, written out as this list $\{g_i\}$, by multiplication from the left (say), to permute the list. That is, there is a permutation $\sigma(g)$ such that $gg_i = g_{\sigma(g)(i)}$. This permutation can be recorded as a matrix,

$$\rho_{Reg}(g) = \sum_{i=1}^{|G|} \epsilon_{i\;\sigma(g)(i)}$$

(where $\epsilon_{ij} \in M_{|G|}(R)$ is the i,j-elementary matrix) and one can check that these matrices form a representation, called the *regular representation.*

Clearly, then, there are unboundedly many representations of any group. However, these constructions also carry the seeds for an organisational scheme...

(**1.2.4**) Firstly, in light of the ρ_S construction, we only seek to classify representations *up to isomorphism* (i.e. up to equivalences of the form $\rho \leftrightarrow \rho_S$).

Secondly, we can go further (in the same general direction), and give a cruder classification, by *character*. (While cruder, this classification is still organisationally very useful.) We can briefly explain this as follows.

1.2.1 Classes and characters; reducible representations

ss:classchar1

Let c_G denote the set of classes of group G. A *class function* on G is a function that factors through the natural set map from G to the set c_G. Thus an R-valued class function is completely specified by a c_G-tuple of elements of R (that is, an element of the set of maps from c_G to R, denoted R^{c_G}). For each representation ρ define a *character* map from G to R

$$\begin{aligned} \chi_\rho : G &\to R && (1.17) \\ g &\mapsto \mathrm{Tr}(\rho(g)) && (1.18) \end{aligned}$$

eq:ch1

(matrix trace). Note that this map is fixed up to isomorphism. Note also that this map is a class function. Fixing G and varying ρ, therefore, we may regard the character map instead as a map χ_- from the collection of representations to the set of c_G-tuples of elements of R.

Note that pointwise addition equips R^{c_G} with the structure of abelian group. Thus, for example, the character of a sum of representations isomorphic to ρ lies in the subgroup generated by the character of ρ; and $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$ and so on.

We can ask if there is a small set of representations whose characters '$\mathbb{N}_0$-span' the image of the collection of representations in R^{c_G}. (We could even ask if such a set provides an R-basis for R^{c_G} (in case R a field, or in a suitably corresponding sense — see later). Note that $|c_G|$ provides an upper bound on the size of such a set.)

(**1.2.5**) Next, conversely to the direct sum result, suppose $R_1 : G \to M_m(R)$, $R_2 : G \to M_n(R)$, and $V : G \to M_{m,n}(R)$ are set maps, and that a set map $\rho_{12} : G \to M_{m+n}(R)$ takes the form

$$\rho_{12}(g) = \begin{pmatrix} R_1(g) & V(g) \\ 0 & R_2(g) \end{pmatrix} \tag{1.19}$$

eq:plus

(a matrix of matrices). Then ρ_{12} a representation of G implies that both R_1 and R_2 are representations. Further, $\chi_{\rho_{12}} = \chi_{R_1} + \chi_{R_2}$ (i.e. the character of ρ_{12} lies in the span of the characters of the smaller representations). Accordingly, if the isomorphism class of a representation contains an element that can be written in this way, we call the representation *reducible*.

(**1.2.6**) For a finite group over $R = \mathbb{C}$ (say) we shall see later that there are only a finite set of 'irreducible' representations needed (up to equivalences of the form $\rho \leftrightarrow \rho_S$) such that every representation can be built (again up to equivalence) as a direct sum of these; and that all of these irreducible representations appear as direct summands in the regular representation.

We have done a couple of things to simplify here. Passing to a field means that we can think of our matrices as recording linear transformations on a space with respect to some basis. To say that ρ is equivalent to a representation of the form ρ_{12} above is to say that this space has a G-subspace (R_1 is the representation associated to the subspace). A representation is irreducible if there is no such proper decomposition (up to equivalence). A representation is *completely reducible* if for every decomposition $\rho_{12}(g)$ there is an equivalent identical to it except that $V(g) = 0$ — the direct sum.

Theorem [Mashke] Let ρ be a representation of a finite group G over a field K. If the characteristic of K does not divide the order of G, then ρ is completely reducible.

Corollary Every complex irreducible representation of G is a direct summand of the regular representation.

Representation theory is more complicated in general than it is in the cases to which Mashke's Theorem applies, but the notion of irreducible representations as fundamental building blocks survives in a fair degree of generality. Thus the question arises:
Over a given R, what are the irreducible representations of G (up to $\rho \leftrightarrow \rho_S$ equivalence)?
There are other questions, but as far as physical applications (for example) are concerned, this is arguably the main interesting question.

(**1.2.7**) Examples: In this sense, of constructing irreducible representations, the representation theory of the symmetric groups S_n over $\mathbb{C}$ is completely understood! (We shall review it.) On the other hand, over other fields we do not have even so much as a conjecture as to how to organise the statement of a conjecture! So there is work to be done.

1.2.2 Unitary and normal representations

A complex representation ρ of a group G in which every $\rho(g)$ is unitary is a *unitary representation* (see e.g. Boerner [12, III§6]). A representation equivalent to a unitary representation is *normal*.

(**1.2.8**) THEOREM. *Let G be a finite group. Every complex representation of G is normal. Every real representation of G is equivalent to a real orthogonal representation.*

1.2.3 Group algebras, rings and modules

The subsequent representation theory of groups is illuminated considerably by the notion of *group algebra*.

de:lset (**1.2.9**) For a set S, a map $\psi : G \times S \to S$ (written $\psi(g,s) = gs$ where no ambiguity arises) such that
$$(gg')s = g(g's),$$
equips S with the property of *left G-set*.

(**1.2.10**) For example, for a group $(G,*)$, then G itself is a left G-set by left multiplication: $\psi(g,s) = g*s$. (Cf. (1.2.3)(III).)

On the other hand, consider the map $\psi_r : G\times G \to G$ given by $\psi_r(g,s) = s*g$. This obeys $\psi_r(g*g',s) = s*(g*g') = (s*g)*g' = \psi_r(g',\psi_r(g,s))$. This ψ_r makes G a *right G-set*: in the notation of (1.2.9) we have
$$(gg')s = g'(gs). \tag{1.20}$$ eq:rset

The map $\psi_- : G\times G \to G$ given by $\psi_-(g,s) = g^{-1}*s$ obeys $\psi_r(g*g',s) = (g*g')^{-1}*s = (g'^{-1}*g^{-1})*s = g'^{-1}*(g^{-1}*s) = \psi_-(g',\psi_-(g,s))$. This ψ_- makes G a *right G-set*.

rem:Rn (**1.2.11**) Remark: When working with a commutative ring K that is a *field* it is natural to view the matrix ring $M_n(K)$ as the ring of linear transformations of vector space K^n expressed with respect to a given ordered basis. The equivalence $\rho \leftrightarrow \rho_S$ corresponds to a change of basis, and so working up to equivalence corresponds to demoting the matrices themselves in favour of the underlying linear transformations (on K^n). In this setting it is common to refer to the linear transformations by which G acts on K^n as the representation (and to spell out that the matrices are a *matrix* representation, regarded as arising from a choice of ordered basis).

Such an action of a group G on a set makes the set a G-set as in 1.2.9. However, given that K^n is a set with extra structure (in this case, a vector space), it is a small step to want to try to take advantage of the extra structure. For example we may proceed as follows.

(**1.2.12**) Continuing for the moment with K a field, we can define KG to be the K-vector space with basis G (see Exercise 2.11.1), and define a multiplication on KG by

$$\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right) = \sum_{ij} (r_i r'_j)(g_i g_j) \tag{1.21}$$ groupalgmult

which makes KG a ring (see Exercise 2.11.2).

One can quickly check that

$$\begin{aligned} \rho : KG &\to M_n(K) && (1.22)\\ \sum_i r_i g_i &\mapsto \sum_i r_i \rho(g_i) && (1.23) \end{aligned}$$

extends a representation ρ of G to a representation of ring KG in the obvious sense.

Superficially this construction is extending the use we already made of the multiplicative structure on $M_n(K)$, to make use not only of the additive structure, but also of the particular structure of 'scalar' multiplication (multiplication by an element of the centre), which plays no role in representing the group multiplication *per se*. The construction *also* makes sense at the G-set/vector space level, since linear transformations support the same extra structure.

de:RG-module (**1.2.13**) The same formal construction of KG works when K is an arbitrary commutative ring (called the *ground ring*), except that KG is not then a vector space. Instead, in respect of the vector-space-like aspect of its structure, it is called a *free K-module with basis G* (see also §8.2.3). The idea of matrix representation goes through unchanged.

If one wants a generalisation of the notion of G-set for KG to act on, the additive structure is forced from the outset. This is called a *(left) KG-module*. A formal definition may be given as follows. (The definition of left module makes sense with KG replaced by an arbitrary ring H, so we state it as such. We keep in mind the ring $H = KG$.) A left H-module is, then, an abelian group $(M, +)$ with a suitable action of H defined on it: $r(x + y) = rx + ry$, $(r + s)x = rx + sx$,

$$(rs)x = r(sx), \tag{1.24}$$ eq:lmodule

$1x = x$ ($r, s \in H$, $x, y \in M$). That is, M is a kind of 'H-set', just as the original vector space K^n was in (1.2.11).

Several examples of modules are given in §1.4.1. One thing that is new at this level is that such a structure may not have a basis (a *free* module has a basis), and so may not correspond to any class of matrix representations.

(**1.2.14**) Exercise. Construct an KG-module without basis.
(Possible hints: With G trivial we have, simply, an K-module. The caveat already applies here — it is enough to look for an K-module without basis for some commutative ring K. 1. Consider $K = \mathbb{Z}$, G trivial, and look at §8.3. 2. Consider the ideal $\langle 2, x\rangle$ in $\mathbb{Z}[x]$.)

(**1.2.15**) Remark. The above exercise concerns a different issue to the formal one which may arise if the module is in fact a vector space. A finite-dimensional vector space has a basis by definition; but it general it is (only) axiomatic that every vector space has a basis. (It can be seen as a consequence of Zorn's Lemma: If a partially ordered set P is such that every chain in P has an upper bound, then P has a maximal element.) Consider the case of $(\mathbb{R}, +)$ regarded as a $\mathbb{Q}$-module.

From this point the study of representation theory may be considered to include the study of both matrix representations and modules.

1.2.4 Algebras

See also Chapter 9.

(**1.2.16**) What other kinds of systems can we consider representation theory for?
A natural place to start studying representation theory is in Physical modeling. Unfortunately we don't have scope for this in the present work, but we will generalise from groups at least as far as rings and algebras.

The generalisation from groups to *group algebras* KG over a commutative ring K is quite natural as we have seen. The most general setting within the ring-theory context would be the study of arbitrary ring homomorphisms from a given ring. However, if one wants to study this ring by studying its modules (the obvious generalisation of the KG-modules introduced above) then the parallel of the matrix representation theory above is the study of modules that are also free modules over the centre, or some subring of the centre. (For many rings this accesses only a very small part of their structure, but for many others it captures the main features. The property that *every* module over a commutative ring is free holds if and only if the ring is a field, so this is our most accessible case. We shall motivate the restriction shortly.) This leads us to the study of *algebras*.

To introduce the general notion of an algebra, we first write $\mathrm{cen}\,(A)$ for the centre of a ring A

$$\mathrm{cen}\,A = \{a \in A \mid ab = ba \ \forall b \in A\}$$

de: alg1 (**1.2.17**) An algebra A (over a commutative ring K), or an K-algebra, is a ring A together with a homomorphism $\psi : K \to \mathrm{cen}\,(A)$, such that $\psi(1_K) = 1_A$.

de:groupalgebra Examples: Any ring is a $\mathbb{Z}$-algebra. Any ring is an algebra over its centre.
The group ring KG is an K-algebra by $r \mapsto r1_G$. The matrix ring $M_n(K)$ is an K-algebra.

Let $\psi : K \to \mathrm{cen}\,(A)$ be a homomorphism as above. We have a composition $K \times A \to A$:

$$(r, a) \mapsto ra = \psi(r)a$$

so that A is a left K-module with

$$r(ab) = (ra)b = a(rb) \tag{1.25}$$ eq: alg12

Conversely any ring which is a left K-module with the property (1.25) for some commutative ring K is an K-algebra.

(**1.2.18**) An K-representation of A is a homomorphism of K-algebras

$$\rho : A \to M_n(K)$$

(**1.2.19**) The study of a group algebra KG depends heavily on K as well as G. The study of such K-algebras takes a relatively simple form when K is an algebraically closed field; and particularly so when that field is $\mathbb{C}$. We shall aim to focus on these cases. However there are significant technical advantages, even for such cases, in starting by considering the more general situation. Accordingly we shall need to know a little ring theory, even though general ring theory is not the object of our study.

Further, as we have said, neither applications nor aesthetics restrict attention to the study of representations of groups and their algebras. One is also interested in the representation theory of more general algebras.

1.3 Group and Partition algebras — some quick examples

ss:pa0001

Our study of representation theory will benefit from plentiful examples. We use algebras such as the *partition algebra* [104, 106, 108] to generate examples.

The objective can be considered to be determining representation theory data, such as (A0-III) from (1.5.1), for various *Artinian algebras* (as in (1.4.25)). (The aim is to illustrate various tools for doing this kind of thing.) We follow directly the argument in [108].

This Section can be skipped at first reading. We start by very briefly recalling the partition algebra construction but, essentially, we assume for now that you know the definition and some notations for the partition algebras (else see §3.2.3 and §16, or [108]).

Implicit in this section are a number of exercises, requiring the proof of the various claims.

1.3.1 Defining an algebra: by basis and structure constants

Let k be a commutative ring. How might we define an algebra over k?

One way to define an algebra is to give a basis and the 'structure constants' — the associative multiplication rule on this basis. (See also §3.2.)

(**1.3.1**) Example. A group algebra for a given group, as in 1.2.17, is a very simple example of this.

1.3.2 Examples: partition algebras

ss:pa000

Here we will assume, in the spirit of A. Grothendieck's "child playing/enfant jouant", the notion of a *graph*. (If the assumption is unsafe then see e.g. §3.5.)

de:Pn (**1.3.2**) For S a set, P_S is the set of partitions of S. Let $n, m \in \mathbb{N}$. Define $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n'} = \{1', 2', ..., n'\}$ and $N(n,m) = \underline{n} \cup \underline{m'}$. We recall the *partition algebra*.

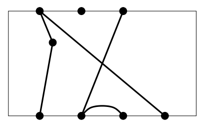
Fix a commutative ring k, and $\delta \in k$. Firstly, the partition algebra $P_n = P_n(\delta)$ over k is an algebra with a basis $\mathsf{P}_{N(n,n)}$. That is, as a k-module,

$$P_n \;=\; k\mathsf{P}_{N(n,n)} \tag{1.26}$$ de:Pn1

In order to describe a suitable multiplication rule on $\mathsf{P}_{N(n,n)}$ it is convenient to proceed as follows. (One can alternatively proceed purely set-theoretically. See e.g. [106].)

de:regu (**1.3.3**) A graph g determines a partition $\pi(g)$ of its vertex set V (into the connected components of g) — and hence determines a partition $\pi_{V'}(g)$ of any subset V' of V by restriction. We may represent a partition of $N(n,m)$ as an (n,m)-*graph*. An (n,m)-graph is a 'regular' drawing d of a graph g in a rectangular box with vertex set including $N(n,m)$ on the frame — unprimed $1,2,...,n$ left-to-right on the northern edge; primed $1',2',...,m'$ on the southern.

'Regular' means in effect that d determines g. We show in (1.3.8) that such drawings exist. Here is an example of a (3,4)-graph:



(1.27) eq:reggrapheg1

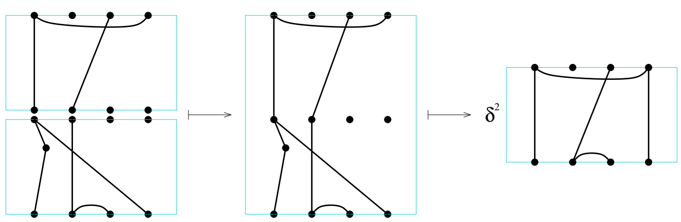
(**1.3.4**) If d is such a graph drawing, then $\pi_{n,m}(d) \in \mathsf{P}_{N(n,m)}$ is the partition with $i,j \in N(n,m)$ in the same part if they are in the same connected component in d.

For us any d such that $\pi_{n,m}(d) = p$, and such that every vertex is in a connected component with an element of $N(n,m)$, serves as a picture of p. A connected component in such a graph is *internal* if it has no vertices on either external edge. A graph d with $l_i(d)$ internal components denotes an element

$$\pi^{\delta}_{n,m}(d) = \delta^{l_i(d)} \pi_{n,m}(d)$$

of $k\mathsf{P}_{n,m}$. (We also extend this k-linearly in the obvious way.)

(**1.3.5**) Note that a suitable (n,m)-graph d will stack over an (m,l)-graph d' to make an (n,l)-graph $d|d'$ in the manner indicated in the first step in (1.28):



(1.28) eq:Ppic1

(the second step shown tidies up, non-uniquely, to a scalar×graph with the same image but the minimal number of edges and vertices). We then compute the product $p * p'$ of $p, p' \in \mathsf{P}_{N(n,n)}$ by

$$p * p' \quad = \quad \delta^{l_i(d|d')}\; \pi_{n,n}(d|d') \tag{1.29}$$

eq:palgx1

where d, d' are pictures for p, p' respectively.

Assuming that the general idea for diagram composition is clear from this picture (else see §3.2.3 or Chapter 16!), then in this approach to P_n we next have to check the following.

(**1.3.6**) Proposition. *The composition $*$ is well-defined and associative.*

For now this is left as an exercise (see §3.2.3 or Chapter 16).

We extend $*$ k-linearly to $k\mathsf{P}_{N(n,n)}$ to obtain P_n.

(**1.3.7**) Remark: By (1.26) the rank of P_n as a free k-module is the Bell number B_{2n}. In particular if k is a field then P_n is Artinian (cf. 1.4.26).

1.3.3 Aside on pictures of partitions

ss:regdraw

In (1.3.3) we said of a drawing d of a graph g that 'Regular' means in effect that d determines g. We show in (1.3.8) that such drawings exist.

de:regdraw (**1.3.8**) Let $\mathcal{G}[S]$ denote the class of finite graphs whose vertex set contains 'external' ordered subset S. A polygonal embedding of $g \in \mathcal{G}[S]$ with full vertex set V is an embedding e in $\mathbb{R}^3$ — vertices to points; edges to polygonal arcs ending at the appropriate points. We also require that y values in $e(g)$ lie in an interval $[0, h]$ for some 'height' h, with the bounds saturated only by the points in $e(S)$; and that external vertex points lie (at WLOG integral points?) on $(x, 0, 0)$ or $(x, h, 0)$.

A *regular embedding* is one such that the projection $p(x, y, z) = (x, y)$ into $\mathbb{R}^2$ is regular in the usual knot theory sense [31]. The point is that one can recover g from the datum $d = (V, \lambda, L)$ consisting of the injective map $\lambda : V$ where $\lambda = p \circ e|_V$, which amounts to a labelling of certain points in the image $L = p(e(g))$; and the image L itself. We call d a regular drawing. (Note that h is not necessarily determined by d and that if $h > 0$ then one can rescale to any other $h > 0$. Note that an analogous finite 'width' of d can be chosen, and is similarly subsidiary to the main datum.)

Note that such an embedding exists for every g (cf. e.g. [31] or §**??**). Let $\mathcal{E}[S]$ denote the class of regular drawings over $\mathcal{G}[S]$.

A regular drawing d is a containing rectangle R in $\mathbb{R}^2$; a set V and an injective map $\lambda : V \to R$; and a subset L of R that is the projection p of a regular embedding of some $g \in \mathcal{G}[S]$ (i.e. a collection of possibly crossing lines). That is (suppressing R) $\quad d = (V, \lambda, L)$.

PROPOSITION. There is a surjective map $\Pi : \mathcal{E}[S] \to \mathcal{G}[S]$. ■

On this basis, when we confuse/identify a drawing with the graph it determines, we mean the graph.

Note that in the case of an (n, m)-graph we can even omit the vertex labels, since these are determined by the ordering on the line for external vertices, and are unimportant for other vertices.

1.3.4 Examples and useful notation for set partitions

dedsidebp (**1.3.9**) See Table 1.1 for examples and notations. Given a partition p of some subset of $N(n, m)$, take $p^\star$ to be the image under toggling the prime. Define partition $p_1 \otimes p_2$ by side-by-side concatenation of diagrams (and hence renumbering the p_2 factor as appropriate). See Table 1.1 for examples.

de:pnotations (**1.3.10**) Let $\mathsf{P}_{n,m} := \mathsf{P}_{N(n,m)}$. We say a part in $p \in \mathsf{P}_{n,m}$ is *propagating* if it contains both primed and unprimed elements. Write $\mathsf{P}_{n,l,m}$ for the subset of $\mathsf{P}_{n,m}$ with l propagating parts; and $\mathsf{P}^l_{n,m}$ for the subset of $\mathsf{P}_{n,m}$ with at most l propagating parts. Thus

$$\mathsf{P}^l_{n,m} = \bigsqcup_{l=0}^{l} \mathsf{P}_{n,l,m} \qquad \text{and} \qquad \mathsf{P}_{n,m} = \bigsqcup_{l=0}^{n} \mathsf{P}_{n,l,m}.$$

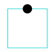
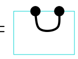
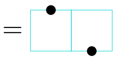
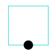
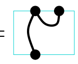
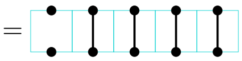
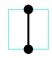
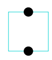
$\mathsf{v} = \{\{1\}\} =$ 	$\mathsf{U} = \{\{1,2\}\} =$ 	$\mathsf{u} = \mathsf{v} \otimes \mathsf{v}^\star =$ 
$\mathsf{v}^\star = \{\{1'\}\} =$ 	$\Gamma = \{\{1,2,1'\}\} =$ 	$\mathsf{u}_1 := \mathsf{u} \otimes 1 \otimes 1 \otimes ... \otimes 1 =$ 
$1 = \{\{1,1'\}\} =$ 	$\sigma = \{\{1,2'\},\{2,1'\}\}$	$\mathsf{u}_2 := 1 \otimes \mathsf{u} \otimes 1 \otimes ... \otimes 1$
$\mathsf{u} = \{\{1\},\{1'\}\} =$ 	$\square = \{\{1,2,1',2'\}\}$	$\mathsf{e} := \mathsf{U} \otimes \mathsf{U}^\star$

Table 1.1: Set partitions: examples and notations tab:part1

E.g. $\mathsf{P}_{2,2,2} = \{1 \otimes 1, \sigma\ \}$, $\mathsf{P}_{2,1,1} = \{\mathsf{v} \otimes 1, 1 \otimes \mathsf{v}, \Gamma\ \}$, $\mathsf{P}_{2,0,0} = \{\mathsf{v} \otimes \mathsf{v}, \mathsf{U}\ \}$ and

$$\mathsf{P}_{2,1,2} = \mathsf{P}_{2,1,1}\mathsf{P}_{1,1,2} = \{\mathsf{u} \otimes 1, 1 \otimes \mathsf{u}, \mathsf{v} \otimes 1 \otimes \mathsf{v}^\star, \mathsf{v}^\star \otimes 1 \otimes \mathsf{v}, \Gamma\Gamma^\star, ...\ \}.$$

Note that $\mathsf{P}_{n,n,n}$ spans a multiplicative subgroup:

$$\mathsf{P}_{n,n,n} \cong S_n \tag{1.30}$$ eq:PnSnsub

Define $L : \mathsf{P}_{n,l,m} \to S_l$ by deleting all but the (top and bottom) leftmost elements in each propagating part, and renumbering consecutively. Define $\mathsf{P}^L_{n,l,m}$ as the subset with $L(p) = 1 \in S_l$.

(**1.3.11**) We have $P_0 \cong k$, $P_1 = k\{1, \mathsf{u}\}$ and

$$P_2 = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \mathsf{P}_{2,0,2}) = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \{\cup \otimes \cup^\star, (\mathsf{v} \otimes \mathsf{v}) \otimes \cup^\star, (\mathsf{v} \otimes \mathsf{v})^\star \otimes \cup, \mathsf{u} \otimes \mathsf{u}\}).$$

We have $\mathsf{u}^2 = \delta\mathsf{u}$ (but see Ch.16 for the definition of the algebra/category composition) and $\mathsf{v}^\star\mathsf{v} = \delta\emptyset$ and $\mathsf{v}\mathsf{v}^\star = \mathsf{u}$.

1.3.5 Defining an algebra: as a subalgebra

(**1.3.12**) Given a ring R with 1 (like P_n) we can consider any subset S and ask what is the ring *generated* by S in R — the smallest subring containing this subset. We can do the same for an algebra A over a commutative ring k. For example, the algebra generated by $\emptyset$ in A is the smallest subalgebra, the ring $k1$.

de:TLn (**1.3.13**) Let $\mathsf{T}_{n,n} \subset \mathsf{P}_{n,n}$ be the subset of non-crossing pair partitions. (Here we follow [104, §9.5].) For example, $\mathsf{e} := \{\{1,2\},\{1',2'\}\} = \mathsf{U} \otimes \mathsf{U}^\star$ is in $\mathsf{T}_{2,2}$; and for given n, $\mathsf{e}_1 := \mathsf{e} \otimes 1 \otimes 1 \otimes ... \otimes 1$, $\mathsf{e}_2 := 1 \otimes \mathsf{e} \otimes 1 \otimes ... \otimes 1$, and so on are in $\mathsf{T}_{n,n}$.

PROPOSITION. The $P_n = P_n(\delta)$ product $*$ from (1.29) closes on $k\mathsf{T}_{n,n}$. ■

Accordingly the subalgebra of P_n generated by $\mathsf{T}_{n,n}$ is also *spanned* k-linearly by $\mathsf{T}_{n,n}$ and we may define T_n as the subalgebra of the k-algebra P_n with basis $\mathsf{T}_{n,n}$:

$$T_n = T_n(\delta) = \ (k\mathsf{T}_{n,n}, *)$$

(**1.3.14**) EXERCISE. Show that there is also a subalgebra J_n of P_n with a basis of arbitrary pair-partitions.

(**1.3.15**) REMARK. Historically the subalgebra J_n of P_n with basis of pair-partitions comes first [15] — the *Brauer algebra* B_n. We look at this in §**??** et seq.

de:fixedring1 (**1.3.16**) Given a ring R with a group of automorphisms G, one can check that the subset R^G of elements fixed under G is a subring — the *fixed ring* of R with respect to G.

For example the lateral flip on partitions in $\mathsf{P}_{n,n}$ (vertex label $i \mapsto n-i$ and so on) extends to an automorphism of P_n, and also of T_n and J_n. This automorphism evidently generates a group Λ of order 2. Thus we have fixed rings P_n^Λ and so on.

1.3.6 Defining an algebra: by a presentation

For R a commutative ring, the free R-algebra on a set S is the R-monoid-algebra of the free monoid on S (all words in S, multiplied by concatentation, as in (1.1.6)). The elements of S are called *generators* of the algebra.

Given an algebra A, the quotient by an ideal I is another algebra, A/I. The quotient by the ideal generated (as an ideal) by an element a has the *relation* $a = 0$. Every algebra is isomorphic to the quotient of some free algebra by (an ideal defined by) some relations.

(**1.3.17**) EXERCISE. (I) Determine a minimal subset of $\mathsf{P}_{n,n}$ that generates P_n.
(II) Determine generators and relations for an algebra isomorphic to P_n.

de:TLiebn (**1.3.18**) For k a commutative ring, and $\delta \in k$, define the Temperley–Lieb algrebra TL_n as the quotient of the free k-algebra generated by the symbols $U_1, U_2, ..., U_{n-1}$ by the relations

$$U_i^2 = \delta U_i$$

$$U_i U_{i\pm1} U_i = U_i$$

$$U_i U_j = U_j U_i \qquad |i-j| \neq 1$$

Thus for example TL_2 has basis $\{1, U_1\}$; while $TL_3 = k\{1, U_1, U_2, U_1U_2, U_2U_1\}$ as a k-space. Note in the case TL_2 that the obvious bijection from this basis/generating set to $\{1, \mathsf{e}\}$ extends to an isomorphism $TL_2 \cong T_2$. We have the following.

(**1.3.19**) THEOREM. (See e.g. [104, Co.10.1]) *Fix a commutative ring k and $\delta \in k$. For each n, $TL_n \cong T_n$.* ■

Hint: check that the map from the generators of TL_n to T_n given by $U_i \mapsto \mathsf{e}_i$ extends to an algebra homomorphism.

:TLbraidquotient (**1.3.20**) Suppose q a unit in k such that $\delta = q + q^{-1}$. The elements $g_i = 1 - qU_i$ in T_n obey the braid relations: $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $g_j g_i = g_i g_j$ $(|i-j| \neq 1)$. This establishes the following.

PROPOSITION. Fix k and δ. Then T_n is a quotient of the group algebra of the braid group $\mathfrak{B}_n$ over k. □

1.3.7 More exercises

(**1.3.21**) PROPOSITION. Assuming δ a unit,

$$P_{n-1} \cong \mathsf{u}_1 P_n \mathsf{u}_1 \tag{1.31}$$ eq:PUPU

$$P_n / P_n \mathsf{u}_1 P_n \cong k S_n. \tag{1.32}$$ eq:PPUPx

■

Remark: Our idea is to determine the representation theory of P_n (over a suitable algebraically closed field k) inductively from that of P_m for $m < n$, using (1.31). To this end we need to connect the two algebras. We will return to this problem shortly.

(**1.3.22**) PROPOSITION. Assuming δ a unit,

$$T_{n-2} \cong \mathsf{e}_1 T_n \mathsf{e}_1 \tag{1.33}$$ eq:UTU2

$$T_n / T_n \mathsf{e}_1 T_n \cong k \tag{1.34}$$ eq:TTeT1

■

(**1.3.23**) Construct more infinite sequences of algebras in the same spirit as those in this section. (See §**??** for more examples.)

1.4 Modules and representations

The study of algebra-modules and representations for an algebra over a field has some special features, but we start with some general properties of modules over an arbitrary ring R. (NB, this topic is covered in more detail in Chapter 8, and in our reference list §2.10.)

A module over an arbitrary ring R is defined exactly as for a module over a group ring — (1.2.13) (NB our ring R here has taken over from KG not the ground ring K, so there is no requirement of commutativity).

We assume familiarity with exact sequences of modules. See Chapter 8, or say [91], for details.

de:ideal0 (**1.4.1**) A *left ideal* of ring R is a submodule of R regarded as a left-module for itself. A subset $I \subset R$ that is both a left and right ideal is a *(two-sided) ideal* of R.

(**1.4.2**) A ring R is *simple* if its only ideals are $\{0\}$ and R.

1.4.1 Preliminary examples of ring and algebra modules

:module examples

ex:ring001 (**1.4.3**) EXAMPLE. Consider the ring $R = M_n(\mathbb{C})$. This acts on the space $M = M_{n,1}(\mathbb{C})$ of n-component column matrices by matrix multiplication from the left. Thus M is a left R-module.

ex:ring01 (**1.4.4**) EXAMPLE. Consider the ring $R = M_2(\mathbb{C}) \oplus^{\cdot} M_3(\mathbb{C}) \subset M_5(\mathbb{C})$ as in §1.1.1. A general element in R takes the form

$$r = r_1 \oplus^{\cdot} r_2 \; = \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus^{\cdot} \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix} \; \in \; R$$

Here, $M = \mathbb{C}\{(1,0)^T, (0,1)^T\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$ is a left R-module with r acting by left-multiplication by $r_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $M'' = M_2(\mathbb{C})$ is a left module with r acting in the same way;

$M' = \left\{ \begin{pmatrix} s \\ t \\ u \end{pmatrix} \mid s,t,u \in \mathbb{C} \right\}$ is a left module with r acting by r_2; and M'' is also a right module by right-multiplication by r_1.

Note that the subset of M'' of form $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ is a left submodule.

(**1.4.5**) Our next example concerns a commutative simple ring, where the distinction between left and right modules is void. Consider the ring $\mathbb{Q}$. This acts on $(\mathbb{R},+)$ in the obvious way, making $(\mathbb{R},+)$ a left (or right) $\mathbb{Q}$-module. Here $(\mathbb{Q},+) \subset (\mathbb{R},+)$ is a submodule — indeed it is a minimal submodule, in the sense that any submodule containing 1 must contain this one. Note that this submodule (generated by 1) and the submodule generated by $\sqrt{2} \in \mathbb{R}$ do not intersect non-trivially. Note that here there is no 'maximal submodule'.

exe:funny1 (**1.4.6**) Exercise. Consider the ring R_χ of matrices of form $\begin{pmatrix} q & 0 \\ x & y \end{pmatrix} \in \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$. (Note that this is not an algebra over $\mathbb{R}$ and is not a finite-dimensional algebra over $\mathbb{Q}$.) Determine some submodules of the left-regular module.

Answer: (See also (1.4.27).) Consider the submodules of the left-regular module R_χ generated by a single element. Firstly:

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$$

— that is, there is a submodule of matrices of the form on the right, with $y \in \mathbb{R}$. Note that this submodule itself has no non-trivial submodules (indeed it is a 1-d $\mathbb{R}$-vector space). Then:

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

is again a 1-d $\mathbb{R}$-vector space. Finally consider

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ x & 0 \end{pmatrix}$$

Note that the submodule generated here, while not an $\mathbb{R}$-vector space, itself has the first case above as a submodule. The quotient has no non-trivial submodule (and indeed is a 1-d $\mathbb{Q}$-vector space).

exa:cfda3 (**1.4.7**) Our next example is a commutative finite dimensional algebra over a field k. As a k-space it is $R_A = k\{1, x, y\}$. The associative commutative ring multiplication is given on the generators by

$*$	1	x	y
1	1	x	y
x	x	0	0
y	y	0	0

Note that $R_A \cong k[x,y]/(x^2, y^2, xy)$.

As always the (left) regular module is generated by 1. Here $k\{x,y\}$ is a 2d submodule. Indeed any nonzero element of form $bx + cy$ spans a 1d submodule (indeed a nilpotent ideal); and the quotient of R_A by this submodule has a 1d submodule. We can construct the (left)-regular representation as follows. We first write the actions out in matrix form:

$$x \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

$$y \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

The matrices give, as usual, the regular antirepresentation. Since R_A is commutative this is also a representation — the 'cv-dual' representation ρ^o. Considering the action of a general element $\rho^o(a.1 + b.x + c.y)$ on the corresponding 3d module we have

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ a \end{pmatrix}$$

Note that the first vector spans a simple submodule (on which x, y act like zero); and that the first and second vectors span a submodule; and the first and third (or the first and any linear combination of the second and third). The 'Loewy structure' is M^o here:

$$M^o = \begin{array}{ccc} \alpha & & \alpha \\ & \diagdown \; \diagup & \\ & \alpha & \end{array}, \qquad\qquad M = \begin{array}{ccc} & \alpha & \\ \diagup & & \diagdown \\ \alpha & & \alpha \end{array}$$

(but we will not explain this notation until §1.5.1). The transposes of these matrices give the regular representation, with the structure M above, as already noted.

pr:simpleinreg (**1.4.8**) Given a ring R and a left R-module M, then consider the set map $f : M \to \mathrm{Hom}_R(R, M)$ given by $f(m)(r) = rm$. Define the map $g : \mathrm{Hom}_R(R, M) \to M$ by $g(\psi) = \psi(1)$. For any $\psi \in \mathrm{Hom}_R(R, M)$ we have $f(\psi(1))(r) = r\psi(1) = \psi(r)$, so $f \circ g(\psi) = f(\psi(1)) = \psi$. Meanwhile $g \circ f(m) = g(r \mapsto rm) = 1m = m$. Thus f and g are inverse. We have shown the following.

PROPOSITION.

$$\mathrm{Hom}_R(R, M) \cong M$$

as sets.

It follows in particular that there is a nonzero module map from the regular module to each nonzero module.

1.4.2 Simple, semisimple and indecomposable modules

(**1.4.9**) A left R-module (for R an arbitrary ring) is *simple* if it has no non-trivial submodules. (See §8.2 for more details.)

In Example 1.4.4 both M and M' are simple; while R is a left-module for itself which is not simple, and M'' is also not simple.

de:semisim (**1.4.10**) A module M is *semisimple* if equal to the sum of its simple submodules.

de:dirsum01 (**1.4.11**) Suppose M', M'' submodules of R-module M. They *span* M if $M' + M'' = M$; and are *independent* if $M' \cap M'' = 0$. If they are both independent and spanning we write

$$M = M' \oplus M''$$

(*(module) direct sum*). A module is *indecomposable* if it has no proper direct sum decomposition.

(**1.4.12**) Example. Suppose $e^2 = e \in R$, then

$$Re \oplus R(1-e) = R \tag{1.35}$$ eq:projid1

as left-module.

Proof. For $r \in R$, $r = re + r(1-e)$ so $Re + R(1-e) = R$; and $re \in R(1-e)$ implies $re = re(1-e) = 0$. □

1.4.3 Jordan–Holder Theorem

de:comp mult (**1.4.13**) Let M be a left R-module. A *composition series* for M is a sequence of submodules $M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_l = 0$ such that the section M_i/M_{i+1} is simple.

In particular if a composition series of M exists for some l then M_{l-1} is a simple submodule.

The sections of a composition series for M (if such exists) are *composition factors*. Their multiplicities up to isomorphism are called *composition multiplicities*. Given a composition series for M, write $(M : L)$ for the multiplicity of simple L.

th:JH (**1.4.14**) **Theorem.** (Jordan–Holder) Let M be a left R-module. (JHA) All composition series for M (if such exist) have the same factors up to permutation; and (JHB) the following are equivalent:
(I) M has a composition series;
(II) every ascending and descending chain of submodules of M stops (these two stopping conditions separately are known as ACC and DCC);
(III) every sequence of submodules of M can be refined to a composition series.

Proof. Obviously (III) implies (I). See §8.3.2 for the rest.

(**1.4.15**) Note that this form of the Theorem does not address the question of conditions for a module to have a composition series. For now note the following.

le:JHkA (**1.4.16**) Lemma. *Suppose A is a finite dimensional algebra over a field. Then every finite dimensional A-module M has a composition series. And, by (JHA), multiplicity $(M : L)$ is well-defined independently of the choice of series. (Exercise.)*

1.4.4 Radicals, semisimplicities, and Artinian rings

de:nilideal0 (**1.4.17**) A *nil ideal* of R is a (left/right/two-sided) ideal in which every element r is nilpotent (there is an $n \in \mathbb{N}$ such that $r^n = 0$). A *nilpotent ideal* of R is an ideal I for which there is an $n \in \mathbb{N}$ such that $I^n = 0$. (So I nilpotent implies I nil.)

de:JacRad0 (**1.4.18**) The *Jacobsen radical* of ring R is the intersection of its maximal left ideals.

th:JL0 (**1.4.19**) Theorem. *The Jacobsen radical of ring R is the subset of elements that annihilate every simple module.* ■

(**1.4.20**) Ring R itself is a *semisimple ring* if its Jacobsen radical vanishes.

Remark: This term is sometimes used for a ring that is semisimple as a left-module for itself (in the sense of (1.4.10)). The two definitions coincide under certain conditions (but not always). See later.

de:lss (**1.4.21**) For the moment we shall say that a ring R is *left-semisimple* if it is semisimple as a left-module ${}_RR$ (cf. e.g. Adamson [2, §22]). There is then a corresponding notion of *right-semisimple*, however: Theorem. A ring is right-semisimple if and only if left-semisimple.

The next theorem is not trivial to show:
Theorem. The following are equivalent:

(I) ring R is left-semisimple.
(II) every module is semisimple (as in (1.4.10)).
(III) every module is projective (every short exact sequence splits — see also 1.4.73).

(**1.4.22**) Theorem. The Jacobsen radical of ring R contains every nil ideal of R. ■[1]

Remark: In general the Jacobsen radical is not necessarily a nil ideal. (But see Theorem 1.4.28.)

(**1.4.23**) An element $r \in R$ is *quasiregular* if $1_R + r$ is a unit. The element $r' = (1_R + r)^{-1} - 1$ is then the *quasiinverse* of r. (See e.g. Faith [?].)

(**1.4.24**) Theorem. If J is the Jacobsen radical of ring R and $r \in J$ then r is quasiregular. ■

1.4.5 Artinian rings

de:artinian (**1.4.25**) Ring R is *Artinian* (resp. *Noetherian*) if it has the DCC (resp. ACC, as in (1.4.14)) as a left and as a right module for itself.

th:fdalgebraa (**1.4.26**) Example: Theorem. A finite dimensional algebra over a field is Artinian.

Proof. A left- (or right-)ideal here is a finite dimensional vector space. A proper subideal necessarily has lower dimension, so any sequence of strict inclusions terminates. □

de:funny ring (**1.4.27**) Aside: We say more about chain conditions in §8.3. Here we briefly show by an example that the left/right distinction is not vacuous (although, as the contrived nature of the example perhaps suggests, it will be largely irrelevant for us in practice). Consider the ring R_χ of matrices of form $\begin{pmatrix} q & 0 \\ x & y \end{pmatrix} \in \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ as in (1.4.6). (Note that this is not an algebra over $\mathbb{R}$ and is

[1] We shall use ■ to mean that the proof is left as an exercise.

not a finite-dimensional algebra over $\mathbb{Q}$.) We claim that R_χ is Artinian and Noetherian as a left module for itself. However we claim that there are an infinite chain of right-submodules of R_χ as a right-module for itself between $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}$. Thus R_χ is left Artinian but not right Artinian.

To prove the left-module claims one can show that all possible candidates are $\mathbb{R}$-vector spaces, and finite dimensional. To prove the infinite chain claim, recall that one can form a set of infinitely many $\mathbb{Q}$-linearly-independent elements in $\mathbb{R}$ (else $\mathbb{R}$ is countable!). Order the beginning of this set as $B_n = \{1, b_1, b_2, ..., b_n\}$ (we have taken the first element as 1 WLOG), for $n = 0, 1, 2, ...$ We have $\mathbb{Q}B_0 = \mathbb{Q}$ and $\mathbb{Q}B_n \subset \mathbb{Q}B_{n+1}$ for all n, thus an infinite ascending chain. On the other hand there is an inverse limit B of the sequence B_n contained in $\mathbb{R}$ (perhaps this requires Zorn's Lemma/the axiom of choice!), so we can define a sequence B^n by eliminating 1 then b_1 and so on from $B = B^0$, giving an infinite descending chain $\mathbb{Q}B^n \supset \mathbb{Q}B^{n+1}$.

(**1.4.28**) THEOREM. If ring R Artinian then the Jacobsen radical is the maximal two-sided nilpotent ideal of R (i.e. it is nilpotent and contains all other nilpotent ideals). ■

th:nilrad0

(**1.4.29**) THEOREM. If ring R Artinian then ideal I nil implies I nilpotent. ■

(**1.4.30**) THEOREM. If a ring is left-semisimple (as in 1.4.21) then it is (left and right) Artinian and left Noetherian, and is semisimple (i.e. has radical zero). ■(See e.g. [2, Th.22.2].)

th:ARLJ

(**1.4.31**) THEOREM. *If ring R is Artinian with radical J then every simple left R-module is also a well-defined simple R/J-module; and this identification gives a complete set of simple R/J-modules.* ■

ss:schur1

1.4.6 Schur's Lemma

Schur's Lemma appears in various useful forms. We start with a general one, then discuss a couple of special cases of particular interest for the representation theory of algebras over algebraically closed fields. (See §**??** for more details.)

lem:Schur

(**1.4.32**) **Theorem.** (Schur's Lemma) Suppose M, M' are nonisomorphic simple R modules. Then the ring $hom_R(M, M)$ of R-module homomorphisms from M to itself is a division ring; and $hom_R(M, M') = 0$.

Proof. (See also 8.2.12.) Let $f \in \hom_R(M, M)$. M simple implies $\ker f = 0$ and $\operatorname{im} f = M$ or 0, so f nonzero is a bijection and hence has an inverse. Now let $g \in \hom_R(M, M')$. M simple implies $\ker g = 0$ and M' simple implies $\operatorname{im} g = M = M'$ or zero, so $g = 0$. □

ex:ring01a

(**1.4.33**) EXAMPLE. Let us return to ring R and module M from Example 1.4.4. In this case $\hom_R(M, M) \subset \hom_{\mathbb{C}}(M, M)$, and $\hom_{\mathbb{C}}(M, M)$ is all $\mathbb{C}$-linear transformations, so realised by $M_2(\mathbb{C})$ in the given basis. We see that $\hom_R(M, M)$ is the subset that commute with the action of R. This is the centre of $M_2(\mathbb{C})$, which is $\mathbb{C}1_2$, which is isomorphic to $\mathbb{C}$.

On the other hand, $\hom(M, M')$ is realised by matrices $\tau \in M_{3,2}(\mathbb{C})$:

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tau_{11}x + \tau_{12}y \\ . \\ . \end{pmatrix}$$

Here in $\hom_R(M, M')$ we look for matrices τ such that

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} r \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for all r, that is

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

but since $a, b, c, d, e, ..., m$ may be varied independently we must have $\tau = 0$.

(**1.4.34**) Remark. Cf. the occurence of the division ring in the general proof with the details in our example. We can consider the occurence of the division ring in Schur's Lemma as one of the main reasons for studying division rings alongside fields.

Next we talk about the specifics of the division ring $\hom_R(L, L)$ from Schur's Lemma, and the case where R is an algebra (over $\mathrm{Cen}(R)$ say), and then specifically an algebra over an algebraically closed field as in Ex.1.4.33.

We start with the case that R is the simplest kind of semisimple ring — a simple ring — which has only one possibility for L.

de:simplering (**1.4.35**) A ring R is a *simple ring* if R is semisimple and has no proper ideals. (Equivalently to the ideal condition we can say that there is only one isomorphism class of simple left modules.)

(**1.4.36**) An algebra that is simple as a ring is a *simple algebra*.

If A is a simple k-algebra and $k = \mathrm{Cen}(A)$ then we call A a *full simple algebra*. (Others call this a *central simple algebra*, see e.g. [**?**].)

If algebra A is division as a ring we call it a *division algebra*.

(**1.4.37**) Suppose A a simple algebra and L a simple A-module. Then the ring $E = \mathrm{Hom}_A(L, L)$ is division (Schur). In fact here one can show that $A \cong \mathrm{Hom}_E(L, L)$. And

$$\mathrm{Cen}(A) \cong \mathrm{Cen}(E) \tag{1.36}$$ eq:cen1

and, writing r for the number of copies of L in ${}_AA$ then

$$A \cong \mathrm{Hom}_E(L, L) \cong M_r(E^{op})$$

And

$$\mathrm{Cen}(E) \cong \mathrm{Cen}(E^{op}) \tag{1.37}$$ eq:cen2

Via (1.36) and (1.37) we have that E is a k-algebra, and finally (cf. (1.7.26))

$$A \cong M_r(k) \otimes_k E^{op}$$

(**1.4.38**) TO DO!

(**1.4.39**) Suppose R is an algebra over an algebraically closed field k (as in Example(1.4.33)). Then $\hom_R(M,M) \cong k$ in Schur's Lemma. It follows that any element of the centre of R acts like a scalar on simple M. Indeed we have the following.

PROPOSITION. Let R be an algebra over an algebraically closed field. Let M be an indecomposable R-module. Then the algebra $\hom_R(M,M)$ has exactly one idempotent element (generating the isomorphism maps). (See e.g. §**??**.)
(I) A central element of R acts like a scalar plus a nilpotent on any indecomposable module (in the sense of 1.4.11 or §8.2.2). (II) A central element of R acts like the same scalar on every simple module in the same block (as defined in 1.4.43). (III) A central element of R that is idempotent acts like a scalar on M.

Proof. The idea is that an idempotent decomposition of 1 in $\hom_R(M,M)$ could be used to split the module as a direct sum. A central element acts on M as part of $\hom_R(M,M)$, so this leads us to (I). Combining with Schur's Lemma we come to (II). For (III) we note (I) and also that in this case the nilpotent must vanish. ■

(**1.4.40**) EXAMPLE. Caveat: The algebra with 1 and a with $a^2 = 0$ has a in the centre. The regular module is indecomposable, but a does not act like a scalar. Rather it acts like a nilpotent.

(**1.4.41**) EXAMPLE. Consider the twist element of the braid group as in [104, §5.7.2]. The double-twist is clearly central. Hence its image is central in a quotient (such as T_n). We can use it to (partially) separate blocks. First we will need some indecomposable T_n-modules to work with. We will use $D_n^{\mathrm{TL}}(l)$ as in (2.4). These modules have extra special properties (a notion of 'generic simplicity') so that the central element even acts as a scalar.

See also, for example, §19.1.1.

1.4.7 Ring direct sum, blocks, Artin–Wedderburn Theorem

de:ringdirectsum (**1.4.42**) Suppose that ring R has a decomposition of 1 into orthogonal central idempotents: $1 = \sum_i e_i$. Then each $R_i = Re_i$ is an ideal of R and a ring with identity e_i. In this case we say that R is a *ring direct sum* of the rings R_i, and write $R = \oplus_i R_i$. (Note that this is consistent with Example (1.4.4).)

de:block01 (**1.4.43**) A refinement of a central idempotent e is a decomposition $e = e' + e''$ where e', e'' are central orthogonal idempotents. A central idempotent e is *primitive central* if it cannot be written $e = e' + e''$ where e', e'' are central orthogonal idempotents.

If $1 = \sum_i e_i$ in (1.4.42) above is a primitive central idempotent (PCI) decomposition then it is unique up to reordering. (Proof: Suppose $1 = \sum_j e'_j$ is another. Since $e_i = \sum_j e_i e'_j$ this is a refinement of e_i unless $e_i e'_k = e_i$ for some k and other summands vanish. Similarly $e_i e'_k = e'_k$.)

(**1.4.44**) EXAMPLE. Consider the algebra T_3 (from (1.3.13)) over the field of rational polynomials. This is an algebra of dimension 5. The element $\frac{1}{\delta}\mathsf{e}_1$ is idempotent, but not central. In fact the PCI decomposition is given by $1 = F + (1 - F)$ where

$$F = \frac{1}{\delta^2 - 1}(\delta(\mathsf{e}_1 + \mathsf{e}_2) - (\mathsf{e}_1\mathsf{e}_2 + \mathsf{e}_2\mathsf{e}_1)) \tag{1.38}$$ eq:T3PCI

(This result is not particularly easy to find, or even check, by brute arithmetic. It helps to know, as we shall later show in (**??**), that every PCI of T_n is fixed under the 'flip' automorphism.)

(**1.4.45**) If R is Artinian then there is a primitive central idempotent decomposition (cf. Th.1.6.7), and the rings R_i for the primitive decomposition are called the *blocks* of R.

A central idempotent acts like 1 or 0 on a simple module L. Thus if R is Artinian then precicely one primitive central idempotent acts like 1 on L. We say L is in block i if $e_i L = L$.

(**1.4.46**) EXAMPLE. In our T_3 example in (1.38) above we see that there are two blocks. This computation of F also works, by evaluation, to give the PCI over any field k in which $\delta^2 - 1$ has an inverse. And in other cases ($k = \mathbb{C}$ and $\delta = 1$ say) we may deduce that there is no possible PCI except 1, and hence only one block.

Note that if a primitive central idempotent such as F lies in a subalgebra then it is also a central idempotent there. But it is not necessarily primitive (since there may be more idempotents that are central in the subalgebra — the test for centrality may require commutation with fewer elements).

In particular note that F lies in the fixed ring of T_3 under the flip automorphism (as in (1.3.16)). It is not primitive there. We have orthogonal idempotents

$$E_\pm = \frac{1}{2(\delta \pm 1)}(\mathtt{e}_1 + \mathtt{e}_2 \pm (\mathtt{e}_1\mathtt{e}_2 + \mathtt{e}_2\mathtt{e}_1))$$

obeying $F = E_+ + E_-$. These idempotents are not central in T_3, but they are in the fixed ring.

(**1.4.47**) On the other hand a PCI, such as F, is also idempotent in a superalgebra (such as T_4 say). However here it may not be primitive or central.

ss:AW3

1.4.8 Artin-Wedderburn Theorem

th:AWI

(**1.4.48**) **Theorem.** (Artin–Wedderburn) Suppose R is semisimple and Artinian. Then R is a direct sum of rings of form $M_{n_i}(K_i)$ ($i = 1, 2, ..., l$, some l) where each K_i is a division ring.

Proof. Exercise. (See also §8.3 or e.g. Benson [7, Th.1.3.5].) ■

(**1.4.49**) Note that a central idempotent decomposition of 1_R leads to an ideal decomposition of R; while an arbitrary orthogonal idempotent decomposition of 1_R leads to a left-module decomposition of R.

Evidently a central idempotent decomposition is an orthogonal idempotent decomposition, but such a decomposition may be refinable once the central condition is relaxed. The matrix algebra $M_n(K)$ has the n elementary matrix idempotents $\{e_i^n\}_i$, which are orthogonal and such that

$$1_{M_n(K)} = \sum_{i=1}^{n} e_i^n$$

so this gives us one way to refine the central idempotent decomposition of 1_R in a semisimple Artinian ring (as in 1.4.48) to an (ordinary) orthogonal idempotent decomposition:

$$1_R = \sum_{i=1}^{l} \sum_{j=1}^{n_i} e_j^{n_i}$$

(here the first sum needs interpretation — it comes formally from the direct sum). We say more about this in §1.6.

(**1.4.50**) With A-W in mind we can consider the ring $M_n(K)$ over division ring K as a left-module for itself. We have

$$_{M_n(K)}M_n(K) \cong nL \; := \; L \oplus L \oplus ... \oplus L$$

(module direct sum as in (1.4.11)) where L is simple. Note from 1.4.8 that this L is the *only* simple module of $M_n(K)$.

th:AW2

(**1.4.51**) Thus a *general* semisimple Artinian ring as in the A-W Theorem becomes, as a left-module for itself, a direct sum of simple modules $\{L_i\}_i$ (n_i copies of L_i for each i). Again by 1.4.8 *every* simple module arises in the left-regular module in this way.

(**1.4.52**) Typically (for us) our Artinian ring R *is* a finite-dimensional algebra over a field k (k lying in the centre of R). What can we say about dimensions?

For a ring of form $M_n(k)$ with k a field, the dimension of L above is n. However if R is a finite-dimensional algebra over a field k it does not follow automatically that the division rings K_i in A-W can be indentified with k.

th:ASTIcaveat

(**1.4.53**) Note therefore that the above does not say, for an k-algebra over a field, that $\dim L_i = n_i$ in 1.4.51. For example, the $\mathbb{Q}$-algebra $A = \mathbb{Q}\{1,x\}/(x^2-2)$ is a simple module for itself of dimension 2. That is, Artin–Wedderburn here is rather trivial: $A = M_1(A)$.

Another perspective on this is that left-module $_AA$ in our example is simple, but it is not 'absolutely irreducible'. A k-algebra module is *absolutely irreducible* if it remains simple when we extend the ground field k (see e.g. §**??**). If we extend $\mathbb{Q} \subset \mathbb{C}$ by adding $\sqrt{2}$ then

$$1 \; = \; (1+\frac{1}{\sqrt{2}}x) + (1-\frac{1}{\sqrt{2}}x)$$

split semisimple

is an orthogonal idempotent decomposition, so $_AA$ is no longer simple.

(**1.4.54**) If every simple module of semisimple k-algebra A is absolutely irreducible then we say A is *split* semisimple.

pr:sumsquares

PROPOSITION. A sufficient condition for $\dim L_i = n_i$ in A–W is that k is algebraically closed. In this case we see that the k-dimension of the algebra is the sum of squares of the simple dimensions.

1.4.9 Artin-Wedderburn and Properties of split semisimple algebras

ss:AWOx

(Here we make extensive use of linearity. See e.g. §10.1.5 *et seq* for related exposition.)

Let A be a finite dimensional algebra over field k. A bilinear form $\langle , \rangle : A \times A \to k$ is called a *cv form* on A (or sometimes an associative form [**?**]) if $\langle xy, z\rangle = \langle x, yz\rangle$.

de:cv form alg

(**1.4.55**) Examples: Let $f : A \to k \; \in A^*$ (recall $A^* = \mathrm{Hom}_k(A,k)$). The map $g_f : A \times A \to k$ given by $g_f(a,b) = f(ab)$ is a cv form.

de:Frobenius

(**1.4.56**) Finite dimensional algebra A as above is a *Frobenius algebra* if there is a left-module isomorphism $\gamma : {}_AA \xrightarrow{\sim} (A_A)^*$.

For each A-module M there is a character χ_M. And characters are certain special elements of A^* ($\chi_M(x) \in k$ for $x \in A$). Thus in a Frobenius algebra the isomorphism γ^{-1} associates an element of A to each character. We can ask what kinds of elements of A are associated to characters (and to simple characters).

(**1.4.57**) PROPOSITION. *An algebra A as above is Frobenius iff there is a nondegenerate cv form on A.* ■

(**1.4.58**) An algebra A as above is *symmetric* if it has a symmetric nondegenerate cv form. (A symmetric form is one for which $\langle a, b\rangle = \langle b, a\rangle$.)

(**1.4.59**) PROPOSITION. (Curtis–Reiner [32, (9.12)]) *If algebra A is symmetric and $e \in A$ a primitive idempotent then the socle (the maximal semisimple submodule - see e.g. 1.5.7 or 8.7.4) of Ae is isomorphic to the head (the quotient by the radical - see e.g. §1.5.1).* ■

(Note that e primitive implies Head(Ae) simple, so here socle also simple.)

de:ndbf (**1.4.60**) Recall that a bilinear form $\langle , \rangle$ on k-space A is *nondegenerate* if $\langle x, a\rangle = 0$ for all $a \in A$ implies $x = 0$.

(**1.4.61**) If $\langle , \rangle$ is a nondegenerate cv form on A (i.e. nondegenerate as a bilinear form) then for a basis $\{b_i\}_i$ of A there exists a *dual* basis with respect to $\langle , \rangle$: a basis $\{c_i\}_i$ such that

$$\langle b_i, c_j\rangle = \delta_{ij}.$$

Example: Let $G = \{g_1, ..., g_l\}$ be a finite group and define $f_1 \in A^*$ by $f_1(\sum_i \alpha_i g_i) = \alpha_1$. Then g_{f_1} as above is nondegenerate, and G is dual to itself wrt g_{f_1}. Specifically $g_{f_1}(g, h^{-1}) = \delta_{gh}$.

Example: If A is split semisimple then by AW (1.4.48) there is a basis of elementary matrices e^l_{ij} (l indexing blocks) so that $e_{ij}e_{i'j'} = \delta_{ji'}e_{ij'}$ (in same block, and zero otherwise). Thus $Tr(e_{ij}e_{i'j'}) = \delta_{ji'}\delta_{ij'}$ and making $e'_{ij} = e_{ji}$ we get a dual basis with $\langle e_{ij}, e'_{i'j'}\rangle = Tr(e_{ij}e'_{i'j'}) = \delta_{(i,j),(i',j')}$.

Example: See (1.4.68).

1.4.10 Aside: Central idempotents and characters

(**1.4.62**) Let $\{e_i\}_{i=1,\dots,\Lambda}$ be the complete set of primitive central idempotent in a finite dimensional k-algebra A as above. We know formally that these idempotents exist — if A is split semisimple then there is one for each isomorphism class of simple modules L_i, by (1.4.48). Can we say anything else about them? Yes, under some circumstances we can construct them using characters as in §1.2.1.

Let $\{L_i\}_{i\in\Lambda'}$ be a complete set of simple modules of A (so that $|\Lambda'| \geq \Lambda$ in general, with equality if A is split semisimple). Let $\chi_i \in A^*$ be the character associated to simple module L_i (cf. (1.17) in §1.2.1). We proceed by using these to construct some central elements in A.

For any cv form $\langle , \rangle$ we have, for $i \neq j$,

$$\langle Ae_i, Ae_j\rangle = \langle A, e_iAe_j\rangle = 0$$

It follows that a nondegenerate $\langle , \rangle$ must be nondegenerate when restricted to Ae_i (any i).

(**1.4.63**) Suppose A is split semisimple. Then the index sets for e_i and χ_j coincide (1.4.48). If $i \neq j$ we have $\chi_i(e_j) = 0$ and hence

$$\chi_i(Ae_j) = 0$$

Now let $\{b_i\}_{i=1,\dots,d}$ and $\{c_i\}_i$ be dual bases wrt a nondegenerate form $\langle , \rangle$ as above. Define

$$e'_i \;=\; \sum_{j=1}^{d} \chi_i(b_j)c_j \quad \in A \tag{1.39}$$ eq:Kilmoyer01

One can check that $\langle e_i', x\rangle = \chi_i(x)$ for all $x \in A$; and hence that e_i' does not depend on the choice of bases.

We have

$$\langle e_i' e_j, Ae_j\rangle = 0$$

so $e_i' e_j = 0$ by restricted nondegeneracy. Thus $e_i' = e_i' 1 = e_i' e_i$.

Since A is semisimple it is also symmetric. Suppose our dual bases are with respect to a symmetric form. Then for $x, y \in A$ we have

$$\langle xe_i', y\rangle = \langle y, xe_i'\rangle = \langle yx, e_i'\rangle \ = \chi_i(yx) = \chi_i(xy) = \langle e_i', xy\rangle = \langle e_i' x, y\rangle$$

hence by nondegeneracy e_i' is central in A.

The centre of Ae_i obeys $Ae_i \cong k$ as a vector space, so it is spanned by $e_i' \propto e_i$. We have $\chi_i(1) = \chi_i(e_i) \propto \chi_i(e_i')$.

Now suppose field k has char.0. Then $\chi_i(1) \neq 0$, so $\chi_i(e_i') \neq 0$. Thus we have the following.

mitive-centralid (**1.4.64**) For a split semisimple algebra over field k of char.0 with elements e_i' constructed using a symmetric (nondegenerate) cv form we have

$$e_i \ = \ \frac{\chi_i(1)}{\chi_i(e_i')} e_i' \qquad (1.40)$$ eq:pci1

Examples — constructing central idempotents

(**1.4.65**) The version of the above construction in the finite group case is somewhat simpler. There we have for finite group G that for each conjugacy class λ

$$s_\lambda = \sum_{g \in \lambda} g$$

is central in kG (since conjugation by any group element fixes such a sum). The elements s_λ are a basis of the centre, so the central idempotents can be expressed in terms of them. Since g^{-1} is the dual of g with respect to the form in (**??**); and since g^{-1} is in the same class as h^{-1} if g, h are in the same class (and χ_i is a class function), we see that a central element of kG has the same coefficient for every group element in the same class.

Specific implementation of (**??**) depends on the irreducible representations of G. But of course there is always one irreducible representation to hand for any G: the trivial representation. The central idempotent associated to the trivial module is

$$e_{triv} = \ \frac{1}{|G|} \sum_{g \in G} g$$

since $\chi_{triv}(g) = \chi_{triv}(g^{-1}) = 1$ for all g. ...

exa:Murphy (**1.4.66**) In particular for the symmetric group S_n, setting

$$m_i = \sum_{j=1}^{i-1} (ij)$$

('Murphy elements') we have

$$s_{(2,1^n)} = \sum_{i=2}^{n} m_i = (12) + ((23) + (13)) + ...$$

See also §**??**. ...

(**1.4.67**) It is interesting to consider (1.4.64) in case A is the 'generic' case of a π-modular system as in §1.8. The denominator in (1.40) will not generally be invertible in arbitrary specialisations, so some idempotents will not be defined in such a specialisation. See §**??**. ...

`exa:t3` (**1.4.68**) Example: For any k, with $\delta \in k$, $T_3(\delta)$ has basis $\{1, U_1, U_2, U_1U_2, U_2U_1\}$. By varying f in (1.4.55)–Example we get various cv forms. With f_1 returning the coefficient of 1 in the given basis we get:

$$\begin{array}{c|ccccc}
\circ & 1 & U_1 & U_2 & U_1U_2 & U_2U_1 \\ \hline
1 & 1 & U_1 & U_2 & U_1U_2 & U_2U_1 \\
U_1 & U_1 & \delta U_1 & U_1U_2 & \delta U_1U_2 & U_1 \\
U_2 & U_2 & U_2U_1 & \delta U_2 & U_2 & \delta U_2U_1 \\
U_1U_2 & U_1U_2 & U_1 & \delta U_1U_2 & U_1U_2 & \delta U_1 \\
U_2U_1 & U_2U_1 & \delta U_2U_1 & U_2 & \delta U_2 & U_2U_1
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c|ccccc}
g_{f_1} & 1 & U_1 & U_2 & U_1U_2 & U_2U_1 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 \\
U_1 & 0 & 0 & 0 & 0 & 0 \\
U_2 & 0 & 0 & 0 & 0 & 0 \\
U_1U_2 & 0 & 0 & 0 & 0 & 0 \\
U_2U_1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

which is clearly degenerate. With f_2 returning the sum of coefficients we get (i) below.

$$(i)\;\begin{array}{c|ccccc}
g_{f_2} & 1 & U_1 & U_2 & U_1U_2 & U_2U_1 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 \\
U_1 & 1 & \delta & 1 & \delta & 1 \\
U_2 & 1 & 1 & \delta & 1 & \delta \\
U_1U_2 & 1 & 1 & \delta & 1 & \delta \\
U_2U_1 & 1 & \delta & 1 & \delta & 1
\end{array}\;, \qquad
(ii)\;\begin{array}{c|ccccc}
g_{f} & 1 & U_1 & U_2 & U_1U_2 & U_2U_1 \\ \hline
1 & \delta^3 & \delta^2 & \delta^2 & \delta & \delta \\
U_1 & \delta^2 & \delta^3 & \delta & \delta^2 & \delta^2 \\
U_2 & \delta^2 & \delta & \delta^3 & \delta^2 & \delta^2 \\
U_1U_2 & \delta & \delta^2 & \delta^2 & \delta & \delta^3 \\
U_2U_1 & \delta & \delta^2 & \delta^2 & \delta^3 & \delta
\end{array}$$

Note from (i) that g_{f_2} is again degenerate.

It is known that T_3 is semisimple over $\mathbb{C}$ for some values of δ, so there is a nondegenerate symmetric cv form, depending on δ. Can one be realised in the g_f construction? How do we find one? Since traces of representations are elements of A^* we can try some of these. Are there conditions on a representation to lead to a 'nondegenerate trace'? It should be a faithful rep, and so have at least one copy of every simple. In the present case we have another possibility — the TL Markov trace[2]. With the TL Markov trace we get (ii) above. Note that (ii) is not degenerate for generic δ.

Starting with the nondegenerate form, we can compute a dual basis. We shall change the basis labelling the columns. This has the effect of changing the matrix by elementary column operations. Since the matrix is nonsingular the reduced form is the unit matrix, as required for a dual basis. Firstly we try to get the first column in reduced form. This is achieved by replacing the first basis element by

$$1 \rightsquigarrow c_1' := 1 - \frac{\delta}{\delta^2 - 1}(U_1 + U_2) + \frac{1}{\delta^2 - 1}(U_1U_2 + U_2U_1)$$

[2]The TL Markov trace is the formal extension of the Potts trace to arbitrary δ (up to an overall factor); or equivalently the diagram-loop trace **??**.

This takes every entry in the first column to zero, except the first entry, which becomes $\delta(\delta^2-2)$. This tells us that there is a basis dual — with respect to the Markov form — to the initial basis, in which the dual of 1 is $c_1 = \frac{1}{\delta(\delta^2-2)}c'_1$.

We are now already in a position to compute the central idempotent associated to the representation given by $\rho(U_i) = 0$:

$$e'_1 = \sum_i \chi(b_i)c_i = \chi(1)c_1 + \chi(U_1)c_2 + \chi(U_2)c_3 + \chi(U_1U_2)c_4 + \chi(U_2U_1)c_5 = \chi(1)c_1 + 0 \ = \ c_1$$

$$e_1 = \frac{\chi(1)}{\chi(e'_1)}e'_1 = c'_1$$

Next we could use the new first column to make the remaining entries in the first row zero. At this point we have

g_f	c_1	U_1	U_2	U_1U_2	U_2U_1
1	$\delta(\delta^2-2)$	δ^2	δ^2	δ	δ
U_1	0	δ^3	δ	δ^2	δ^2
U_2	0	δ	δ^3	δ^2	δ^2
U_1U_2	0	δ^2	δ^2	δ	δ^3
U_2U_1	0	δ^2	δ^2	δ^3	δ

However the other idempotent is evidently $1 - c_1$, so we stop here. This is not an easy way to construct central idempotents in general.

(**1.4.69**) On the other hand $T_3(1)$ is not Frobenius. What happens when we 'tune' $\delta \rightsquigarrow 1$? Evidently the form becomes degenerate, so we cannot construct a dual basis, and in particular the idempotent c_1 ceases to be well-defined. We may infer from this that there is no non-trivial central idempotent decomposition of 1 in T_3 when $\delta = 1$.

If we relax the normalisation condition for dual basis then we can get a little further. First, while still working over $\mathbb{Z}[\delta]$, we can rescale the form so that c_1 becomes $n = (\delta^2-1)c_1$:

$$n = (\delta^2-1) - \frac{\delta}{1}(U_1+U_2) + \frac{1}{1}(U_1U_2 + U_2U_1)$$

which is well-defined at $\delta = 1$ and obeys $n^2 = 0$ as well as $U_i n = 0$. That is, n lies in the radical. ...

ex:Hecke0 (**1.4.70**) Example. The Hecke algebra H_n over the field k of rational functions over the complex/rational polynomial ring in indeterminate q. See e.g. §12.3. This is essentially a flat deformation of the group algebra of the symmetric group S_n. We can regard S_n (up to isomorphism) as the quotient of Artin's braid group on generators $\langle t_i \rangle_{i=1,2,\dots,n-1}$ by $t_i^2 = 1$. Passing to the group algebra we can deform this relation to

$$(t_i+q)(t_i-1) = 0 \qquad\qquad (t_i^2 = (1-q)t_i + q) \qquad (1.41)$$ eq:Hq1

(or some rescaling thereof), that is, $t_1^{-1} = \frac{t_1+(q-1)}{q}$.

It is not obvious that this makes sense, but it does. The same set of reduced words as make a basis for kS_n work for H_n. For example

$$H_3 = k\{1, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1\}$$

In the S_n case this is a basis, and the list of inverses is the *same* basis just written out in slightly different order. In the H_n case the list of inverses is not the same basis, but it is a basis.

...Is it a dual basis?? Furthermore we have a form that lifts the form from the group case... ...OR DOES IT!? We have here a couple of attempts at a dual basis (the left columns here):

g_g	1	t_1	t_2	t_1t_2	t_2t_1	$t_1t_2t_1$
1	1	0	0	0	0	0
t_1^{-1}	?	1	0	0	0	0
t_2^{-1}	?	0	1	0	0	0
t_1t_2	0		?	?		
t_2t_1	0		?	?	?	0
$t_1t_2t_1$	0	0	0	0	0	q^3

g_g	1	t_1	t_2	t_1t_2	t_2t_1	$t_1t_2t_1$
1	1	0	0	0	0	0
t_1	0	q	0	0	0	0
t_2	0	0	q	0	0	0
t_2t_1	0	0	0	q^2	0	0
t_1t_2	0	0	0	0	q^2	0
$t_1t_2t_1$	0	0	0	0	0	q^3

It feels like $\langle t_1^{-1}, 1\rangle = g_1(t_1^{-1}) = \frac{q-1}{q}$, so the left-hand try won't work. The right-hand try just needs renormalising, then it will be ok.

From (1.39) we have

$$\begin{aligned} e'_i \;=\; & \chi_i(1)\,1 \;+\; \chi_i\left(\frac{1}{q}t_1\right)\,t_1 \;+\; \chi_i\left(\frac{1}{q}t_2\right)\,t_2 \\ & +\chi_i\left(\frac{1}{q^2}(t_1t_2)^t\right)t_1t_2 \;+\; \chi_i\left(\frac{1}{q^2}(t_2t_1)^t\right)t_2t_1 \;+\; \chi_i\left(\frac{1}{q^3}(t_1t_2t_1)\right)t_1t_2t_1 \end{aligned}$$

where t just means reverse the word order. For example noting (1.41) we can take $\chi_0(g) = 1$, giving

$$e'_0 = 1 + \frac{1}{q}t_1 + \frac{1}{q}t_2 + \frac{1}{q^2}t_1t_2 + \frac{1}{q^2}t_2t_1 + \frac{1}{q^3}t_1t_2t_1$$

Acting with t_1 we get

$$t_1e'_0 = t_1 + \frac{1}{q}t_1^2 + \frac{1}{q}t_1t_2 + \frac{1}{q^2}t_1^2t_2 + \frac{1}{q^2}t_1t_2t_1 + \frac{1}{q^3}t_1^2t_2t_1$$

$$= t_1 + \frac{1}{q}(1-q)t_1 + \frac{1}{q}q + \frac{1}{q}t_1t_2 + \frac{1}{q^2}(1-q)t_1t_2 + \frac{1}{q^2}qt_2 + \frac{1}{q^2}t_1t_2t_1 + \frac{1}{q^3}(1-q)t_1t_2t_1 + \frac{1}{q^3}qt_2t_1$$

$$= 1 + \frac{1}{q}t_1 + \frac{1}{q}t_2 + \frac{1}{q^2}t_1t_2 + \frac{1}{q^2}t_2t_1 + \frac{1}{q^3}t_1t_2t_1$$

as required. Thus

$$e'_0e'_0 = \left(1 + \frac{2}{q} + \frac{2}{q^2} + \frac{1}{q^3}\right)e'_0 = \frac{1+2q+2q^2+q^3}{q^3}e'_0$$

We can make $\chi_{1^3}(t_1) = -q$. Here then

$$e'_{1^3} \;=\; 1 - t_1 - t_2 + t_1t_2 + t_2t_1 - t_1t_2t_1$$

Here

$$e'_{1^3}e'_{1^3} = \left(1 + 2q + 2q^2 + q^3\right)e'_{1^3}$$

Check: .....

1.4.11 Krull–Schmidt Theorem over Artinian rings

The next few sections consider projective modules in general, and in particular consider an Artinian ring R as a left-module for itself.

Krull **(1.4.71) Theorem.** (Krull–Schmidt) If R is Artinian then as a left-module for itself it is a finite direct sum of indecomposable modules (as in (1.4.11) or §8.2.2); and any two such decompositions may be ordered so that the i-th summands are isomorphic.

Proof. Exercise. (See also §8.3.2.)

1.4.12 Projective modules over arbitrary rings

ss:proj0001

(**1.4.72**) If $x : M \to M'$, $x' : M' \to M$ are R-module homomorphisms such that $x \circ x' = 1_{M'}$ then x is a *split surjection* (and x' a split injection).

de:iproj (**1.4.73**) An R-module is *projective* if it is a direct summand of a free module (an R-module with a linearly independent generating set).

(**1.4.74**) EXAMPLE. $e^2 = e \in R$ implies left-module Re projective, since it is a direct summand of free module R, by (1.35).

th:proj intro (**1.4.75**) **Theorem.** TFAE
(I) R-module P is projective;
(II) whenever there is an R-module surjection $x : M \to M'$ and a map $y : P \to M'$ then there is a map $z : P \to M$ such that $x \circ z = y$;
(III) every R-module surjection $t : M \to P$ splits.

Proof. Exercise. (See also §8.6.)

1.4.13 Structure of Artinian rings

:structArtinian1

th:ASTI (**1.4.76**) If R is Artinian and J_R its radical then R/J_R is semisimple so by (1.4.48):

$$R/J_R \;=\; \oplus_{i \in l(R)} M_{n_i}(R_i)$$

for some set $l(R)$, numbers n_i and division rings R_i. There is a simple R/J_R-module (L_i say) for each factor, so that *as a left module*

$$R/J_R \cong \oplus_i n_i L_i$$

(i.e. n_i copies of L_i). There is a corresponding decomposition of 1 in R/J_R:

$$1 = \sum_i e_i$$

into orthogonal idempotents. One may find corresponding idempotents in R itself (see later) so that $1 = \sum_i e'_i$ there. This gives left module decomposition

$$R = \oplus_i n_i P_i$$

where (by (1.4.71)) the P_is are a complete set of indecomposable projective modules up to isomorphism.
(See also §8.7.)

1.4.14 Finite dimensional algebras over algebraically closed fields

(**1.4.77**) Let A be a finite dimensional algebra over an algebraically closed field k. Let $\{L_i\}_{i\in\Lambda}$ be a set of isomorphism classes of simple A-modules L_i. Then $\dim A \;\geq\; \sum_{i\in\Lambda}(\dim L_i)^2$; with equality iff the set is complete and A semisimple.

Proof. Cf. Prop.1.4.54 and 1.4.51. Exercise.

a2 (**1.4.78**) THEOREM. *For A as above, and J_A the radical, suppose ${}_AA$ filtered by a set $\{S_i\}$. Then $\sum_i(\dim S_i)^2 \;\geq\; \dim(A/J_A)$ with equality iff $\{S_i\}$ a (necessarily complete) set of simples.*

1.5 Nominal aims of representation theory

ss:NAoRT

So, what are the aims of representation theory? For Artinian algebras they are, broadly and roughly speaking, to describe the (finite dimensional) modules, and their homomorphisms. One might also be looking for representations (i.e. module bases) with special properties (perhaps motivated by physics). But in any case, it is worth being a bit more specific about this 'description'.

Typically, to start with, one is looking for *invariants* — properties of modules that would be manifested by any isomorphic algebra; so that one can, say, determine from representation theory whether two algebras are isomorphic (or more easily, that two algebras are *not* isomorphic).

An example of an invariant would be the number of isomorphism classes of simple modules — this would be the same for any isomorphic algebra... See (1.3.18) for a specific example.

de:fund inv (**1.5.1**) Given an Artinian algebra R (let us say specifically a finite dimensional algebra over an algebraically closed field k, so that each $R_i = k$ in (1.4.76)), we are called on
(A0) to determine a suitable indexing set $l(R)$ as in (1.4.76),
(A0') to determine the blocks as a partition of $l(R)$,
(AI) to compute the fundamental invariants $\{n_i \;:\; i \in l(R)\}$,
(AII) to give a construction of the simple modules L_i,
(AIII) to compute composition multiplicites for the indecomposable projective modules P_i,
(AIV) to compute Jordan-Holder series for the modules P_i.
(AV) to compute some further invariants (see e.g. (1.5.9) below).

(**1.5.2**) Note that (AI) contains (A0), and completely determines the maximal semisimple quotient algebra up to isomorphism (by the Artin–Wedderburn Theorem). Aim (AII) is not an invariant, so does not have a unique answer; but having at least one such construction is clearly desirable in studying an algebra (and any answer for (AII) contains (AI)).

Of course there are unboundedly many nonisomorphic algebras with the same maximal semisimple quotient in general, so we need more information to classify non-semisimple algebras.

The aim (AIII) is an invariant, and tells us more about a non-semisimple algebra. Aim (AIV) contains (AIII). But still, (AIV) is not enough to classify algebras in general. It is very useful partial data, however. And we will usually consider this to be 'enough' for most purposes (applications, for example). We will say a little next about futher (and possibly complete) invariants; before returning to study the above aims in detail.

(**1.5.3**) At a further level, we might also try the following. To investigate the isomorphism classes of indecomposable modules (beyond projective modules).

(**1.5.4**) Some invariants are invariants of isomorphism classes of algebras. Some are invariants of 'Morita' equivalence classes of algebras (see §1.7.2). This latter is a weaker (but very useful) notion. The number $l(R)$ is an invariance of Morita equivalence. The multiset $\{n_i\}$ is an invariance of isomorphism.

1.5.1 Radical series and socle of a module

ss:Loewy1

(**1.5.5**) Fix an algebra A. Given an A-module M, its *radical* $\mathrm{Rad}(M)$ is the intersection of maximal submodules. The *radical series* of M is

$$M \supset \mathrm{Rad}\ M \supset \mathrm{Rad}\ \mathrm{Rad}\ M \supset ...$$

The sections $\mathrm{Rad}\ ^i M/\mathrm{Rad}\ ^{i+1}M$ are the *radical layers*. In particular

$$\mathrm{Head}(M) \ = \ M/\mathrm{Rad}\ M$$

$$\mathrm{Shoulder}(M) \ = \ \mathrm{Rad}\ M/\mathrm{Rad}\ ^2M \ = \ \mathrm{Head}(\mathrm{Rad}\ M)$$

pr:mradM

(**1.5.6**) PROPOSITION. *(I) Module M is semisimple (of finite length) iff Artinian and Rad $M = 0$. (II) If a module M is Artinian then $M/\mathrm{Rad}\ M$ is semisimple.* ■

de:socle0

(**1.5.7**) (See also 8.7.4.) The *socle* $\mathrm{Soc}(M)$ of a module is the maximal semisimple submodule. One can form socle layers: $\mathrm{Soc}(M)$, $\mathrm{Soc}(M/\mathrm{Soc}(M))$, $\mathrm{Soc}((M/\mathrm{Soc}(M))/\mathrm{Soc}(M/\mathrm{Soc}(M)))$, ... in the obvious way. These layers do not agree, in general, with the reverse of the radical layers; but the lengths of sequences agree if defined.

(**1.5.8**) Let A be a finite dimensional algebra over an algebraically closed field. (Then the radical series of any finite dimensional module terminates; and the sections are semisimple modules, by Prop.1.5.6.) Here we put indexing set $l(A) = \Lambda(A)$. For the indecomposable projective A-modules $\{P_i\}_{i\in\Lambda(A)}$ then

$$\{P_i\}_{i\in\Lambda(A)} \leftrightarrow \{S_i = \mathrm{Head}(P_i)\}_{i\in\Lambda(A)}$$

is a bijection between indecomposable projectives and simples. In general we have

$$\mathrm{Head}(M) \ \cong \ \bigoplus_{i\in\Lambda(A)} \ \underbrace{m_i^0(M)}_{multiplicity} \ S_i$$

$$\mathrm{Shoulder}(M) \ \cong \ \bigoplus_{i\in\Lambda(A)} m_i^1(M) \ S_i$$

(and so on) for some multiplicities $m_i^l(M) \in \mathbb{N}_0$.

A *radical Loewy diagram* of an Artinian module M gives the radical layers:

$$\begin{array}{ccllllll} M & = & S_{0,1} & S_{0,2} & S_{0,3} & ... & S_{0,l_0} & \\ & & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & ... & S_{1,l_1} \\ & & S_{2,1} & S_{2,2} & ... & & & \\ & & ... & & & & & \end{array}$$

(the multiset of simple modules $\{S_{0,1}, S_{0,2}, ...\}$ encodes $\mathrm{Head}(M)$ and so on). We give some examples in §1.5.2.

ss:quiv00

1.5.2 The ordinary quiver of an algebra

de:quiv1

(**1.5.9**) The ordinary quiver of an algebra. (...See §3.6 for details.)

How *do* we classify finite dimensional algebras (over an algebraically closed field) up to isomorphism; or up to Morita equivalence?

(**1.5.10**) An algebra is *connected* if it has no proper central idempotent. Every algebra is isomorphic to a direct sum of connected algebras, so it is enough to classify connected algebras (and then, for an arbitrary algebra, give its connected components).

de:basicalg0

(**1.5.11**) An algebra is *basic* if every simple module is one-dimensional. (See also (1.6.9).) Every algebra is Morita equivalent to (i.e. has an equivalent module category to) a basic algebra. So it is enough to classify basic connected algebras.

(**1.5.12**) The *Ext-matrix* $\mathcal{M}(A)$ of algebra A is given by the 'shoulder data'

$$\mathcal{M}(A)_{ij} \;=\; m^1_i(P_j)$$

A necessary condition for algebra isomorphism $A \cong B$ is that there is an ordering of the index sets such that $\mathcal{M}(A) = \mathcal{M}(B)$.

The *Ext-quiver* or *ordinary quiver* $Q(A)$ of algebra A is the matrix $\mathcal{M}(A)$ expressed as a graph. Note that $Q(A)$ is connected as a graph if A is connected as an algebra. Isomorphism $A \cong B$ implies isomorphic Ext-quivers, but not v.v.. However one can characterise any connected basic algebra A up to isomorphism using a quotient of the *path algebra* $kQ(A)$ of $Q(A)$ (given a quiver Q, then kQ is the k-algebra with basis of walks on Q and composition on walks by concatenation where defined, and zero otherwise [3]), as we describe in §**??**. Specifically we have the following.

(**1.5.13**) Theorem. [51, §4.3] *For any connected basic algebra A there is an ideal I_A in $kQ(A)$ (contained in $I_{\geq 2}$ and containing $I_{\geq m}$ for some m) such that*

$$A \cong kQ(A)/I_A$$

Proof. First note that there is a surjective algebra homomorphism $\Psi : kQ(A) \to A$. The walks of length-0 pass to a set of idempotents such that $P_i = Ae_i$. The walks of length-1 from i to j pass to a basis for $e_i J_A e_j / e_i J_A^2 e_j$.

Next we need to show that the kernel of Ψ has the required form. See e.g. [7, Prop.1.2.8]. ∎

(**1.5.14**) Thus we can determine (characterise up to isomorphism) such a connected basic A by computing $Q(A)$ and then giving elements of $kQ(A)$ that generate I_A. (Note however that generators for I_A are not unique in general.)

More generally then, one can determine an arbitrary algebra A by giving the corresponding data for its connected components; together with the dimensions of the simple modules.

(**1.5.15**) Given $A \cong kQ(A)/I_A$, we can recover structural data about the indecomposable projective modules as follows. Write e_a for the path of length 0 from vertex a (sometimes we just write $a = e_a$ for this). This is an idempotent in $kQ(A)$. Then

$$P_a = Ae_a$$

[3]Note that walks of length at least l span an ideal in kQ. Write $I_{\geq l}$ for this ideal.

(identifying A with $kQ(A)/I_A$ here without loss of generality). Thus a basis for P_a is the set of all paths from a 'up to the quotient'. This is the path of length 0 (corresponding to the head); and all the paths of length 1 (the shoulder); and some paths of length 2; and so on.

Note that (the image of) $I_{\geq 1}$ lies in the radical of kQ/I_A, since the m-th power lies in $I_{\geq m} \equiv 0$. Hence the image of $I_{\geq 1}$ *is* the radical.

(**1.5.16**) Let us give some low-dimensional examples of algebras of form Q/I_A, where $I_A \subset I_{\geq 2}$ and $I_A \supset I_{\geq m}$ for some m.

For Q a single point then kQ is one-dimensional and $I_{\geq 2} = 0$. Indeed any kQ with $I_{\geq 1} = 0$ is semisimple — the quiver is just a collection of points. Let us give some non-semisimple examples. For

$$\begin{tikzcd} a \arrow[loop above, "u"] \end{tikzcd} \qquad \text{with relation } u^2 = 0$$

we have a 2d algebra with 1 simple S_a. The corresponding projective P_a is $P_a = Aa = k\{a, ua\}$ (it terminates here since $aua = ua = u$ and $u^2 a = 0$ and so on), in which $k\{ua\}$ is a submodule (of, in a suitable sense, length-1 elements) isomorphic to S_a. That is, a radical Loewy diagram for P_a is

$$P_a \quad = \quad \begin{array}{c} S_a \\ S_a \end{array}$$

There is a 1-simple algebra in each dimension obtained by replacing $u^2 = 0$ by $u^d = 0$.

Alternatively in 3d, we can take the quiver with 1 vertex and two loops u, v, together with the relations $uu = uv = vu = vv = 0$. The quiver

$$\begin{tikzcd} a & b \arrow[l, "x"'] \end{tikzcd} \qquad \text{with no relations}$$

(again $I_{\geq 2} = 0$ here) gives another 3d algebra, this time with 2 simples.

The quiver

$$\begin{tikzcd} a \arrow[r, bend left, "x"] & b \arrow[l, bend left, "s"] \end{tikzcd} \qquad \text{with } sx = 0$$

has basis $\{a, b, xa, sb, xsb\}$. (Note that the given relation is sufficient to make kQ/I_A finite, but otherwise an arbitrary choice for an example here.) The indecomposable projective Aa is generated by walks out of a: a, xa, $sxa = 0$, that is, it terminates after one step. The projective $P_b = Ab$ has walks b, sb, xsb, $sxsb = 0$.

(**1.5.17**) What about this?:

$$\begin{tikzcd} a \arrow[r, bend left, "x_{ab}"] & b \arrow[l, bend left, "x_{ba}"] \arrow[r, bend left, "x_{bc}"] & c \arrow[l, bend left, "x_{cb}"] \end{tikzcd} \qquad \text{with } x_{bc}x_{ab},\ x_{ba}x_{cb},\ x_{ba}x_{ab} \text{ and } x_{ab}x_{ba} - x_{cb}x_{bc} \text{ in } I_A.$$

(These relations are another arbitrary finite choice here. However these particular relations will appear 'in the wild' later.) We have $P_a = Aa = k\{a, x_{ab}a\}$. Next $P_b = Ab = k\{b, x_{ba}b, x_{bc}b, x_{ab}x_{ba}b\}$.

Finally $P_c = Ac$. Note the submodule structure of P_b. As ever there is a unique maximal submodule Rad $P_b = k\{x_{ba}b, x_{bc}b, x_{ab}x_{ba}b\}$. The intersection of the maximal submodules of this, in turn, is spanned by $x_{ab}x_{ba}b$. Thus the radical layers of the projectives look like this:

$$P_a = \begin{array}{c} S_a \\ S_b \\ \end{array} \qquad P_b = \begin{array}{ccc} & S_b & \\ S_a & & S_c \\ & S_b & \end{array} \qquad P_c = \begin{array}{c} S_c \\ S_b \\ S_c \end{array}$$

Remark. This case exemplifies a very interesting point: that the presence of a simple module as a compostion factor for a module always allows for a corresponding homomorphism from the indecomposable projective cover of that simple module. Here in particular there is no homomorphism from S_a to P_b, say, but there is a homomorphism from P_a to P_b. See later.

(**1.5.18**) What about this?:

$$\begin{tikzcd} a \arrow[r, bend left, "x"] & b \arrow[l, bend left, "s"] \arrow[loop above, "u"] \end{tikzcd}$$

Determine some conforming relations to make a finite quotient of kQ. ...

1.6 Idempotents, Morita hints, primitive idempotents

ss:xxid

1.6.1 Morita hints

We started by thinking about matrix representations of groups, and this has led us naturally to consider modules over algebras. Two components of this progression have been (i) the passage to natural new algebraic structures (from groups to rings to algebras) on which to study representation theory; and (ii) the organisation of representations into equivalence classes (de-emphasising the basis). Representation theory studies algebras by studying the structure preserving maps between algebras (a map from the algebra under study to a known algebra gives us the modules for the known algebra as modules for the new algebra). We could go further and de-emphasise the modules in favour of the maps between them. This is one route into using 'category theory' (cf. §1.7).

(**1.6.1**) Let A be an algebra over k and $e^2 = e \in A$ (e not necessarily central, cf. 1.4.42). The *Peirce decomposition* (or Pierce decomposition! [33, 34, §6]) of A is

$$A = \; eAe \oplus (1-e)Ae \oplus eA(1-e) \oplus (1-e)A(1-e) \;\; = \;\; \bigoplus_{i,j} e_i A e_j$$

where $e_1 = e$ and $e_2 = 1-e$. (Question: What algebraic structures are being identified here? This is an identification of vector spaces; but the algebra multiplication is also respected. On the other hand not every summand on the right is unital.)

This decomposition is non-trivial if $1 = e + (1-e)$ is a non-trivial decomposition. Set $A(i,j) = e_i A e_j$. These components are not-necessarily-unital 'algebras', and non-unit-preserving subalgebras of A. The cases $A(i,i)$ are unital, with identity e_i.

Can we study A by studying the algebras $A(i,i)$?

(**1.6.2**) EXAMPLE. Consider $M_3(\mathbb{C})$ and the idempotent $e_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have the corresponding vector space decomposition (not confusing $\oplus$ with $\oplus^{\cdot}$)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

(which is not necessarily a particularly interesting decomposition, but see later).

de:primid1 (**1.6.3**) If we can further decompose e into orthogonal idempotents then there is a corresponding further Peirce decomposition. This decomposition process terminates when some $e = e_\pi$ has no decomposition in A (it is 'primitive'). What special properties does $e_\pi A e_\pi$ have then?

(**1.6.4**) Later we will provide detailed answers to the questions raised above. For now, our next objective will be to construct some interesting examples. We return to this discussion in (8.6.13) and §9.4.1 and §13.4.2.

1.6.2 Primitive idempotents

(**1.6.5**) An orthogonal decomposition of 1 into primitive idempotents (in the sense of 1.6.3) is called a 'complete' orthogonal decomposition.

For examples see §9.3.1.

(**1.6.6**) Aside: Let $1 = \sum_{i \in H} e_i$ be an orthogonal idempotent decomposition, and extend the definition of $A(i,j)$ to this case. Note that we have a composition $A(i,j) \times A(k,l) \to A(i,l)$ given by $a \circ b = ab$ in A. But in particular $ab = 0$ unless $j = k$. Thinking along these lines we see that the orthogonal idempotent decomposition of $1 \in A$ gives rise to a category (see §1.7,§6.1) 'hiding' in A. The category is $A_H = (H, A(i,j), \circ)$.

th:eRe-Re1 (**1.6.7**) THEOREM. *If a ring R is left or right Artinian then it has a complete orthogonal idempotent decomposition of* 1, $1 = \sum_{i=1}^{l} e_i$ *say, with $e_i R e_i$ a local ring.*

If $e_i R e_i$ is local then e_i is primitive and Re_i is indecomposable projective. ■

(**1.6.8**) EXAMPLE. Fix a field k and $\delta \in k^*$. Recall the algebra T_n and idempotents $\frac{1}{\delta}\mathsf{e}_1$, $\frac{1}{\delta}\mathsf{e}_2 = \frac{1}{\delta} 1 \otimes \mathsf{e} \otimes 1 \otimes 1 \otimes ...$ and so on. Consider the quotient algebra $T'_n = T_n / T_n \mathsf{e}_1 \mathsf{e}_3 T_n$.

PROPOSITION. (1) For $n > 3$ the element $\frac{1}{\delta}\mathsf{e}_1$ is idempotent but not primitive in T_n.
(2) For $i = 1, 2, ..., n-1$ the (image of the) idempotent $\frac{1}{\delta}\mathsf{e}_i$ is primitive in T'_n.
(3) We have a left-T'_n-module isomorphism $T'_n \mathsf{e}_i \cong T'_n \mathsf{e}_j$ for all i, j.

Proof. (1) We can see that $\mathsf{e}_1 T_n \mathsf{e}_2 \cong T_{n-2}$ as an algebra. But T_{n-2} is not local ring for $n > 3$.
(2) Note that $\mathsf{e}_1 T'_n \mathsf{e}_1 = k\mathsf{e}_1 \cong k$.
(3) Exercise. ■

On the other hand we have the following. Consider the fixed subring $\overline{T}_n$ of T_n with respect to the left-right diagram flip involutive automorphism. What can we say about analogous quotient algebras and analogous primitive idempotents in this case.

de:basicalgebra (**1.6.9**) An Artinian ring R, with complete set $\{e_1, e_2, ..., e_l\}$ of orthogonal idempotents, is *basic* if $Re_i \cong Re_j$ as left-R-modules implies $i = j$. (Cf. also (1.5.11).)

(**1.6.10**) EXAMPLE. The k-subalgebra of $M_2(k)$ given by $A_{1,1} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k\}$ has a complete set $\{e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. One easily checks that $A_{1,1}e_1 \ncong A_{1,1}e_2$ (consider the action of e_1 on each side, say), so $A_{1,1}$ is basic.

On the other hand $M_2(k)$ has the same complete set, but $M_2(k)e_1 \cong M_2(k)e_2$, so $M_2(k)$ is not basic.

(**1.6.11**) One can check that if a finite-dimensional k-algebra A is basic then every simple R-module is 1-dimensional.

(**1.6.12**) (We will see shortly that) For every finite-dimensional k-algebra there is a basic algebra having an 'equivalent module category'.

1.6.3 General idempotent localisation

If $e^2 = e \in A$ and M an A-module, then eM is an eAe-module.

pr:eMsimple (**1.6.13**) PROPOSITION. *If M is a simple A-module; and $e^2 = e \in A$. Then eM is a simple eAe-module or zero.* ■*(See e.g. §13.4.2.)*

pr:eMJH (**1.6.14**) PROPOSITION. [Jordan–Holder localisation] *Let k be a field, and A a finite dimensional k-algebra. Let M be an A-module. Let $M \supset M_1 \supset ...$ be a Jordan–Holder series for M, with simple factors $L_i = M_i/M_{i+1}$. Let $e^2 = e \in A$. Then*
(I) $eM \supseteq eM_1 \supseteq ...$ becomes a JH series for ${}_{eAe}eM$ on deleting the terms for which $eM_i/eM_{i+1} = eL_i = 0$. In particular if $eL \neq 0$ for some simple L then composition multiplicity

$$(M : L) = (eM : eL).$$

Thus in particular, (II) if ${}_{eAe}eM$ is simple then the composition factors of M include a factor L_e, such that $eM = eL_e$, appearing once, and any other factors L obey $eL = 0$.

(III) If ${}_{eAe}eAe$ is simple (i.e. if eAe is a copy of the ground field k) then the composition factors of $M = Ae$ are a simple head factor $L_e = eL_e$ appearing once, and any other factors L obey $eL = 0$.

Proof. (I,II) See e.g. (13.9). (III) Note that eAe simple as a left-module implies that it is local as a ring, so Ae is indecomposable projective, so has a unique maximal submodule M_1. Noting (II), we need only show that the head M/M_1 is not killed by e. For a contradiction suppose $e(M/M_1) = 0$. For any $M \supset M'$ we have $e(M/M') = eM/eM'$ (just unpack the definitions). Thus $e(M/M_1) = 0$ implies $eM/eM_1 = 0$, which implies $eM = eM'$. But $AeM = AeAe = Ae$ while $AeM_1 \subset Ae$, giving a contradiction. ■

In particular for the proof of (I) it will be convenient to have a category theoretic context... see §1.7.

We can also show that if eM has simple head then so does M [**?**] (again it will be convenient to introduce some 'globalisation' category theory first).

(**1.6.15**) THEOREM. [Green localisation theorem, [57, §6.2]] *Let k be a field, let A be a k-algebra, and $e \in A$ idempotent. Let $\Lambda(A), \Lambda(eAe)$ and $\Lambda(A/AeA)$ be index sets for classes of simple modules* *of the indicated algebras. Then there is a bijection*

$$\Lambda(A) \xrightarrow{\sim} \Lambda(eAe) \sqcup \Lambda(A/AeA)$$

■

1.7 Small categories and categories

ss:cat0001

See §6.1 for more details and references (or see Adamek [1], or Jacobson [72] for now).

Categories are useful from at least two different perspectives in representation theory. One is in the idea of de-emphasising modules in favour of the (existence of) morphisms between them. Another is in embedding our algebraic structures (our objects of study) in yet more general settings.

(**1.7.1**) A *small category* is a quadruple, $\mathcal{A}$ say, of the form $(ob\,\mathcal{A},\ \mathcal{A}(-,-),\ 1_-,\ \circ_{\mathcal{A}})$ consisting of
a set $ob\,\mathcal{A}$ (or $\mathcal{A}_0$, of 'objects'); and
for each element $(a,b) \in ob\,\mathcal{A} \times ob\,\mathcal{A}$ a set $\mathcal{A}(a,b)$ (or $hom_{\mathcal{A}}(a,b)$, of 'arrows' or 'morphisms'); and
for each $a \in ob\,\mathcal{A}$ an element $1_a \in \mathcal{A}(a,a)$ (called 'identity'); and
for each element $(a,b,c) \in ob\,\mathcal{A}^{\times 3}$ a composition: $\circ_{\mathcal{A}} : \mathcal{A}(a,b) \times \mathcal{A}(b,c) \to \mathcal{A}(a,c)$, satisfying associativity and identity conditions (writing $\circ_{\mathcal{A}}(f,g) = f \circ_{\mathcal{A}} g$ then $1_a \circ_{\mathcal{A}} f = f = f \circ_{\mathcal{A}} 1_b$ whenever these make sense).

Jacobson [72, p.9] organises the composition as above, but writes $(\alpha,\beta) \mapsto \beta\alpha$ (for the popular juxtaposition-is-composition notation, which might be more naturally written $(\alpha,\beta) \mapsto \alpha\beta$). In this way collections of sets, set maps and function composition can form a category.

We generally write $\mathcal{A}$ for the category as above. If we confuse $\mathcal{A}$ notationally with a set it is usually the class of all morphisms. (But sometimes $\mathcal{A}$ might be confused with its object set. Obviously these two conventions are not consistent.)

(**1.7.2**) A *category* is a similar structure allowing larger classes of objects and arrows.

(**1.7.3**) Example: A monoid is a (small) category with one object.

(**1.7.4**) Example: $ob\,\mathcal{A} = \mathbb{N}$ and $\mathcal{A}(m,n)$ is $m \times n$ matrices over a ring R.

(**1.7.5**) Example: $ob\,A$ is a set of R-modules and $A(M,N)$ is the set of R-module homomorphisms from M to N. (The category $R-$mod is the category of all left R-modules.)

de:Pcat1

(**1.7.6**) The product in (1.28) generalises to a category P in an obvious way, with object set $\mathbb{N}_0$. There is a corresponding T subcategory.

(**1.7.7**) We may construct an 'opposite' category A^o from category A, with the same object class, by setting $A^o(a,b) = A(b,a)$ and reversing the compositions.

1.7.1 Functors

(**1.7.8**) A *functor* is a map $F : A \to B$ between (small) categories (consisting of an object map F_0 say, and a collection of morphism maps F_1) that preserves composition and identities.

de:functoreg0001 (**1.7.9**) Example: (I) If R is a ring and $e^2 = e \in R$ then there is a map $F_e : R\text{−mod} \to eRe - \text{mod}$ given by $M \mapsto eM$ that extends to a functor.

de:homfunctintro (**1.7.10**) (II) If R is a ring and N a left R-module then there is a map

$$\text{Hom}(N, -) : R\text{−mod} \to \mathbb{Z} - \text{mod}$$

given by $M \mapsto \text{Hom}(N, M)$. This extends to a functor by $L \xrightarrow{f} M \mapsto (N \xrightarrow{g} L \mapsto N \xrightarrow{f \circ g} M)$.

de:homfunctproj (**1.7.11**) The functor $\text{Hom}(N, -)$ has some nice properties. Consider a not-necessarily short-exact sequence $0 \longrightarrow M' \xrightarrow{\mu} M \xrightarrow{\nu} M'' \longrightarrow 0$ and its not-necessarily exact image

$$0 \longrightarrow \text{Hom}(N, M') \stackrel{\mu_N = \text{Hom}(N,\mu)}{\longrightarrow} \text{Hom}(N, M) \stackrel{\nu_N = \text{Hom}(N,\nu)}{\longrightarrow} \text{Hom}(N, M'') \longrightarrow 0.$$

$$N \xrightarrow{f} M' \qquad \mapsto \qquad N \xrightarrow{\mu \circ f} M$$

We can ask (i) if exactness at M' implies $\ker \mu_N = 0$; (ii) if exactness at M implies $\text{im}\, \mu_N = \ker \nu_N$; (ii') if $\nu \circ \mu = 0$ implies $\nu_N \circ \mu_N = 0$; (iii) if exactness at M'' implies $\text{im}\, \nu_N = \text{Hom}(N, M'')$?

(i) Since μ injective, $\mu \circ f = \mu \circ g$ implies $f = g$. But then $\mu \circ f = 0$ implies $f = 0$, so $\ker \mu_N = 0$.

(ii) See (8.5.6). (The answer if yes if exact at M' *and* M.)

(ii') $\text{Hom}(N, \nu) \circ \text{Hom}(N, \mu) = \text{Hom}(N, \nu \circ \mu) = 0$.

(iii) This does not hold in general. However if N is projective then by Th.1.4.75(II), given exactness at M'', every $\gamma \in \text{Hom}(N, M'')$ can be expressed $\nu \circ g$ for some $g \in \text{Hom}(P, M)$, so then (iii) holds.

We will give some more examples shortly — see e.g. (1.7.12).

ex:functy (**1.7.12**) Let $\psi : A \to B$ be an map of algebras over k. We define functor

$$\text{Res}_\psi : B\text{−mod} \to A\text{−mod}$$

by $\text{Res}_\psi M = M$, with action of $a \in A$ given by $am = \psi(a)m$ for $m \in M$; and by $\text{Res}_\psi f = f$ for $f : M \to N$.
We need to check that Res_ψ extends to a well-defined functor, i.e. that every B-module map $f : M \to N$ is also an A-module map. We have $bf(m) = f(bm)$ for $b \in B$ and $m \in M$. Consider $af(m) = \psi(a)f(m) = f(\psi(a)m)$, where the second identity holds since $\psi(a) \in B$. Finally $f(\psi(a)m) = f(am)$ and we are done.

See §2.3.2 for properties of Res_ψ.

(**1.7.13**) In order to develop a useful notion of equivalence of categories we need the notion of a natural transformation — a map between functors.

1.7.2 Natural transformations, Morita equivalence, adjoints

ss:ME0

For now see (6.1.40) for natural transformations. A natural isomorphism is a natural transformation whose underlying maps are isomorphisms.

Two categories A, B are equivalent if there are functors $F : A \to B$ and $G : B \to A$ such that the composites FG and GF are naturally isomorphic to the corresponding identity functors.

(**1.7.14**) Two categories are equivalent if there are functors between them whose composite is in a suitable sense isomorphic to the identity functor. We talk about making this precise later. For now we will rather aim to build some illustrative examples.

`de:adjointI` (**1.7.15**) Consider functors $C \leftrightarrows^{F}_{G} C'$. Then (F, G) is an *adjoint pair* if for each suitable object pair M, N there are natural bijections $\mathrm{Hom}(FM, N) \mapsto \mathrm{Hom}(M, GN)$.

1.7.3 Aside: Special objects and arrows

(**1.7.16**) An arrow f is *epi* if $gf = g'f$ implies $g = g'$ (see e.g. Mitchell [**?**]).

Given a category $\mathcal{A}$ we write $A \overset{f}{\twoheadrightarrow} B$ if f is epi.

(**1.7.17**) An arrow f is *mono* if $fg = fg'$ implies $g = g'$.

Given a category $\mathcal{A}$ we write $A \overset{f}{\hookrightarrow} B$ if f is mono.

If $A \overset{f}{\hookrightarrow} B$ then we say A is a *subobject* of B.

(**1.7.18**) Next we should define the notions of isomorphism; isomorphic subobject; and balanced category.

`de:projincat1` (**1.7.19**) An object P is *projective* if for every $P \overset{h}{\to} B$ and $A \overset{f}{\twoheadrightarrow} B$ then $h = ff'$ for some $P \xrightarrow{f'} A$. (Cf. (1.4.75)(II).)

(**1.7.20**) A category $\mathcal{A}$ *has enough projectives* if there is an $P \overset{f}{\twoheadrightarrow} A$, with P projective, for each object A.

`de:zeroobject` (**1.7.21**) An object O in category $\mathcal{A}$ is a *zero object* if every $\mathcal{A}(M, O)$ and $\mathcal{A}(O, M)$ contains a single element.

If there is a unique zero object we denote it 0. In this case we also write $M \xrightarrow{0} 0$ and $0 \xrightarrow{0} M$ for all the 'zero-arrows' (even though they are distinct); and $M \xrightarrow{0} N$ for the arrow that factors through 0.

`de:kernelI` (**1.7.22**) Here we suppose that $\mathcal{A}$ has a unique zero-object.

A *prekernel* of $A \xrightarrow{f} B$ is any pair $(K, K \xrightarrow{k} A)$ such that $fk = 0$.

A *kernel* of $A \xrightarrow{f} B$ is a prekernel $(K, K \xrightarrow{k} A)$ such that if $(K', K' \xrightarrow{k'} A)$ is another prekernel then there is a unique $K' \xrightarrow{g} K$ such that $kg = k'$.

(**1.7.23**) Note that if $(K, K \xrightarrow{k} A)$ is a kernel of f then k is mono, and K is an isomorphic suboject of A to every other kernel object of f (see later).

Exercise: consider the existence and uniqueness of kernels.

(**1.7.24**) Next we should define normal categories and exact categories; define exact sequences. —FINISH THIS SECTION!!!—

(**1.7.25**) A category of modules has a lot of extra structure and special properties compared to a generic category (see Freyd [48] or §**??** for details). For example: (EI) The arrow set $A(M, N) = \mathrm{Hom}(M, N)$ is an abelian group; composition of arrows is bilinear. (An *additive* functor between such categories respects this extra structure.) (SII) There is a unique object 0 such that $\mathrm{Hom}(M,0) \cong \mathrm{Hom}(0,M) \cong \{0\}$ for all M (by $0 : M \to 0$ we mean this zero-arrow — an abuse of notation!). (SIII) Given objects M, N there is a categorical notion of an object $M \oplus N$, and these objects exist. (SIV) There is a function ker associating to each arrow $f \in \mathrm{Hom}(M,N)$ an object K_f and an arrow $k_f \in \mathrm{Hom}(K_f, A)$ such that $f \circ k_f = 0$ (in the sense above), and (K_f, k_f) is in a suitable sense universal (see later).

This extra structure is useful, and warrants the treatment of module categories almost separately from generic categories. This raises the question of what aspects of representation theory are 'categorical' — i.e. detectable from looking at the category alone, without probing the objects and arrows as modules and module morphisms per se.

For example, the property of projectivity is categorical. (Exercise. Hint: consider $\mathrm{Hom}(P,-)$ and short exact sequences.) The property of an object being a set is not categorical (although this concreteness is a safe working assumption for module categories, fine details of the nature of this set are certainly not categorical).

1.7.4 Aside: tensor products

e:tensorprod0001 (**1.7.26**) Let R be a ring and $M = M_R$ and $N = {}_R N$ right and left R-modules respectively. Then there is a *tensor product* — an abelian group denoted $M \otimes_R N$ constructed as follows. Consider the formal additive group $\mathbb{Z}(M \times N)$, and the subgroup S_{MN} generated by elements of form $(m+m', n) - (m,n) - (m',n)$, $(m, n+n') - (m,n) - (m,n')$ and $(mr, n) - (m, rn)$ (all $r \in R$). We set $M \otimes_R N = \mathbb{Z}(M \times N)/S_{MN}$. (In essence $M \otimes_R N$ is equivalence classes of $M \times N$ under the relation $(mr,n) = (m,rn)$. See §8.4 for details.)

This construction is useful because it gives us, for each M_R, a functor $M_R \otimes -$ from R-mod to the category $\mathbb{Z}$-mod (of abelian groups). This has many useful generalisations.

1.7.5 Functor examples for module categories: globalisation

ss:glob1

de:GF1 (**1.7.27**) Let A be an algebra over k and $e^2 = e \in A$ as in §1.6 above. We define functor $G = G_e$

$$G_e : eAe{-}\mathrm{mod} \;\to\; A{-}\mathrm{mod}$$

by $G_e M = Ae \otimes_{eAe} M$ (as defined in §8.4) and $F_e : A - \mathrm{mod} \to eAe - \mathrm{mod}$ by $F_e N = eN$. (Exercise: check that there are suitable mappings of module maps.)

ex:GF1 (**1.7.28**) Exercise. Show the following.
(I) Pair (G_e, F_e) is an adjunction (as in (6.3.7)).
(II) Functor F_e is exact.
(III) Functor G_e is right exact, takes projectives to projectives and indecomposables to indecomposables. (See Th.8.5.19 et seq.)
(IV) The composite $F_e \circ G_e : eAe{-}\mathrm{mod} \;\to\; eAe{-}\mathrm{mod}$ is a category isomorphism.

Note from these facts that there is an embedded image of $eAe{-}\mathrm{mod}$ in $A{-}\mathrm{mod}$ (the functorial version of an inclusion). Cf. Fig.1.1. Functor G_e does not take simples to simples in general. (One can see this either from the construction or 'categorically'.) However since simples and indecomposable projectives are in bijective correspondence, we can effectively 'count' simples in A-mod by counting those in eAe-mod and then adding those which this count does not include. It

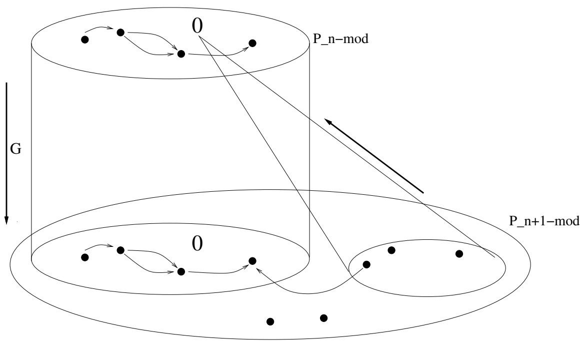


Figure 1.1: Schematic for the G-functor. fig:Pnmodembed1

is easy to see the following.
PROPOSITION. Functor F_e takes a simple module to a simple module or zero. ■

th:simp0001 (**1.7.29**) THEOREM. Let us write $\Lambda(A)$ for some index set for simple A-modules (up to isomorphism); and $\Lambda_e(A)$ for the subset on which e acts as zero. It follows from (1.7.28) that we may take $\Lambda(A) \setminus \Lambda_e(A)$ as index set $\Lambda(eAe)$, and hence

$$\Lambda(A) = \Lambda(eAe) \sqcup \Lambda_e(A).$$

Of course simples on which e acts as zero are also the simples of the quotient algebra A/AeA, so $\Lambda_e(A) = \Lambda(A/AeA)$. ■

Let us consider some examples.

pr:lams (**1.7.30**) PROPOSITION. *Recall the partition algebra P_n from (1.3.2); and T_n from (1.3.13).*
For $\delta \in k$ a unit, we may take $\Lambda(P_n) = \Lambda(P_{n-1}) \sqcup \Lambda(kS_n)$. Thus

$$\Lambda(P_n) = \sqcup_{i=0,1,\ldots,n} \Lambda(kS_i).$$

Similarly $\Lambda(T_n) = \Lambda(T_{n-2}) \sqcup \Lambda(k)$. Thus

$$\Lambda(T_n) = \sqcup_{i=n,n-2,\ldots,1/0} \Lambda(k).$$

Proof. Consider in particular the functor $G = G_{\mathsf{u}_1}$ and use (1.31) (resp. 1.33) and (1.7.29). □

de:PDelt (**1.7.31**) Note that every simple module of P_n is associated to a symmetric group S_i irreducible for some $i \leq n$. Symmetric group irreducibles can be found in the heads of symmetric group Specht modules $\Delta^S_\lambda := kS_i v_\lambda$ (suitable $v_\lambda \in S_i$; these are classical constructions for irreducible modules over $\mathbb{C}$ that are well defined over any ground ring). Accordingly we define P_n-module $\Delta(\lambda) = \Delta_n(\lambda)$ by applying G-functors to Δ^S_λ as many times as necessary:

$$\Delta_n(\lambda) \;=\; G^{n-i} \Delta^S_\lambda \qquad (\lambda \vdash i)$$

Note that it follows that

$$F\Delta_n(\lambda) \;= \mathsf{u}_1 \Delta_n(\lambda) \;\cong\; \Delta_{n-1}(\lambda)$$

and hence (by the Jordan-Holder localisation Theorem) that

$$(\Delta_n(\lambda) : L) \; = \; (\Delta_{n-1}(\lambda) : \mathsf{u}_1 L)$$

whenever the RHS makes sense (i.e. whenever $\mathsf{u}_1 L \neq 0$).

(**1.7.32**) If $k \supset \mathbb{Q}$ then v_λ can be chosen idempotent (indeed primitive). It follows that $\Delta(\lambda)$ is indecomposable projective in a suitable quotient algebra of P_n. Thus it has simple head. It follows that every module's structure can be investigated by investigating morphisms from these modules.

(**1.7.33**) Remark. The preceeding example will be very useful for analysing P_n-mod by induction on n. But first we think about some other examples, and how module categories and functors work with representation theory in general.

1.8 Modular representation theory

ss:mod0001

Sometimes an algebra is defined over an arbitrary commutative ring k (for example a monoid algebra). We may focus on the representation theory over the cases of k a field in particular - hence perhaps benefiting from the organisational structure of Artinian representation theory. But the idea of considering all cases together provides us with some useful tools. (This follows ideas of Brauer [16]. See also, for example, Curtis–Reiner [33, Ch.2], [34, Ch.2]? Benson [7, Ch.1]. And §9.3 later. And for some illustrative specific cases see also for example §17.5.1 and 16.7.1 later.)

Let R be a commutative ring with a field of fractions (R_0) and quotient field k (quotient by some given maximal ideal). (Ring R a complete rank one discrete valuation ring would be sufficient to have such endowments.) Let A be an R-algebra that is a free R-module of finite rank. Let $A_0 = R_0 \otimes_R A$ and

$$A_k = k \otimes_R A$$

(we call these constructions 'base changes' from R to R_0 and to k respectively).

The working assumption here is that A_0 is relatively easy to analyse. (The standard example would be a group algebra over a sufficiently large field of characteristic zero; which is semisimple by Mashke's Theorem.) And that A_k is the primary object of study.

In particular, suppose that we have a complete set of simple modules for A_0. One can see (e.g. in (**??**)) that:

lem:liftem

(**1.8.1**) Lemma. For every A_0-module M there is a finitely generated A-module (that is a free R-module) that passes to M by base change. ■

Remark: Note that there can be multiple non-isomorphic A-modules all passing to M. (We will give examples shortly.)

pa:indexm

(**1.8.2**) Let

$$\mathcal{D} \; = \; \{ D^R(l) \; : \; l \in \Lambda = \{1, 2, ..., m\} \; \}$$

be an ordered set of A-modules that passes by base change to a complete ordered set $\mathcal{D}_0$ of m simple A_0-modules $D(l) = A_0 \otimes_A D^R(l)$. Let $D^k(l) = k \otimes D^R(l)$. Write

$$\mathcal{L}^k \; = \; \{ L^k_\lambda \; : \; (\lambda \in \Lambda^k) \}$$

for a complete ordered set of simple A_k-modules.

de:moddecompmat (**1.8.3**) Fix k, and the ordering of Λ^k. There is then a decomposition matrix for any ordered set of modules. In particular, the choice of ordering of Λ gives us a $\mathcal{D}$-decomposition matrix D:

$$\mathsf{D}_{i\lambda} = [D^k(i) : L^k_\lambda]$$

(note that the index sets $\Lambda = \{1, 2, ..., m\}$ and Λ^k are not the same in general).
Remark: because all possible choices for $\mathcal{D}$ come from $\mathcal{D}_0$ we will see that the matrix D does not depend on $\mathcal{D}$ (althought there is potentially plenty of choice in $\mathcal{D}$). We call it the *modular decomposition matrix* of A_k.

Note that A_k is Artinian. Write P^k_λ for the projective cover of L^k_λ (the indecomposable projective with head L^k_λ), and e^k_λ for a corresponding primitive idempotent. One can show the following.

th:liftee (**1.8.4**) PROPOSITION. *(We assume suitable conditions on our base rings — see later.) There is a primitive idempotent in A that passes to e^k_λ, and an indecomposable projective A-module, $P^{k,R}_\lambda$ say, that passes to P^k_λ by base change.* (Caveat: A is not Artinian in general.)

For examples see §9.3.1.

(**1.8.5**) Since P^k_λ is projective, $\mathsf{D}_{i\lambda} = \dim \hom(P^k_\lambda, D^k(i))$. (Proof: For any indecomposable projective P^k_λ we have $\dim \hom(P^k_\lambda, M) = [M : L^k_\lambda]$ by the exactness property (as in (1.7.11)) of the functor $\mathrm{Hom}(P^k_\lambda, -)$. For example one can use exactness and an induction on the length of composition series.)

On the other hand the free R-module $\hom(P^{k,R}_\lambda, D^R(i))$ has a basis which passes to a basis of $\hom(P^k_\lambda, D^k(i))$; and to a basis of $\hom(A_0 \otimes P^{k,R}_\lambda, A_0 \otimes D^R(i))$. Now suppose that A_0 is semisimple. A basis of the latter hom-set is the collection of maps, one for each simple factor of the direct sum $A_0 \otimes P^{k,R}_\lambda$ isomorphic to the simple module $A_0 \otimes D^R(i)$. That is, the dimension is the multiplicity of the A_0-simple module in $A_0 \otimes P^{k,R}_\lambda$. We have the following.

pr:mod recip (**1.8.6**) PROPOSITION. *(Modular reciprocity) Let (A, A_0, A_k) be as above, with A_0 semisimple (indeed split semisimple as in 1.4.53). Then*

$$[D^k(i) : L^k_\lambda] \;=\; [A_0 \otimes P^{k,R}_\lambda : A_0 \otimes D^R(i)].$$

■

(**1.8.7**) REMARK. (I) The Prop. does not say that P^k_λ has a filtration by $\{D^k(l)\}_l$. Indeed $\mathcal{D}$ could be a mixture of Specht and coSpecht modules, so that such a filtration would be unlikely. (While on the other hand such filtrations are certainly sometimes possible.)
(II) However D does not depend on the choice of $\mathcal{D}$.
(III) The Prop. does not determine any decomposition numbers. However, we have the following.

(**1.8.8**) For given k this says in particular that the *Cartan decomposition matrix* of algebra A_k (with rows and columns indexed by Λ^k) is

$$C \;=\; \left([P^k_\lambda : L^k_\mu]\right) \;=\; \left(\sum_i \underbrace{(P^k_\lambda : D^k(i))}_{*}[D^k(i) : L^k_\mu] \right) \;=\; \mathsf{D}^T \mathsf{D} \qquad (1.42)$$ eq:Cartan0001

(here $*$ is undefined, but can be understood here as in the Prop.). For an example see §2.6.4.

1.8.1 Modularity and localisation together

ss:malt1

Now suppose there is an idempotent e in the algebra A in §1.8. With the 'localised' algebra $B = eAe$ we also have algebras $B_0 = eA_0e$ and $B_k = eA_ke$. With the quotient algebra

$$A^{(e)} = A/AeA$$

we have $A_0^{(e)} = A_0/A_0eA_0$ and so on.

Write Λ for the index set $\underline{m}$ from (1.8.2) here. Let the set

$$\Lambda_e \; := \; \{l \in \Lambda \mid eD(l) \neq 0\}$$

and $\Lambda_e^k \; = \{\lambda \in \Lambda^k \mid eL_\lambda^k \neq 0\}$. By (1.7.28) we have a complete set of simple B_0-modules

$$\mathcal{D}_0^e \; = \; \{eD(l) \mid l \in \Lambda_e\}$$

and a complete set of simple B_k-modules $\mathcal{L}^{k(e)} \; = \; \{eL_\lambda^k \mid \lambda \in \Lambda_e^k\}$.

The triple B, B_0, B_k and the sets $\mathcal{D}_0^e$ and $\mathcal{L}^{k(e)}$ obey the conditions in §1.8 so we can define

$$\mathsf{D}_{i\lambda}^e \; = \; [eD^k(i) : eL_\lambda^k]$$

whenever $i \in \Lambda_e$ and $\lambda \in \Lambda_e^k$. This gives a decomposition matrix for the B_k-modules $eD^k(i)$.

th:modlocal

(**1.8.9**) Theorem. [Modular localisation] *Let* (A, A_0, A_k) *and* $e \in A$ *be as above. Then* $\mathsf{D}_{i\lambda}^e = \mathsf{D}_{i\lambda}$ *(i.e., whenever* $i \in \Lambda_e$ *and* $\lambda \in \Lambda_e^k$*).* ■

In other words the modular decomposition matrix of A_k is given in part by

$$\mathsf{D} = \; i \left(\begin{array}{c|c} \mathsf{D}^e & \cdots \\ \hline \cdots & \cdots \end{array} \right) \; \begin{array}{l} i \in \Lambda_e \\ \\ \end{array}$$

That is, the multiplicities we do not know in terms of D^e include those of the modules $D^k(l)$ with $eD^k(l) = 0$. These are also modules for the quotient algebra $A_k^{(e)}$. Indeed any module obeying $eM = 0$ is also a module for the quotient.

See e.g. Pr.(18.6.12).

Note the following.

(**1.8.10**) Lemma. *Suppose* L *a composition factor of* M*, a module for an Artinian algebra. Then* $eL \neq 0$ *implies* $eM \neq 0$. ■

Therefore $eM = 0$ implies $eL = 0$ and so the lower block (giving composition factors L obeying $eL \neq 0$ of $D^k(i)$'s obeying $eD^k(i) = 0$) is zero:

$$\mathsf{D} = \; i \left(\begin{array}{c|c} \mathsf{D}^e & \cdots \\ \hline 0 & \check{\mathsf{D}}^e \end{array} \right) \; \begin{array}{l} i \in \Lambda_e \\ \\ \end{array} \qquad (1.43)$$

eq:DDD1

Meanwhile $\mathsf{D}^{\check{e}}$ encodes the multiplicities of simples L obeying $eL = 0$ in $D^k(i)$'s obeying $eD^k(i) = 0$. Note that these are all modules of the quotient algebra $A_k^{(e)}$. So $\mathsf{D}^{\check{e}}$ can be considered as a decomposition matrix for certain modules of this algebra.

1.8.2 Quasi quasiheredity

ss:qqh1

In Sec.1.8 we are looking at axiomatic settings that can help with representation theory for algebras with certain properties. Historically an early example is Brauer modular rep theory - well suited to addressing finite groups over fields of finite characteristic (making extended use, in a sense, of Mashke's Theorem). Quasiheredity is an axiomatisation well-suited to algebras arising in Lie theory. 'Diagram algebras' manifest properties common to both finite groups and to Lie theory, so some mix-and-match of methods is appropriate.

(**1.8.11**) Now suppose that the quotient algebra $A_k^{(e)} = A_k/A_k e A_k$ is semisimple. Then its simple modules are also projective.

CLAIM: there is an ordering so that $\mathsf{D}^{\check{e}}$ is the identity matrix.

Proof: 1. The Cartan decomposition matrix is the identity matrix by semisimplicity. 2. It follows from 1. and modular reciprocity that the modular decomposition matrix is the unit matrix.

(**1.8.12**) We say that our system is *semihereditary* if there is a nested chain of idempotents as above, $e = e_1, e_2, e_3, ...$ say, so that the idempotent subquotient algebras $e_i A_k e_i / e_i A_k e_i e_{i+1} e_i A_k e_i$ are all semisimple. (In other words there is an idempotent e' in $B = eAe$ such that $B/Be'B$ is semisimple and so on. NB Since $e' \in eAe$ we see that $ee' = e'e = e'$.) Then we have the following.

th:uutr1 (**1.8.13**) THEOREM. *If (A, A_0, A_k) is semihereditary then D is upper-unitriangular.*

Proof. Iterate the construction as in (1.43) with the bottom block in each iteration given by a unit matrix. □

(**1.8.14**) Remark/Exercise: Quasiheredity requires also that there is a bijection $Ae \otimes_{eAe} eA \to AeA$. Construct examples where this is not the case. See §**??** for more on this.

Chapter 2

Introduction II

ch:basic2

In this Chapter we study examples in support of Chapter 1. In §2.1 we study a particular 5-dimensional algebra. In §2.2 we study various infinite 'towers' of finite-dimensional algebras (Temperley–Lieb algebras). In §2.4 we study these towers from an 'alcove geometric' perspective. In §2.6 we prove a Temperley–Lieb structure Theorem. (We study these algebras futher in Chapters 12, 13.) In §2.7 we begin the parallel study of a more complex tower (partition algebras, continued in Chapter 16).

Temperley–Lieb algebras are so named for Temperley and Lieb [146], who studied them from a two-dimensional statistical mechanics perspective, although they are natural subalgebras of Brauer algebras, introduced much earlier by Brauer with invariant theory in mind. A fairly comprehensive treatment (and set of references) can be found in [104], which again starts from statistical mechanical motivations. It is the generalisation there to higher-dimensions that yields the partition algebras, which themselves have Brauer algebras as subalgebras.

2.1 Example of almost everything: $TL_3(1)$

ss:TL31

Here we look at a small Artinian ring which is non-commutative with non-zero radical. (This Artinian ring example is not entirely 'generic', however. It is isomorphic to its opposite. And it is basic. For other small examples see e.g. §4.1.4.)

Fix commutative ring k. Set $A = TL_3(1)$, i.e. TL_3 with $\delta = 1$. Recall from (1.3.18) that

$$TL_3(1) = k\langle 1, U_1, U_2, \rangle / \sim$$

where $\sim$ is the relations $U_i^2 = U_i$, $U_1U_2U_1 = U_1$ and $U_2U_1U_2 = U_2$.

2.1.1 Generalities

It will be clear from the relations that A is spanned by the five words 1, U_1, U_2, U_1U_2, U_2U_1. Thus if k is a field we have an Artinian ring/k-algebra. (Exercise: Show that these words are independent in A.)

(**2.1.1**) We have (as usual, see e.g. (10.1.5) for details) the contravariant functor $\mathrm{Hom}_k(-, k)$: $A - \mathrm{mod} \to \mathrm{mod} - A$: For every left-$A$-module N there is a dual right-module $N^* = \mathrm{Hom}_k(N, k)$.

For a basis B_N of N the dual basis is the set of linear maps $\{f_b \mid b \in B_N\}$ given by $f_b(a) = \delta_{a,b}$ for $a \in B_N$. Alternatively N^* can be viewed as a left-module for the opposite algebra.

Consider the representation ρ_N of A afforded by B_N. The transpose matrices give a representation of the opposite algebra.

de:cv-functor1 (**2.1.2**) If A is isomorphic to its opposite then this gives an action of A on N^* again — we write N° for this contravariant dual module. Note that this construction lifts to a contravariant functor on $A-$mod.

de:cv-rep1 (**2.1.3**) In our case A is isomorphic to its opposite under the map $i_A : A \to A^{op}$ that fixes the generators U_i (with $i_A(U_1U_2) = U_2U_1$ and so on). Thus the map from A to matrices given by the map on generators U_i to the transpose matrices $\rho_N(U_i)^t$ is also a representation of A. We write ρ°_M for this (the representation afforded by the contravariant dual module).

2.1.2 Regular module, basis and representation

ss:TL211

de:rep afforded (**2.1.4**) We may encode the linear action of $a \in A$ on a k-basis of a left A-module as a matrix $M(a)$, as follows. We arrange the basis as a column vector V (merely for convenience), on which a acts pointwise, then there is a unique $M(a)$ such that $aV = M(a)V$. The *representation afforded by ordered basis* V of the left module is given by the transposes of the matrices $M(a)$ (one easily sees that $a \mapsto M(a)$ is an antirepresentation, cf. also §4.1.4 and **??**).

In our case we have, for the left regular module:

$$U_1 \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix}, \quad U_2 \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix}$$

The representation afforded by this basis of the left regular module ${}_AA$ is given by the transposes of the above 5×5 matrices $M(U_i)$. This is the left-regular representation ρ_A (cf. (1.2.3)).

(**2.1.5**) Note that if we use row vectors V^t the corresponding matrices appear on the right: $U_iV^t = (U_iV)^t = (M(U_i)V)^t = V^t(M(U_i))^t = V^t\rho_A(U_i)$, and do not require transposition. (Note that we are still encoding the left action here. If we encode the right action on V: $VU_i = N(U_i)V$ then the matrices $N(U_i)$ again give a representation without transposition — the right-regular representation.)

(**2.1.6**) On the other hand, the algebra is isomorphic to its opposite via a map that fixes these generators (and transpose maps a matrix ring to its opposite), so these two matrices $M(U_i)$ themselves also give rise to a representation, the k-dual of the right-regular representation: $\rho_{A^*_A} : A \to M_5(k)$.

It is interesting to note that ρ_A and $\rho_{A^*_A}$ are not isomorphic in this case.

(**2.1.7**) It will be clear from the definition of A that there are two one-dimensional representations: $\rho_0(U_i) = 0$ and $\rho_1(U_i) = 1$.

lem:informa (**2.1.8**) By reciprocity the composition multiplicity of a simple module L_λ in the regular module is equal to $\dim P_\lambda$, and so at least equal to $\dim L_\lambda$. The bound is saturated for all simples if and only if A is semisimple — the dimension of the radical is $\dim(A) - \sum_\lambda \dim(L_\lambda)^2$. It follows that

(1) the 1d modules above are a complete set of simple A-modules;
(2) the dimension of the radical is 3;
(3) ${}_A A \cong P_0 \oplus P_1$, with dimensions 3 and 2 respectively.

2.1.3 Morphisms, bases and Intertwiner generalities

ss:IntertwineI

Let A be a k-algebra. Let $Rep(A)$ denote the collection of matrix representations of A. Note that we get a matrix representation for each basis of an A-module, but two different (but isomorphic) A-modules may give rise to the same matrix representation.

[TASK: make Int below a category. Give the example of the identity intertwiner on reg module and show how to extract another example by restriction. Do more examples and show gram-matrix - intertwiner connection.]

e:intertwinermat (**2.1.9**) An *intertwiner matrix* corresponding to a left-A-module map

$$\psi : M \to N$$

for the representations ρ_M, ρ_N afforded by given bases, is a matrix X such that

$$X\ \rho_M(a) \quad = \quad \rho_N(a)\ X \qquad \forall\, a \in A \tag{2.1}$$ eq:intertwined

(N.B. I think that Curtis–Reiner [33, §29] have this the wrong way round.)

intertwinerspace (**2.1.10**) Define $\mathrm{Int}(\rho, \rho')$ as the k-space of intertwiners from representation ρ to ρ'.

(**2.1.11**) Note that to verify $X \in \mathrm{Int}(\rho, \rho')$ it is sufficient to check (2.1) on generators of A.

(**2.1.12**) Example. The identity matrix intertwines a representation with itself.

de:cvimagemat (**2.1.13**) In cases (like $TL_3(1)$) with a generator-fixing opposite isomorphism we can effectively look simultaneously for $X \in \mathrm{Int}(\rho, \rho')$ and for $Y \in \mathrm{Int}(\rho'^{\circ}, \rho^{\circ})$, since the transpose of (2.1) (on generators) gives the latter — the contravariant image of X.

(**2.1.14**) Note that there is a submodule (a left ideal) $M = \Delta_1$ of ${}_A A$ as in §2.1.2 spanned by $B_M = \{U_1, U_2U_1\}$. We have

$$U_1 \begin{pmatrix} U_1 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2U_1 \end{pmatrix}, \qquad U_2 \begin{pmatrix} U_1 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2U_1 \end{pmatrix}$$

affording a corresponding representation ρ_M and cv dual ρ_M°.

(**2.1.15**) Exercise: look for intertwiners for ρ_M and ρ_M° to and from the 1d representations. (We do this shortly.)

2.1.4 Intertwiners between $M = \Delta_1$ and ${}_A A = TL_3(1)$

(**2.1.16**) Exercise: look for intertwiners corresponding to the module map

$$\psi : M \hookrightarrow {}_A A$$

and other module maps $\mu : M \to {}_A A$; and for possible maps $\phi : M^{\circ} \to {}_A A^{\circ}$.

de:IntX1 (**2.1.17**) We can look for ψ directly or (just because we can!) by looking for the cv image $\psi^\circ : {}_AA^\circ \to M^\circ$ (and then taking transpose). We have for $\rho^\circ_M X = X \rho^\circ_{{}_AA}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $X \in \mathrm{Int}(\rho^\circ_{{}_AA}, \rho^\circ_M)$; and $X^t \in \mathrm{Int}(\rho_M, \rho_{{}_AA})$. (Note from the form of X^t how it specifically realises the inclusion ψ of M.)

(**2.1.18**) Are there other independent intertwiners in $\mathrm{Int}(\rho^\circ_M, \rho^\circ_{{}_AA})$? We have to simultaneously solve

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

These give $a = c = e = f = g = i = 0$, $b = j$ and $d = h$, so $\mathrm{Int}(\rho_M, \rho_{{}_AA})$ is spanned by X above and

$$X' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(**2.1.19**) Is there an intertwiner in the other direction? Is there a splitting idempotent?

(**2.1.20**) Now for maps $\psi : A \to M$ we can look directly or at $\psi^\circ : M^\circ \to {}_AA^\circ$. For the latter we have

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix}$$

The space of intertwiners is thus the space of matrices of form X

$$X = \begin{pmatrix} a & b \\ a+b & 0 \\ 0 & a+b \\ a+b & 0 \\ 0 & a+b \end{pmatrix}$$

What about composite maps $M \to A \to M$? Note

$$\begin{pmatrix} 0 & x & 0 & y & 0 \\ 0 & 0 & y & 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ a+b & 0 \\ 0 & a+b \\ a+b & 0 \\ 0 & a+b \end{pmatrix} = \begin{pmatrix} (x+y)(a+b) & 0 \\ 0 & (x+y)(a+b) \end{pmatrix}$$

That is, there are maps whose composite is the identity, and maps whose composite is zero.

2.1.5 Structure of $M = \Delta_1$ (maps between M and L_0 and L_1)

(**2.1.21**) Module M has a simple submodule $L_0 = k\{U_2U_1 - U_1\}$ (spanned by a single element) giving rise to the representation $\rho_0(U_i) = (0)$. We have

$$\phi : L_0 \hookrightarrow M$$

$$\rho_M(U_1) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} (0)$$

$$\rho_M(U_2) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} (0)$$

which give $a + b = 0$. That is, the hom-space is 1d.

Evidently (for example from the trace) the composition factors of M are L_0 and L_1. However if we look for an intertwiner for $L_1 \to M$ (replace (0) by (1) above) we get $a + b = a$, $0 = b$, $0 = a$ and $a + b = b$, that is $a = b = 0$, so there is no intertwiner. It follows that M is non-split:

$$0 \to L_0 \to M \to L_1 \to 0$$

de:IntY1 (**2.1.22**) We can confirm that we also have $\mu : M^\circ \to L_0$ with intertwiner Y such that $\rho_0 Y = Y \rho^\circ_M$:

$$(0)(1, -1) = (1, -1) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$(0)(1, -1) = (1, -1) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

(**2.1.23**) Next we can look for $L_1 \to M^\circ$, or (its cv image) $M \to L_1$.

...

(**2.1.24**) We can combine Y from (2.1.22) with, say, X from (2.1.17):

$$\rho_0\, YX \;=\; Y\rho^\circ_M X \;=\; YX\rho^\circ_{A}{}_{A}$$

2.1.6 Structure of A/M

Consider

$$U_1 \begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix}$$

$$U_2 \begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix}$$

Exercise: complete this analysis.

2.1.7 Irreducible content of TL representations

Recall that quite generally the simple modules ($\{L_\lambda : \lambda \in \Lambda\}$, say) of an algebra are a basis for the Grothendieck group. This means that the character of rep ρ determines its irreducible content.

How does this work in practice?

The character of ρ is the map from the algebra to scalars given by trace. Given the basis theorem above the character is determined by the images of $|\Lambda|$ elements that are independent in this sense. Picking such a subset of elements, the character becomes a vector. We then have

$$\chi_\rho = \sum_\lambda m_\lambda \chi_\lambda$$

where m_λ is the multiplicity.

Example: The characters of TL standard modules are easy. Recall that the index set in case n (n strings) is the set $(n), (n-1,1), (n-2,2), ...$ There are roughly $n/2$ of these and it follows that a suitable independent set of elements is $1, U_1, U_1U_3, ...$ We have

$$\chi_\lambda = \begin{pmatrix} tr_\lambda(1) \\ tr_\lambda(U_1) \\ ... \end{pmatrix}$$

For example, setting $\delta = \sqrt{Q}$, the characters of the standard modules are

$$\chi_{(4)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \chi_{(3,1)} = \begin{pmatrix} 3 \\ \sqrt{Q} \\ 0 \end{pmatrix}, \qquad \chi_{(2,2)} = \begin{pmatrix} 2 \\ \sqrt{Q} \\ Q \end{pmatrix}$$

Meanwhile

$$\chi_{Potts} = \begin{pmatrix} Q^2 \\ \sqrt{Q}Q \\ Q \end{pmatrix}$$

so we have

$$m_{(4)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + m_{(3,1)} \begin{pmatrix} 3 \\ \sqrt{Q} \\ 0 \end{pmatrix} + m_{(2,2)} \begin{pmatrix} 2 \\ \sqrt{Q} \\ Q \end{pmatrix} = \begin{pmatrix} Q^2 \\ \sqrt{Q}Q \\ Q \end{pmatrix}$$

This is easy to solve. Exercise.

We get $m_{(4)} = Q^2 - 3Q + 1$, $m_{(3,1)} = Q - 1$ and $m_{(2,2)} = 1$. It is interesting to consider some specific cases.

In case $Q = 3$ we get mutliplicities 1,2,1, which have nice physical interepretations.

In case $Q = 2$ we get -1,1,1. This requires mathematical interpretation! The point is that we used characters for standard modules, and these are not the simple modules in this case. Specifically we already know from [104] that the (2,2) standard is not simple. Indeed it contains (4) as a submodule. Thus we have (from the minus 1) the cancellation we need! We get only the simple part of the standard module.

2.2 Modules and ideals for the algebra T_n

ss:ModidTn

Let k be a commutative ring, and $\delta \in k$. Recall the definition (1.3.13) of T_n over k. (Recall also the notations for elements of T_n in (1.3.13).)

(**2.2.1**) Fix n. Set $\mathsf{e}_1^{2l-1} = \mathsf{e}_1\mathsf{e}_3...\mathsf{e}_{2l-1}$ (l factors). Thus $\mathsf{e}_1^{2l-1} \in \mathsf{T}_{n,n-2l,n}$ (partitions with $n-2l$ propagating parts, cf. 1.3.10). If $\delta \in k^*$ set $\bar{\mathsf{e}}_1^{2l-1} = \delta^{-l}\mathsf{e}_1\mathsf{e}_3...\mathsf{e}_{2l-1}$. Then the ideal $T_n\mathsf{e}_1\mathsf{e}_3...\mathsf{e}_{2l-1}T_n$ has basis $\mathsf{T}_{n,n}^{n-2l}$ ($n-2l$ or fewer propagating parts, cf. 1.3.10). Write

$$T_n^{/n-2l} \;:=\; T_n/(T_n\mathsf{e}_1\mathsf{e}_3...\mathsf{e}_{2l-1}T_n)$$

for the quotient algebra by this ideal (with a basis of diagrams with more that $n-2l$ propagating lines). In particular, (1.34) becomes $T_n^{/n-2} \cong k$.

Note that $\mathsf{e}_1T_n^{/n-4}\mathsf{e}_1 \cong T_{n-2}^{/n-4} \cong k$ and $\mathsf{e}_1\mathsf{e}_3T_n^{/n-6}\mathsf{e}_1\mathsf{e}_3 \cong T_{n-4}^{/n-6} \cong k$ and so on. By 1.6.7 this says that $\frac{1}{\delta}\mathsf{e}_1$ is a primitive idempotent in $T_n^{/n-4}$ and $\bar{\mathsf{e}}_1^3$ is primitive in $T_n^{/n-6}$ and so on:

pr:idqT1 (**2.2.2**) PROPOSITION. *Suppose $\delta \in k^*$. The image of $\bar{\mathsf{e}}_1^{2l-1}$ is a primitive idempotent in the quotient algebra $T_n^{/n-2l-2}$.* $\square$

2.2.1 Propagating ideals

Let $\mathsf{T}_{n,m}^l$ denote the subset of $\mathsf{T}_{n,m}$ of partitions with $\leq l$ propagating lines as above. Note

$$k\mathsf{T}_{n,m}^l = k\mathsf{T}_{n,l} * k\mathsf{T}_{l,m}. \tag{2.2}$$ eq:catfilti

Analogously to the P_n case (2.18) we have an ideal filtration:

$$T_n = k\mathsf{T}_{n,n}^n \supset k\mathsf{T}_{n,n}^{n-2} \supset ... \supset k\mathsf{T}_{n,n}^{0/1}$$

Similarly $k\mathsf{T}_{n,m}^l \supseteq k\mathsf{T}_{n,m}^{l-2}$ for any l, m, n. Write

$$\mathfrak{T}_{n,m}^l = k\mathsf{T}_{n,m}^l / k\mathsf{T}_{n,m}^{l-2}$$

for the section bimodule, with basis $\mathsf{T}_{n,l,m}$. Note that for $l \leq n, m$ we have a bijection

$$* : \mathsf{T}_{n,l,l} \times \mathsf{T}_{l,l,m} \xrightarrow{\sim} \mathsf{T}_{n,l,m} \tag{2.3}$$ eq:cartax

The inverse is called 'polar decomposition' of a TL diagram.

2.2.2 C-modules ('half-diagram modules')

As a *left*-module $\mathfrak{T}^l_{n,n}$ decomposes as a direct sum:

$$ {}_{T_n}\mathfrak{T}^l_{n,n} \;\cong\; \bigoplus_{w \in \mathsf{T}_{l,l,n}} k\mathsf{T}_{n,l,l}w $$

where each $k\mathsf{T}_{n,l,l}w$ is a left-module by the algebra action on the quotient; and these modules are pairwise isomorphic. In other words we have a filtration of the regular module T_n by the modules

$$ C^{\mathrm{TL}}_n(l) \;=\; \mathfrak{T}^l_{n,l} \;=\; k\mathsf{T}_{n,l,l} \;, $$

$l = n, n-2, ..., 1/0$.

th:Clemma (**2.2.3**) THEOREM. *For each n we have the following. (0) The left-regular module T_n is filtered by C-modules (cf. (8.3.23)). (I) $\sum_l (\dim C^{\mathrm{TL}}_n(l))^2 \;= \dim T_n$. (II) If k a field and T_n semisimple then $\{C^{\mathrm{TL}}_n(l)\}_l$ is a complete set of simples.*

Proof. (0) By construction. (I) Consider (2.3) and the analysis preceeding it. (II) Cf. (I) and Th.(1.4.78). □

Next we will show that these modules $\{C^{\mathrm{TL}}_n(l)\}_l$ are indecomposable.

2.2.3 D-modules ('standard modules')

By Prop.2.2.2 the $T_n^{/n-4}$-module $D^{\mathrm{TL}}_n(n-2) = T_n^{/n-4}\mathsf{e}_1$ is indecomposable projective (we assume $\delta \in k^*$ for now); and hence also indecomposable with simple head as a T_n-module. Generalising, for $l = n, n-2, n-4, ..., 0/1$ define $D^{\mathrm{TL}}_n(l)$ by

$$ D^{\mathrm{TL}}_n(n-2j) \;:=\; T_n^{/n-2j-2}\mathsf{e}_1^{2j-1} \tag{2.4} $$ eq:DTe

We have:

pr:DTL1 (**2.2.4**) PROPOSITION. *If $\delta \in k^*$, or $l \neq 0$, then each $D^{\mathrm{TL}}_n(l)$ is indecomposable with simple head as a T_n-module. Furthermore, by Prop.1.6.14 all the factors below the head obey $\mathsf{e}_1^{2l-1}L = 0$.* ■

Note that this says that the multiplcity of the simply head factor $L(l)$ in $D^{\mathrm{TL}}_n(l)$ is 1; and that no simple $L(m)$ with $m < l$ is a factor of $D^{\mathrm{TL}}_n(l)$. This is called the *upper-triangular property* for D-modules, since it means that the decomposition matrix of the set of D-modules can be written as an upper-triangular matrix.

co:DTL1 (**2.2.5**) COROLLARY. *Every projective T_n-module has a filtration by D-modules. (We will see shortly that the multiplicities are well-defined.)* ■

pr:basisDTL (**2.2.6**) PROPOSITION. *(I) $\mathsf{T}_{n,l,l}$ is a basis for $D^{\mathrm{TL}}_n(l)$. (II) $D^{\mathrm{TL}}_n(l) \cong C^{\mathrm{TL}}_n(l)$.* ■

A construction for all such bases is given in Fig.2.1 (n increases top to bottom; l left to right). Map $\iota : \mathsf{T}_{n,l,l} \to \mathsf{T}_{n+1,l+1,l+1}$ adds a line on the right. Map $\rho : \mathsf{T}_{n,l,l} \to \mathsf{T}_{n+1,l-1,l-1}$ bends the bottom of the last propagating line back to the top.

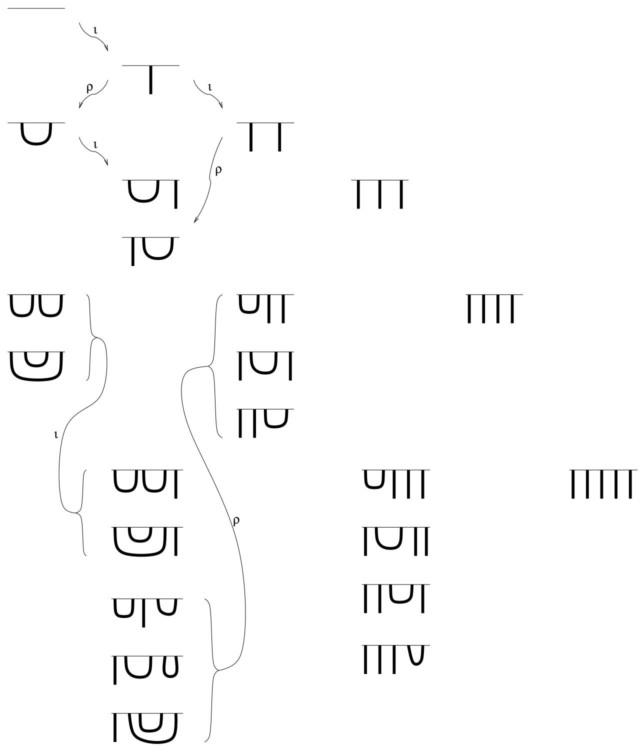


Figure 2.1: Truncated Pascal triangle enumerating sets $\mathsf{T}_{n,l,l}$. Here we have only drawn the northern edge of the frame rectangle for each diagram.

fig:bratt001

2.2.4 Flips, right D-modules and contravariant duals

de:flippy

(**2.2.7**) Note that the flip map $t \mapsto t^\star$ from (1.3.9) obeys $(t_1 t_2)^\star = t_2^\star t_1^\star$. It follows that the flip $\star$, extended k-linearly defines an involutive antiautomorphism of T_n. That is, we have a k-space map $\star : T_n \to T_n^{op}$ and this is an algebra isomorphism to the opposite algebra.

Quite generally a right-module for an algebra A becomes a left-module for the opposite algebra A^{op}. Thus in our case, via the isomorphism, we can convert right T_n-modules into left T_n-modules.

Cf. §4.1.4 for examples of algebras *not* isomorphic to their opposite.

(**2.2.8**) Note that there is a directly corresponding construction to (2.4) of indecomposable *right*-modules using the same idempotents, with analogous properties.

Note that the flip map fixes the idempotent used in the construction. It follows that the eA construction 'mirrors' the Ae construction. In particular the image of a morphism of left-modules from the first construction would be a morphism of right-modules from the right-version.

The flip conversion from (2.2.7) takes each module Ae (as it were) into its right-version eA.

de:vsdual3 (**2.2.9**) There is also the construction of right-modules from the $D_n^{\text{TL}}(l)$ themselves by taking the ordinary duals, i.e. by applying the contravariant functor $()^*$ as follows.

For A a k-algebra we have a contravariant functor (an arrow reversing functor)

$$()^* : A-\text{mod} \;\; \rightarrow \;\; \text{mod}-A$$

$$()^* : M \mapsto \text{Hom}_k(M, k)$$

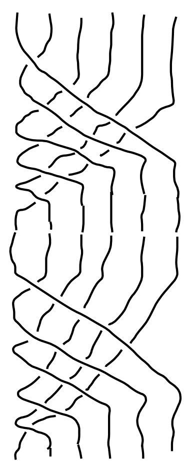
— see e.g. (2.3.1) for proof that M^* is a right module.

(**2.2.10**) The ordinary dual right modules $(D_n^{\text{TL}}(l))^*$ are also indecomposable on general grounds; but they need not have the other 'standard' properties from Prop.2.2.4 in general. We give a concrete example in (2.3.3).

One can ask how these two kinds of right modules are related. In general they are *not* isomorphic (but do have the same composition factors), as we shall see. In §2.3 we shall construct yet another kind of right modules from left modules (and vice versa), using the flip $\star$.

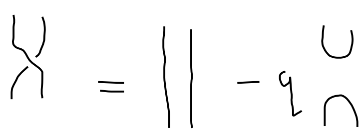
2.2.5 Aside: action of a central element in T_n on D-modules

Note from Prop.1.3.20 (the special feature) that T_n (with $\delta = q+q^{-1}$ for some $q \in k^*$) is a quotient of the braid group B_n. We consider the action of the central double-twist braid element M^2

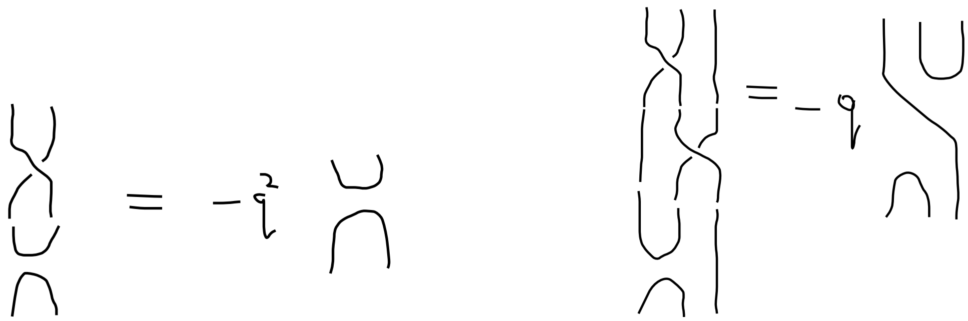


on our indecomposable D-modules.

This action can be computed using some hybrid diagrammatic rules, where crossings are understood as linear combinations of TL diagrams. First recall that the quotient map takes the braid generator g_i to $g_i \mapsto 1 - qU_i$. Informally we can generalise our diagrams for TL elements to incorporate this as:

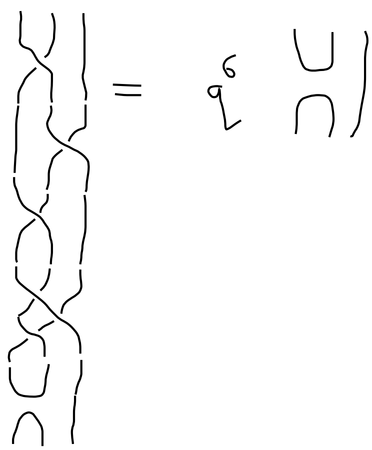


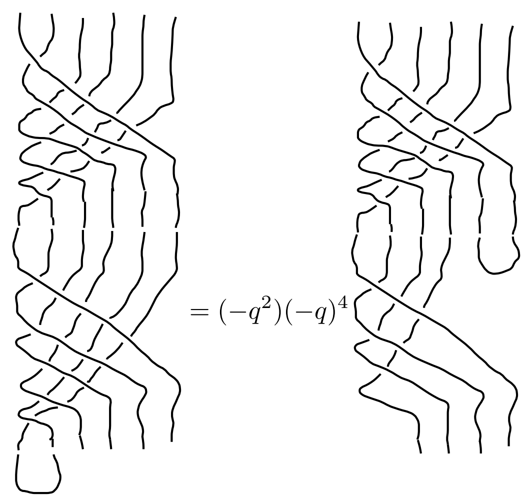
This gives us actions of braids on TL diagrams (and half-diagrams). In particular we have 'move 1' and 'move 2':



Note that the braids *look* like partition diagrams, but we *cannot* consider these as partition diagrams any more!

Applying the moves we get, for example,





We can think of the computation for the action of M^2 as passing the 'U' from the bottom-left through the various braids, first using move-1 ($-q^2$); then move-2 ($n-2$) times ($(-q)^{(n-2)}$); then a 'right-to-left over' version of move-2 ($n-2$) times ($(-q)^{(n-2)}$); then move-1 again. This gives a factor q^{2n} altogether; and what is left to act is M^2_{n-2} — the double-twist from B_{n-2} — on the remaining part of the basis element. (Thus if there is another 'U' then we will get a factor $q^{2(n-2)}$, and so on.)

In this way we can easily compute the action of M^2 on a basis element for any one of our modules from Fig.2.1. Besides the moves, the other feature is that because of the quotient by which the modules are defined, a braid acts like 1 on parallel lines in a module basis element.

The results are given in Fig.2.2. For $b \in D^{\mathrm{TL}}_n(l)$ we have:

$$M^2 b = q^{(n-l)(n+l+2)/2} b \tag{2.5}$$

eq:TLblock01

$n \setminus l$	0	1	2	3	4	5	6
0	1						
1		1					
2	q^4		1				
3		q^6		1			
4	q^{12}		q^8		1		
5		q^{16}		q^{10}		1	
6	q^{24}		q^{20}		q^{12}		1

Figure 2.2: Scalars by which M^2 acts on indecomposable T_n-modules $\Delta_n^{TL}(l)$. fig:Mact0001

Note in particular that the actions are all by powers of q, and that for given n they are all by different powers of q. By (**??**) this tells us that no two D-modules are in the same block (in the sense of 1.4.43) unless q is a root of unity.

pr:TLgensimp01 (**2.2.11**) Proposition. *The algebra T_n over a field k is semisimple unless ($\delta = q + q^{-1}$ where) q is a root of unity.*

Proof. Exercise.

Hints: Observe from the M^2 action that if q is not a root of unity then every D-module is in a different block, by Schur's Lemma. On the other hand, their heads yield all simple modules in the algebra; and the head simple does not reoccur in any D-module (consider Prop.2.2.4), so each must in fact be simple. Now one could use the Cartan matrix property (see e.g. §1.8), or do a dimension count.

(**2.2.12**) The same method can be used to treat other algebras. Consider for example Fig.2.3. This allows us to study the fixed rings of the algebras T_n under the left-right flip symmetry. (As opposed to the blob algebra — roughly speaking, the subalgebra spanned by left-right symmetric diagrams.)

2.3 Some module morphisms: standard and costandard modules

ss:smm3

In this section we consider co-standard modules (with simple socles instead of simple heads), both generally and in the T_n context.

$n \setminus l$	$+0$	-0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$+5$	-5	$+6$	-6	$+7$	-7
0	1															
1			1													
2	$-q^2$	q^2			1											
3			q^3	$-q^3$			1									
4	q^6	$-q^6$			$-q^4$	q^4			1							
5			q^8	$-q^8$			q^5	$-q^5$			1					
6	$-q^{12}$	q^{12}			q^{10}	$-q^{10}$			$-q^6$	q^6			1			
7			q^{15}	$-q^{15}$			q^{12}	$-q^{12}$			q^7	$-q^7$			1	
8	q^{20}	$-q^{20}$			$-q^{18}$	q^{18}			q^{14}	$-q^{14}$			$-q^8$	q^8		

fig:tab11

Figure 2.3: Scalars by which M acts on indecomposable modules $\Delta_n(l)$ of the fixed subring of T_n under the left-right diagram flip (see **??**).

Remark. The term co-standard arises in a couple of contexts. Here we will focus on the setting of algebras with involutive antiautomorphism, and hence on contravariant duality, but for example dualising quasi-heredity constructions is also of interest. See later. Indeed the term 'standard' is already overloaded. We can use it to mean modules that are the convergence of multiple useful properties in a modular representation theory setting; or again in a strictly quasi-hereditary setting.

pr:TLcvdual

(**2.3.1**) Proposition. *Let A be a k-algebra with involutive antiautomorphism $\star : A \to A^{op}$.*
(I) Every right A-module M can be made into a left A-module $\Pi_\star(M)$ by allowing A to act via the $\star$-map (e.g. the flip map in the T_n case).
(II) Note that a submodule of M passes to a submodule of $\Pi_\star(M)$. Indeed map $\Pi_\star$ extends to a covariant functor between the categories of modules (in either direction):

$$\Pi_\star : mod{-}A \;\;\leftrightarrow\;\; A{-}mod$$

In particular, every exact sequence of right modules passes to an exact sequence of left modules.
(III) Furthermore, for given $\star$, each A-module M has a contravariant (c-v) dual[1]*, here denoted $\Pi^o(M)$:*

$$\Pi^o(M) \;:=\; \Pi_\star(Hom_k(M,k)) \;=\; \Pi_\star(M^*) \tag{2.6}$$ eq:cvfunctor3

Proof. (I)-(II) are clear. For (III) we next note that M^* is indeed a right A-module. Given a basis $\{b_1, b_2, ..., b_r\}$ of M, the usual choice of basis of the ordinary dual vector space $M^* = \mathrm{Hom}_k(M,k)$ is the set of linear maps f_i such that

$$f_i : b_j \mapsto \delta_{i,j}. \tag{2.7}$$ eq:dualbasis3

[1] The c-v dual of a module M over such a k-algebra is the ordinary dual right-module $M^* = \mathrm{Hom}_k(M,k)$ made into a left-module via $\star$.

The right-action of $a \in A$ on M^* is given by $(f_i a)(b_j) = f_i(ab_j)$. Thus $((f_i a)a')(b_j) = (f_i a)(a'b_j) = f_i(a(a'b_j))$ and $(f_i(aa'))(b_j) = f_i((aa')b_j) = f_i(a(a'b_j))$, so $((f_i a)a')(b_j) = (f_i(aa'))(b_j)$ as required. $\square$

(**2.3.2**) It follows from (2.2.7) that $A = T_n$ has functor $\Pi_\star$ and contravariant functor Π^o.

exa:422 (**2.3.3**) Example: What does the c-v dual $\Pi^o(M)$ of $M = D_n^{\text{TL}}(l)$ look like? By construction the cv dual of any M is 'like' M but a JH series is obtained by replacing simple factors by *their* cv duals and reversing the series order. It is good to do an explicit example. As a k-module $\Pi^o(M)$ is $\text{Hom}_k(M, k)$.

In our case let us order the basis of $M = D_n^{\text{TL}}(l)$ as in Fig.2.4. Then our basis for the dual is $\{f_1, f_2, ..., f_{n-1}\}$ as in (2.7).
Exercise: What is the right action of T_n on this k-module? For example, what is $f_1 U_1$?

2.3.1 Contravariant form; Gram matrix

ss:cvform101

See also §10.1.8.

de:cvform3 (**2.3.4**) Given a k-algebra A with $\star$ as in (2.3.1) above, a *contravariant form* on A-module M is a bilinear form such that $< x, ay > = < a^\star x, y >$, as in (**??**)).

Such forms on M are in bijection with A-module morphisms from M to $\Pi^o(M)$.

(**2.3.5**) Suppose a contravariant form exists for some M — write ψ for the corresponding module morphism. Then at least one head factor of M is not in the kernel of ψ, and hence is also a factor of $\Pi^o(M)$. In particular if M has simple head L then this factor also appears in $\Pi^o(M)$. If L is not the simple socle factor L^o in $\Pi^o(M)$ then L appears above this factor, so M necessarily contains both L and L^o.

Now suppose there is a module N with L^o as head, and a cv form. Then by the same argument this also contains L as a factor.

de:headshot (**2.3.6**) Now suppose there exists a cv form on each $D_n^{\text{TL}}(l)$. It follows from the above observations and from (2.2.4) that the only copy of the simple head L_l (say) of $D_n^{\text{TL}}(l)$ occuring in the c-v dual lies in the simple socle (note that e_1^{2l-1} is fixed under $\star$). It then follows from Schur's Lemma 1.4.32 that there is a unique T_n-module map, up to scalars, from $D_n^{\text{TL}}(l)$ to its contravariant dual — taking the simple head to the simple socle. (In theory the socle, which is the simple dual of the simple head, might not be isomorphic to it; allowing no map. But we will show the existence of at least one map explicitly.)

As we will see, it follows from this abstract representation theoretic argument that $D_n^{\text{TL}}(l)$ has a contravariant form defined on it that is unique up to scalars.

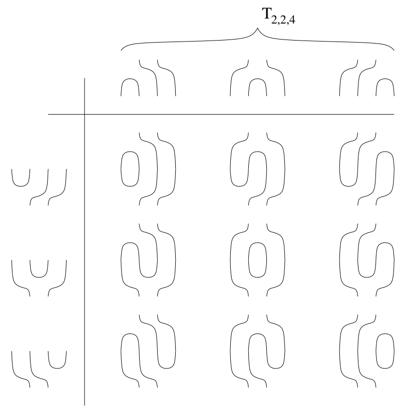
Actually *finding* the explicit c-v form could be difficult in general. But in fact we can construct such a form here for all δ simultaneously (over a ring with δ indeterminate, as it were). We can use this to determine the structure of the module.

Strictly speaking, to evaluate at $\delta = 0$ we must proceed a little differently to the other cases (part of the direct argument requires that δ has an inverse and so an extra step is needed).

(**2.3.7**) For a, b in the basis $\mathsf{T}_{n,l,l}$ (from (2.2.6)) then define $\alpha(a, b) \in k$ as follows. Note that $a^\star b \in \mathsf{T}_{l,l}$ (up to a scalar), thus either $a^\star b = \alpha(a, b)c$ with $c \in \mathsf{T}_{l,l,l}$ (indeed $c = 1_l$) for some $\alpha(a, b) \in k$; or $a^\star b \in k\mathsf{T}_{l,l}^{l-2}$, in which case set $\alpha(a, b) = 0$. Define an inner product on $k\mathsf{T}_{n,l,l}$ by

$$< a, b > = \alpha(a, b)$$

Figure 2.4: The array of diagrams $a^\star b$ over the basis $\mathsf{T}_{4,2,2}$. fig:epud



and extending linearly.

Observe that this is a cv form, as in (2.3.4), by construction. (The only issue is that it might be zero. This of itself does not indicate that there is no head-socle map, since our construction is not established to be unique in this case. Among our examples are instances that will address this point.) The construction works formally for $k = \mathbb{Z}[\delta]$. We can potentially therefore compute over this field and then reduce/base-change to our field of interest.

ex:gramTL1 (**2.3.8**) Example: Fig.2.4. The corresponding matrix of scalars is called the *gram matrix* with respect to this basis. From our example in the Figure we have (in the handy alternative parameterisation $\delta = q + q^{-1}$):

$$\mathrm{Gram}_n(n-2) \;=\; \begin{pmatrix} [2] & 1 & 0 & & \\ 1 & [2] & 1 & 0 & \\ 0 & 1 & [2] & 1 & \\ & & & \ddots & \\ 0 & \dots & 0 & 1 & [2] \end{pmatrix} \qquad \text{so} \quad |\mathrm{Gram}_n(n-2)| = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (2.8)$$ eq:TLgram0001

For more details on this example, and in particular devices such as the Smith normal form, see e.g. §10.1.8 (for generalities see also §8.2.4).

pa:Gramcare (**2.3.9**) Strictly speaking the notation $\mathrm{Gram}_n(l)$ relies both on the fixing of the form and on the fixing of the ordered basis, so care is needed.

(**2.3.10**) Aside. A couple of handy identities derived from $\delta = [2] = q + q^{-1}$:

$$[3] = (\delta^2 - 1), \qquad [4] = \delta(\delta^2 - 2), \qquad [5] = (\delta^4 - 3\delta^2 + 1), \qquad [6] = \delta(\delta^4 - 4\delta^2 + 3) = \delta(\delta^2 - 3)(\delta^2 - 1),$$

so

$$[3][4][5] = \delta(\delta^2 - 1)(\delta^2 - 2)(\delta^4 - 3\delta^2 + 1) = \delta(\delta^8 - 6\delta^6 + 12\delta^4 - 9\delta^2 + 2)$$

$$\frac{[4][5][6]}{[2]} \;=\; \frac{\delta(\delta^2 - 2)\;(\delta^4 - 3\delta^2 + 1)\;\delta(\delta^4 - 4\delta^2 + 3)}{\delta} \;=\; \delta(\delta^{10} - 9\delta^8 + 30\delta^6 - 45\delta^4 + 29\delta^2 - 6)$$

$$
\begin{array}{cccccccc}
1 &&&&&&& \\
& 1 &&&&&& \\
[2] && 1 &&&&& \\
& \begin{pmatrix} [3] \\ 1 \end{pmatrix} && 1 &&&& \\
\begin{pmatrix} [3][2] \\ [2] \end{pmatrix} && \begin{pmatrix} [4] \\ 1 \\ 1 \end{pmatrix} && 1 &&& \\
& \begin{pmatrix} \frac{[4][3]}{[2]} \\ [3] \\ [3] \\ [3] \\ 1 \end{pmatrix} && \begin{pmatrix} [5] \\ 1 \\ 1 \\ 1 \end{pmatrix} && 1 && \\
\begin{pmatrix} [4][3] \\ [3][2] \\ [3][2] \\ [3][2] \\ [2] \end{pmatrix} && \begin{pmatrix} [5][4] \\ ([4],3) \\ \frac{[4]}{[2]} \\ (1,4) \end{pmatrix} && \begin{pmatrix} [6] \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} && 1 & \\
& \begin{pmatrix} \frac{[5][4][3]}{[2]} \\ (\frac{[4][3]}{[2]},5) \\ ([3],7) \\ 1 \end{pmatrix} && \begin{pmatrix} \frac{[6][5]}{[2]} \\ ([5],5) \\ (1,8) \end{pmatrix} && \begin{pmatrix} [7] \\ (1,5) \end{pmatrix} && 1 \\
\begin{pmatrix} [5][4][3] \\ ([4][3],5) \\ ([3][2],7) \\ [2] \end{pmatrix} && \begin{pmatrix} \frac{[6][5][4]}{[3][2]} \\ ([5][4],6) \\ ([4],7) \\ (\frac{[4]}{[2]},6) \\ (1,8) \end{pmatrix} && \cdots && \begin{pmatrix} [8] \\ (1,6) \end{pmatrix} & 1 \\
& \begin{pmatrix} \frac{[6][5][4]}{[2]^2} \\ (\frac{[5][4][3]}{[2]},7) \\ (\frac{[4][3]}{[2]},18) \\ ([3],15) \\ 1 \end{pmatrix} &&&&&& \\
\begin{pmatrix} \frac{[6][5][4]}{[2]} \\ ([5][4][3],7) \\ ([4][3],18) \\ ([3][2],15) \\ [2] \end{pmatrix} && \cdots &&&&&
\end{array}
$$

Figure 2.5: Table of Smith normal forms - each given as the list of invariant factors - for gram matrices of cv-forms on Temperley–Lieb D-modules, working over (say) $\mathbb{Q}[\delta]$. The notation $([4],3)$ means entry [4] occurs 3 times. (This figure is from unpublished notes at stripped3.tex, correcting one typo and adding part of a new line.)

fig:SmithFormTL

(Job: explain relationship TL to noncrossing-P; and hence δ to Q_1 — essentially $Q_1 = \delta^2$, and overall factors — essentially ????!?! — hmmm not so easy here.)

$$[4][5] \;=\; \delta(Q_1-2)\,(Q_1^2-3Q_1+1) \;=\; \delta(Q_1^3-5*Q_1^2+7*Q_1-2)$$

$$\frac{[4][5][6]}{[2][3]} \;=\; \delta(Q_1-2)\,(Q_1^2-3Q_1+1)\,\frac{\delta(Q_1-1)(Q_1-3)}{\delta(Q_1-1)} \;=\; \delta(Q_1^4-8Q_1^3+22Q_1^2-23Q_1+6)$$

(**2.3.11**) As we will see below, the example in (2.3.8) above is the only one we need for the Temperley-Lieb structure theorem. But, for curiosity's sake... For another example we can consider $\mathrm{Gram}_6(0)$. Ordering the basis as $\{(12)(34)(56),(12)(36)(45),(14)(23)(56),(16)(23)(45),(16)(25)(34)\}$ we have:

$$\mathrm{Gram}_6(0) = \begin{pmatrix} \delta^3 & \delta^2 & \delta^2 & \delta & \delta^2 \\ \delta^2 & \delta^3 & \delta & \delta^2 & \delta \\ \delta^2 & \delta & \delta^3 & \delta^2 & \delta \\ \delta & \delta^2 & \delta^2 & \delta^3 & \delta^2 \\ \delta^2 & \delta & \delta & \delta^2 & \delta^3 \end{pmatrix}$$

The corresponding Smith form (and indeed the first few of these) is given in Fig.2.5.[2][3]

For $\mathrm{Gram}_5(1)$, ordering the basis as $\{(12)(34)(5),(12)(3)(45),(14)(23)(5),(1)(23)(45),(1)(25)(34)\}$ (in this case it is simply the basis as above but with the vertex 6 'forgotten') we have:

$$\mathrm{Gram}_5(1) = \begin{pmatrix} \delta^2 & \delta & \delta & 1 & \delta \\ \delta & \delta^2 & 1 & \delta & 1 \\ \delta & 1 & \delta^2 & \delta & 1 \\ 1 & \delta & \delta & \delta^2 & \delta \\ \delta & 1 & 1 & \delta & \delta^2 \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} \delta^3 & \delta^2 & \delta^2 & \delta & \delta^2 \\ \delta^2 & \delta^3 & \delta & \delta^2 & \delta \\ \delta^2 & \delta & \delta^3 & \delta^2 & \delta \\ \delta & \delta^2 & \delta^2 & \delta^3 & \delta^2 \\ \delta^2 & \delta & \delta & \delta^2 & \delta^3 \end{pmatrix}$$

with determinant $\frac{[4]}{[2]}[3]^4$. Observe that nominally (and essentially coincidentally) $\mathrm{Gram}_6(0) = \delta\,\mathrm{Gram}_5(1)$, and indeed $\mathrm{Gram}_{2n}(0) = \delta\,\mathrm{Gram}_{2n-1}(1)$ for all n. But of course the comparison is strictly only up to units, and in our simplified setup δ is taken to be a unit.

pa:noncrossn3

(**2.3.12**) Remark. It is interesting in passing to compare with the corresponding calculations in the setting of noncrossing partitions (the isomorphic incarnation of the algebra that we started with). There the rank $n = 3$ corresponds to the rank 6 here. The basis for the corresponding $D_3(0)$ is $\{(1)(2)(3),(1)(23),(12)(3),(13)(2),(123)\}$ and we have

$$\mathrm{Gram}'_3(0) = \begin{pmatrix} Q^3 & Q^2 & Q^2 & Q^2 & Q \\ Q^2 & Q^2 & Q & Q & Q \\ Q^2 & Q & Q^2 & Q & Q \\ Q^2 & Q & Q & Q^2 & Q \\ Q & Q & Q & Q & Q \end{pmatrix}$$

[2] An aside on calculation using GAP[**?**]. GAP code for calculating these gram matrices can be found in the file GRamPa33-24.g (originally used for gram matrices for ramified partition algebras, derived itself from some older partition algebra code). This uses the isomorphic algebra of noncrossing partitions - see e.g. §**??**. One can then for example move the matrix to Maple[**?**] to compute the Smith form efficiently. (Although as noted all this data can be reverse-engineered directly from the known representation theory in this case.) For more on computation see §**??**.

[3] Aside (author's note to self): This data is adapted from earlier notes at stripped3.tex. Note that there are typos and open questions with regard to the figure there which are answered here.

The isomorphism uses $Q = \delta^2$ and ... The determinant here is $Q^5(Q-1)^4(Q-2)$. Recall that in the first form it was $[4][3]^4[2]^4 = \delta^5(\delta^2-1)^4(\delta^2-2)$. Thus these differ by a factor δ^5. (Cf. also §2.7 and in particular §2.7.4.)

(**2.3.13**) We will explain the entries in Figure 2.5 in terms of D-module morphisms in §2.6 and **??**. For now let us briefly look at the spine module in rank 10.
The common factor $\delta = [2]$ in all 42 terms can be seen (in one way) as due to the basis choice.
There are 41 terms with factor (δ^2-1), corresponding to a socle of this size - and hence head of size 1 - when $\delta^2 = 1$. ...
There are 26 terms with factor (δ^2-2), corresponding to a socle of this size - and hence head of size 16 - when $\delta^2 = 2$. ...
There are 8 terms with factor $(\delta^4 - 3\delta^2 + 1)$, corresponding to a socle of this size - and hence head of size 34 - when δ^2 =golden mean/inverse. ...
There is 1 term with factor (δ^2-3), corresponding to a socle of this size - and hence head of size 41 - when $\delta^2 = 3$. ...
The dimensions of the D-modules in this layer are:

$$42 \qquad 90 \qquad 75 \qquad 35 \qquad 9 \qquad 1$$

Next we give a schematic for module morphisms in the various specialisations. The first row gives a 'ρ-shifted weight-space label for the weight-space to which these modules can be associated - so morphisms are associated to reflections in affine walls (see §2.4 and §**??** for further details):

0	1	2	3	4	5	6	7	8	9	10	11
|	42	|	90	|	75	|	35	|	9	|	1

For reflections in walls congruent to 0 mod. 3 we thus have head dimension $1 = 42-(75-(35-1))$, and so on. For the $[4]/[2] = 0$ case we have:

0	1	2	3	4	5	6	7	8	9	10	11
|	42	|	90	|	75	|	35	|	9	|	1

so ...
There are terms with factor (δ^2-2),

Let us now return from the example to the more general case.

pr:innprodcov1 (**2.3.14**) PROPOSITION. *The inner product defined by $<-,->$ is a contravariant form on $D_n^{\text{TL}}(l)$.* ■

(**2.3.15**) Consider the k-space map

$$\phi_{\langle\rangle} : D_n^{\text{TL}}(l) \rightarrow \Pi^o(D_n^{\text{TL}}(l)) \tag{2.9}$$
$$\phi_{\langle\rangle} : m \mapsto \phi_{\langle\rangle}(m) \tag{2.10}$$

where $\phi_{\langle\rangle}(m) \in \hom(D_n^{\text{TL}}(l), k)$ is given by

$$\phi_{\langle\rangle}(m)(m') \;=\; <m|m'> .$$

(**2.3.16**) Proposition. *The map $\phi_{\langle\rangle}$ is a T_n-module homomorphism (unique up to scalars) from $D_n^{\text{TL}}(l)$ to its contravariant dual.*

Proof. This map is a module morphism by Prop.2.3.14. To show uniqueness note that by (2.2.4) the contravariant dual must have the simple head of $D_n^{\text{TL}}(l)$ as its simple socle (and only in the socle). Thus a head-to-socle map is the only possibility. □

(**2.3.17**) Example. In our example we have (from the gram matrix, using (2.3.3))

$$\phi_{\langle\rangle} : \cup|| \;\mapsto\; [2]f_1 + f_2$$

$$\phi_{\langle\rangle} : |\cup| \;\mapsto\; f_1 + [2]f_2 + f_3$$

$$\phi_{\langle\rangle} : ||\cup \;\mapsto\; f_2 + [2]f_3$$

and for instance

$$\phi_{\langle\rangle} : \;\cup|| - [2]\,|\cup| - [3]\,||\cup \quad\mapsto\quad [4]f_3$$

The point of this case is to show that the module map $\phi_{\langle\rangle}$ has a kernel when $[4] = 0$. Obviously, in general,

Proposition. If a T_n-module map has a kernel then the kernel is a submodule of the domain.

Thus in our case, when $[4] = 0$, the domain is not simple.

It will also be clear from the example that if the rank of the gram matrix is maximal then the morphism $\phi_{\langle\rangle}$ has no kernel, and so is an isomorphism. This does not, of itself, show that the domain is a simple module, but we already showed in (2.3.6) that in our case the image must be simple, so the domain *is* simple.

de:gramdetzero (**2.3.18**) If $D_n^{\text{TL}}(l)$ is in fact simple then $\phi_{\langle\rangle}$ is an isomorphism and the contravariant form is non-degenerate. Otherwise the form is degenerate.

It will be clear from our example that if the determinant of the gram matrix is non-zero then $D_n^{\text{TL}}(l)$ is simple; and otherwise it is not. (Note that the case $\delta = 0$ is excluded here, for brevity. It is easy to include it if desired, via a minor modification.) In particular if the determinant is zero then $D_n^{\text{TL}}(n-2)$ has composition length 2; and the other composition factor is the simple module $D_n^{\text{TL}}(n)$.

(**2.3.19**) Proposition. *Given a c-v form (with respect to involutive antiautomorphism $\star$) on A-module M and $Rad_{<>}M = \{x \in M \;: <y,x> = 0 \;\forall y\}$ then*
(I) $Rad_{<>}M$ is a submodule, since $x \in Rad_{<>}M$ implies $<y, ax> = <a^\star y, x> = 0$.
(II) Thus $\dim Rad_{<>}M \;=\; corank\, \text{Gram}_{<>}M$. ■

(**2.3.20**) In our example rows 2 to $(n-1)$ of the $(n-1)\times(n-1)$ matrix $\text{Gram}_n(n-2)$ are clearly independent, while replacing $\boxed{\cup||...|}$ (the basis element in the first row) by

$$w = \boxed{\cup||...|} - [2]\boxed{|\cup|...|} + [3]\boxed{||\cup...|} - ...$$

(a sequence of elementary row operations adding to the first row multiples of each of the subsequent rows) replaces the first row of $\mathrm{Gram}_n(n-2)$ with $(0,0,...,0,[n])$. That is, $\mathrm{Rad}_{<>}D_n^{\mathrm{TL}}(n-2)=0$ unless $[n]=0$. If $[n]=0$ then w spans the Rad.

Explicit check in case $n=4$: $U_1 w = ([2]-[2]+0)\boxed{\cup||} = 0$; $U_2 w = (1-[2]^2+[3])\boxed{|\cup|} = 0$; $U_3 w = (0-[2]+[2][3])\boxed{||\cup}$.

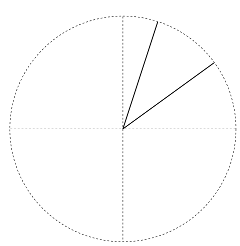
(**2.3.21**) Proposition. The T_n-module $D_n^{\mathrm{TL}}(n-2)$ is simple unless $[n]=0$, in which case $D_n^{\mathrm{TL}}(n) \hookrightarrow D_n^{\mathrm{TL}}(n-2)$ and the quotient is simple.

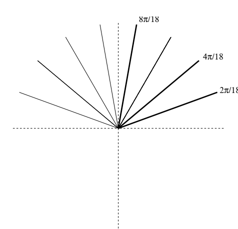
The condition $[n]=0$ is satisfied when q is a solution to $q^{2n}=1$ excluding $q=\pm 1$. One should compare this with the block data in Fig.2.2.

What values of q do we need to consider, to capture all possible algebra structures arising up to isomorphism in case $k=\mathbb{C}$? (1) Complex conjugation of q of magnitude 1 does not change δ, so it is enough to consider cases of q of nonnegative imaginary part. (2) It is easy to see that the algebra with $\delta \to -\delta$ is isomorphic to the original (via the invertible map $U_i \to -U_i$), and hence that $q \to -q$ also gives an isomorphism. Thus the algebras with $q^r=1$ with r odd (satisfying $q^{2r}=1$) can be treated with $q^r=-1$ and hence in the primitive $q^{2r}=1$ cases. We will obfuscate this slightly by using the sign change to take representatives all in the non-negative real part region (some of which will not then be primitive $2r$-th roots), and hence give representatives in the positive (nonnegative) quadrant.

Cases:
$q^4=1$ yields $q=i$ and $\delta=0$
$q^6=1$ yields $q=\frac{1\pm\sqrt{-3}}{2}$ and $\delta=1$
$q^8=1$ yields $q=\frac{1\pm\sqrt{-1}}{\sqrt{2}}$ and $\delta=\sqrt{2}$
$q^{10}=1$ yields two new positive δ values $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$
$q^{12}=1$ yields $q=\frac{\sqrt{3}\pm\sqrt{-1}}{2}$ and $\delta=\sqrt{3}$ ...

'Principle' q values for $q^{10}=1$: and for example $q^{18}=1$:





In the 10 case the coprime numbers are 1,3,7,9. We discard 7,9 as complex conjugates and make $3 \to 2$.

We will see in Th.2.6.5 that the *structure* of the algebra depends only on the r for which q is a primitive $2r$th root, if any, and not directly on the value of q. (This does not imply that the explicit construction of simple modules, say, only depends on r. For a possible meta-example note that there are some constructions of representations that only work when δ^2 is a natural number. See **??**.)

(**2.3.22**) It is easy to write down the cv form explicitly, particularly for $l = n - 2$, and compute the determinant. We can use this to determine the structure of the algebra. First we will need a couple of functors.

(**2.3.23**) REMARK. In case M is a matrix over a PID, the *Smith form* of M (see e.g. [5]) is a certain diagonal matrix equivalent to M under elementary operations.

One sees from the proposition and example that the rank, or indeed a Smith form, of Gram$D$ is potentially more useful than the determinant. However note that working over $\mathbb{Z}[\delta]$ as we partly are, a Smith form may not exist until we pass specifically to $\mathbb{C}$, say (or at least to a PID $k[\delta]$ with k a field); and they are harder to compute than the determinant when they do exist.

See §13.1 for more on this. (Also §8.2.4, and §10.1.2.)

2.3.2 Aside on Res-functors (exactness etc)

ss:aside res

(**2.3.24**) Note the limits of what functor Res_ψ (from (1.7.12)) says about A-modules in practice. For each B-module there is an A-module identical to it as a k-space. And for each B-module homomorphism there is an A-module homomorphism. It *does not* say that if $\mathrm{Hom}_B(M, N) = 0$ then so is $\mathrm{Hom}_A(M, N) = 0$.

In the particular case when ψ is surjective then M simple implies $\mathrm{Res}_\psi M$ simple — i.e. M simple as an A-module (any A-submodule M' of M would also be a B-submodule, since in this case the B action is contained in the A action).

(**2.3.25**) We can also think about what happens to exact sequences under this functor Res_ψ. Suppose $M' \hookrightarrow M \twoheadrightarrow M''$ is a short-exact sequence of B-module maps. As we have just seen, it is again a sequence of A-module maps. The sequence is of the form $M' \hookrightarrow M \twoheadrightarrow M''$ since injection and surjection are properties of the underlying k-modules; but such a sequence is short-exact if $dim(M') + dim(M'') = dim(M)$ — again a property of the underlying k-modules. In other words Res_ψ is an *exact* functor on finite dimensional modules.

We can also ask about split-ness. If the B-module sequence is split (i.e. $M = M' \oplus M''$) then there is another SES reversing the arrows, which again passes to an A-module sequence. If the B-module sequence is *non-split* what happens? Suppose that the A sequence is split. This means that there is an A-submodule of M isomorphic to M'', i.e. (up to isomorphism) $aM'' \in M''$ for all a. Note that *if ψ is surjective* [4] then every B action can be expressed as an A action (via ψ), so M'' is also a B-submodule, contradicting non-splitness. That is,

LEMMA. If algebra map ψ surjective then Res_ψ takes a non-split extension to a non-split extension. □

2.3.3 Functor examples for module categories: induction

(**2.3.26**) Functor Res_ψ makes B a left-A right-B-bimodule; and there is a similar functor making B a left-B right-A-bimodule. Hence define

$$\mathrm{Ind}_\psi : A - \mathrm{mod} \to B - \mathrm{mod}$$

by $\mathrm{Ind}_\psi N = B \otimes_A N$ (cf. 1.7.27).

[4] needed?

REMARK. This construction is typically used in case $\psi : A \to B$ is an inclusion of a subalgebra (in which case Res is called restriction).

(**2.3.27**) EXERCISE. Investigate these functors for possible adjunctions. Hints: Consider the map

$$a : \mathrm{Hom}_B(\mathrm{Ind}_{\psi} M, N) \to \mathrm{Hom}_A(M, \mathrm{Res}_{\psi} N)$$

given as follows. For $f \in \mathrm{Hom}_B(\mathrm{Ind}_{\psi} M, N)$ we define $a(f) \in \mathrm{Hom}_A(M, \mathrm{Res}_{\psi} N)$ by $a(f)(m) = f(1 \otimes m)$. Given $g \in \mathrm{Hom}_A(M, \mathrm{Res}_{\psi} N)$ we define $b(g) \in \mathrm{Hom}_B(\mathrm{Ind}_{\psi} M, N)$ by $b(g)(c \otimes m) = cg(m)$. One checks that $b = a^{-1}$, since $b(a(f)) = b(f(1 \otimes -)) = 1f = f$.

(**2.3.28**) EXAMPLE. We have in (1.32) above a surjective algebra map $\psi : P_n \to S_n$. It follows that every S_n-module is also a P_n-module via ψ. Of course every S_n-module map is also a P_n-module map.

pr:pr ind pr (**2.3.29**) PROPOSITION. *The functor Ind_{ψ} takes projectives to projectives.* ■

2.4 Alcove geometry and representation theory: first view

ss:TLAlc

Here we look at a useful manifestation of alcove geometry in TL representation theory, that determines 'geometric' conditions for the trivial module to be in a singleton block. The idea is to note the following. (1) The tensor space representation R_W (defined in **??**) is faithful [105]. (2) The image of the idempotent associated to the trivial module for generic q takes a particular form in each Young module in R_w [105]. (3) This form follows a pattern related to the Pascal triangle, and in particular to the projection of the triangle onto the horizontal line.

It is worth considering the pattern in char.p, but in char.0 is takes a particularly simple form. We obtain the following (also proved by direct calculation in [104]).

th:primidsing (**2.4.1**) THEOREM. *Let q be a primitive $2l$-th root of unity. In char.0 the T_n preidempotent associated to the trivial module can be normalised as an idempotent when l divides $n+1$.*

Hecke algebra

See also e.g. (1.4.70).

2.4.1 Basic Definitions

$H_n^{\mathbb{Z}}(q = x^2)$ is the $\mathbb{Z}[x, x^{-1}]$–algebra with generators $\{g_1, g_2, .., g_{n-1}\}$ and relations

$$(g_i - x^{-1})(g_i + x) = 0 \qquad \text{(equiv. } g_i^2 = (x^{-1} - x)g_i + 1)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

$$g_i g_j = g_j g_i \qquad |i - j| \neq 1.$$

(The 'Lusztig' form is $(t_i - q)(t_i + 1) = 0$ — a simple rescaling.) It will also be useful to have

$$U_i = x^{-1} - g_i \qquad \text{(so } U_i^2 = [2]U_i)$$

Let $\mathcal{R}$ be a ring equipped with the property of $\mathcal{A}$–algebra. Then $H_n^{\mathcal{R}} = \mathcal{R} \otimes_{\mathcal{A}} H_n^{\mathbb{Z}}(q)$. Examples: the field of fractions $\mathcal{A}^0 \supset \mathcal{A}$; and $\mathbb{C}$ is an $\mathcal{A}$–algebra for each choice of $q_c \in \mathbb{C}$, via $y \otimes q = yq_c \otimes 1$. Since the choice of q_c is not manifest in the bare field $\mathbb{C}$ we will usually write $H_n(q_c)$ for this case.

Recall that S_n is the symmetric group. We have $H_n^{\mathbb{Z}}(1) = \mathbb{Z}S_n$. For any object which passes sensibly to an S_n object at $q = 1$ we will use the S_n terminology in general. Let $l : S_n \to \mathbb{N}$ be the usual length function. Note that this coincides with the function in (**??**) via $l(w) = l(w12\ldots n)$ ($w \in S_n$ acting by permutation).

(**2.4.2**) For each sequence $i_1..i_l$ such that $\sigma_{i_1}..\sigma_{i_l}$ is a reduced expression for some $w \in S_n$ define $T_{i_1..i_l} = g_{i_1}..g_{i_l}$. Note that $T_{i_1..i_l} = T_{j_1..j_l}$ iff $\sigma_{j_1}..\sigma_{j_l} = w$ and define $T_w = T_{i_1..i_l}$ accordingly. A basis for $H_n^{\mathbb{Z}}(q = x^2)$ is $\{T_w | w \in S_n\}$.

(**2.4.3**) It is easy to see [95, (1.4.1)] that $f : H \to \mathbb{Z}[x, x^{-1}]$ given by

$$f(\sum_w a_w T_w) = a_1$$

obeys $f(T_x T_y) = \delta_{x,y^{-1}}$, so primitive central idempotents are accessible via proposition **??**.

For suitable characters to use consider the one–dimensional representations given by $R_\pm(g_i) = \pm x^{\mp 1}$ (and hence $R_+(T_w) = (x^{-1})^{l(w)}$). An element inducing R_+ is

$$e'_{(n)} = \sum_{w \in S_n} x^{l(w_0)-l(w)} T_w = x^{l(w_0)} \sum_w R_+(T_w) T_w$$

and the idempotent (over the field of fractions $\mathcal{A}^0$) is $e_{(n)} = \frac{1}{[n]!} e'_{(n)}$.

(**2.4.4**) The element $e'_{(n)}$ is amenable to various useful expansions. Define

$$\mathcal{T}_{(n-1)} = \left(1 + x^{-1} g_{n-1} + x^{-2} g_{n-2} g_{n-1} + \ldots + x^{-(n-1)} g_1 g_2..g_{n-1}\right)$$

$$\mathcal{T}^u_{(n-1)} = x^{1-n}\left([n] - [n-1]U_{n-1} + [n-2]U_{n-2}U_{n-1} - \ldots \pm [1]U_1 \ldots U_{n-1}\right)$$

Then

$$e'_{(n)} = \mathcal{T}_{(n-1)} e'_{(n-1)} = \mathcal{T}^u_{(n-1)} e'_{(n-1)} \tag{2.11}$$ eYoung

For example,

$$e'_{(3)} = \left(1 + x^{-1} g_2 + x^{-2} g_1 g_2\right)\left(1 + x^{-1} g_1\right)$$

And

$$\begin{aligned}
e'_{(n-1)} e'_{(n)} = [n-1]!\, e'_{(n)} &= e'_{(n-1)} \left(1 + x^{-1} g_{n-1} + x^{-2} g_{n-2} g_{n-1} + .. + x^{-(n-1)} g_1 g_2..g_{n-1}\right) e'_{(n-1)} \\
&= e'_{(n-1)} \left(1 + x^{-1}(1 + x^{-2} + .. + x^{-2(n-2)}) g_{n-1}\right) e'_{(n-1)} \\
&= e'_{(n-1)} \left(1 + x^{-1} x^{-(n-2)} (x^{n-2} + x^{n-4} + .. + x^{-(n-2)}) g_{n-1}\right) e'_{(n-1)} \\
&= e'_{(n-1)} \left(1 + x^{-(n-1)} [n-1] g_{n-1}\right) e'_{(n-1)} \\
&= e'_{(n-1)} \left(1 + x^{-(n-1)} [n-1] g_{n-1}\right) e'_{(n-2)} \mathcal{T}^o_{(n-2)}
\end{aligned}$$

$$= e'_{(n-1)} \left(1 + x^{-(n-1)}[n-1]g_{n-1}\right)\ [n-2]!\ \mathcal{T}^o_{(n-2)}$$

which is to say

$$[n-1]\ e'_{(n)} = e'_{(n-1)} \left(1 + x^{-(n-1)}[n-1]g_{n-1}\right)\ \mathcal{T}^o_{(n-2)} \qquad (2.12)$$ eTemperley

genwedge **Proposition 2.1.** *If $X \in H_n^{\mathbb{Z}}$ such that $g_i X = x^{-1} X$ $(i = 1, 2, .., n-2)$ then*

$$g_i \mathcal{T}_{(n-1)} X = x^{-1} \mathcal{T}_{(n-1)} X \qquad\qquad i = 1, 2, .., n-1$$

2.4.2 Specht modules

We now describe a version for $H_n^{\mathbb{Z}}$ of S_n Specht modules (as in, for example, [57, §6.3], [65]).

For Y a row standard Young tableau of degree n let P_Y (Q_Y) denote the row (column) stabilizing Young subgroup of S_n. Thus $P_Y = Q_{Y'}$.

Let Y^0 be the lexicographically lowest row standard Young tableau (e.g. (1234)(567)(8)) of shape ν, and Y^v the lexicographically highest standard Young tableau (e.g. (1468)(257)(3)). Set

$$\mathbf{f}^\nu = \sum_{w \in P_{Y^0}} R_+(T_w) T_w \qquad\qquad \mathbf{g}^\nu = \sum_{w \in Q_{Y^v}} R_-(T_w) T_w$$

unique w **Proposition 2.2.** *For given ν there is a unique shortest w such that*

$$\mathbf{f}^\nu H_n^{\mathbb{Z}} \mathbf{g}^\nu = \mathcal{A} \mathbf{f}^\nu T_w \mathbf{g}^\nu$$

With this w define

$$\mathbf{h}_\nu = \mathbf{f}^\nu T_w \mathbf{g}^\nu.$$

Define $H_n^{\mathbb{Z}}$–modules

$$\Delta_\nu = H_n^{\mathbb{Z}} (\mathbf{h}_\nu)^o \qquad\qquad \nabla_\nu = H_n^{\mathbb{Z}} \mathbf{h}_\nu$$

(**2.4.5**) Define elements v_s, v'_s of Δ_ν for each standard sequence s in B_ν iteratively on the (inverse) step order, with $(\mathbf{h}_\nu)^o$ as base, as follows. For $a < b$ define

$$\mathrm{ht}_i^{ab}(w) = \mathrm{hash}^a(\mathrm{trunc}_i(w)) - \mathrm{hash}^b(\mathrm{trunc}_i(w)).$$

Let s^ν be the lex highest standard sequence in B_ν. Set $v_{s^\nu} = v'_{s^\nu} = (\mathbf{h}_\nu)^o$. Fix a step path from s^ν to w. If sequences $w, (i)w$ on this path differ by $..ab.. \to ..ba..$ then define

$$v'_w = \ g_i\ v'_{(i)w}$$

$$v_w = ([h] - [h+1]U_i)\, v_{(i)w}$$

where $h = \mathrm{ht}_{i-1}^{ab}(w)$. (The definitions depend on the path to w, but a direct calculation shows that v_w, v'_w do not.) For example, consider $(2)w = 121$ and $w = 112$. We have $\mathrm{ht}_1^{12}(112) = 1$, so

$$v_{112} = (1 - [2]U_2)\, v_{121} = -x^{-2} \left(1 - x^2[2]g_2\right) v_{121}$$

gbasis **Proposition 2.3.** *The set $V'_\nu = \{v'_w \mid w \in B_\nu^{stan}\}$ is an $\mathcal{A}$–basis for Δ_ν.*

The set $V_\nu = \{v_w \mid w \in B_\nu^{stan}\}$ is NOT an $\mathcal{A}$–basis for Δ_ν in general (see Example **??**). However,

U=0 **Proposition 2.4.** *If $\sigma_i w = w$ then $U_i v_w = 0$, i.e.*

$$g_i v_w = x^{-1} v_w.$$

U=[h+2] *If $h = ht^{ab}_{i-1}(w)$ and $(i)w$ is defined*

$$U_i v_w = -[h+2] U_i v_{(i)w}.$$

Proof: See for example [104, §9.3.1].

For $\mathcal{R}$ an $\mathcal{A}$–algebra define $H_n^{\mathcal{R}}$-modules $\nabla_\nu^{\mathcal{R}} = \mathcal{R} \otimes_{\mathcal{A}} \nabla_\nu$, $\Delta_\nu^{\mathcal{R}} = \mathcal{R} \otimes_{\mathcal{A}} \Delta_\nu$.

t indecomposable **Proposition 2.5.** *For any ν, both ∇_ν^k and Δ_ν^k are indecomposable over any suitable field k.*

Proof: It is straightforward to show that a suitable scalar multiple of h_ν is a primitive idempotent over $\mathcal{A}^0$. Any non–trivial idempotent in $\mathrm{End}(\Delta_\nu^k)$ would lift contradicting this primitivity. □

It follows that Δ_ν is simple over $\mathcal{A}^0$ and over $\mathbb{C}$ for all but a closed subset of choices for q_c.

2.4.3 Tensor space

(**2.4.6**) The R_W tensor space representations of $H_n^{\mathbb{Z}}$ require only a mild generalisation of the S_n case from section **??**. For example the $W = \{1,2\}$ tensor space representation is

$$R_W(U_i) = 1_2 \otimes 1_2 .. \otimes \begin{pmatrix} 0 & & & \\ & x & -1 & \\ & -1 & x^{-1} & \\ & & & 0 \end{pmatrix} .. \otimes 1_2$$

$$R_W(g_i + x = [2] - U_i) = 1_2 \otimes 1_2 .. \otimes \begin{pmatrix} [2] & & & \\ & x^{-1} & 1 & \\ & 1 & x & \\ & & & [2] \end{pmatrix} .. \otimes 1_2$$

For any W there is an *immediate* direct sum decomposition into permutation representations R_ν (and permutation modules $\mathtt{M}_\nu$) exactly as for S_n:

$$R_W = \oplus_\nu R_\nu. \tag{2.13}$$ direct sum decomp H

We will again use the bases B_ν.

perm module H **Proposition 2.6.** *There is an isomorphism of left $H_n^{\mathbb{Z}}$–modules*

$$H_n^{\mathbb{Z}} \mathbf{f}^\nu \cong \mathtt{M}_\nu \tag{2.14}$$ left ideal

$$\mathbf{f}^\nu \mapsto s^0$$

Example: In case $\nu = (2,1)$, $\{\mathbf{f}^Y, T_{\sigma_2}\mathbf{f}^Y, T_{\sigma_1\sigma_2}\mathbf{f}^Y\}$ is a basis for the left ideal, with ordered image $\{112, 121, 211\}$.

As consequences we have:

self dual H

Proposition 2.7. $\mathtt{M}_\nu$ *is contravariant self–dual.*

Proof: This follows from the usual involutive antiautomorphism (cf. [57, §2.7], [58]) on noting that $\mathbf{f}^\nu = (\mathbf{f}^\nu)^o$.

specht in perm H

Proposition 2.8. *For each* ν *there is a module* $\mathtt{M}_\nu^\nu$ *and a short exact sequence*

$$0 \longrightarrow \Delta_\nu \longrightarrow \mathtt{M}_\nu \longrightarrow \mathtt{M}_\nu^\nu \longrightarrow 0.$$

Proof: Comparing the construction of the module Δ_ν with (2.14) we have

$$\Delta_\nu \cong H_n^{\mathbb{Z}} \mathbf{g}^\nu \mathtt{M}_\nu \hookrightarrow \mathtt{M}_\nu$$

(NB this sequence *defines* $\mathtt{M}_\nu^\nu$).

Considering $H_n(q) = k \otimes_{\mathbb{Z}[x,x^{-1}]} H_n^{\mathbb{Z}}(q)$ for some field k, the further direct sum decomposition of R_ν itself depends on k. For any given k let the (!) indecomposable summand containing Δ_ν be denoted T_ν.

[illegible]

Proposition 2.9. *For given* k *the complement of* T_ν *in* R_ν *will be a direct sum of modules of form* $T_{\nu'}$ *with* $\nu' \rhd \nu$.

(**2.4.7**) The map given by

$$\Phi_\nu(11..1) = \sum_{s \in B_\nu} q^{l(s)} s$$

defines a hom from $\mathtt{M}_{(n)} \hookrightarrow \mathtt{M}_\nu$. For example

$$\Phi_{(2,2)}(1111) = 1122 + q1212 + q^2(1221 + 2112) + q^3 2121 + q^4 2211.$$

NB, $\Phi_\nu(11..1) \propto e'_{(n)} s_0$.

2.5 q-dimensions and flagged morphisms

(**2.5.1**) For given ν let $(s^0, s^1, .., s^{last})$ be the sequence of sequences written in lexicographically increasing order. Following [105, Appendix] define a vector $l(\nu) = (x^{l(s^0)}, x^{l(s^1)}, ..)$ and a matrix

$$D_x(\nu) = x^{-l(s^{last})} (l(\nu))^t l(\nu)$$

E.g.

$$D_q((2,2)) = \begin{pmatrix} q^{-2} \\ q^{-1} \\ 1 \\ 1 \\ q \\ q^2 \end{pmatrix} \begin{pmatrix} q^{-2} & q^{-1} & 1 & 1 & q & q^2 \end{pmatrix} = \begin{pmatrix} q^{-4} & q^{-3} & q^{-2} & q^{-2} & q^{-1} & q^0 \\ q^{-3} & q^{-2} & q^{-1} & q^{-1} & q^0 & q^1 \\ q^{-2} & q^{-1} & q^0 & q^0 & q^1 & q^2 \\ q^{-2} & q^{-1} & q^0 & q^0 & q^1 & q^2 \\ q^{-1} & q^0 & q^1 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^2 & q^3 & q^4 \end{pmatrix}$$

Note that for a permutation basis element which is a sequence with ab in the i^{th}, $(i+1)^{th}$ positions

$$(g_i + x)..ab.. \propto \begin{cases} x^{-1}..ab.. + ..ba.. & a < b \\ x \quad ..ab.. + ..ba.. & a > b \\ [2] \; ..ab.. & a = b \end{cases}$$

and that

$$(g_i + x)e'_{(n)} = [2]e'_{(n)}$$

It follows (see also [118, §4]) that

Proposition 2.10.

$$R_\nu(e_{(n)}) = \frac{\prod_i [\nu_i]!}{[n]!} D_x(\nu)$$

Proof: Up to the overall scalar this is a consequence of the action noted above. For the scalar, we note that the trace must be 1 as for S_n (it is a discretely valued continuous function of x which takes the value 1 at $x = 1$). Computation of the unnormalized trace is an exercise in the properties of $D_x(\nu)$.

(**2.5.2**) Now fix a field k of characteristic p in which x is a primitive l^{th} root of unity. We are interested in $H_n(q) = k \otimes_{\mathbb{Z}[x,x^{-1}]} H_n^{\mathbb{Z}}(q)$.

Let δ_λ be the formal l,p–valuation of $\dim_q R_\lambda$. It follows from proposition (2.9) and proposition (2.10) that we may apply equation (**??**) to obtain

Proposition *There is a homomorphism*

$$0 \to \Delta_{(n)} \to \Delta_\lambda \tag{2.15}$$ crucial

over k whenever $\delta_\lambda > \delta_{\lambda'}$ for all $\lambda' \rhd \lambda$.

NB, using the kernel intersection theorem [74] James obtains

Theorem [James] *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition of n with $\lambda_h > 0$. Then the trivial module is a submodule of Δ_λ if and only if for each $d \in \{1, 2, \ldots, h-1\}$*
(i) l divides $(\lambda_d + 1)$; and
(ii) $\left[\frac{\lambda_d+1}{l}\right] \equiv 0$ modulo $p^{l_p(\left[\frac{\lambda_{d+1}}{l}\right])}$,
where $[x]$ denotes the integer part, and (for $b \in \mathbb{N}_0$) $l_p(b)$ denotes the leasy nonnegative integer i such that $b < p^i$.
(For $q = 1$ see Chapter 24 of James [74]; while a general q version can be derived by the methods of [76].)

Fixing r, the array of l,p–valuations δ_λ for those λs embedded in A_r weight space gives an interesting pattern, which the above application motivates us to study. We will begin with the case $W = \{1, 2\}$. Consider figure 2.6.

There are several examples of bases for Specht and coSpecht modules in section **??**.

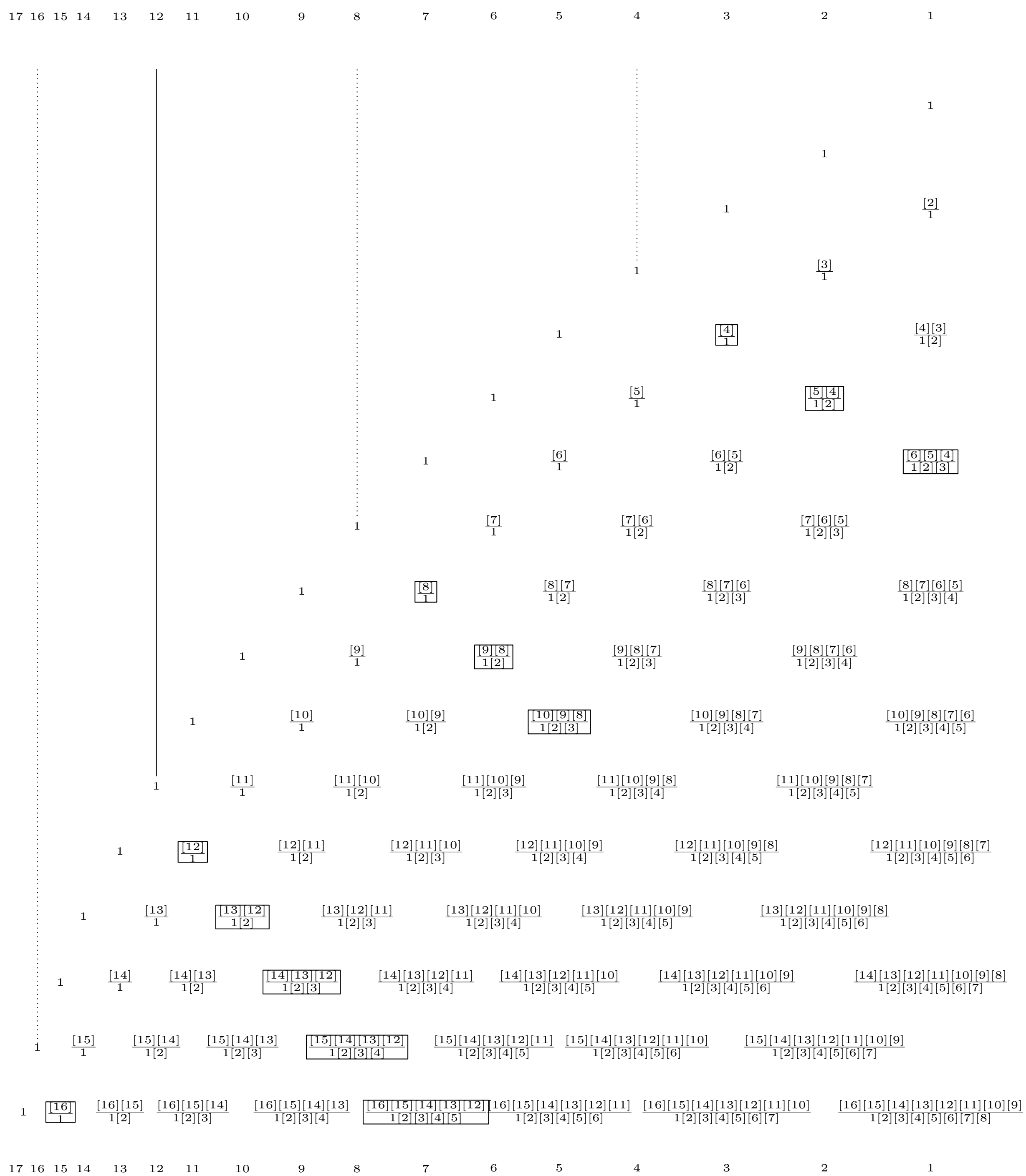
(**2.5.3**) Note in particular that we have proved Thm.2.4.1. □

2.6 Structure theorem for T_n over $\mathbb{C}$

ss:TLST

We are interested in determining the representation theory, in the sense of §1.5, of P_n and similar algebras. These are towers of algebras A_n, say, derived from diagram categories. That is to say, roughly speaking, that for each n there is an idempotent $e = e_n \in A_n$ such that $eA_ne \cong A_{n-1}$, embedding $A_{n-1}-$mod in A_n-mod, as in §1.7.5, and the quotient A_n/A_neA_n has known structure. To introduce this study it is convenient to begin with T_n.

In this section we give a quick illustrative summary of T_n. (We do not take particular care here of the case $\delta = 0$.) For details and alternative approaches to T_n see Ch.12 and references therein. The overarching strategy is roughly as follows.



qpascal

Figure 2.6: The beginning of the q–Pascal triangle. Each q–dimensions whose l, p–adic valuation dominates all to its left is boxed (case $l = 4$, $p = 3$).

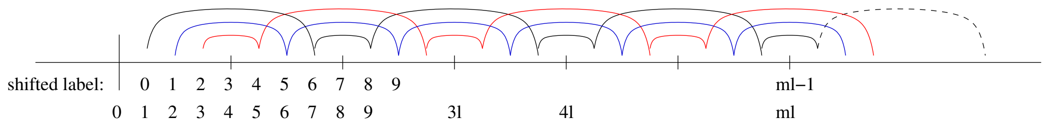


Figure 2.7: Orbits of an affine reflection group on $\mathbb{Z}$ giving blocks for T_n with $l = 4$. fig:TLalcoves1

Step 1: construct, and show to be isomorphic, certain key classes of modules. Each construction has distinct useful properties, so the isomorphism means that these 'Δ-modules' have all the useful properties. Roughly speaking the classes are as follows.
Specht modules (modules defined integrally, and generically simple, as useful for π-modular systems);
global-standard modules (images of simple modules under globalisation functors); and possibly some others such as
standard modules (for a quasihereditary algebra — indec. projective modules for certain special quotient algebras).

Step 2 is to state a theorem giving the simple composition factors for the Δ-modules (NB this assumes we know them, or have a conjecture!). By the Specht property and Brauer reciprocity **??** this determines the Cartan decomposition matrix.

Step 3 is to set up an inductive proof using the global-standard property to move data directly up the ranks, and Frobenius reciprocity (induction and restriction between ranks) and the block decomposition to build 'translation functors' that determine the remaining data.

2.6.1 Global-Standard modules are standard

For $n \in \mathbb{N}_0$ set $\Lambda_n^T = \{n, n-2, ..., 1/0\}$. Consider T_n over an arbitrary field k with $\delta \neq 0$.

(**2.6.1**) Set $\Delta_n^T(n) = k$ (the trivial T_n-module). Then for $l \in \Lambda_{n-2}^T = \Lambda_n^T \setminus \{n\}$ define T_n-modules by iterating from T_{n-2}:

$$\Delta_n^T(l) \;=\; G_{\mathsf{e}_1} \Delta_{n-2}^T(l)$$

as in §1.7.5.

exa:TLsbas (**2.6.2**) Example. $G_{\mathsf{e}_1} \Delta_{n-2}^T(n-2) = T_n \mathsf{e}_1 \otimes_{T_{n-2}} \Delta_{n-2}^T(n-2)$ (using the isomorphism to confuse $T_{n-2} \cong \mathsf{e}_1 T_n \mathsf{e}_1$). Noting $T_n \mathsf{e}_1 = k\mathsf{T}_{n,n-2} \otimes \cap$ (as in (1.3.9)); this is spanned by $\mathsf{T}_{n,n-2} \otimes_{T_{n-2}} 1_{n-2}$, where $\{1_{n-2}\}$ is acting as a basis for $\Delta_{n-2}^T(n-2)$. Note that $\mathsf{T}_{n,n-4,n-2} \otimes_{T_{n-2}} 1_{n-2} = 0$, so a basis is $\mathsf{T}_{n,n-2,n-2} \otimes_{T_{n-2}} 1_{n-2}$.

pr:DelD (**2.6.3**) Lemma. *For $l \in \Lambda_n^T = \{n, n-2, ..., 1/0\}$*

$$\Delta_n^T(l) \;\cong\; D_n^{\mathrm{TL}}(l)$$

(as defined in §2.2).

Proof. As illustrated by the example (2.6.2) above, a basis of $\Delta_n^T(l)$ is $\mathsf{T}_{n,l,l} \otimes_{T_l} 1_l$. Now cf. (2.2.6) and consider the obvious bijection between bases. The actions of $a \in T_n$ are the same — if (in the

T category) $a * b \in k\mathsf{T}_{n,l,l}$ then $ab = a * b$ in both cases; otherwise $ab = 0$ in $\Delta_n^T(l)$ by the balanced map, and in $D_n^{\mathrm{TL}}(l)$ by the quotient. □

...See §?? for more details and treatment of the $\delta = 0$ case.

2.6.2 Weights: geometrical index schemes for standard modules

TLwallnotation

(**2.6.4**) Consider Fig.2.7. Fix $r \in \mathbb{N}$. We give the positive real line two labellings for integral points: the natural labelling (with the origin labelled 0); and the *shifted* labelling. Points of form mr in the natural labelling ($mr - 1$ in the shifted labelling) are called *walls*. The regions between walls are called *alcoves*. Write $\sigma_{(m)} : \mathbb{R} \to \mathbb{R}$ for reflection in the m-th wall. Write

$$\Sigma^{(r)} = \langle \sigma_{(0)}, \sigma_{(1)} \rangle$$

for the group of (affine) reflections. Write $l^{\Sigma^{(r)}}$ for the dominant (non-negative) part of the orbit of point l (in the shifted labelling) under $\Sigma^{(r)}$. Thus for example $0^{\Sigma^{(r)}} = \{0, 2r-2, 2r, 4r-2, ...\}$.

2.6.3 The structure theorem over $\mathbb{C}$

th:TLoverC

(**2.6.5**) Theorem. [104, §7.3 Th.2] (Structure Theorem for T_n over $\mathbb{C}$.) Set $k = \mathbb{C}$ and fix $\delta \in k$; or set $k = \mathbb{C}(\delta)$. The T_n-modules $\{L_n(\lambda) = \text{head } \Delta_n^T(\lambda)\}_{\lambda \in \Lambda_n^T}$ are a complete set of simple T_n-modules. The simple content of the modules $\{\Delta_n^T(\lambda)\}_\lambda$ determines the structure of T_n, and is given depending on δ as follows.

(I) In case there is no $r \in \mathbb{N}$ such that δ is of the form $\delta = q + q^{-1}$ with $q^r = 1$, the Δ-modules are simple, and absolutely irreducible, and T_n is split semisimple.

(II) Fix $r \in \mathbb{N}$ (here we take $r \geq 3$ for now) and let $q \in \mathbb{C}$ be a primitive $2r$-th root of unity. Suppose $\delta = q + q^{-1}$.

For given $\lambda \in \mathbb{N}_0$ determine m and b in $\mathbb{N}_0$ by $\lambda + 1 = mr + b$ with $0 \leq b < m$ (so b is the position of $\lambda + 1$ in the alcove above mr, in the sense of (2.6.4)). For $b > 0$ set $\sigma_{(m+1)}.\lambda = \lambda + 2m - 2b$ — the image of λ under reflection in the wall above.

1) If $b = 0$ then $\Delta_n^T(\lambda) = L_n(\lambda)$.

2) Otherwise

$$0 \longrightarrow L_n(\lambda + 2m - 2b) \longrightarrow \Delta_n^T(\lambda) \longrightarrow L_n(\lambda) \longrightarrow 0 \qquad (2.16)$$ eq:sesTLa

Here $L_n(\lambda + 2m - 2b)$ is to be understood as 0 if n is too small.

In particular the orbits of the reflection action describe the 'regular' blocks (blocks of points not fixed by any non-trivial reflection); while the singular blocks (of points fixed by a non-trivial reflection) are singletons.

(III) We leave the cases $r = 1, 2$ ($q = 1, -1, i, -i$) as an exercise for now. See §??.

(**2.6.6**) An informal way to present Theorem 2.6.5, following [104], is that the simple content of standard T_n-modules (arranged as in Fig.2.1, and cf. Fig.2.7) is indicated by the example in Figure 2.8.

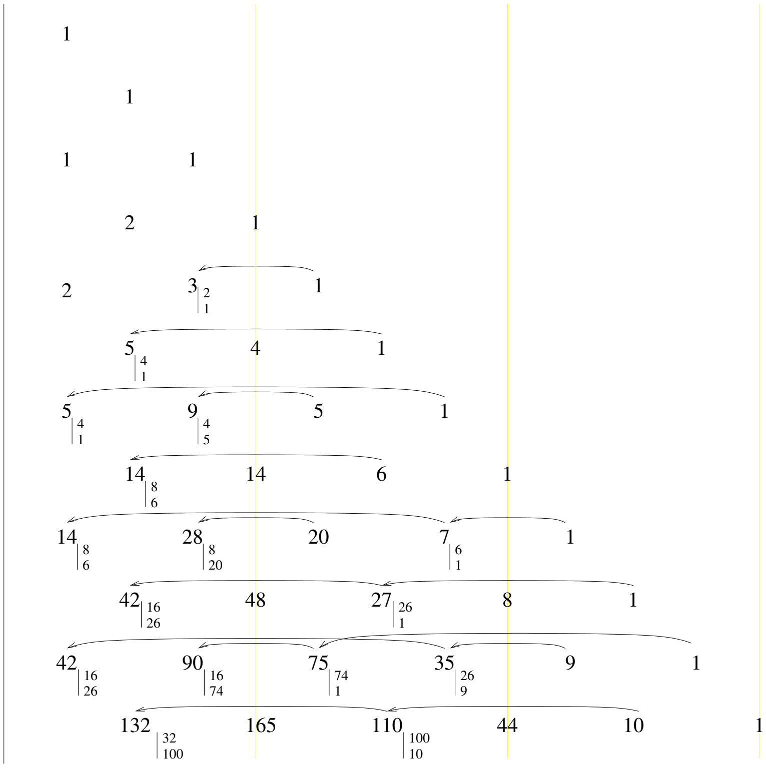


fig:TLbratthop1

Figure 2.8: Simple content, dimensions and morphisms of standard T_n-modules (in case $k = \mathbb{C}$, $r = l = 4$).

2.6.4 The decomposition matrices of T_n over $\mathbb{C}$

ss:decompmatex1

Note that the decomposition matrices (from §1.8 and (1.42)) are determined by the structure Theorem 2.6.5. The standard-decomposition matrix for a single regular block $l^{\Sigma^{(r)}}$ (starting from the low-numbered weight) is of form

$$D_{block} = \begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$$

(this should be thought of as the n-dependent truncation of a semiinfinite matrix continuing down to the right), that is $\Delta^T(0)$ (say, from the first row) contains $L(0)$ and the next simple in the block, and so on. This gives the block Cartan decomposition matrix:

$$C_{block} = D_{block}^T D_{block} = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix}$$

2.6.5 Proof of Theorem: set up the induction — translation functors

Proof. Firstly, by construction the modules $D_n^{\text{TL}}(\lambda)$ give a filtration of the left-regular T_n-module. Thus by Jordan–Holder(III) (1.4.14) every simple module appears in (the head of) some $D_n^{\text{TL}}(l)$. The completeness of $\{L_n(\lambda)\}_\lambda$ follows by, say, 2.2.4 and 2.6.3.

To proceed we will need some lemmas.

(**2.6.7**) Lemma. [Δ-filtration Lemma] *Projective T_n-modules have filtrations by Δ modules; and the corresponding composition multiplicities are well defined.*

Proof. Filtration was proved in Cor.2.2.5. We can see well-definedness in various different ways. For now we note from §1.8 (specifically (1.8.4) and (1.8.1) respectively) that both kinds of modules have lifts to the integral case $k = \mathbb{C}[\delta]$, and hence corresponding modules in the ordinary case $k = \mathbb{C}(\delta)$. But in the ordinary case the Δ-modules are simple, with well-defined multiplicities by Jordan–Holder. ■

lem:wol0 (**2.6.8**) Lemma. [Upper-unitriangular Lemma] *The composition multiplicities*

$$(\Delta_n^T(\mu) : L_n(\lambda)) \;=\; 0 \quad \textit{unless } \lambda \geq \mu$$

(and $(\Delta_n^T(\lambda) : L_n(\lambda)) = 1$*).*

Proof. Otherwise we can localise until $\Delta_m^T(\mu) \cong\ e\Delta_n^T(\mu)$ (some e) is simple and get a contradiction using (1.6.14).

lem:wol (**2.6.9**) Lemma. [Weight-order Lemma] *Once $n \geq \lambda$, so indecomposable projective $P_n(\lambda)$ is defined, then the multiplicity $(P_n(\lambda) : \Delta_n^T(\lambda)) = 1$; $(P_n(\lambda) : \Delta_n^T(\mu)) = 0$ if $|\mu| \geq |\lambda|$ $(\mu \neq \lambda)$; and otherwise $(P_n(\lambda) : \Delta_n^T(\mu))$ does not depend on n.*

Proof. By (1.8.6) and (1.6.14). Note from 2.6.8 that $(\Delta_n^T(\mu) : L_n(\lambda)) = 0$ unless $\lambda \geq \mu$ (and $(\Delta_n^T(\lambda) : L_n(\lambda)) = 1$). We can express this as saying that the corresponding decomposition matrix is lower-unitriangular. Then apply Brauer reciprocity, as in 1.8.6, in case base ring $R = \mathbb{C}[\delta]$ or $R = \mathbb{C}[q, q^{-1}]$. (NB Reciprocity assumes that T_n is semisimple over the field R_0. This follows from Case (I) in the Theorem.) □

(**2.6.10**) Proof of Theorem in a case of type-(I). Method 1: Note from 2.3.14 that there is always a T_n-module map from a Δ-module to its contravariant dual (so that they have at least one simple factor in common); and that if δ is indeterminate then this map is an isomorphism. Since each Δ contains only one copy of its head-simple (Lem.2.6.8), a single isomorphic factor must be both the head and socle of the cv dual. That is, both modules are simple. If $\delta \in \mathbb{C}$ then this argument shows specifically that $\Delta^T_n(n-2)$ is simple for all n unless $\delta = q + q^{-1}$ with q some root of unity. One may then show that all the other Δs are simple using 2.6.13 and Frobenius reciprocity. (See later.)

Method 2: If $1 \notin q^{\mathbb{N}}$ then the Δs are in different blocks (by 2.2.11) and so none contains a composition factor in common with another. Thus each is simple by (the parenthetical result in) Lem.2.6.8.

(**2.6.11**) Proof in a case of type-(II). We proceed by induction on n. Let $\mathsf{A}(n)$ denote the proposition that the Theorem holds in level n and below. In case (I) we assume $\mathsf{A}(mr-1)$, i.e. we assume level $n = mr-1$ and below. (And will work through a 'cycle' $n = mr, mr+1, ..., mr+r-1$. That is, the inductive step is from m to $m+1$.) It is an exercise to check the base cases. By $\mathsf{A}(mr-1)$ we have $\Delta^T_{mr-1}(mr-1) = L_{mr-1}(mr-1) = P_{mr-1}(mr-1)$.

(Thus, if $n' \equiv mr-1$ mod.2, we have $\Delta^T_{n'}(mr-1) = L_{n'}(mr-1) = P_{n'}(mr-1)$.

Why? Firstly, we have some organisational Lemmas.)

(**2.6.12**) Remark: By Lem.2.6.9 if $\Delta^T_{n=mr-1}(mr-1) = P_{n=mr-1}(mr-1)$ this identification holds for all higher n. (NB this does not of itself guarantee that the module is *simple* for all n.)

pr:resDeTL (**2.6.13**) PROPOSITION. [Δ-restriction Lemma] Let $\psi : T_{n-1} \hookrightarrow T_n$ and Res = Res$_\psi$. We have

$$0 \longrightarrow \Delta^T_{n-1}(l-1) \longrightarrow \mathrm{Res}\Delta^T_n(l) \longrightarrow \Delta^T_{n-1}(l+1) \longrightarrow 0$$

Proof. Hint: consider Fig.2.1. ■

pr:indresG (**2.6.14**) PROPOSITION. *The functors Ind$_\psi$ and Res$_\psi G$ are naturally isomorphic.*

Proof. Ind $-$ is $T_{n+1} \otimes_{T_n} -$ while $G-$ is $k\mathsf{T}(n+2,n) \otimes_{T_n} -$. But $T_{n+1} = k\mathsf{T}(n+1,n+1)$ and $k\mathsf{T}(n+2,n)$ are isomorphic as left-T_{n+1} right-T_n-modules (by the 'disk bijection', which draws partitions on a disk instead of a rectangular frame). ■

(**2.6.15**) By 2.6.13 and 2.6.14 (and the definition of $\Delta^T(l)$) we have

$$\mathrm{Ind}\,\Delta^T(l) = \Delta^T(l+1) + \Delta^T(l-1),$$

So for example if the inductive assumption holds we have

$$\mathrm{Ind}\,P(mr-1) = \Delta^T(mr) + \Delta^T(mr-2). \tag{2.17}$$ eq:PDD-2

On the other hand, by Lem.2.6.9,

lem:Phwt1 (**2.6.16**) LEMMA. *Any projective T_n-module is a direct sum of indecomposable projectives including those with the highest shifted label among those appearing in its Δ^T factors.* □

(**2.6.17**) Define Pr_l as the projection functor onto the block of $L(l)$. (This is to be considered formally for the moment — we make no intrinsic assumptions about which other simples lie in this block.) Define the 'translation functor' $\mathrm{Ind}_l - = Pr_l\mathrm{Ind}-$.

We have for example $\mathrm{Ind}_l\, P(l-1) = P(l) + Q$, where Q is a (possibly zero) 'lower' projective in the block of l.

2.6.6 Starting the induction

We now proceed with the induction. The first step is to show that $\mathsf{A}(mr-1)$ implies $\mathsf{A}(mr)$. For this it is sufficient to 'compute' the Δ-content of $P(mr)$.

(**2.6.18**) By (2.17) (i.e. by the inductive assumption) and (2.6.16) we have that $\operatorname{Ind} P(mr-1)$, which is projective since $\operatorname{Ind}-$ preserves projectivity (Prop.2.3.29), contains $P(mr)$ as a direct
pa:step0 summand.

Suppose (for a contradiction) that $P(mr) = \Delta^T(mr)$. Then in particular (i) $P_{mr}(mr) = \Delta^T_{mr}(mr) = L_{mr}(mr)$ and the module would be in a simple block here. Next note that the remaining factor in $\operatorname{Ind} P(mr-1)$ would also be projective, so (again by 2.6.16) $P(mr-2) = \Delta^T(mr-2)$. But this would imply (ii) $\Delta^T(mr-2) = L(mr-2)$ by the argument in the proof of (2.6.9), since the only other possible factor is $L(mr)$, but the working assumptions place this in a different block. Finally this contradicts the fact (iii) from 2.3.8 that the gram determinant $||\Delta^T_{mr}(mr-2)|| = [mr] = 0$ when $q^{2r} = 1$, which implies that $\Delta^T_{mr}(mr-2)$ has a submodule in this case.

Thus $\operatorname{Ind} P(mr-1) = P(mr)$. Thus $P(mr) = \Delta^T(mr) + \Delta^T(mr-2)$.

Remark. Appart from case $m = 1$ the supposition above (specifically the implication $P(mr-2) = \Delta^T(mr-2)$) also contradicts the inductive assumption. That is, we only strictly need the argument above in case $m = 1$.

(**2.6.19**) Next (to verify $\mathsf{A}(mr+1)$) we need to compute $P(mr+1)$. We have

$$\operatorname{Ind} P(mr) = \Delta^T(mr+1) + \Delta^T(mr-1) + \Delta^T(mr-1) + \Delta^T(mr-3)$$

Again this contains $P(mr+1)$ and the game is to determine which of the factors are in $P(mr+1)$.

Step 1: If $\Delta^T(mr-1)$ is in $P(mr+1)$ then $L(mr+1)$ would be in $\Delta^T(mr-1)$ by modular reciprocity (necessarily in the socle); in particular $\Delta^T_{mr+1}(mr-1)$ would have a submodule, which would imply a degenerate unique contravariant form, and hence $||\Delta^T_{mr+1}(mr-1)|| = 0$ — a contradiction since by (2.8) $||\Delta^T_{mr+1}(mr-1)|| = [mr+1] = 1$ when $q^{2r} = 1$.

Remark. Alternatively it is very easy to show using Schur's Lemma and a suitable central element of T_n (such as the image in T_n of the double-twist braid) that indecomposables $\Delta^T(mr-1)$ and $P(mr+1)$ are not in the same block — see (2.5).

de:TL901 (**2.6.20**) Step 2: Next we will show by a contradiction that $P(mr+1) = \Delta^T(mr+1) + \Delta^T(mr-3)$. Suppose this sum splits. Then this would imply $P(mr-3) = \Delta^T(mr-3)$ and hence $L(mr-3) = \Delta^T(mr-3)$, arguing as in (2.6.18)(I-II). However, for a contradiction consider the following (method for avoiding computing the analogue of (2.6.18)(III) by hand!).

de:TL902 (**2.6.21**) By Frobenius reciprocity (8.5.16) we have

$$\operatorname{Hom}(\operatorname{Ind} A, B) \cong \operatorname{Hom}(A, \operatorname{Res} B)$$

in particular in the case in Fig.2.9: [5]

$$\operatorname{Hom}(\operatorname{Ind} \Delta^T_{ml}(ml),\ \Delta^T_{ml+1}(ml-3)) \ \cong\ \operatorname{Hom}(\Delta^T_{ml}(ml),\ \operatorname{Res}\Delta^T_{ml+1}(ml-3))$$

[5] caveat: $l = r$!!!

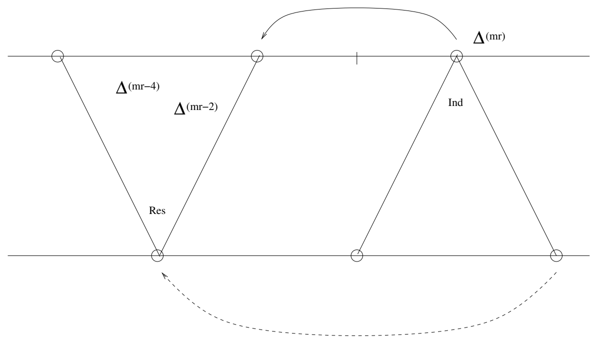


Figure 2.9: Δ-module maps by Frobenius reciprocity. $\boxed{\texttt{fig:FRTL1}}$

Note that $\mathrm{Res}\Delta^T_{ml+1}(ml-3) = \Delta^T_{ml}(ml-2) \oplus \Delta^T_{ml}(ml-4)$ (a direct sum by the block assumption in $\mathsf{A}(ml)$, unless $r=2$), so that the RHS is nonzero by assumption (noting, say, (2.16)). Thus the LHS is nonzero. There is no map from $\Delta^T(ml+1)$ to $\Delta^T(ml-1)$, as already noted in Step 1, so there is a map from $\Delta^T(ml+1)$ to $\Delta^T(ml-3)$. This demonstrates the contradiction needed in 2.6.20. Thus

$$P(mr+1) = \Delta^T(mr+1) + \Delta^T(mr-3)$$

(**2.6.22**) Step 2 (alternate approach): Suppose again for a contradiction that $\mathrm{Ind}_{mr+1} P(mr) = \Delta^T(mr+1) \oplus \Delta^T(mr-3)$. This would imply $\mathrm{Ind}\,\mathrm{Ind}_{mr+1} P(mr) = \; (\Delta^T(mr+2) + \Delta^T(mr)) \oplus (\Delta^T(mr-2) + \Delta^T(mr-4))$. This would imply that either $P(mr+2) = \Delta^T(mr+2)$ and $P(mr) = \Delta^T(mr)$ — contradicting $\mathsf{A}(mr)$ — or $P(mr+2) = \Delta^T(mr+2) + \Delta^T(mr)$. The latter would imply $(\Delta^T(mr) : L(mr+2)) = 1$ by Brauer reciprocity, but $\Delta^T_{mr+2}(mr)$ is simple (unless $r=2$) by the determinant calculation (2.8) (which gives determinant $[mr+2] = \frac{q^{mr+2} - q^{-mr-2}}{q-q^{-1}} = \pm[2]$ when $q^{2r}=1$) — a contradiction.

(**2.6.23**) Next we have to verify $\mathsf{A}(mr+2)$. We have

$$\mathrm{Ind}\, P(mr+1) = \Delta^T(mr+2) + \Delta^T(mr) + \Delta^T(mr-2) + \Delta^T(mr-4)$$

We have $P(mr+2) = \Delta^T(mr+2) + ...$ The question is, which of the factors above should be included? If we include $\Delta^T(mr)$ then $L(mr+2)$ is in $\Delta^T(mr)$ by modular reciprocity. We can eliminate this possibility in a couple of ways. For example, we can compute a central element of T_n and show using this that the two shifted labels are in different blocks. Alternatively we can compute $||\Delta^T_{mr+2}(mr)||$ and check that it is nonzero in this case.

So far, then, we have that $\mathrm{Ind}\, P(mr+1) = P(mr+2) \oplus P(mr) \oplus ....$ However since $P(mr) = \Delta^T(mr) + \Delta^T(mr-2)$ we have $P(mr+2) = \Delta^T(mr+2) + X$ where $X = \Delta^T(mr-4)$ or zero.

In the latter case we would have $P(mr-4) = \Delta^T(mr-4)$. This contradicts the inductive assumption for every m value except $m=1$. For $m=1$ (or in general) we note instead that

$$\mathrm{Hom}(\mathrm{Ind}\,\Delta^T_{mr+1}(mr+1), \Delta^T_{mr+2}(mr-4)) \cong \mathrm{Hom}(\Delta^T_{mr+1}(mr+1), \mathrm{Res}\Delta^T_{mr+2}(mr-4))$$

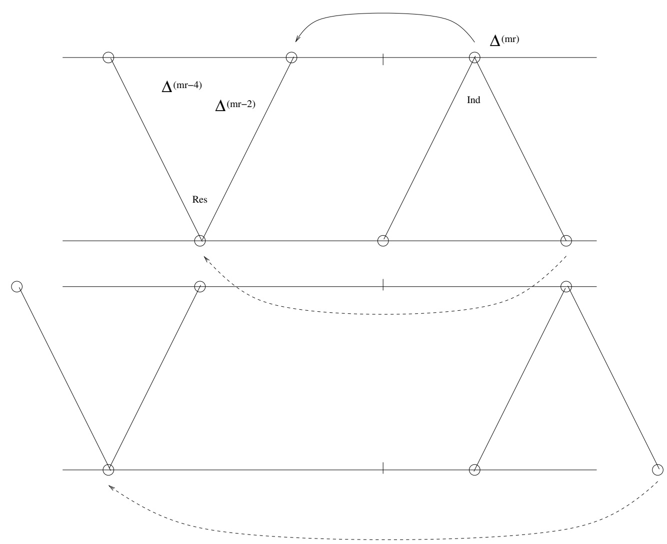


Figure 2.10: $\boxed{\texttt{fig:FRTL3}}$

and that the RHS is nonzero (for $r > 3$) by the inductive assumption (indeed we just showed this in 2.6.21 above) — see also the schematic in Fig.2.10. Thus the LHS is nonzero. But there is no map $\Delta^T(mr) \to \Delta^T(mr-4)$ by the inductive assumption, so there is a map $\Delta^T(mr+2) \to \Delta^T(mr-4)$. This provides the required contradiction, so $X \neq$ zero. That is

$$P(mr+2) \;=\; \Delta^T(mr+2) + \Delta^T(mr-4) \quad=\quad \Delta^T(mr+2) + \Delta^T(\sigma_{(m)}(mr+2))$$

(**2.6.24**) We may continue in the same way until we come to show $\mathsf{A}(mr+r-1)$, by stepping up from $P(mr+(r-2)) = \Delta^T(mr+(r-2)) + \Delta^T(mr-r)$. Thus $\mathrm{Ind}\,P(mr+(r-2)) = P(mr+r-1) \oplus ... = \Delta^T(mr+(r-1)) + \Delta^T(mr+(r-3)) + \Delta^T(mr-r+1) + \Delta^T(mr-r-1)$. Analogously to before we rule out $\Delta^T(mr+(r-3))$ from $P(mr+r-1)$ by modular reciprocity and $||\Delta^T_{mr+r-1}(mr+(r-3))|| = [mr+r-1] \neq 0$ (and hence also rule out $\Delta^T(mr-r+1)$). But this time we can also rule out $\Delta^T(mr-r-1)$ by modular reciprocity (if it exists, i.e. if $m > 1$), since this is simple by the inductive assumption and (!!) Thm.2.4.1.
(Remark. At this point $\mathrm{Res}\Delta^T(mr-r-1)$ is not a direct sum (indeed it is indecomposable projective) and the argument for a nonzero RHS in Frobenius reciprocity fails. This tells us that this time there is not necessarily map on the LHS. Indeed we have just shown that there is no map. This then tells us that $P(mr-r)$ has simple socle. In fact it is cv self-dual and injective. See **??**.)
So $P(mr+(r-1)) = \Delta^T(mr+(r-1))$ and we have completed the main inductive step. $\square$

2.6.7 Odds and ends

(**2.6.25**) By 1.6.14 and 1.8.6 the $\Delta_n(l)$ content of $P_n(m)$ does not depend on n (once n is big enough for these modules to make sense). Thus $P_n(0) = \Delta_n(0)$; $P_n(1) = \Delta_n(1)$.

For $P_n(2)$ we have $\mathrm{Ind}\, P_n(1) = \Delta_n(0) + \Delta_n(2)$; and $\mathrm{Ind}\, P_n(1)$ contains $P_n(2)$ as a direct summand. If this is a proper direct sum then this is true in particular at $n = 2$ and there is a primitive idempotent decomposition of 1 in T_2. It is easy to see that this depends on δ, but it true unless $\delta = 0$. *(We shall assume for now that $k = \mathbb{C}$ for definiteness.)*

Another way to look at the decomposition of $\mathrm{Ind}\, P_n(1)$ is as follows. If it does not decompose then by **??** there is a homomorphism $\Delta(2) \to \Delta(0)$, so that the gram matrix of $\Delta(0)$ must be singular.

Let us assume $\delta \neq 0$. Proceeding to $P_n(3)$ we have $\mathrm{Ind}\, P_n(2) = \Delta_n(1) + \Delta_n(3)$. Again this splits if and only if the gram matrix for $\Delta(1)$ is singular.

(**2.6.26**) TO DO:
Grothendieck group

2.7 Modules and ideals for the partition algebra P_n

ss:ModidPn

2.7.1 Ideals

We continue to use the notations as in (1.3.10) and so on.

(**2.7.1**) Note that the number of propagating components cannot increase in the composition of partitions in P_n (the 'bottleneck principle'). Hence $k\mathsf{P}^m_{n,n}$ is an ideal of P_n for each $m \leq n$, and we have the following ideal filtration of P_n

$$P_n = k\mathsf{P}^n_{n,n} \supset k\mathsf{P}^{n-1}_{n,n} \supset ... \supset k\mathsf{P}^0_{n,n}. \tag{2.18}$$ eq:Pstar01

Note that the sections $\mathfrak{P}^m_{n,n} \; := \; k\mathsf{P}^m_{n,n}/k\mathsf{P}^{m-1}_{n,n}$ of this filtration are bimodules, with bases $\mathsf{P}_{n,m,n}$.

(**2.7.2**) Write

$$P_n^{/m} \;:=\; P_n/k\mathsf{P}^m_{n,n}$$

for the quotient algebra.

(**2.7.3**) Note the natural inclusion

$$\mathsf{P}_{n,l,m} \otimes \mathsf{v}^\star \hookrightarrow \mathsf{P}_{n,l,m+1}$$

lem:natdecomp (**2.7.4**) LEMMA. *For any $l \leq n$ there is a natural bijection*

$$\mathsf{P}_{n,l,n} \xrightarrow{\sim} \mathsf{P}^L_{n,l,l} \times \mathsf{P}_{l,l,l} \times \mathsf{P}^L_{l,l,n}$$

(the inverse map is essentially category composition in P as in 1.7.6).

2.7.2 Idempotents and idempotent ideals

(**2.7.5**) Lemma. If $\delta \in k^*$ then $\mathsf{u}_1 \in P_n$ is an unnormalised idempotent and

(I) The ideal $k\mathsf{P}^m_{n,n} \; = \; P_n(\mathsf{u}^{\otimes(n-m)} \otimes 1_m)P_n$

(II) $k\mathsf{P}_{n,m} = k\mathsf{P}^m_{n,m} \cong \; P_n(\mathsf{u}^{\otimes(n-m)} \otimes 1_m)$ as a left P_n-module.

(**2.7.6**) Note that $k\mathsf{P}^m_{n,l}$ is a left P_n-module (indeed a $P_n - P_l$-bimodule) for each l, m, and $k\mathsf{P}^{m-1}_{n,l} \subset k\mathsf{P}^m_{n,l}$ (assuming $n \geq l \geq m$). Hence there is a quotient bimodule

$$\mathfrak{P}^l_{n,l} = k\mathsf{P}^l_{n,l}/k\mathsf{P}^{l-1}_{n,l}$$

with basis $\mathsf{P}_{n,l,l}$.

There is a natural right action of the symmetric group S_l on this module (NB $S_l \subset P_l$), which we can use. Let $v_\lambda \in kS_l$ be such that $kS_l v_\lambda$ is a Specht S_l-module (an irreducible S_l-module over $\mathbb{C}$). Then define left P_n-module

$$D_\lambda \; = \; k\mathsf{P}_{n,l,l} \; v_\lambda.$$

(**2.7.7**) If $k \supset \mathbb{Q}$ then v_λ can be chosen idempotent, and this module D_λ is a quotient of an indecomposable projective module, and hence has simple head. It follows that if P_n is semisimple then the modules of this form are a complete set of simple modules.

(**2.7.8**) Exercise. What can we say about $\mathrm{End}_{P_n}(D_\lambda)$?

(**2.7.9**) Exercise. Construct some examples. What about contravariant duals?

(**2.7.10**) The case $n = 1$, $k = \mathbb{C}$. Fix δ. Artinian algebra P_1 has dimension 2. By (1.4.76) and (1.4.53) this tells us that either it is semisimple with two simple modules, or else it has one simple module.

Unless $\delta = 0$ then u/δ is idempotent so there are two simples. If $\delta = 0$ then u lies in the radical $J(P_1)$, and $P_1/J(P_1)$ is one-dimensional (semi)simple.

(**2.7.11**) The case $n = 2$, $k = \mathbb{C}$. Fix δ. Artinian algebra P_2 has dimension 15.

As we shall see, for most values of δ we have $P_2 \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$.

(**2.7.12**) We have $P_n \subset P_{n+1}$ via the injection given, say, by $p \mapsto p \cup \{\{n+1, (n+1)'\}\}$, which it will be convenient to regard as an inclusion.

2.7.3 Back to P_n G-functors for a moment

(**2.7.13**) Fix n. It follows from the results assembled in §1.7.5 (e.g. 1.7.31) that for each $\lambda \vdash l \in \{n, n-1, ..., 0\}$ we have a P_n-module $\Delta_\lambda = G^{n-l} S_\lambda$, where S_λ is a symmetric group Specht module. (Note that this notation omits n, so care is needed. We can write Δ^n_λ to emphasise n.)

Fix $k = \mathbb{C}$, so that every S_λ is simple. It follows from 1.7.28(III) and 1.7.30 that if P_n is semisimple for some given choice of δ (and some given n) then the set of Δ_λ modules is a complete set of simple modules for this algebra.

(**2.7.14**) More generally, if P_n is non-semisimple then at least one Δ_λ is not simple. Further, if Δ^n_λ is not simple, then Δ^{n+1}_λ is not simple. Thus, for fixed δ, we may think of the 'first' non-semisimple case (noting that P_0 is always simple), and hence a 'first' (one or more) non-simple Δ_λ — at level

n say. We note (from 1.6.14, say) that this first non-simple case is manifested by a homomorphism from some Δ_ν with $\nu \vdash n$.

There are a number of ways we can 'detect' these homomorphisms. One approach starts by noting another adjunction: the (ind,res) adjunction corresponding to the inclusion $P_n \hookrightarrow P_{n+1}$. One can work out Res$\Delta_\lambda$ by constructing an explicit basis for each Δ_λ. One can work out Ind Δ_λ by using the formula Ind $=$ Res$G$. It then follows from the (ind,res) adjoint isomorphism that any such homomorphism implies a homomorphism to Δ_λ with $|\lambda|$ 'close' to n. These modules take a relatively simple form, and it is possible to detect morphisms to them explicitly by direct calculation.

Let D be the *decomposition matrix* for the Δ-modules (ordered in any way consistent with $\lambda > \mu$ if $|\lambda| > |\mu|$). It follows that D is upper unitriangular. It also follows that the *Cartan decomposition matrix* C is $C = DD^T$.

2.7.4 A first look at gram matrices and alcove geometry for P_n

ss:gramPn

... We already saw a gram matrix for a partition algebra in (2.3.12) above. ...

2.8 Lie algebras

ss:Liealg0

We include a brief discussion of Lie algebras here,
(a) to provide some contrast with and hence context for our 'associative' algebras; and
(b) as a certain partner notion to the special case of (associative) finite group algebras.
See 23.3.1 for a more detailed exposition. Here k is a field.

(**2.8.1**) A Lie algebra A over field k is a k-vector space and a bilinear operation $A \times A \to A$ denoted $[a, b]$ such that $[a, a] = 0$ and

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \qquad \text{('Jacobi identity')}$$

(**2.8.2**) From an associative algebra T we obtain a Lie algebra $Lie(T)$ by $[a, b] = ab - ba$.

(**2.8.3**) In particular, for V a vector space, the space of endomorphisms, sometimes denoted $gl(V)$, is a Lie algebra with $[a, b] = ab - ba$ (where ab is the composition of endomorphisms).

A representation of a Lie algebra A over k is a Lie algebra morphism $\rho : A \to gl(V)$ for some V.

An A-module is a space V and a map $A \times V \to V$ with

$$[a, b]v = a(bv) - b(av).$$

(**2.8.4**) Let V, V' be A-modules. Then the tensor product $V \otimes_k V'$ has a 'diagonal' action of A:

$$a(v \otimes v') = av \otimes v' + v \otimes av'$$

that makes $V \otimes_k V'$ an A-module.

Check: $[a, b](v \otimes v') = [a, b]v \otimes v' + v \otimes [a, b]v' = (a(bv) - b(av)) \otimes v' + v \otimes (a(bv') - b(av')) = ...$

(**2.8.5**) The tensor algebra of Lie algebra A is the vector space

$$\tau = \bigoplus_{n \geq 0} A^{\otimes n}$$

with multiplication given by $(a \otimes b)(c \otimes d) = a \otimes b \otimes c \otimes d$ and so on. Set H to be the ideal in τ generated by the elements of form $a \otimes b - b \otimes a - [a, b]$, with $a, b \in A$. Define

$$U_A = \tau / H$$

(**2.8.6**) A *universal enveloping algebra* (UEA) of Lie algebra A is an associative algebra U together with a Lie algebra homomorphism $I : A \to Lie(U)$ such that every Lie algebra homomorphism of form $h : A \to Lie(B)$ has a unique 'factorisation through $Lie(U)$', that is, a unique morphism of associative (unital) algebras $f : U \to B$ such that $h = f \circ I$.

(**2.8.7**) U_A is a UEA for A, with the homomorphism $I : A \to Lie(U_A)$ given by $a \mapsto a + H$. It is unique as such up to isomorphism.

(**2.8.8**) There is a vector space bijection

$$Hom_{Lie}(A, Lie(B)) \cong \mathrm{Hom}(U, B).$$

(**2.8.9**) Let V be an A-module and $\rho : A \to gl(V)$ the corresponding representation. Then ρ extends to a representation of a UEA U. This lifts to an 'isomorphism' of the categories of A-modules and U-modules (as subcategories of the category of vector spaces).

(**2.8.10**) THEOREM. *(Poincare–Birkoff–Witt) Let $J = \{j_1, j_2, ...\}$ be an ordered basis of A. Then the monomials of form $I(j_{i_1})I(j_{i_2})...I(j_{i_n})$ with $i_1 \leq i_2 \leq ...$ and $n \geq 0$ are a basis for U_A.*

(**2.8.11**) Recalling that k is fixed here, write sl_n for the Lie algebra of traceless $n \times n$ matrices. For example, sl_2 has k-basis:

$$x^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These obey $[x^+, x^-] = h$, $[h, x^+] = 2x^+$, $[h, x^-] = -2x^-$.

2.9 Eigenvalue problems

—

(2.9.1) **Operators acting on a space; their eigenvectors and eigenvalues.**
Here we remark very briefly and generally on the kind of Physical problem that can lead us into representation theory.
A typical Physical problem has a linear operator Ω acting on a space H, with that action given by the action of the operator on a (spanning) subset of the space. One wants to find the eigenvalues of Ω.
The eigenvalue problem may be thought of as the problem of finding the one-dimensional subspaces

of H as an $\langle\Omega\rangle$-module, where $\langle\Omega\rangle$ is the (complex) algebra generated by Ω. That is, we want to find elements h_i in H such that:

$$\Omega h_i = \lambda_i h_i$$

— noting only that, usually, the object of primary physical interest is λ_i rather than h_i. If H is finite dimensional then (the complex algebra generated by) Ω will obey a relation of the form

$$\prod_i (\Omega - \lambda_i)^{m_i} = 0$$

Of course the details of this form are *ab initio* unknown to us. But, proceeding formally for a moment, if any $m_i > 1$ (necessarily) here, so that $S = \prod_i (\Omega - \lambda_i) \neq 0$, then S generates a non-vanishing nilpotent ideal (we say, the algebra has a radical). Obviously any such nilpotent object has 0-spectrum, so two operators differing by such an object have the same spectrum. In other words, the image of Ω in the quotient algebra by the radical has the same spectrum $\{\lambda_i\}$. An algebra with vanishing radical (such as the quotient of a complex algebra by its radical) has a particularly simple structural form, so this is a potentially useful step.

However, gaining *access* to this form may require enormously greater arithmetic complexity than the original algebra. In practice, a balance of techniques is most effective, even when motivated by physical ends. This balance can often be made by analysing the regular module (in which every eigenvalue is manifested), and thus subquotients of projective modules, but not more exotic modules. (Of course Mathematically other modules may well also be interesting — but this is a matter of aesthetic judgement rather than application.)
It may also be necessary to find the subspaces of H as a module for an algebra generated by a set of operators $\langle\Omega_i\rangle$. A similar analysis pertains.

A particularly nice (and Physically manifested) situation is one in which the operators Ω_i (whose unknown spectrum we seek to determine) are known to take the form of the representation matrices of elements of an abstract algebra A in some representation:

$$\Omega_i = \rho(\omega_i)$$

Of course any reduction of Ω_i in the form of (1.19) reduces the problem to finding the spectrum of $R_1(\omega_i)$ and $R_2(\omega_i)$. Thus the reduction of ρ to a (not necessarily direct) sum of irreducibles:

$$\rho(\omega_i) \cong +_\alpha \, \rho_\alpha(\omega_i)$$

reduces the spectrum problem in kind. In this way, Physics drives us to study the representation theory of the abstract algebra A.

2.10 Notes and references

ss:refs

The following texts are recommended reading: Jacobson[71, 72], Bass[6], Maclane and Birkoff[97], Green[57], Curtis and Reiner[33, 34], Cohn[26], Anderson and Fuller[3], Benson[7], Adamson[2], Cassels[22], Magnus, Karrass and Solitar[98], Lang[91], and references therein. .

2.11 Exercises

exe:gr01 (**2.11.1**) Let R be a commutative ring and S a set. Then RS denotes the 'free R-module with basis S', the R-module of formal finite sums $\sum_i r_i s_i$ with the obvious addition and R action. Show that this is indeed an R-module.

exe:gr1 (**2.11.2**) Let R be a commutative ring and G a finite group. Show that the multiplication in (1.21) makes RG a ring.

Hints: We need to show associativity. We have

$$\left(\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right)\right)\left(\sum_k r''_k g_k\right) = \left(\sum_{ij}(r_i r'_j)(g_i g_j)\right)\left(\sum_k r''_k g_k\right) = \sum_{ijk}((r_i r'_j)r''_k)((g_i g_j)g_k) \tag{2.19}$$ groupalgmult2

and

$$\left(\sum_i r_i g_i\right)\left(\left(\sum_j r'_j g_j\right)\left(\sum_k r''_k g_k\right)\right) = \left(\sum_i r_i g_i\right)\left(\sum_{jk}(r'_j r''_k)(g_j g_k)\right) = \sum_{ijk}(r_i(r'_j r''_k))(g_i(g_j g_k)) \tag{2.20}$$ groupalgmult3

These are equal by associativity of multiplication in R and G separately.

(**2.11.3**) Show that RG is still a ring as above if G is a not-necessarily finite monoid and RG means the free module of finite support as above.

Hints: Multiplication in monoid G is also associative.

2.11.1 Radicals

ss:radical0001

Write J_R for the radical of ring R.

(**2.11.4**) A ring is *semiprime* if it has no nilpotent ideal.

(**2.11.5**) Theorem. A ring is left-semisimple if and only if every left ideal is a direct summand of the left regular module. ■

Show:

(**2.11.6**) Theorem. If S a subring of ring R such that, regarded as an S-bimodule, R contains S as a direct summand, then R left-semisimple implies S is left-semisimple.

Hint:

Let S' be an S-bimodule complement of S in R: that is, $R = S \oplus S'$ as an S-bimodule. (For example if $R = \mathbb{C}$ and $S = \mathbb{R}$ then we can take $S' = \mathbb{R}z$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.) If I is any left ideal of S then it is in particular a subset of R and RI makes sense as a left R-module, and hence as a left S-module by restriction. We claim $RI = (S \oplus S')I = SI \oplus S'I = S \oplus S'I$ as a left S-module. Now RI is a direct summand of ${}_R R$ by left-semisimplicity, so I is a direct summand of ${}_S S$.

(**2.11.7**) Let G be a finite group of automorphisms of ring R. Write r^g for the image of $r \in R$ under $g \in G$. Show that

$$R^G := \{r \in R \mid r^g = r \;\forall\, g \in G\}$$

is a subring of R.

Show:

(**2.11.8**) THEOREM. Suppose that $|G|$ is invertible in R. If R is semisimple Artinian (e.g. a semisimple algebra over a field) then R^G is semisimple Artinian. ■

Hints:

Show that $J_R \cap R^G \subseteq J_{R^G}$.

2.11.2 What is categorical?

ss:whatcat

(**2.11.9**) Prove: THEOREM. Let A be an Artinian algebra and I an ideal. Then A/I non-semisimple implies A non-semisimple. ■

Solution: (There are many ways to prove this. Here is one close to the idea of indecomposable matrix representations.) If A/I non-semisimple then not every module is a direct sum of simple modules (by definition), so there are a pair of modules with a non-split extension between them. That is, there is a short exact sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

such that there is no sequence with the arrows reversed. This sequence, indeed any sequence involving these modules, is also 'in' A-mod via $\psi : A \to A/I$. Now suppose (for a contradiction) that there is a sequence in A-mod involving the images of these modules but with the arrows reversed. This means that some $N \subset M$ obeys $N \cong M''$ as an A-submodule of M, i.e. $AN = N$ (keep in mind that the action of A on M and hence N comes by $am = \psi(a)m$, and the A/I-module property of M). But ψ is surjective, so every $x \in A/I$ is $\psi(a)$ for some a, so $(A/I)N = AN = N$ so N is also an A/I-submodule. This is a contradiction. Thus the original sequence is non-split in A-mod. □

(**2.11.10**) Write $\mathrm{Res}_\psi : A/I - \mathrm{mod} \to A - mod$ for the functor associated to $\psi : A \to A/I$.

Let B be any algebra. Note that given a sequence of B-module maps

$$L \xrightarrow{f} M \xrightarrow{g} N$$

there is, trivially, an underlying sequence of maps of these objects as abelian groups. The exactness property at M, $im(f) = ker(g)$, is defined at the level of abelian groups. Thus the sequence is exact for any B if and only if it is exact at the level of abelian groups.

Use this to show that Res_ψ is exact.

Chapter 3

Basic definitions, notations and examples

ch:defs

3.1 Preliminaries

3.1.1 Definition summary

ss:defsum

There follows a list of definitions in the form
Algebraic system $A = (A$ a set, $n-$ary operations), axioms.
(The selection of a special element $u \in A$, say, counts as a 0-ary operation.)
Extended examples are postponed to the relevant sections.

Semigroup	$S = (S, \square)$, $\square$ a closed associative binary operation on S.
Monoid	$M = (M, \square, u)$, $(M, \square)$ a semigroup, $u \in M$ a *unit element* (i.e. $au = a = ua \ \forall a \in M$). Example: $(\mathbb{N}_0, +, 0)$.
Group	$G = (G, ., u)$, G a monoid, $\forall a \in G \exists a'$ such that $aa' = u = a'a$.
Abelian group	$G = (G, +, 0)$, G a group, $a + b = b + a$.
Ring	$R = (R, +, ., 1, 0)$, $(R, +, 0)$ an abelian group, $(R, ., 1)$ a monoid, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.
Division ring	D, D a ring, every non–zero element has a multiplicative inverse.
Local ring	A, A a ring, sum of two nonunits is a nonunit (a a nonunit means there does not exist b such that $ab = ba = 1$). [1]
Domain	K, K a ring, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$.
Integral domain	K, K a ring, . commutative, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$. (I.e. an integral domain is a commutative domain.)
Principal ideal domain	K, K an integral domain, every ideal $J \subseteq K$ is principal (i.e. $\exists$ $a \in K$ such that $J = aK$).
Field	F, F an integral domain, every $a \neq 0$ has a multiplicative inverse.

Our other core definitions are, for S a semigroup, R a ring as above:

S–IDEAL J: $J \subset S$ and $rj, jr \in J$ for all $r \in S$, $j \in J$.
R–IDEAL J: $J \subset R$ and $rj, jr \in J$ for all $r \in R$, $j \in J$.

(LEFT) R–MODULE M: M an abelian group with map $R \times M \to M$ (we write rx for the image of (r, x)) such that $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$ ($r \in R$, $x, y \in M$). Right modules defined similarly, but with $(rs)x = s(rx)$.

(LEFT) R–MODULE HOMOMORPHISM : Ψ from left R-module M to N is a map $\Psi : M \to N$ such that $\Psi(x + y) = \Psi(x) + \Psi(y)$, $\Psi(rx) = r\Psi(x)$ for $x, y \in M$ and $r \in R$.

(**3.1.1**) EXERCISE. $\mathbb{Z}$ is a ring. Form examples of as many of the other structures as possible from this one. (And some non-examples.)

In the following table k is a field and $\mathbb{H}$ is the ring of real quaternions (see §4.2).

	$DivR$	LR	ID	PID
$\mathbb{Z}$	$\times$	$\times$	$\surd$	$\surd$
$\mathbb{Z}[x]$	$\times$	$\times$	$\surd$	$\times$
$k[x]$	$\times$	$\times$	$\surd$	$\surd$
$k[x, y]$	$\times$	$\times$	$\surd$	$\times$
$\mathbb{H}$	$\surd$	$\surd$	$\times$	$\times$

ss:glossary

3.1.2 Glossary

		alternatives and references
$M_N(R)$	ring of $N \times N$ matrices over ring R	[6]
$GL(N)$	general linear group on $\mathbb{C}^N$	GL_N, $GL(N,\mathbb{C})$
$SL(N)$	special (det=1) linear group on $\mathbb{C}^N$	SL_N, $SL(N,\mathbb{C})$
$O(N)$	orthogonal ($g^T g = 1$) group on $\mathbb{R}^N$	$O(N,\mathbb{R})$
$O(N,\mathbb{C})$	orthogonal ($g^T g = 1$) group on $\mathbb{C}^N$	$O(N,\mathbb{C})$
$SO(N,\mathbb{C})$	special orthogonal group on $\mathbb{C}^N$	$SO(N,\mathbb{C})$
$U(N)$	unitary ($g^\dagger g = 1$) group on $\mathbb{C}^N$	U_N
$SU(N)$	special unitary group on $\mathbb{C}^N$	SU_N
Λ	set of integer partitions	$\mathcal{P}$ [96, I.1]
Λ_n	set of integer partitions of n	$\mathcal{P}_n$ [96, I.1]
P_S	partitions of a set S	
J_S	pair partitions of a set S	
$\mathtt{P}(S)$	power set (lattice) of a set S	
$\mathtt{P}_n(S)$	subset of $\mathtt{P}(S)$ of sets of order n	
$U_{S,T}$	the set of relations $\mathtt{P}(S \times T)$	
E_S	set of equivalence relations on set S	
$\underline{n}$	$\{1, 2, ..., n\}$	[57, §2]
$\underline{l}^{\underline{n}}$	set of functions $f : \underline{n} \to \underline{l}$	$I(l, n)$ [57, §2]
Σ_n, S_n, $\mathfrak{S}_n$	symmetric group $\subset (\underline{n}^{\underline{n}}, \circ)$	S_n [96, I.7], $G(n)$ [57, §2]
$\Lambda(l, n)$	S_n orbits of $\underline{l}^{\underline{n}}$ / compositions of n into l parts	[57, §3.1]
$\Lambda^+(l, n)$	S_l orbits of $\Lambda(l, n)$ / partitions of n into l parts	[57, §3.1]
Set	category of sets and set maps	
$\mathsf{C}_\mathbb{N}$	skeleton category in **Set**	§4.1.1
$B_n(\delta)$	Brauer algebra	§**??** [14]
$P_n(\delta)$	partition algebra	
P	partition category	§16.7.1
$T_n(\delta)$	Temperley–Lieb algebra	
T_n	full transformation semigroup	§4.1.1
T'_n	presented monoid $T'_n \cong T_n$	§4.1.1
T	diagonal category in $\mathsf{C}_\mathbb{N}$	

$Z(\mathbb{H})$	polyhedral complex defined by set of hyperplanes $\mathbb{H}$	
$\Gamma(G, S)$	Cayley graph of group G over subset S	
$G(W, S)$	directed Cayley graph of Coxeter system (W, S)	
$W_\mathbb{H}$	reflection group generated by set of hyperplanes $\mathbb{H}$	
$D(\mathbb{H})$	dual graph of complex defined by hyperplanes $\mathbb{H}$	
$\mathcal{C}_\mathbb{H}$	set of chambers of defined by $\mathbb{H}$	
$\mathcal{C}_W$	set of chambers of reflection group W	
$\mathcal{A}_\mathbb{H}$	set of alcoves of defined by $\mathbb{H}$	
$\mathbb{H}_a$	subset of hyperplanes, walls of chamber $a \in \mathcal{C}_\mathbb{H}$	

3.2 Elementary set theory notations and constructions

ss:est

As in Green [57], let

$$\underline{n} \ := \ \{1, 2, .., n\}$$

Similarly here $\underline{n}' \ := \ \{1', 2', .., n'\}$, $\underline{n}'' \ := \ \{1'', 2'', .., n''\}$ and so on.

de:rel1

(**3.2.1**) For S a set, let $\mathtt{P}(S)$ denote the *power set*, the set of subsets of S. (We may consider $\mathtt{P}(S)$ to be partially ordered by inclusion. As such it is a lattice (see 3.4.8).) Let $\mathtt{P}_n(S) \subset \mathtt{P}(S)$ be the subset of elements of order n.

For S, T sets, let $S \times T$ denote the Cartesian product (the set of ordered pairs (s, t)), and let $U_{S,T}$ denote the set of relations on S to T. That is,

$$U_{S,T} = \mathtt{P}(S \times T).$$

Let $U_S = U_{S,S}$. Even in this case we may consider the left-hand 'input' set to be distinct from the right-hand 'output' set — elements are distinguished by their position in the ordered pair.

(**3.2.2**) A relation ρ on S to T is *'simple'* if no element of the left-hand set S appears twice. That is, $\rho \in \mathtt{P}(S \times T)$ is simple if $(s, t), (s, t') \in \rho$ implies $t = t'$.

(**3.2.3**) We assume familiarity with the notions of reflexive, symmetric, antisymmetric and transitive relations in $\mathtt{P}(S \times S)$. An RST relation is a reflexive, symmetric and transitive relation; an RS relation is reflexive, symmetric; and so on. (See also e.g. §3.4.)

Remark. We have the usual bootstrapping issue with introducing basic definitions. We assume familiarity with naive set theory (subsets and so on). We take the notion of set itself to be implicit. See perhaps Mac Lane [90], Cameron [19], Kamke [**?**], or the author's Lecture Notes on Topology [**?**] for background references. But because we use them so heavily, we will include a discussion of relations, functions and so on.

3.2.1 Functions

(**3.2.4**) For $(a, b) \in S \times T$, set $\mathfrak{p}_1(a, b) = a$. For $\rho \in U_{S,T}$ let $\mathrm{dom}(\rho) \ := \ \mathfrak{p}_1(\rho)$, that is, the subset of elements $s \in S$ such that $(s, t) \in \rho$ for some t.

Let $T^S \ = \ \hom(S, T) \ \subset \ U_{S,T}$ be the subset of simple relations with $\mathrm{dom}(\rho) = S$, or *functions*. For example

$$\underline{2}^{\underline{2}} \ = \ \hom(\underline{2}, \underline{2}) \ = \ \{\{(1, 1), (2, 1)\}, \{(1, 1), (2, 2)\}, \{(1, 2), (2, 1)\}, \{(1, 2), (2, 2)\}\} \tag{3.1}$$ eq:eg01

(**3.2.5**) Each element f of $\hom(S, \underline{2})$ defines an element $I(f)$ of $\mathtt{P}(S)$ by $s \in I(f)$ if $f(s) = 1$. This map is a set bijection:

$$\mathtt{P}(S) \ \cong \ \hom(S, \underline{2}) \ = \ \underline{2}^S$$

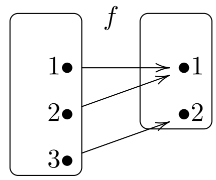
(**3.2.6**) For I a set and S_i a non-empty set for each $i \in I$, a choice function is a function that takes an element $i \in I$ as input and returns an element from S_i. (The axiom of choice says that such a function exists for any such family of sets.) Then $\prod_{i \in I} S_i$ is the set of all such functions.

3.2.2 Composition of functions

(**3.2.7**) It will be useful to have in mind the *mapping diagram* realisation of finite functions such as in (3.1). For example

$$f = \{(1,1),(2,1),(3,2)\} \in \underline{2}^{\underline{3}}$$

can be represented as



(Such a diagram uses a lot of heavy geometric-topological machinery - for example an embedding of $\underline{3}$ in the plane, but we will rely on intuition for now. See also e.g. (3.2.18).)

:monoid notation (**3.2.8**) If T, S finite it will be clear that any total order on each of T and S puts T^S in bijection with $\underline{|T|}^{\underline{|S|}}$. We may represent the elements of T^S as S-ordered lists of elements from T. Thus

$$\underline{2}^{\underline{2}} = \{11, 12, 21, 22\}, \qquad \underline{2}^{\underline{3}} = \{111, 112, 121, 122, 211, 212, 221, 222\}$$

(for example $22(1) = 2$, since the first entry in 22 is the image of 1).

(**3.2.9**) A *composition* of n is a finite sequence λ in $\mathbb{N}_0$ that sums to n. We write $\lambda \vDash n$.

We define the *shape* of an element f of $\underline{m}^{\underline{n}}$ as the composition of n given by

$$\lambda(f)_i = |f^{-1}(i)|$$

Example: for $111432525 \in \underline{6}^{\underline{9}}$ we have $\lambda(111432525) = (3,2,1,1,2,0)$.

If $\lambda \vDash n$ we write $|\lambda| = n$.

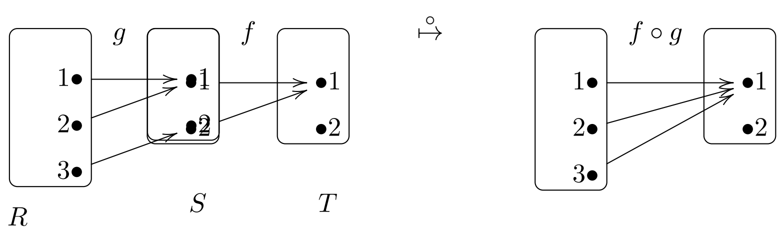
(**3.2.10**) Composition of functions defines a map

$$\hom(S,T) \times \hom(R,S) \quad \to \quad \hom(R,T) \tag{3.2}$$ x

$$(f,g) \mapsto f \circ g \tag{3.3}$$

where as usual $(f \circ g)(x) = f(g(x))$. For example $11 \circ 22 = 11$ (since $11(22(1)) = 11(2) = 1$; and so on).

The *mapping diagram realisation* of composition is to first juxtapose the two functions so that the two instances of the set S coincide, then define a direct path from R to T for each path of length 2 so formed:



(3.4) eq:diagram comp1

Aside: In general pictures such as those in (3.4) are part of a bridge of ideas between algebra and geometry. This is useful at face value; and it is also useful if one interrogates more closely the 'bricks' of the bridge. See later. For now we observe, as represented more directly on the picture side, that embeddings of intervals can be chained together to make embeddings of intervals. And that similar pictures can be used to represent other relations than functions. (An element ρ of $\mathsf{P}(S \times T)$ can be represented by embedding S and T-indexed sets of points in the plane and embedding an interval between points when $(s,t) \in \rho$ — our present picture is a special case. Alternatively, an element ρ of $\mathsf{P}(S \times S)$ can be represented by embedding an S-indexed set of points in the plane and embedding a directed interval between points when $(s,t) \in \rho$. Cf. also §3.5.)
In particular the notion of (directed) path-connectedness of points in a topological space is transitive.

(**3.2.11**) If the image $f(S)$ of a map $f : S \to T$ is of finite order we shall say that f has order $|f(S)|$ (otherwise it has infinite order).

For $R \xrightarrow{f} S \xrightarrow{g} T$ we have the *bottleneck principle*

$$|(g \circ f)(R)| \;\leq\; \min(|g(S)|, |f(R)|)$$

To see this note that evidently $g(S) \supseteq g \circ f(R)$, from which the first inequality follows; meanwhile clearly $|f^{-1}(R)| = |f(R)|$ for any $f \in \hom(R, -)$, leading to the second inequality.

pr:bottleneck1

(**3.2.12**) PROPOSITION. *(i) For S a set, $S^S = \hom(S,S)$ is a monoid under composition of functions.*
(ii) For each $d \in \mathbb{N}$ then set $\hom^d(S,S) \;:= \{f \in S^S \mid |f(S)| < d\}$ is an ideal (hence a subsemigroup) of S^S.

Proof. (i) Hint: $(f \circ (g \circ h))(x) = f(g(h(x))) = ((f \circ g) \circ h)(x)$
Exercise: explain this argument in terms of mapping diagrams.
(ii) Consider $g \circ f$, say. Evidently $g(S) \supseteq g \circ f(S)$. Thus $\hom^d(S,S) \circ f \subset \hom^d(S,S)$ for all f. Meanwhile $f(s) = f(t)$ implies $g \circ f(s) = g \circ f(t)$ so the partition $p = f^{-1}(S)$ of S implied by f cannot be refined in passing to the partition implied by $g \circ f$. Of course $|f^{-1}(S)| = |f(S)|$ for any f. Thus $g \circ \hom^d(S,S) \subset \hom^d(S,S)$ for all g. □

3.2.3 Set partitions

ss:set part

Here we essentially follow [106]. See also Ch.16.

(**3.2.13**) Let $\mathsf{E}_S \subset U_S$ denote the set of equivalence relations (reflexive, symmetric, transitive/RST relations) on set S. Let P_S denote the set of partitions of S. Note the natural bijection

$$\begin{tikzcd} \mathsf{E}_S \arrow[r, bend right, "\kappa"'] & \mathsf{P}_S \arrow[l, bend right, "\epsilon"'] \end{tikzcd}.$$

Specifically, $i \overset{\epsilon(p)}{\sim} j$ if i, j in the same part in p.

For $\rho \in U_S$ let $\bar{\rho} \in U_S$ be the smallest transitive relation containing ρ. The relation $\bar{\rho}$ is called the *transitive closure* of ρ.

(**3.2.14**) Let a, b be RS relations on any two finite sets. Then $a \cup b$ is an RS relation on the union. Let $ab := \overline{a \cup b}$ be the transitive closure of $a \cup b$.

Note that $\overline{a \cup b}$ is an equivalence relation on the union of the two finite sets. Note that

$$\overline{\overline{a} \cup b} \;=\; \overline{a \cup b} \tag{3.5}$$

eq:acupbs

If a, b are partitions then $\epsilon(a), \epsilon(b)$ are RS (indeed RST), and we will understand by ab the partition given by $ab = \kappa(\epsilon(a)\epsilon(b))$.

pr:assocxxx

(**3.2.15**) PROPOSITION. *For a, b, c RS relations*

$$a(bc) = (ab)c$$

Proof.

$$(ab)c \;=\; \overline{\overline{(a \cup b)} \cup c} \;\stackrel{(3.5)}{=}\; \overline{(a \cup b) \cup c} \;=\; \overline{a \cup b \cup c} = \overline{a \cup \overline{b \cup c}} \;=\; a(bc)$$

□

(**3.2.16**) Let $\mathsf{P}_{n,m} = \mathsf{P}_{\underline{n} \cup \underline{m}'}$; and $\mathsf{P}_n = \mathsf{P}_{n,n}$. Let $\mathsf{E}_{n,m} = \mathsf{E}_{\underline{n} \cup \underline{m}'}$ similarly. For $a \in \mathsf{P}_{n,m}$ let a' be the partition of $\underline{n}' \cup \underline{m}''$ obtained by adding a prime to each object in every part.

For $a \in \mathsf{P}_{l,m}$, $b \in \mathsf{P}_{m,n}$ partitions (and hence $\epsilon(a)$, $\epsilon(b)$ equivalence relations) note that $\epsilon(a)\epsilon(b')$ is an equivalence relation on $\underline{l} \cup \underline{m}' \cup \underline{n}''$. Restricting to $\underline{l} \cup \underline{n}''$ this equivalence relation gives again a partition, call it $r(ab')$ (indeed if a, b are pair-partitions then so is $r(ab')$).

For $x \in \underline{l} \cup \underline{n}''$ let $u(x) \in \underline{l} \cup \underline{n}'$ be the image under the action of replacing double primes with single.

We may define a map

$$\circ : \mathsf{P}_{l,m} \times \mathsf{P}_{m,n} \to \mathsf{P}_{l,n}$$

by

$$a \circ b \;=\; u(r(ab')) \in \mathsf{P}_{l,n}$$

— the image under the obvious application of the u map.

partition monoid

(**3.2.17**) PROPOSITION. *For each $n \in \mathbb{N}$ the map $\circ : (a, b) \mapsto u(r(ab'))$ defines an associative unital product on P_n, making it a monoid, with identity $1_n \;=\; \{\{1, 1'\}, \{2, 2'\}, ..., \{n, n'\}\}$.*

Proof. To show associativity note that ab' encodes $a \circ b$ directly, except that it is encoded via the unprimed and double-primed 'vertices'. Thus $(ab')c''$ encodes $(a \circ b) \circ c$ via the unprimed and triple-primed vertices. Meanwhile $b'c''$ encodes $b \circ c$ via the primed and triple-primed vertices; thus $a(b'c'')$ encodes $a \circ (b \circ c)$ via the unprimed and triple-primed vertices. But by Prop.3.2.15 we have $a(b'c'') = (ab')c''$.

To show unital with identity 1_n: Exercise. ■

pa:pic01

(**3.2.18**) A convenient pictorial realisation of such a set partition p, i.e. a realisation as a picture in the plane, is as follows. (See also (1.3.3) and §1.3.3. The extent to which the page and markings thereon can represent the plane and points thereon is limited, but workable for constructions that are sufficiently algorithmic.)

Firstly, a digraph G (as in 3.5.1) on vertex set V determines a relation on V in the obvious way. In particular a graph determines a symmetric relation. Hence a graph G determines an equivalence relation on V (take the RT closure); and hence also an equivalence relation on (or partition of) any subset of V. Thus it is enough to realise a suitable graph G of p as a picture.

To depict such a G one draws a set of points for the vertices V, and specifies an injective map from the underlying set of p to V; and then draws a 'regular' collection of 'edges'. Here a picture edge is a representation of a piecewise smooth line between two vertices. A collection is *regular* if two lines never meet at points where they do not have distinct tangents. The collection consists of one picture edge for each vertex pair that are associated by an edge in G. ('Incidental' vertices in the picture are those *not* associated to the underlying set.)

Note that two elements from the underlying set are in the same part in p if there is a path between their vertices.

(**3.2.19**) For a picture in particular of a partition in P_n one may arrange the underlying-set vertices naturally as two parallel rows of vertices (if there are incidental vertices these are drawn between the two rows). In this realisation the product $\circ$ may be computed, schematically, by concatenating the two pictures so as to identify certain vertices in pairs between two rows — one row from each picture (thus forming a 'middle' row).

(**3.2.20**) So far here we ignored the primed elements discarded in ab'. Recall that in 1.29 we used these to introduce a parameter into a linearised version of the construction. It is an exercise here to introduce a corresponding extension of the monoid construction, and in particular to prove well-definedness and associativity. Indeed further extensions are also of interest. See Ch.16 for some details and references.

(**3.2.21**) Let $J_S \subset \mathsf{P}_S$ be the set of pair-partitions of S. Let

$$J_{n,m} = J_{\underline{n} \cup \underline{m}'} \subset \mathsf{P}_{n,m}$$

pa:brauer1

Set $J_n = J_{n,n}$.

brauer monoid

(**3.2.22**) Proposition. *The compostion $\circ$ restricts to make J_n a monoid.*

Proof. Exercise. ■

(**3.2.23**) A partition is *non-crossing* if there is such a pictorial realisation having the property that all lines are drawn in the interior of the interval defined by the two rows, and no two lines cross. Let T_n denote the subset of non-crossing pair partitions.

One easily checks that the product above restricts to make T_n a monoid. This is sometimes called the n-th *Temperley–Lieb monoid*.

(**3.2.24**) One could similarly imagine drawing a realisation of a partition on a cylinder. This leads us to a notion of cylinder-non-crossing pair partitions. There are several further subsets of the set of partitions that are characterised in terms of geometrical embeddings.
Exercise: Find some more submonoids of P_n.

3.2.4 Exercises on closed binary operations

s:exe binary ops

(**3.2.25**) A closed binary operation on a (finite) set S (of degree n) may be given by a multiplication table — an element of $S^{S\times S}$. There are $|S^{S\times S}| = n^{(n^2)}$ of these. Note that an ordering of S induces an ordering on the set of closed binary operations (read order the entries in the multiplication table and dictionary order the ordered lists).

Define a natural notion of isomorphism of closed binary operations on S, and determine the number of isomorphism classes for $n = 2$. Is commutativity a class property? If so, how many of these classes are commutative?

Which of the following are semigroups/ monoids / groups?:

$$\begin{array}{c|cc} & a & b \\ \hline a & aa & ab \\ b & ba & bb \end{array} \quad : \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & b \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & b \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a \end{array}$$

(Hint: S,M,X $(b(ab) = b)$,S,G,X $((aa)b = a)$.)

Explain the following statement: "For $n = 3$, most binary operations are not associative." (Hint: 113 of them are associative.)

3.3 Basic tools: topology

ss:topo1

See e.g. Mendelson [124], Hartshorne [63].

(**3.3.1**) A *sigma-algebra* over a set S is a subset Σ of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under countable unions, and complementation in S.

Any subset S' of $\mathtt{P}(S)$ defines a sigma-algebra — the smallest sigma-algebra generated by S'. For example $\{\{1\}\} \subset \mathtt{P}(\{1,2,3\})$ generates $\Sigma = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$.

(**3.3.2**) A *topological space* is a set S together with a subset T of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under unions and finite intersections.

The set T is called a *topology* on S. The elements of T are called the *open sets* of this topology. A set is *closed* if it is the complement in S of an open set.

(**3.3.3**) A function between topological spaces is *continuous* if the inverse image of every open set is open. Two spaces are *homeomorphic* if there is a bijection between them, continuous in both directions.

(**3.3.4**) Example. Consider the set $\mathbb{R}^n$ together with the set of subsets 'generated' by (unions and finite intersections of) the set of open balls. This is a topological space.

In particular this makes $M_n(\mathbb{R})$ a topological space — as a topological space it is $\mathbb{R}^{n^2}$. The subgroup $GL_n(\mathbb{R})$ of invertible matrices may be considered as a topological space by restriction. Note that $GL_n(\mathbb{R})$ is open and not closed (its complement is not open) in $M_n(\mathbb{R})$, but it is open and closed in the restricted topology.

(**3.3.5**) Given a topological space (S, T), the restriction of T to $S' \subset S$ is a topology on S', called the *subspace topology.*

A subset S' of a topological space (S, T) is *irreducible* if $S' = S_1 \cup S_2$ with S_1 closed implies S_2 not closed.

e:affine variety

(**3.3.6**) Let k be a field. A polynomial $p \in k[x_1, ..., x_r]$ determines a map from k^r to k by evaluation. For $P = \{p_i\}_i \subset k[x_1, ..., x_r]$ define

$$Z(P) = Z(\{p_i\}_i) = \{x \in k^r \ : \ p_i(x) = 0 \ \forall \ i\}$$

An *affine algebraic set* is any such set, in case k algebraically closed. An *affine variety* is any such set, that cannot be written as the union of two proper such subsets. (See for example, Hartshorne [63, I.1].)

(**3.3.7**) EXAMPLE. $Z(x_1x_2-1) = Z(\{p(x_1,x_2) = x_1x_2-1\})$ is a variety in k^2. Its points $(x_1,x_2) = (\alpha,\beta)$ may be given by a free choice of α (say) from $k^\times$, with β then determined. (Note that this latter characterisation looks like an open subset of k (specifically the complement of $Z(x)$), but the original formulation makes it clear that it is closed in k^2.)

(**3.3.8**) The set of affine varieties in k^r satisfy the axioms for closed sets in a topology. This is called the *Zariski topology.* The Zariski topology on an affine variety is simply the corresponding subspace topology.

The set $I(P) \in k[x_1,...,x_r]$ of all functions vanishing on $Z(P)$ is the ideal in $k[x_1,...,x_r]$ generated by P. We call

$$k_P = k[x_1,...,x_r]/I(P)$$

the *coordinate ring* of $Z(P)$.

(**3.3.9**) Let Z be an affine variety in k^r and $f : Z \to k$. We say f is *regular* at $z \in Z$ if there is an open set containing z, and $p_1, p_2 \in k[x_1,...,x_r]$, such that f agrees with p_1/p_2 on this set.

(**3.3.10**) A morphism of varieties is a Zariski continuous map $f : Z \to Z'$ such that if V is open in Z' and $g : V \to k$ is regular then $g \circ f : f^{-1}(V) \to k$ is regular.

(**3.3.11**) Given affine varieties X, Y then $X \times Y$ may be made in to an algebraic variety in the obvious way.

(**3.3.12**) An *algebraic group* G is a group that is an affine variety such that inversion is a morphism of algebraic varieties; and multiplication is a morphism of algebraic varieties from $G \times G$ to G.

3.4 Partial orders, lattices (and graphs)

ss:poset1

3.4.1 Posets and lattices

General references on posets and lattices include Birkhoff [9], and Burris and Sankappanavar [18, §1].

(**3.4.1**) A *relation* on a set S is a subset of $S \times S$ as in (3.2.1). Thus the intersection of any set of relations on S is a relation. Indeed the intersection of any set of transitive relations is transitive.

The *transitive closure* of a relation ρ on S is the intersection of all transitive relations containing ρ. (This transitive relation is non-empty since $S \times S$ is a transitive relation.)

(**3.4.2**) A *poset* is a reflexive, antisymmetric, transitive relation.

An acylic (no cyclic chains) relation ρ on S defines a partial order, by taking the transitive reflexive closure $TR(\rho)$. Note that every relation in the interval $[\rho, TR(\rho)]$ (with respect to the inclusion partial order) is acyclic.

We may consider the set of all relations having the same transitive reflexive closure. If the closure is a poset then all the relations 'above' it are acyclic. A minimal such relation is a *transitive reduction* (of any element of this set).

If S is finite then there is a unique transitive reduction of an acylic relation. Otherwise there may be no (or one, or multiple) transitive reductions.

If there is a unique transitive reduction of an acyclic relation we call this the *covering relation*.

(**3.4.3**) If we use the notation $(S, \geq)$ for a poset then we may write $a > b$ for $a \geq b$ and $a \neq b$. In this case the relation $(S, >)$ induces the same poset. Further we may write $(S, \leq)$ for the opposite relation, which is another poset.

de:cover (**3.4.4**) Let $(S, \geq)$ be a poset, and $s, t \in S$. We say *s covers t* if $s > t$ and there does not exist $s > u > t$.

(**3.4.5**) The notion of cover/covering relation leads to the notion of *Hasse diagram*, as for example in [9, 18].

(**3.4.6**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

(**3.4.7**) By convention if we declare a poset $(S, \leq)$ then $a \leq b$ can be read as *a is less than or equal to b* (although the opposite relation is a perfectly good poset, and we could in principle have associated the relation symbol $\leq$ to that).

de:lattice (**3.4.8**) With the above convention, a poset S is a *join semilattice* if every pair $s, t \in S$ has a least upper bound (join) in S.
A poset is a *lattice* if both it and its opposite are join semilattices.

eg:lattice (**3.4.9**) Example. The power set $\mathtt{P}(S)$ of a finite set with the inclusion order is a lattice. An upper bound of $s, t \in \mathtt{P}(S)$ is any set containing sets s, t; and the least upper bound is the union. That is

$$s \vee t \;=\; s \cup t.$$

(**3.4.10**) A lattice is *modular* if

$$S \wedge (T \vee U) \;=\; (S \wedge T) \vee U \qquad \Rightarrow \qquad S \geq U.$$

(**3.4.11**) For P a lattice, the interval

$$[a, b] := \;\{c \in P \mid a \leq c \leq b\}$$

is sometimes called a *quotient* (see e.g. Faith [**?**]).

3.5 Digraphs and graphs

ss:digraph

See also for example Spanier [143], Lando and Zvonkin [89], Giblin [53], and references therein.

Foreword. Although directed graphs and graphs are fundamental structures, their definition in existing literature is not an exact science! The underlying concepts are ubiquitous and useful (see §1.3.2 for example), but detailed formalties depend on context. For example, sometimes the names of edges are important, and sometimes they are not. Sometimes the multiplicities of parallel edges are important, sometimes not. (Examples:
Giblin [53] defines an abstract graph simply as a pair V, E with E a set of unordered pairs from set V - a set of order-2 subsets of V.

Lando–Zvonkin [89] define a graph as a triple (V, E, I), where V, E are sets and I is a relation on $V \times E$ such that for each e there are either one or two pairs of form (v, e) in I.
Bondy–Murphy [**?**] define a graph similarly, except that I is replaced by an incidence function from E to unordered pairs of not-necessarily distinct elements of V.
Spanier [143] defines a graph to be a 1d simplicial complex — that is, a set V together with a set of subsets of V including all the singeltons and then some subsets of order two.
Massey [**?**] defines a graph to be a 1d CW complex — that is, a Hausdorff space X and a subspace V obeying certain axioms, notably that V is a discrete open subset; and $X - V$ is a disjoint union of open subsets e_i where each e_i is homeomorphic to an open interval of the real line.)
Furthermore, finitary graphs have natural (but not unique) pictorial representations, which blur the distinction from *embedded graphs* (also known as fat graphs, ribbon graphs, maps, ...). Indeed the term 'graph' highlights that the set-theoretic formulation should not dominate over the pictorial (it is rather that the pictorial requires more work to lift from the intuitive to the mathematical).

Of course for each definition there is one or more corresponding notion of isomorphism; and these again can vary from author to author, even for the same notion of graph.

The choice of which formulation to use in practice depends on the specific use. One important use is in setting up computation for statistical mechanical models. These are models of physical systems, hence in 3d. Degrees of freedom, 'atoms', are (or can be considered to be) localised at points in $\mathbb{R}^3$. The Hamiltonian describes energetically how these degrees of freedom interact. Pairwise interaction of 'near' atoms is often a good approximation. A graph is then one way of summarizing these interactions. See §15. For this use, a (concrete) graph is a set V of atoms; a set E indexing interactions; and hence a function $\mathbf{E} : E \to \mathtt{P}_2(V)$. Indeed $\mathbf{E}$ might be an inclusion. Now say that $\underline{Q}^V = \hom(V, \underline{Q})$ is the set of configurations of the system. An example of a Hamiltonian

$$H_G : \hom(V, \underline{Q}) \to \mathbb{R}$$

for a graph $G = (V, E, \mathbf{E})$ is given by

$$H_G(\sigma) = \sum_{\langle ij \rangle \in E} \delta_{\sigma(i),\sigma(j)}$$

— here we write $\langle ij \rangle \in E$ to indicate an element $e \in E$ such that $\mathbf{E}(e) = \{i, j\}$ (of course this can hold for multiple elements here). In this setting vertices i, j are of equal standing, so their pairing is naturally $\{i, j\}$ rather than (i, j). In some settings 'directing' the interactions can be natural, so we allow a generalisation of this G as in §3.5.1.

3.5.1 Concrete digraphs

ss:digraph0

de:digraph

(**3.5.1**) A *concrete digraph* (or directed graph) is a triple (V, E, f) where V, E are (finite) sets and f a function $f : E \to V \times V$.

(**3.5.2**) Equivalently a concrete digraph is a quadruple (V, E, s, t) where V, E are sets and $s, t : E \to V$ (called source and target maps).

(**3.5.3**) Observe that even if we fix V, the collection of concrete digraphs on V is too big to be a set, because of the (generally rather uninteresting) dependence on E.

(**3.5.4**) Given a concrete digraph $G = (V, E, f)$, we say V is the set of 'vertices' and E the set of 'edges'. An edge $e \in E$ with $f(e) = (a, a)$ is a 'loop'. If $f(e) = (a, b)$ then e is an edge 'on' (a, b) or from a to b.

(**3.5.5**) A concrete digraph can be represented by a picture with a labeled node for each vertex and a directed labeled arc for each edge. Examples:

$$G_1 = \begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, "y"'] & b \\ c \arrow[u, "z"] \arrow[ur, "t"'] & \end{tikzcd} \qquad G_2 = \begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, "y"'] & b \arrow[loop above, "u"] \arrow[dl, bend left, "s"] \\ c \arrow[u, "z"] \arrow[ur, "t"] & \end{tikzcd} \tag{3.6}$$ eq:graphs12

The arcs in such a representation are allowed to intersect pairwise, so long as they do so transversally (i.e. both have a tangent at the intersection point; and these tangents differ).

adjacency matrix (**3.5.6**) Let $G = (V, E, f)$ be a concrete digraph. The *adjacency function* $\mu_G : V \times V \to \mathbb{N}_0$ is given by $\mu_G(i, j) = |f^{-1}(i, j)|$, the number of edges from i to j in G.

If V is countable (and hence ordered), the *adjacency matrix* $M^G = \mathsf{A}d(G)$ is a vertex indexed square array such that entry M^G_{ij} is the number of edges from i to j in G.

Example from (3.6) above:

$$\mathsf{A}d(G_1) = \left(\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 2 & 0 \\ b & 0 & 0 & 0 \\ c & 1 & 1 & 0 \end{array} \right)$$

N.B., Strictly speaking, for a matrix we require a total order on V. We could adjust this imposed total order without changing the vertex labels per se. So two distinct matrices (conjugates under permutation) can represent the exact same digraph.

de:edgeequiv (**3.5.7**) Fix an order on V. We way that two concrete digraphs on vertex set V are *edge-equivalent* if they have the same adjacency function (and matrix, if the order on V is preserved).
Remark. Note that this amounts to forgetting the edge labels.
Remark. Note that here vertex relabellings are not allowed.
Remark. Edge-equivalence is not the same as isomorphism as defined in (3.5.12).

Digraph.

(**3.5.8**) A *digraph* is a set V together with a function $\mu_G : V \times V \to \mathbb{N}_0$.

For V a set, write $\mathbf{\Gamma}(V)$ for the set of digraphs on V.

Observe that there is a bijection between $\mathbf{\Gamma}(V)$ and the set of classes of concrete digraphs on V up to edge-equivalence.

(**3.5.9**) Given a digraph $G = (V, \mu_G)$ define

$$\mu_G^{(2)} : V \times V \to \mathbb{N}_0$$

by $\mu_G^{(2)}(i, j) = \sum_{v \in V} \mu_G(i, v)\mu_G(v, j)$ (in the matrix form this is simply the matrix square).

(**3.5.10**) A *simple concrete digraph* is a digraph (V, E, f) in which $f : E \to V \times V$ is an inclusion.

This amounts to saying that we can use a subset of $V \times V$ as the edge set. Thus we do not need labels on edges in a picture. That is, a simple concrete digraph is just a relation on V.

This notion for digraphs is slightly cumbersome - there can still be zero, one or two edges between any two vertices. It is more straightforward for graphs.

(**3.5.11**) Two simple concrete digraphs (V, E, f) and (V', E', f') are *isomorphic* if there is a bijection $\psi : V \to V'$ such that (a, b) is in $f(E)$ iff $(\psi(a), \psi(b))$ is in $f'(E')$.

de:digraphiso

(**3.5.12**) We will say that two (not necessarily simple) concrete digraphs are isomorphic if there is a bijection $\psi : V \to V'$ such that $f^{-1}(a, b)$ has the same order as $f'^{-1}(\psi(a), \psi(b))$ for all a, b (thus each of these pairs of sets could be placed in explicit bijection, but such a set of bijections is not necessarily given).

That is, two digraphs are isomorphic if their pictures can be 'morphed' into each other, using ψ, but ignoring the edge labels.

(**3.5.13**) Alternatively (cf. Bondy–Murphy [**?**], for example) we could impose that digraphs are isomorphic if there are bijections both $V \to V'$ and (now explicitly) $E \to E'$, taking G into G'.

And in fact we sometimes want to ignore edge labels but not 'ignore' vertex labels. Hence the notion of edge-equivalence in (3.5.7).

(**3.5.14**) Given a digraph, if there is a proper path (along directed edges) from a to b then the *'distance'* from a to b is the minimum number of edges in such a path. (Note that this is not a true distance function. The distance from b to a may be different, for example.)

A digraph is *acyclic* if there is no proper path (along directed edges) from a to a for any $a \in V$.

(**3.5.15**) A digraph is *rooted* with root $r \in V$ if there is a vertex $r \in V$ such that every vertex is reachable by a directed path from r.

Note that if a digraph is acyclic then it has at most one root.

(**3.5.16**) Remark. We do not require the *finite* set condition for digraphs here. In practice our digraphs are either finite or inverse limits of sequences of finite graphs. This means in particular that there are only finitely many edges associated to any given pair of vertices, i.e. $f^{-1}(v, w)$ is always finite.

(**3.5.17**) An *edge colouring* of a digraph is a map from E to a set of 'colours', and hence a partition of E into same-coloured subsets.

(**3.5.18**) The *opposite concrete digraph* of a concrete digraph $G = (V, E, f)$ has the same V and E but $f^{op}(e) = f(e)^{op}$ (i.e. if $f(e) = (a, b)$ then $f^{op}(e) = (b, a)$).

(**3.5.19**) The adjacency matrix of the opposite concrete digraph is the transpose matrix.

Graph.

de:congraph01

(**3.5.20**) A *concrete graph* is a concrete digraph that is edge-equivalent to its opposite.

Equivalently, a concrete graph on V is a pair of sets V, E together with a map $\mathbf{E} : E \to \mathtt{P}_2(V)$.

de:graph01

(**3.5.21**) A *graph* is a digraph (V, μ_G) with symmetric adjacency function.

Equivalently, a graph is a class of concrete graphs under edge-equivalence.

:connected graph

(**3.5.22**) We say a graph is *connected* if for any pair of vertices there is a finite chain of edges connecting them.

(**3.5.23**) Example. There will be many examples, for instance in §16.1.2.

de:Cayley

(**3.5.24**) Example. Let G be a group and S a set of elements. The *Cayley graph* $\Gamma(G, S)$ is the digraph with vertex set G and an edge s_a on (a, b) whenever $b = as$ for some $s \in S$.

(**3.5.25**) Notes:

1. $s = a^{-1}b$ so there is at most one edge on (a, b), i.e. $\Gamma(G, S)$ is simple.
2. If $s \in S$ is an involution then edges involving s are effectively undirected. Some workers define S to include inverses (write this as $S = S^{-1}$), so again $\Gamma(G, S)$ is undirected.
3. We consider that S excludes the identity, so $\Gamma(G, S)$ is loop-free.
4. Some workers require that S generates G. Then $\Gamma(G, S)$ is connected. If $S = G$ then $\Gamma(G, S)$ is the complete graph.
5. If all generators are involutions (or $S = S^{-1}$) then the graph is effectively undirected by construction. However one can sometimes 'direct' such a $\Gamma(G, S)$, by using a *length function...* The root vertex is the identity 1, and there is a well-defined distance (really minimum distance, since there are undirected adjacent pairs) from 1 to any vertex g, denoted $l(g)$. If there is no edge between vertices of equal distance then we can 'direct' edges away from the root.

(**3.5.26**) Example. For $\Gamma(S_n, S)$ where S is the set of adjacent pair permutations, one can show that $l(gs) \neq l(g)$. See §5.2.2.

(**3.5.27**) Observe that the adjacency matrix of a graph as in (3.5.21) is a symmetric matrix.

de:graph02

(**3.5.28**) A concrete graph as in (3.5.21) above is equivalently a pair V, E of sets together with a relation $\rho \subset E \times V$ with the property that each $e \in E$ occurs in ρ as the left-hand side of either two pairs or one pair.

3.6 Aside on quiver algebra characterisations of algebras

ss:quiv0

(**3.6.1**) A *quiver* is a quadruple

$$Q = (Q_v, Q_a, f_{\text{init}}, f_{\text{fin}})$$

consisting of a set Q_v of *points* or *vertices*; a set Q_a of *arrows*; a function $f_{\text{init}} : Q_a \to Q_v$ and a function $f_{\text{fin}} : Q_a \to Q_v$.

Thus a quiver is a digraph (V, E, f) via the correspondence $V = Q_v$, $E = Q_a$, $f(e)_1 = f_{\text{init}}(e)$ and $f(e)_2 = f_{\text{fin}}(e)$. That is

$$f(e) = (f_{\text{init}}(e), f_{\text{fin}}(e)).$$

(This correspondence is trivial. 'Quiver' and 'digraph' are essentially synonymous. The term quiver seems to be favoured by pure algebraists.)

(**3.6.2**) We write Q_l for the set of paths of length l on quiver Q.

Here we may confuse Q_0, the set of paths of length 0, with Q_v. To emphasise the *path* of length 0 from $v \in Q_v$ we write e_v.

(**3.6.3**) Let k be an algebraically closed field and Q a quiver. The *path algebra* kQ is the k-space with basis the set of paths on Q; together with a product on paths defined by concatenation of paths where this makes sense, and zero otherwise, extended k-linearly.

Note that the concatenation product is associative; that kQ has a 1 iff Q_v is finite

$$1 = \sum_{v \in Q_v} e_v$$

(indeed the e_v's are a complete set of primitive orthogonal idempotents in this case); and that kQ is finite dimensional iff Q_v, Q_a are finite and Q has no cycles.

The k-space $R_n := \oplus_{l \geq n} kQ_l$ is an ideal of kQ for each n. Note that if Q is finite and acyclic then the ideal R_1 is the radical of kQ.

(**3.6.4**) If Q is finite (with $d := |Q_v|$ say), acyclic and connected; simple and underlying-graph-acyclic, then the space $e_v(kQ)e_{v'}$ is at most 1-dimensional and

$$kQ \hookrightarrow \mathrm{Tri}_d^-(k)$$

where $\mathrm{Tri}_d^-(k)$ is the k-algebra of lower-triangular $d \times d$ matrices. (Later we adopt the notation $\mathrm{Tri}_d(k)$ for upper-triangular matrices. Either kind can be used here.)

3.6.1 Ordinary quivers of algebras

connectedalgebra (**3.6.5**) An algebra is *connected* if it has no proper central idempotents. Every basic connected k-algebra of finite dimension is isomorphic to the quotient of a quiver algebra by an ideal H such that

$$R_m \subseteq H \subseteq R_2$$

for some $m \geq 2$.

e:orindaryquiver (**3.6.6**) Let X be a basic connected algebra of finite dimension, with $E = \{e_1, e_2, ..., e_d\}$ a complete set of primitive orthogonal idempotents. Define a quiver $Q(X, E) = Q(X) = (Q(X)_v, Q(X)_a, ...)$ (the *ordinary quiver* of X) as follows. First $Q(X)_v = \underline{d}$. For $v, v' \in Q(X)_v$ let $b_{v,v'}$ be a basis for $e_v(J/J^2)e_{v'}$ where J is the radical of X. Then the collection in $Q(X)_a$ of arrows that start at v and end at v' is identified with $b_{v,v'}$.

The dropping of E from the notation can be justified by showing that $Q(X)$ does not depend on the choice of E.

(**3.6.7**) Theorem. *If X is a basic connected algebra of finite dimension over k then there is a surjective algebra homomorphism $\psi : kQ(X) \to X$, and indeed an ideal of the form of H above such that* $\ker \psi = H$*; so $X \cong kQ(X)/H$.*

See Gabriel [**?**] and elsewhere for proofs.

ex:Xlowertrisub (**3.6.8**) Example. Consider the k-algebra

$$X = \left\{ \begin{pmatrix} a & 0 & 0 \\ x & b & 0 \\ z & y & a \end{pmatrix} \mid a, b, x, y, z \in k \right\}$$

(We note in passing that this is a 5-dimensional algebra. Since the elements are matrices over k then X is a faithful representation for itself, but it is not the regular representation.) Note that X has two simple modules, L_a, L_b, corresponding to the representations ρ_a and ρ_b, given by $\rho_a(m) = (a)$ and $\rho_b(m) = (b)$, where $m = m(a, b, x, y, z)$ is the general element of X parameterised as in the definition.

A complete set of idempotents in X is

$$e_a = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \qquad e_b = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

while

$$J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix} \mid x, y, z \in k \right\}, \qquad J^2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \mid z \in k \right\}$$

so

$$e_a(J/J^2)e_b \sim \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ & y & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & y & 0 \end{pmatrix}$$

$$Q(X) = \circ \underset{p}{\overset{q}{\longleftrightarrow}} \circ$$

and H is generated by pq. The basis for the path algebra is $e_a, e_b, p, q, qp, \ldots$ (all subsequent paths 0 in the quotient).

(**3.6.9**) REMARK. Our algebra X is an example of what is known as a *quasihereditary* algebra (see §9.5), which means that it has certain special properties (usefully generalising semisimplicity).

3.6.2 Aside on centraliser algebras

Given an algebra A and a faithful module M, we have the *centraliser algebra*

$$B = \mathrm{End}_A(M)$$

What can we say about the structure of B based on known properties of A and M? (What are the 'right' questions to ask? One could ask when $A = \mathrm{End}_B(M)$? When this happens it is known as the *double-centraliser property*.)

Evidently a central idempotent decomposition of A is also such a decomposition of B. Thus we can restrict attention to connected (single block) algebras.

For example if k is algebraically closed and M is a simple module, so A is a simple k-algebra, then so is B; and so if A is semisimple then so is B.

(**3.6.10**) Note that M is an $A \otimes_k B$-module. This gives us a tensor functor between mod $-$ A (hence A^{op} $-$ mod) and B $-$ mod; and v.v.. Similarly $\hom_A(M, -)$ is a functor from A $-$ mod to B^{op} $-$ mod.

(**3.6.11**) EXERCISE. Give examples and consider the covariance of these functors.

Hint: $\hom_A(M, M)$ can be seen as a B-module in two ways. This corresponds to $B = \mathrm{End}_A(M) = \hom_A(M, M)$ regarded as an algebra and hence as a B, B-bimodule.

(**3.6.12**) Next, as an exercise, we give a little thought to the centraliser algebra of the action of X on its defining representation in (3.6.8) above. Note that this is not the regular representation, which is 5-dimensional. The underlying module for X as a representation for itself is 3-dimensional — we call this module M.

Since X regarded as a representation has just one indecomposable component, i.e. M is indecomposable, then $B = \mathrm{End}_X(M)$ has just one simple module. Indeed B is spanned by the identity matrix and

$$\begin{pmatrix} 0 & & \\ & 0 & \\ 1 & 0 & 0 \end{pmatrix}$$

That is, B is a 2-dimensional algebra with a single indecomposable projective module. One readily checks the double-centraliser property

$$X \cong \mathrm{End}_B({}_B M)$$

(**3.6.13**) Remark. Our algebra B is *not* a quasihereditary algebra.

(**3.6.14**) What happens to B when we change M to be a larger X-module containing M as a direct summand (hence still faithful, of course)?

We consider $B' = \mathrm{End}_X(M')$ where $M' = M \oplus L_a$, giving the representation

$$\begin{pmatrix} a & 0 & 0 & \\ x & b & 0 & \\ z & y & a & \\ & & & a \end{pmatrix}.$$

In this case the following elements lie in (and span) the 'dual' B':

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \\ & 0 & 0 & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix}$$

In this case X and B' are Morita equivalent (indeed isomorphic, since they are both basic).

(**3.6.15**) If we consider $M'' = M \oplus 2L_a$ then we get another Morita equivalent (but no longer isomorphic) algebra $B'' = \mathrm{End}_X(M'')$. Increasing the multiplicity of L_a further does not affect the Morita equivalence class.

An interesting question is: To what extent can the original case M be seen as a 'degenerate' case of the collection of Morita equivalent cases?

3.7 Notes and references

For more on posets and lattices see e.g. Birkhoff [9], Burris and Sankappanavar [18, §1].

Chapter 4

Initial examples in representation theory

4.1 Initial examples in representation theory: category $\mathsf{C}_{\mathbb{N}}$

4.1.1 The monoid $\hom(\underline{n},\underline{n})$ - the full tranformation semigroup T_n

ss:n^n

A useful counterpoint to diagram and group algebras is provided by the monoids in the category SET itself - the full transformation semigroups T_n. Let us start with a quick summary of rep-theory-relevant facts.

From the category theory setting we see that irreducible reps of T_n are naturally indexed by integer partitions of 1 up to n (so, not so different from the partition algebra).

However T_n is not isomorphic to its opposite, so it allows us to illustrate the role this feature plays.

(**4.1.1**) A presentation of T_n is given as follows. First note that the symmetric group S_n has a group presentation (i.e. with each relation of form $w = 1$, where w is a word in generators and inverses, given simply as w):

$$S_n \cong S'_n \;=\; \langle a, b \mid a^2, b^n, (ba)^{n-1}, (ab^{-1}ab)^3,\; (ab^{-j}ab^j)^2 (j = 2, 3, ..., n-2) \;\rangle$$

The isomorphism $S'_n \to S_n$ is given (in cycle notation) by

$$a \mapsto (12), \qquad\qquad b \mapsto (12...n)$$

(this being one possible presentation among many). And note that there is an obvious 'monoidification' of presentation S'_n (with b^{-1} replaced by b^{n-1} and so on), thus $\langle a, b|R\rangle$ with relation set R. Then Aizenstat's monoid presentation is

$$T'_n = \langle a, b, \tau \mid R, Q\rangle$$

where relation set Q is:

$$a\tau = \tau = b^{n-2}ab^2\tau b^{n-2}ab^2 = ...$$

This obeys $T_n \cong T'_n$, with the isomorphism $T'_n \to T_n$ given by

$$\tau \;\mapsto\; 1134...n \;\;=\;\; 11 \otimes 12...n{-}2$$

- see (**??**) for notion for elements of T_n. (For the monoidal notation, see e.g. §4.1.2.)

There follow some illustrative explicit calculations in low ranks.

First we discuss various calculi for writing elements and compositions in T_n.

(**4.1.2**) It is convenient to think of the collection of these monoids T_n as a diagonal monoidal category T in the spirit of the Young monoidal category $\underline{\Sigma}$ of symmetric groups (which it contains).

This T is the full diagonal subcategory of the skeleton category $\mathsf{C}_{\mathbb{N}}$ in the category of finite sets. See e.g. Prop.6.3.4.

Recall that the category $\mathsf{C}_{\mathbb{N}}$ has a total bottleneck order on objects, so that each $T_n = \mathsf{C}_{\mathbb{N}}(n,n)$ is filtered by 'ideals':

$$\mathsf{C}^m_{\mathbb{N}}(n,n) = \mathsf{C}_{\mathbb{N}}(m,n) \circ \mathsf{C}_{\mathbb{N}}(n,m)$$

$$\mathsf{C}^1_{\mathbb{N}}(n,n) \subseteq ... \subseteq \mathsf{C}^m_{\mathbb{N}}(n,n) \subseteq \mathsf{C}^{m+1}_{\mathbb{N}}(n,n) \subseteq ... \subseteq \mathsf{C}^{n-1}_{\mathbb{N}}(n,n) \subseteq \mathsf{C}_{\mathbb{N}}(n,n) \tag{4.1}$$

eq:CCCbott

NB here we use function composition not diagram composition.

Fix a commutative ring k. Note that each of the above constructions extends k-linearly, whereupon our ideals are indeed ideals in the monoid algebra.

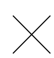
ss:calcTn

4.1.2 Calculi

In §1.3.2 and §6.4.2, for example, we encode the monoidal category $\underline{\Sigma}$ by drawing composition as vertical stacking of vertically directed (let us say downward directed) mapping graphs; so that the *monoidal* composition is side-by-side concatenation. (The version for in-line sequence representation as in (3.2.8) is slightly less elegant: $12 \otimes 12 = 1234$ and so on.) We can use a similar calculus here.

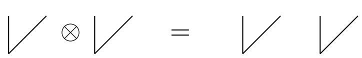
But there are several possible convention choices. For example we can compose left-to-right or right-to-left - and indeed it is sometimes convenient to compose top-to-bottom or bottom-to-top (where there are also, perhaps, fewer 'handedness' biases).

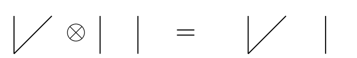
One simple representation is that given in (3.2.8). And one visual alternative is using mapping diagrams, drawn mapping from top to bottom. For example, arranging number-symbols in natural order and then stripping off the labels (and even un-hit targets etc):

11 $\mapsto$  12 $\mapsto$ 21 $\mapsto$  22 $\mapsto$ 

Composition is then by stacking pictures and following the connection from top to bottom (whether top-to-bottom is this version of left-to-right depends on the core convention).

In this scheme the monoidal composition is





Observe here that one must be careful with the target positions (targets that are not hit may simply not be drawn, so position is important).

Note that via the monoidal composition/Young embedding we may understand the idempotent element $11 \in T_2$ as an idempotent element in T_n. Indeed we can see that the category is monoidally generated by 11 and 21.

(**4.1.3**) Note that

$$kT_n\ 11\ kT_n \ = \ k\mathsf{C}_{\mathbb{N}}^{n-1}(n,n)$$

and

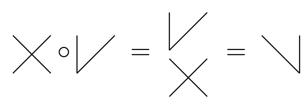
$$0 \to \ kT_n\ 11\ kT_n \ \to kT_n \to kS_n \to 0$$

It follows in particular that irreducible representations of S_n extend to irreducible representations of T_n, in which the image of 11 is zero.

None of the above is the trivial representation, so we already have three one-dimensional representations of T_2 here.

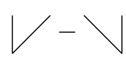
4.1.3 Some example calculations

As an illustration we can observe some idempotents in case $n = 2$. We first fix our conventions by:

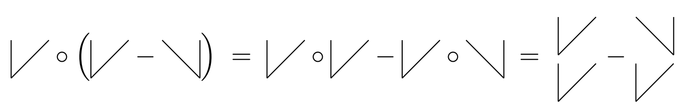
$21 \circ 11 =$  $= 22$

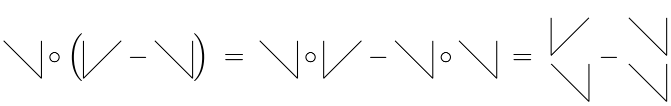
- notice that here the leftmost factor passes to the bottom, and in the composite we follow paths from top to bottom to compute the composition.

In the monoid algebra we have elements like

$11 - 22 =$ 

With this we have various pieces of arithmetic. For the left ideal generated by $11 - 22$:

 $= 0$

 $= 0$

(so far we see that $11 - 22$ is nilpotent);

$$21 \circ (11 - 22) = 22 - 11 = -(11 - 22)$$

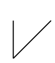
so the left ideal is one-dimensional. For the right ideal:

$$(11{-}22){\circ}11 = 11{-}22, \qquad (11{-}22){\circ}22 = 11{-}22, \qquad (11{-}22){\circ}21 = 11{-}22,$$

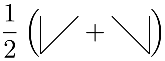
so the two-sided ideal is one-dimensional. In fact this nilpotent ideal is the radical as we will see.

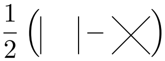
Observe that $11 - 22$ induces a different one-dimensional irreducible representation on the left (that corresponding to $\lambda = 1^2$) and on the right (the trivial).

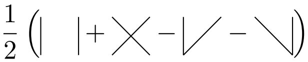
Let

$v = \ 11 \ =$ 

Assuming $\mathbb{Q} \subseteq k$, let e_λ denote the primitive central idempotent in the symmetric group algebra kS_n corresponding to $\lambda \vdash n$. We have $S_n \hookrightarrow T_n$ and hence $kS_n \hookrightarrow kT_n$. Assuming at least that the characteristic of k is not 2, define

$\epsilon_1 \quad = \mathsf{e}_2 v \quad = \frac{1}{2}$  , (4.2)

$\epsilon_{1^2} \quad = \mathsf{e}_{1^2} \quad = \frac{1}{2}$  , (4.3)

$\epsilon_2 \quad = \mathsf{e}_2 - \mathsf{e}_2 v \quad = \frac{1}{2}$  . (4.4)

in kT_2 (or indeed in kT_n by the Young embedding). Observe that these are idempotent, and, with $11-22$, span kT_2.

Let us consider the left and right ideals generated by these elements, which are thus the corre-ponding left and right projective modules. We have

$$kT_2\,\epsilon_1 \;=\; k\{11,22\} \;=\; k\{\epsilon_1,\; 11-22\} \qquad\qquad \epsilon_1\,kT_2 \;=\; k\epsilon_1 \tag{4.5}$$ eq:LRproj

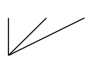
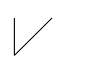
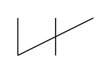
(cf. e.g. ([?])). Observe that $11-22$ spans a submodule of the left projective, while the right projective is simple. It is interesting to note how different the left and right projective covers of the 'same' simple module are here (the simple module is associated also to the maximal semisimple quotient, where the left and right versions are necessarily paired).

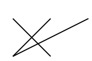
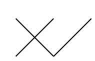
Observe that ϵ_1 is primitive. For later use we observe that we can use it to define a form which induces a map from the left projective to the corresponding injective (dual of right projective).

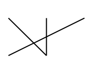
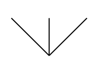
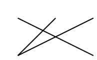
...

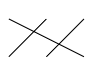
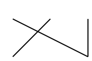
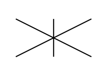
Next
$$kT_2\,\epsilon_{1^2} \;=\; k\epsilon_{1^2} \qquad\qquad \epsilon_{1^2}\,kT_2 \;=\; k\{\epsilon_{1^2},\; 11-22\}$$
Finally
$$\epsilon_2\,kT_2 \;=\; k\epsilon_2$$
In fact in the semisimple quotient these idempotents are primitive orthogonal (indeed they are the unique set as such and the algebra is basic). Thus they are also primitive here - but here neither orthogonal nor unique. We have
$$\epsilon_{1^2}\,kT_2\,\epsilon_1 = \;...$$

(**4.1.4**) Case $n=3$:

111 $\mapsto$  112 $\mapsto$  113 $\mapsto$  121 $\mapsto$ 

122 $\mapsto$  211 $\mapsto$  212 $\mapsto$ 

221 $\mapsto$  222 $\mapsto$  223 $\mapsto$  311 $\mapsto$ 

312 $\mapsto$  313 $\mapsto$  321 $\mapsto$ 

and so on.

4.1.4 The monoid $\hom(\underline{2},\underline{2})$

ss:2^2

Two matrices A, B are conformable to a product AB if (i) the number of columns of A equals the number of rows of B; (ii) they have entries in the same ring R. By convention, if R is a K-algebra, then a matrix over K is considered a matrix over R by the homomorphism ψ (see (1.2.17)), taking elements of K to scalar multiples of 1_R.

ex:mon22

(**4.1.5**) Consider the monoid $M = \underline{2}^{\underline{2}}$, and the free $\mathbb{Z}$-module $\mathbb{Z}M$ with basis M. This is a $\mathbb{Z}$-algebra (by virtue of the monoid multiplication). Totally ordering this (or any other) basis we may encode $x \in \mathbb{Z}M$ by

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

(here we have used the shorthand for monoid elements give in (3.2.8)). This organisational scheme yields a generalisation of the regular representation construction mentioned in the Introduction. Indeed there is both a left and a right regular construction. We shall consider both.

(**4.1.6**) Firstly consider the encoding of multiplication by

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * (11,\ 12,\ 21,\ 22) = \begin{pmatrix} 11\circ 11 & 11\circ 12 & 11\circ 21 & 11\circ 22 \\ 12\circ 11 & 12\circ 12 & 12\circ 21 & 12\circ 22 \\ 21\circ 11 & 21\circ 12 & 21\circ 21 & 21\circ 22 \\ 22\circ 11 & 22\circ 12 & 22\circ 21 & 22\circ 22 \end{pmatrix} = \begin{pmatrix} 11 & 11 & 11 & 11 \\ 11 & 12 & 21 & 22 \\ 22 & 21 & 12 & 11 \\ 22 & 22 & 22 & 22 \end{pmatrix}$$

(we put the $*$ in on the left, to emphasise that this is matrix multiplication over a non-commutative ring) and hence

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * m = \begin{pmatrix} 11\circ m \\ 12\circ m \\ 21\circ m \\ 22\circ m \end{pmatrix} \qquad m \in \underline{2}^{\underline{2}}$$

That is

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * 11 = \begin{pmatrix} 11\circ 11 \\ 12\circ 11 \\ 21\circ 11 \\ 22\circ 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 22 \\ 22 \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} \tag{4.6}$$

eq:22rightaction

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * 12 = \begin{pmatrix} 11\circ 12 \\ 12\circ 12 \\ 21\circ 12 \\ 22\circ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

and so on. By this (general) construction we have a map $R_r : M \to M_4(\mathbb{Z})$

$$R_r(11) = \begin{pmatrix} 1&0&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix},\ R_r(12) = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ R_r(21) = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix},\ R_r(22) = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix}$$

These matrices give a representation.

(**4.1.7**) We do not yet have the tools for a systematic analysis of representations of a monoid, but a couple of observations are in order. This representation is, up to a reordering of the basis, in the form of (1.19):

$$R_{r'}(11) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad R_{r'}(21) = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This corresponds to the fact that the free $\mathbb{Z}$-submodule of $\mathbb{Z}M$ with basis $\{11, 22\}$ is also invariant under this action of M from the right.

This representation does not have a manifest direct sum decomposition, but we can ask if such a decomposition can be manifested by basis change. However the possibilities for basis change beyond reordering depend on the choice of ring.

(**4.1.8**) *Provided we pass to a ground ring k in which 2 is invertible*, another basis is $\{-11+12+21-22, 11, 11-22, 12-21\}$. (Questions: Where did this come from?! How did the restriction arise?) Using this ordered basis we get another representation:

$$R'_r(11) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R'_r(12) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R'_r(21) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad R'_r(22) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In other words there is a direct sum decomposition:

$$R' = R_1 \oplus R_{1'} \oplus R_2$$

Note that R_2 is not irreducible, but it is not amenable to a direct sum decomposition in any basis over any ring. It is, however, of the form in Equation (1.19). In this sense it 'contains' two one-dimensional (hence irreducible) representations:

$$R_2 = R_{1'} + R_{1''}$$

(**4.1.9**) Exercise. In case k a field, compute the maximal left ideals of kM and hence determine their intersection (the Jacobson radical J_{kM}). Hint: Write $B = \{b_1, b_2, b_3, b_4\}$ for our new basis. Then $k\{b_1, b_2, b_3\}$ is a left ideal of dimension 3 (and hence maximal).

(**4.1.10**) Remark. Note that there is *no* semisimple case, so we cannot use *Brauer reciprocity* (see later) to infer a symmetric *Cartan decomposition matrix* here.

(**4.1.11**) Alternatively, we may encode multiplication by

$$m * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} m \circ 11 \\ m \circ 12 \\ m \circ 21 \\ m \circ 22 \end{pmatrix} \qquad m \in \underline{2}^{\underline{2}}$$

That is

$$11 * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 11 * 11 \\ 11 * 12 \\ 11 * 21 \\ 11 * 22 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

$$12 * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 12*11 \\ 12*12 \\ 12*21 \\ 12*22 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

and so on. By this construction we have another map $R^r : M \to M_4(\mathbb{Z})$

$$R^r(11)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; R^r(12)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; R^r(21)=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; R^r(22)=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These matrices give an antirepresentation. That is

$$R^r(a)R^r(b) = R^r(ba)$$

This becomes a representation once we compose with the matrix transpose map. Note however that there is no similarity transformation between R_r and $(R^r)^t$ (the map from the algebra to the ring defined for each representation by matrix trace is not changed by similarity, and differs between the two), so they are not equivalent representations.

Quite generally, if a representation can be expressed in the form of Equation (1.19):

$$\rho_{12}(g) = \begin{pmatrix} \rho_1(g) & V(g) \\ 0 & \rho_2(g) \end{pmatrix} \tag{4.7}$$ eq:plusnew

then

$$\mathrm{Tr}(\rho_{12}(g)) = \mathrm{Tr}(\rho_1(g)) + \mathrm{Tr}(\rho_2(g)) \qquad \text{(any } g\text{)}$$

If we assume that $(R^r)^t$ is a (not necessarily direct) sum of the irreducible representations we have already seen, then we can deduce immediately that this sum contains two copies of $R_{1''}$, since $\mathrm{Tr}(R_{1''}(21)) = -1$ and $\mathrm{Tr}(R_1(21)) = \mathrm{Tr}(R_{1'}(21)) = 1$, and so this is the only way to get $\mathrm{Tr}(R^r(21)) = 0$. Considering $\mathrm{Tr}(R^r(11)) = 1$ we then see that

$$(R^r)^t = R_1 + R_{1'} + R_{1''} + R_{1''}$$

(under the stated assumption). In other words $(R^r)^t$ does not even have quite the same irreducible summands as R_r — at least the multiplicities are different.

That was Too much linear algebra! How can we be more slick? We shall shortly begin to address this question.

(**4.1.12**) For K a given commutative ring, and M a left K-module write $End(M)$ for the set of linear transformations of M. For any subset $S \in End(M)$ we define $End_S(M)$ as the subset of linear transformations that commute with every element of S.

(**4.1.13**) Exercise. Consider R'_r as a subset of $End(\mathbb{Z}\underline{2}^{\underline{2}})$. What is $End_{R'}(\mathbb{Z}\underline{2}^{\underline{2}})$?

4.1.5 The monoid $\hom(\underline{3}, \underline{3})$ and beyond

ss:3^3

How can we use what we know about $\underline{2}^{\underline{2}}$? How can we use Proposition 3.2.12?

Note that our initial block diagonalisation for the regular $\underline{2}^{\underline{2}}$ module comes from ordering the basis according to the bottleneck principle. For example 11 spans a submodule with regard to the right action as defined in (4.6). Indeed 22 spans an isomorphic submodule. We have immediately:

(**4.1.14**) Proposition. *Let k be any field and $n > 1$. The algebra $k\underline{n}^{\underline{n}}$ is not semisimple.*

Proof. Write $1^n = 11...1 \in M_n = \underline{n}^{\underline{n}}$ and so on. Then 1^n and 2^n each span isomorphic one-dimensional submodules of the regular module. But by the Artin-Wedderburn Theorem this cannot happen for a semisimple algebra. □

(**4.1.15**) Indeed 11 is an idempotent. Thus there is an *idempotent submonoid* $11M_2 11$ (i.e. a non-identity-preserving submonoid). Note that this is isomorphic to the trivial monoid. Indeed the same holds for $1^n M_n 1^n$. The DSI

$$M_2 11 M_2 = \{11, 22\}$$

and the quotient algebra

$$kM_2/kM_2 11 kM_2 \;\cong\; kS_2$$

Note that we immediately read off the three simple modules of M_2 (in case 2 invertible in k) from this.

Remark. Suppose that k is a field. As we will see later, the property that $k11M_2 11$ is one-dimensional implies that 11 is a *primitive idempotent*, and hence induces indecomposable left and right projective modules. Note that the left projective module $kM_2 \circ 11 = k\{11, 22\}$ is quite different from the right projective module $k11 \circ M_2 = k\{11\}$.

(**4.1.16**) Similarly to 1^n, 121^{n-2} is idempotent, as is 212^{n-2} and $\frac{1}{2}(121^{n-2} \pm 212^{n-2})$. We claim

$$121^{n-2} M_n 121^{n-2} = \{1^n, 2^n, 121^{n-2}, 212^{n-2}\} \cong M_2 \tag{4.8}$$ eq:precat1

(**4.1.17**) Lemma. *In our notation for elements of M_n, the left action of $S_n \subset M_n$ is given by the corresponding permutation of the symbol set. The right action is given by the corresponding (inverse) place permutation (e.g. $121 \circ 312 = 112$).* □

Note that the right ideal $121^{n-2} kM_n$ is naturally also a left kS_2-module (the image is $\{1, 2\}$, so we can restrict the range to this subset). Thus it is a left kS_2- right kM_n-bimodule. These properties survive the quotient by $1^n, 2^n$. We can thus decompose the quotient of the projective module $121^{n-2} kM_n$ as a right kM_n-module by attaching a kS_2 idempotent decomposition of 1 on the left. From this we get two bases:

$$\{121 \pm 212, 211 \pm 122, 112 \pm 221\}$$

The full S_3 action on the left (in the appropriate quotient of the regular module) generates a total of three such pairs in kM_3. Quotienting by all these submodules we get a final kS_3 subalgebra.

In this case, then, we have altogether modules of dimensions $1_1, 3_+, 3_-, 1, 2, 1$ (three of each of the first three) as subquotients of the regular module. Each module here is projective modulo those lower in the bottleneck order, by construction (since generated by an idempotent). One can deduce from this and the AW Theorem (and a little category theory using (4.8) to show that the full projective associated to 3_- contains a copy of 1_1) that at least one of the 3s must be reducible, hence containing a submodule isomorphic to one of the modules *higher* in the bottleneck order. ...

In light of the above, we should order our basis $\underline{3}^{\underline{3}}$ according to the bottleneck principle: 111,222,333, 112,121,211, 122,212,221, 113,131,311, ..., 123,132,213,231,312,321.

Here 111 spanss a submodule (as does 222, and 333). Also $\{112, 121, 211, 122, 212, 221, 111, 222\}$ spans a submodule (for example $112 \circ 333 = 222$, $112 \circ 133 = 122$, $112 \circ 112 = 111$), and $\{112, 121, 211, 122, 212, 221\}$ form a basis for a quotient of this module.

(**4.1.18**) Exercise. Determine the structure of $\mathbb{C}\underline{3}^{\underline{3}}$.

(**4.1.19**) If you get stuck, see Mazorchuk's book [52] for the general case.

4.2 Quaternions

ss:quat

Set

$$\mathbf{i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

We define $\mathbb{H}$ as the subring of $M_4(\mathbb{R})$ generated as an $\mathbb{R}$-algebra by these matrices (as an $\mathbb{R}$-vector space it is spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{ii}, \mathbf{ij}\}$). This is a noncommutative division ring.

Chapter 5

Reflection groups and geometry

A number of important tools in representation theory can be characterised as *geometrical* or *combinatorial*. Here we set up some of the basic underlying machinery. We assume the reader knows linear algebra, what is a Euclidean space, a topological space (see §3.3), and so on.

In §5.1 we consider some basic geometry. In §5.2 we consider *reflection group*, *Coxeter group* and *chamber geometry* basics. (In particular we construct several specific groups here. However they appear here in a geometrical context, and not yet as objects of study in their own right.) In §5.4 we look at *parabolic subgroups* (needed later for representation theory) and *alcove geometry*.

In §5.6 we look at the basics of Kazdhan–Lusztig polynomials (considered combinatorially). In §5.7,§5.8 we change tack and look at Young diagram combinatorics (and *its* connection to alcove geometry).

Background reading examples: Humphreys [68], Lefschetz [**?**] (I am privileged to have inherited Goldie's copy at Leeds `:-)`).

5.1 Some basic geometry

ss:reflect0

Recall the definition of a topological space from §3.3, and that $\mathbb{R}^n$ is a topological space via the metric topology. In particular $\mathbb{R}$ is a topological space with open sets generated by the fundamental set of open intervals.

A topological space T is *Hausdorff* if every pair of distinct points are contained in distinct open sets of T (see e.g. [**?**, §6]).

5.1.1 Affine spaces and simplicial complexes

Fix any $n \in \mathbb{N}$. An *affine subspace* of $\mathbb{R}^n$ is a subset of form $A = v + S$ where S is a subspace and v is a vector not in S. Note that if $w \in A$ then $A - w = S$, so A determines the underlying space S. The *dimension* of A is the dimension of S. Note that every affine space defines a fibre of affine spaces over its underlying space, and that this fibre is a partition of $\mathbb{R}^n$.

A subspace of codimension 1 is a *hyperplane*, $H = v + H_0$ say. A hyperplane in $\mathbb{R}^n$ defines a partition of $\mathbb{R}^n$ into three: H itself; the part of the fibre 'above' H relative to v (the union of hyperplanes of form $H_x = xv + H_0$ with $x > 1$ — note that we have used the ordered field property of $\mathbb{R}$ here); and the part below. (The designation above/below depends on v, but the partition does not.) These unions are called *open half-spaces.* The union of hyperplanes of form $H_x = xv + H_0$ with $x \geq 1$ is a closed half-space. Thus we have a pair of closed half-spaces, whose intersection is the hyperplane.

A hyperplane *supports* a subset T of $\mathbb{R}^n$ if T lies entirely in one of its closed half-spaces.

(**5.1.1**) The *affine hull* of a set of points in $\mathbb{R}^n$ is the intersection of all affine spaces containing the set.

A set of $t+1$ points in $\mathbb{R}^n$ is *affinely independent* if every proper subset has a strictly smaller affine hull. Thus the vertices of a triangle are affinely independent, but the vertices of a square (or plane quadrilateral) are not.

(**5.1.2**) A *t-simplex* is the convex hull of an affinely independent set of $t+1$ points in $\mathbb{R}^n$.

A hyperplane supports a simplex if it supports it as a subset of the underlying space. A *face* of a simplex is the intersection with a supporting hyperplane.

Note that a face is a simplex (of dimension not greater than that of the original simplex). Note that there are in general infinitely many supporting hyperplanes for a simplex. If the simplex is of maximal dimension then there is precisely one hyperplane defining each face of codimension 1. The lower-dimensional faces may then be characterised as the intersections of the simplex with more than one of these defining hyperplanes.

(**5.1.3**) A *simplicial complex* (see e.g. [55, Ch.15]) is a set Z of simplices that is (i) closed under including faces; (ii) if $A, B \in Z$ then $A \cap B$ is a face of both.

A complex Z is *pure* if the maximal faces (faces that are not faces of simplices in Z other than themselves) are all of the same dimension.

Given a complex Z we write Z^i for the subset of i-dimensional faces; and $Z^{\leq i}$ for the *i-skeleton* (the obvious union; which is a subcomplex).

5.1.2 Hyperplane geometry, polytopes etc

Let V be a euclidean space and $H \subset V$ a hyperplane. This H separates V into three connected components: H and the two open components of $V \setminus H$. The complement of one of these two open components is called a half-space (this includes affine half-spaces; to emphasise a non-affine case we can say 'linear half-space').

`de:chyp` (**5.1.4**) Let $\mathbb{H}$ be a set of hyperplanes in V (not necessarily closed under the reflection group they generate). We assume that $\mathbb{H}$ is finite (or in a suitable sense locally finite, so that the set of points of V not lying on any hyperplane is euclidean-topology-open; a generic point on a hyperplane does not lie on any other hyperplane, and so on — we will give not-strictly-finite examples of this later).

The $\mathbb{H}$-singularity of $v \in V$ is

$$s_{\mathbb{H}}(v) = \#\{H \in \mathbb{H} \mid v \in H\}$$

The set $\mathbb{H}$ separates V into various maximal connected components of fixed singularity, called *facets.* Singularity zero facets are *chambers of* $\mathbb{H}$. We write $\mathcal{C}_{\mathbb{H}}$ for the set of chambers.

(**5.1.5**) For f a facet, the euclidean-space closure $\bar{f}$ is the union of certain facets. Besides f, all the other facets in $\bar{f}$ are of higher singularity.

(**5.1.6**) Write $\bar{\mathbb{H}}$ for the closure of $\mathbb{H}$ under the reflection group it generates.

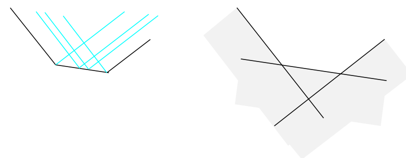


Figure 5.1: Polyhedron as (a) $H + C$; (b) intersection of half-spaces (unshaded region). fig:polyh1

(**5.1.7**) Fix $n \in \mathbb{N}$. Given a set $P \subset \mathbb{R}^n$ of points, its *hull* is the smallest convex set containing P. Given a finite set $P \subset \mathbb{R}^n$ of vectors (NB points define vectors relative to 0), its *cone* is the set of $\mathbb{R}^{\geq 0}$-linear combinations. Note that a cone necessarily contains 0, but a hull does not.

A *(convex) polytope* is the hull of a finite set of points. The dimension of a polytope Π is the smallest dimension of an affine space containing Π.

A *polyhedron* is a subset of $\mathbb{R}^n$ of form $\Pi = H + C$ where H is a finitely generated hull and C is a cone.

Example: Fig.5.1

A *ray* is a polyhedron $\Pi = H + C$ where H and C are each generated by a single point/vector. A polyhedron is *bounded* if it contains no ray. Thus a bounded polyhedron is a polytope.

(**5.1.8**) Theorem. *(I) A subset $\Pi \subset \mathbb{R}^n$ is a polytope (a finitely-generated hull) if and only if it is a bounded intersection of half-spaces, that is: $\Pi = \{x \mid Mx \leq c\}$ for some matrix M and vector c; and Π contains no ray.*
(II) A subset $\Pi \subset \mathbb{R}^n$ is a polyhedron if and only if it is an intersection of half-spaces, that is: $\Pi = \{x \mid Mx \leq c\}$ for some matrix M and vector c.

Proof. See e.g. Ziegler [151, §1.1]. □

(**5.1.9**) Using the theorem we have the following definition.

A *proper face* of a polyhedron (or polytope) Π is the intersection with any hyperplane defining a half-space entirely containing Π. For example, any hyperplane which misses Π entirely defines the *empty face*. The vertices and edges are also faces. A *face* is either a proper face or is the polyhedron itself.

Note that a face is itself a polyhedron (resp. polytope).

(**5.1.10**) A *polyhedral (polytopal) complex* is a (locally) finite set Z of polyhedra (resp. polytopes) such that (i) $A, B \in Z$ implies $A \cap B$ is a face of both; (ii) $A \in Z$ implies all its faces are in Z; (iii) $\emptyset \in Z$.

A complex is *pure* if the maximal faces are all of the same dimension. We will also assume that they are of maximal dimension (i.e. of dimension n in $\mathbb{R}^n$).

(**5.1.11**) Example. Any locally finite set $\mathbb{H}$ of hyperplanes in $V = \mathbb{R}^n$ defines a complex (typically polyhedral) $Z(\mathbb{H})$. The (closures of the) chambers of $V \setminus \mathbb{H}$ are the maximal faces.

de:dual graph (**5.1.12**) The *dual graph* of a pure polyhedral complex in $\mathbb{R}^n$ has a vertex for each polyhedron of dimension n (codimension 0) and an edge (A, B) whenever $A \cap B$ is a face of dimension $n - 1$ (codimension 1).
(See e.g. Kalai [55, §17.3].)

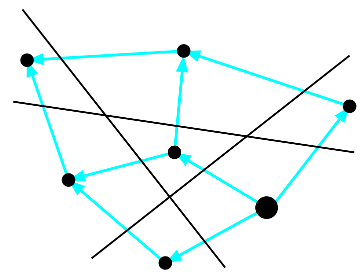


Figure 5.2: Directed rooted dual graph $D(\mathbb{H})$ of a hyperplane complex $Z(\mathbb{H})$. fig:polyh2

(**5.1.13**) Define $D_-(\mathbb{H})$ as the dual graph of hyperplane complex $Z(\mathbb{H})$. Given a 'root' chamber/maximal-face, this graph becomes a directed graph $D(\mathbb{H})$: every hyperplane, hence codimension 1 face, has one half-space containing the root; the edge is directed towards the half-space not containing the root.

Example: Fig.5.2

(**5.1.14**) The digraph $D(\mathbb{H})$ is clearly acyclic. Indeed it has a length function on vertices: $l(v)$ is the number of hyperplanes between v and the root, so that every directed edge corresponds to increasing l by 1. It follows here that any two directed paths from any vertex A to B have the same number of edges.

Note that the dual graph of an arbitrary pure complex does not have a length function since its faces do not necessarily extend to separate the underlying space.

5.2 Reflections, hyperplanes and reflection groups

ss:reflect

Here we follow the approach of Humphreys [68] (cf. Bourbaki) in significant part. See also Jacobson [69], Fulton–Harris [50, §21.1], Moise [125], Ziegler [151], and various chapters from Goodman–O'Rourke [55].

A *reflection* in $\mathbb{E}^n$ is a linear map that fixes a hyperplane pointwise (or the corresponding map for an affine hyperplane).

de:refsex1 (**5.2.1**) Examples of reflections acting on a Euclidean space ($\mathbb{R}^n$ with the usual dot product):

$$\begin{aligned} (ij) : \mathbb{R}^n &\to \mathbb{R}^n && (5.1) \\ (x_1, x_2, ..., x_i, ..., x_j, ...x_n) &\mapsto (x_1, x_2, ..., x_j, ..., x_i, ...x_n) && (5.2) \end{aligned}$$

eq:ref00ij

with reflection hyperplane

$$H_{(ij)} = \{x \in \mathbb{R}^n \mid x_i = x_j\};$$

$$\begin{aligned} (i)_- : \mathbb{R}^n &\to \mathbb{R}^n && (5.3) \\ (x_1, x_2, ..., x_i, ..., x_j, ...x_n) &\mapsto (x_1, x_2, ..., -x_i, ..., x_j, ...x_n) && (5.4) \end{aligned}$$

eq:ref00i-with reflection hyperplane

$$H_{(i)_-} = \{x \in \mathbb{R}^n \mid x_i = 0\}.$$

These reflections make sense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, and also in $\mathbb{R}^{\mathbb{N}}$, the space of infinite sequences (the dot product is not defined here for arbitrary pairs of vectors, but it is defined for an arbitrary vector with one of the perpendicular vectors for the given reflections).

(**5.2.2**) In the examples (ij) and $(i)_-$ the hyperplane contains the origin of coordinates, i.e. they are subspaces. A hyperplane not containing the origin is an *affine* hyperplane — a translate of a non-affine hyperplane by some vector. Any non-affine hyperplane may be characterised as the subspace of vectors perpendicular to a given non-zero vector v. For example $H_{(i)_-}$ is perpendicular to e_i. Any affine hyperplane can then be characterised by a perpendicular vector and a distance along that vector.

If v is the vector defining H the corresponding reflection is the linear map

$$y \mapsto y - \frac{2y.v}{v.v} v \tag{5.5}$$

eq:reflect

(which obviously fixes H and sends v to $-v$). If H is affine with vector v and distance d we have reflection

$$y \;\mapsto\; ((y - d\bar{v}) - \frac{2(y - d\bar{v}).v}{v.v} v) + d\bar{v} \;=\; y - \frac{2(y - d\bar{v}).v}{v.v} v$$

where $\bar{v}$ is the unit vector. Defining $\check{v} = \frac{2}{v.v} v$ this is

$$y \;\mapsto\; y - (y.v - d\bar{v}.v)\check{v}.$$

A reflection group is a group generated by reflections (affine and or non-affine).

(**5.2.3**) A *non-affine reflection group on a space* E is a subgroup of the group of orthogonal transformations on E that is generated by reflections.

Any set $\mathbb{H}$ of hyperplanes H_t in a space E specifies a set of reflections t and hence a reflection group — the group $W_{\mathbb{H}}$ generated by reflection in these hyperplanes. Thus if $\mathbb{H} = \{H_t \mid t \in S\}$, say, we have, for some set of relations $\sim$, an abstract presentation

$$W_{\mathbb{H}} \;\; \cong \;\; \langle t \in S \mid \; \sim \rangle \tag{5.6}$$

eq:precox

where the relations $\sim$ include $t^2 = 1$ but otherwise depend on the details (see §5.2.2).

(**5.2.4**) Some other elements of the group $W_{\mathbb{H}}$ may also be reflections. Indeed the image of any hyperplane H_s in another, H_t say, is a hyperplane H_{tst} corresponding to a reflection tst in $W_{\mathbb{H}}$ (NB for t, s reflections then $(tst)(tst) = 1$). We write $\bar{\mathbb{H}}$ for the closed set of hyperplanes obtained from $\mathbb{H}$.

(**5.2.5**) EXAMPLE. Consider the group W_S generated by reflection set $S = \{(12), (23), (34)\}$ in $\mathbb{R}^4$. We have $(12)(23)(12) = (13)$ (indeed $(23)(12)(23) = (13)$). Also $(12)(34)(12) = (34)$, $(34)(12)(34) = (12)$, and $(34)(13)(34) = (14)$. Meanwhile neither $(12)(34)$ nor $(12)(23)$ in W_S are reflections. See §11.1.4 for more examples.

(**5.2.6**) REMARK. We might say $W_{\mathbb{H}}$ on space V is 'locally finite' if only finitely many elements of $\bar{\mathbb{H}}$ pass through a finite interval of V. (We will effectively restrict ourselves to such groups here, along with certain finitary versions. The study of *Coxeter systems* (see 5.2.10) gives a way of selecting locally finite cases from the more general context, but we shall not focus on this aspect of their use here.)

5.2.1 Reflection group root systems

de:root system

(**5.2.7**) Let W be a finite reflection group on space V and $\mathbb{H}$ its set of hyperplanes in V. Recall that a (non-affine) hyperplane can be characterised by a perpendicular vector (any vector in the same direction will do). Reflection of a vector in a hyperplane produces a vector for the corresponding reflected hyperplane. Closing this process, however, produces at least two vectors per hyperplane, since reflecting vector v in its own hyperplane gives $-v$. (We can characterise a specific half-space by a vector into this half-space, so then we have two vectors for the two half-spaces associated to a hyperplane.)

A *root system* Φ is a finite set of non-zero vectors closed under the reflections it defines, and minimal as such in that it contains precisely two vectors on each of its lines, v and $-v$.

Notes: The combination of closure and finiteness strongly constrains the possibilities.
It is of interest (for example in Lie theory, see later) to impose the further 'crystallographic' constraint that $n_{v,v'} = 2v.v'/(v'.v') \in \mathbb{Z}$ for all $v, v' \in \Phi$. (This amounts to saying that W fixes some lattice of integral combinations of a basis in V.)
Note that

$$n_{v,v'} n_{v',v} = 4\cos^2(\theta)$$

where θ is the angle between the vectors. (One can think of the constraints in terms of this angle.) Such an angle *may* be defined even between pairs of affine reflections — excepting of course those reflections that are parallel.

: simple system

(**5.2.8**) We can characterise a root system by giving the numbers $n_{v,v'}$. This amounts to a description of the system via certain subgroups, each generated by a certain pair of reflections $s_v, s_{v'}$. (Finiteness implies $(s_v s_{v'})^m = 1$ for some m.) It is enough to give this data for one of each pair $v, -v$ (since $s_v = s_{-v}$), and indeed for a spanning subset of Φ. (Exercise: why is this enough?)

A *positive root system* Π is a choice of one of $v, -v$ over all Φ (a transversal of the partition of Φ into $v, -v$ pairs). A *simple root system* Δ is a subset that is a basis of $\mathbb{R}\Phi$. (It is not meant to be obvious that such a thing exists.)

(**5.2.9**) For given Δ the *Cartan matrix* of Φ is $(n_{v,v'})_{v,v' \in \Delta}$. The constraints limit the possibilities for this matrix to a countable collection of 'types'. Evidently $n_{v,v} = 2$, but also off-diagonal entries are restricted (in strictly finite cases) to $\{0, -1, -2, -3\}$. The zero entries appear symmetrically, and otherwise at least one of each opposing pair is -1. For example:

$$C_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad C_{A_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad C_{B_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix} \tag{5.7}$$

eq:Cartan mat eg

The angle θ for a pair 0,0 is $\theta = \pi/2$. It follows that the product st of the reflections involved acts as a rotation by $2\theta = \pi$; so the pair of reflections obeys $stst = 1$, i.e. $(st)^2 = 1$.
The angle θ for a pair -1,-1 is $\theta = 2\pi/3$ (so the pair of reflections obeys $ststst = 1$, i.e. $(st)^3 = 1$). In fact, in a suitable sense, almost every case is of the 0,0 form (see later), and almost every other of the -1,-1 form.
For a pair -1,-2 the angle is $\theta = 3\pi/4$ (so the pair of reflections obeys $stststst = 1$, i.e. $(st)^4 = 1$); for $-1, -3$ is it $\theta = 5\pi/6$.

The singular case of parallel walls s, t gives a translation st rather than a reflection.

ss:coxeter1

5.2.2 Coxeter systems and reflection groups by presentation

Recall that presentations for groups are useful things, for various reasons. A systematic approach to the presentation problem (5.6) — finding abstract presentations for reflection groups — is given by Coxeter systems.

de:Coxeter

(**5.2.10**) A *Coxeter system* is a pair (W, S) consisting of a group W, and a set of generators $S \subset W$, that satisfy the *Coxeter-generators condition.* This means that there is a symmetric matrix M indexed by S such that $M_{s,s} = 1$ and $M_{s,s'} \geq 2$ (with every entry in $\mathbb{N} \cup \{\infty\}$), defining relations by

$$(ss')^{M_{s,s'}} = 1$$

(here $M_{s,s'} = \infty$ means no relation; $M_{s,s'} = 3$ means $(ss')^3 = 1$, i.e. $ss's = s'ss'$).

(**5.2.11**) Example. 1. Every $(W, \{s\})$ has $M = (1)$ and group W of order 2.
2. The system defined (up to isomorphism) by

$$M = M_{\tilde{A}_1} := \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$$

has group W of infinite order. (See (5.2.16) for a reflection group realisation.)

(**5.2.12**) Remark. The $\tilde{A}_1$ above refers to the 'type' classification of certain integer matrices. See e.g. [68] and (5.2.7) *et seq.*

de:parab

(**5.2.13**) A *parabolic subsystem* of (W, S) is a system generated by some $I \subset S$.

Coxeter systems are closely related to reflection groups. Indeed (see [68, §6.4]):

th:coxeterr

(**5.2.14**) Theorem. *Coxeter group W is finite if and only if it is a finite reflection group.*

And for our present purposes their main role is as an organisational device for the study of reflection groups. A consequence of the definition is that the generating set S is minimal; and that there is a reflection group realisation of S as a set of reflections (thus corresponding to a set of hyperplanes). Thus we can realise (W, S) as some pair $(W_{\mathbb{H}}, \mathbb{H})$.

de:Clength

(**5.2.15**) Fix a Coxeter system (W, S). The *length* $l(w)$ of $w \in W$ is the smallest number of factors in an expression of w as a product of generators.

pa:hyper1

(**5.2.16**) Let V be a Euclidean space. We want to think for a moment about the reflection groups on V, and Coxeter systems (W, S) with an action generated by reflections on V, that coincide with these reflection groups.

Any single hyperplane in V generates a reflection group of order 2. Starting with this example, one may add in a second hyperplane and ask about the group G generated by reflections in the two hyperplanes.

The nature of this group G depends on the relationship of the two hyperplanes; and the possibilities for this relationship depend on the dimension of V. In dimension 2 or greater reflection in one hyperplane may fix the other; or they may be parallel; or they may generate a finite number of other hyperplanes; or (if the second hyperplane is in generic position with respect to the first) an infinite number. (Here we are interested in all but the generic position case.)

In dimension 1 of course, any two distinct hyperplanes (i.e. points) are necessarily parallel. The associated Coxeter system is necessarily $M = M_{\tilde{A}_1}$.

(**5.2.17**) Let (W, S) be a Coxeter system. The Coxeter relation matrix M can be represented by a graph, with a vertex for each generator and $M_{ss'} - 2$ edges between s and s'. (As far as the Coxeter system is concerned these edges are undirected, although there is a related role in Lie theory where more data is used.) Some of these graphs have names, or 'types'.

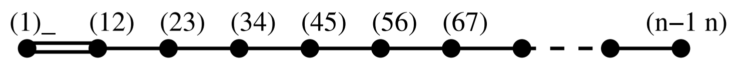
5.2.3 Some finite and hyperfinite examples and exercises

ex:Dgroup (**5.2.18**) Example. Fix $n \in \mathbb{N}$ and consider the reflections in $\mathbb{R}^n$ defined in (5.2.1).
(1) The reflection group $\mathcal{D}_+ = \mathcal{D}_+(n)$ is the group generated by the set of reflections $S_{A_n} := \{(i\ i{+}1) : i = 1, 2, ..., n-1\}$. We have $\mathcal{D}_+ \cong S_n$.

Proposition. The generators S_{A_n} obey Coxeter relations of type-A. ■
(2) The reflection group $\mathcal{D}^-$ generated by $S_{B_n} := \{(1)_-, (i\ i{+}1) : i = 1, 2, ..., n-1\}$ obeys Coxeter relations including:

$$(1)_-(12)(1)_-(12) = (12)(1)_-(12)(1)_-$$

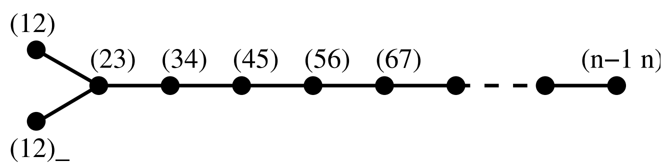
of type-B:



Let $(12)_-$ denote reflection in the hyperplane with $x_1 = -x_2$. Define $(ij)_-$ similarly. Note that $(12)_- = (1)_-(12)(1)_-$. Note that $(12)(1)_-(12) = (2)_-$. Thus $\mathcal{D}^-$ includes all the reflections of form $(ij), (i)_-$ and $(ij)_-$.
(3) The reflection group $\mathcal{D}$ generated by $S_{D_n} := \{(12)_-, (i\ i{+}1) : i = 1, 2, ..., n-1\}$ obeys relations including

$$(12)_-(12)(12)_-(12) = 1 \qquad \text{and} \qquad (12)_-(23)(12)_- = (23)(12)_-(23)$$

of type-D:



(**5.2.19**) Note that $\mathcal{D}_+ \subset \mathcal{D} \subset \mathcal{D}^-$. Note that $\mathcal{D}_+(n) \hookrightarrow \mathcal{D}_+(n{+}1)$ (take the map which corresponds to inclusion in the corresponding abstract Coxeter system), so we may also define a $n \to \infty$ limit.

(**5.2.20**) Exercise. By construction an automorphism of the Coxeter graph induces an automorphism of the corresponding group that fixes the set of Coxeter generators. It can be shown that the subgroup of elements fixed by the automorphism is also a Coxeter group. For example, the graph for $\mathcal{D}_+$ is a chain, which has a chain order reversal automorphism, call it γ. Construct the fixed point group $\mathcal{D}_{+\gamma}$.

Hint: We represent permutations as diagrams, lets say drawn from top to bottom. Then an element is fixed under γ if it is left-right symmetric. (The resultant Coxeter group is of type-B.)

(**5.2.21**) Exercise. Consider the inner automorphism of $\mathcal{D}^-$ given by $w \mapsto (1)_-w(1)_-$. This interchanges (12) and $(12)_-$ (not a Coxeter generator of $\mathcal{D}^-$) but leaves the other Coxeter generators alone. This map then passes to an outer automorphism ω (say) of $\mathcal{D}$, corresponding to the obvious automorphism of the type-D Coxeter graph. 1. Compute the subgroup $\mathcal{D}_\omega$ of elements fixed by this outer automorphism. 2. Show that this is again a Coxeter group (see e.g. Nanba [**?**]).

1. This subgroup includes (and is generated by) the Coxeter generators excluding (12) and $(12)_-$, together with the element $(12)(12)_- = (1)_-(2)_-$.

2. Evidently $(1)_-$ commutes with this fixed group and with $(2)_-$, giving

$$(1)_-(2)_-(23)(1)_-(2)_-(23) = (2)_-(23)(2)_-(23) = (23)(2)_-(23)(2)_- = (23)(1)_-(2)_-(23)(1)_-(2)_-$$

so there is a map given by $(1)_-(2)_- \mapsto (1)_-$ and $(i\ i+1) \mapsto (i-1\ i)$ (for $i = 2, 3, ...$) that is a bijection between the fixed group $\mathcal{D}_\omega(n)$ and $\mathcal{D}^-(n-1)$.

Remark: Since $\mathcal{D}(n-1) \subset \mathcal{D}^-(n-1)$ this gives an injection of $\mathcal{D}(n-1) \subset \mathcal{D}(n)$ different from the obvious inclusion. Specifically one takes $(12) \mapsto (23)$ and $(12)_- \mapsto (12)(12)_-(23)(12)(12)_- = (1)_-(2)_-(23)(1)_-(2)_- = (23)_-$ and then $(23) \mapsto (34)$ and so on. (Of course $(23)_-$ is not a Coxeter generator on the right, but otherwise this still seems fairly obvious and uninteresing...)

D--ex (**5.2.22**) It will be clear that the point

$$v_- \ := \ (-1, -2, -3, ...) \in \mathbb{Z}^n \subset \mathbb{R}^n$$

($n \in \mathbb{N}$ or $n = \infty$) is not fixed by any nontrivial element of $\mathcal{D}^-$ (it is a *regular* point). The orbit of this point is the set of signed perms of $(-1, -2, -3, ...)$ (with finitely many $+$'s).

D--ex2 (**5.2.23**) It will be useful later to note that there are several descending ('dominant') elements, such as $(-1, -2, -3, ...)$, $(1, -2, -3, ...)$, $(2, -1, -3, ...)$, $(2, 1, -3, ...)$, $(3, -1, -2, ...)$, ... in the orbit $\mathcal{D}^- v_-$. We denote the set of these by $\mathcal{D}^-(-1, -2, -3, ...)^A$ (this is a transversal of the partition of $\mathcal{D}^-(-1, -2, -3, ...)$ into $\mathcal{D}_+$-orbits). A convenient representation of these descending cases is to list the positive entries (the order in the list does not matter).

A pair of these signed perms are adjacent (on either side of a reflection hyperplane, with no other hyperplane between them) if they are related by 'right' action by a generator.

See Fig.5.5 for the subgraph of descending elements and edges labelled by the corresponding generator.

(**5.2.24**) On the other hand certain points such as $(0, -2, -3, ...)$ lie on reflection hyperplanes of $\mathcal{D}^-$ (in this case the point lies on the hyperplane corresponding to $(1)_-$), and so are fixed by the corresponding reflections.

Do the orbits of such points have different looking such graphs?

We have $\mathcal{D}^-(0, -2, -3, ...)^A = \{(0, -2, -3, ...), (2, 0, -3, ...), (3, 0, -2, -4, ...), (3, 2, 0, -4, ...),$
$(4, 0, -2, -3, ...), (5, 0, -2, -3, -4, ...), (4, 2, 0, -3, ...), (6, 0, -2, -3, -4, -5, ...), \}$

We will return to this example in §5.5.

(**5.2.25**) Exercise. Do something similar to (5.2.23) for type-A, restricting to the subgraph of elements descending in all but possible the first position: $(-1, -2, -3, ...)$, $(-2, -1, -3, ...)$, $(-3, -1, -2, ...)$, and so on. (Note that this is much simpler.)

5.3 Reflection group chamber geometry

ss:Hum

We continue with W a reflection group on space V. By (5.2.14) we may assume there is an associated Coxeter system (W, S). The group W and its action defines a closed set of hyperplanes $\mathbb{H} = \bar{\mathbb{H}}$ in V. This is irrespective of S. However the closed set of hyperplanes $\bar{\mathbb{H}}$ will be generated by the set $\mathbb{H}_S$ corresponding to S.

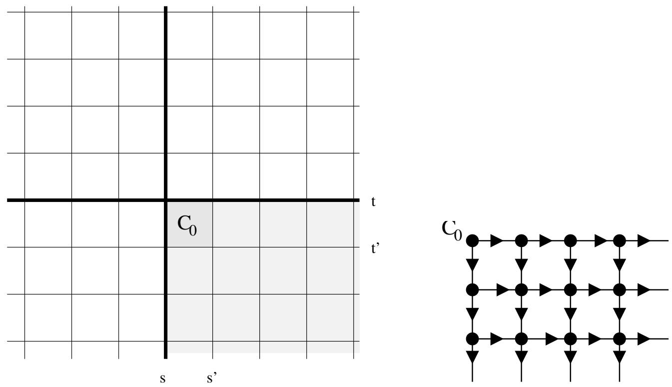


fig:Youngreflect
ig:Youngreflect2

Figure 5.3: (a) Chamber geometry of type $\tilde{A}_1 \times \tilde{A}_1$ with $S = \{s, s', t, t'\}$, and a fundamental region for the $W_{\{s,t\}}$ parabolic. (b) The digraph G_a for $W' = W_{\{s,s',t,t'\}}$ and $W = W_{\{s,t\}}$ from Figure 5.3(a).

(**5.3.1**) Let H_s be the reflection hyperplane of $s \in S$, or indeed of any reflection $s \in W$ generated by these. For T any subset of S let $[T] \subset W$ be the set of reflections generated by T. Set

$$\mathbb{H}_T = \bigcup_{t \in [S] \setminus [T]} H_t$$

(**5.3.2**) A *chamber* of W on V is a chamber of $\mathbb{H}$ or $\mathbb{H}_\emptyset$ as in (5.1.4) — a maximal connected component of $V \setminus \mathbb{H}_\emptyset$. Write $\mathcal{C}_W$ for the set of chambers here. That is, $\mathcal{C}_W = \mathcal{C}_{\mathbb{H}}$.

(**5.3.3**) Example. In our $\tilde{A}_1$ example in (5.2.16) above the chambers are simply a collection of disjoint open intervals of the line.
For an example generalising this we may think of the group generated by two parallel lines in $\mathbb{R}^2$, together with two other parallel lines, perpendicular to the first two:

$$M = \begin{pmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 2 & 2 \\ 2 & 2 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$

The chambers here are a collection of disjoint open rectangles (which may be arranged to be squares without loss of generality here). See Figure 5.3.

(**5.3.4**) Note that the set $H_t \backslash \mathbb{H}_{\{t\}}$ is the subset of hyperplane H_t that intersects no other hyperplane — or equivalently the subset of H_t of elements with singularity 1. This set may similarly be broken up into connected components. At most one of these components intersects any given chamber closure $\overline{C}$. If H_t intersects $\overline{C}$ in this way it is called a *wall* of C. The intersection itself is a *wall-facet* of C. That is, singularity-1 facets (in the sense of (5.1.4)) are wall-facets.

lem:coxeter gens (**5.3.5**) LEMMA. *Let W be a reflection group acting on V. For any given chamber $C \in \mathcal{C}_W$, the set of hyperplanes, hence reflections t, that make up its walls functions as a choice of a Coxeter generating set S in W.*

(**5.3.6**) The choice of a prefered chamber C_0 corresponds to the choice of a simple system (in the sense of (5.2.8)) in V, and the associated reflections are *simple* reflections. (Given a non-commuting pair of these, the conjugate of one by the other is also a reflection, but not 'simple' in this choice.)

(**5.3.7**) NB The choice of C_0 determines S, but S does not necessarily determine C_0. We will usually fix C_0 along with W. Thus we may consider a *reflection system* (W, C_0), which determines a Coxeter system.

(**5.3.8**) A reflection s in W is *simple for chamber B* if its hyperplane H_s makes a wall of B (NB 'simple for B' is not the same as simple, unless $B = C_0$). For our purposes it will be convenient to think specifically of the intersection of the hyperplane with the chamber closure (i.e. this facet) as the wall (thus we distinguish the walls of distinct chambers in general, even if they come from the same hyperplane).

para:reflectx (**5.3.9**) LEMMA. *The reflection action of W on V acts to permute $\mathcal{C}_W$. This action on $\mathcal{C}_W$ is transitive (i.e. for any $C \in \mathcal{C}_W$ the orbit $WC = \mathcal{C}_W$) and indeed regular (simply transitive) (i.e. for $C, D \in \mathcal{C}_W$ there is exactly one $w \in W$ such that $wC = D$).*

Proof. See for example [68, §1.12]. □

(**5.3.10**) Note that W does *not* act transitively on V, or specifically, on the set of walls.

le:orb1 LEMMA. The wall-facets of any chosen chamber C_0 are representatives for the W orbits of the set of all wall-facets.

pa:zeta (**5.3.11**) Regularity of $\mathcal{C}_W$ as a W-set says that for each choice of chamber C_0 we may make an identification

$$\zeta_{C_0} : \mathcal{C}_W \xrightarrow{\sim} W,$$

so that the action of W becomes the left-action on itself. In particular we identify the chosen facet C_0 with $1 \in W$ and write

$$A = w_A C_0 \tag{5.8}$$ idW

to identify the chamber A with group element $\zeta_{C_0}(A) = w_A$.

de:right-act (**5.3.12**) Note that it follows from the identification ζ_{C_0} that there is another commuting action of W on $\mathcal{C}_W$, corresponding to the right-action of W on itself.

(**5.3.13**) With W and C_0 given, define a length function on $\mathcal{C}_W$: $l_W(A)$ as the number of hyperplanes separating A from C_0. (If W is clear from context we shall write simply $l = l_W$.)

(**5.3.14**) LEMMA. *This geometric length function l_W on $\mathcal{C}_W$ for (W, C_0) agrees with the Coxeter length function on W for Coxeter system $(W, \mathbb{H}_{C_0})$ from (5.2.15) via ζ.*

Proof. Firstly $l_W(C_0) = l(1) = 0$. Then for $a \in \mathcal{C}_W$, $l_W(sa) = l_W(a) \pm 1$, since we reflected over a hyperplane, while $l(s\zeta(a)) = l(\zeta(a)) \pm 1$ by definition. It remains to check the signs. Exercise: finish! ???

5.3.1 S_n as a reflection group, permutahedra, etc

See §11.1.4.

5.3.2 Cayley and dual graphs, Bruhat order

para:r-action

(**5.3.15**) We define a digraph $G(W,S)$ with vertex set $\mathcal{C}_W$ by (A,B) an edge if $B = tA$ with t simple for A and $l(B) = l(A)+1$.

How does this depend on S?

(**5.3.16**) We call t the *left-action label* of edge (A,tA).

By (5.8) the edge (A,tA) may also be written $(w_A C_0, t w_A C_0)$. The image under w_A of a particular 'initial' edge (C_0, sC_0) $(s \in S)$ is

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A s w_A^{-1} w_A C_0) = (A, w_A s w_A^{-1} A)$$

Our t is such a $w_A s w_A^{-1}$, by Lem.5.3.10. Using the right-action this can be expressed as

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A C_0 s) = (A, As)$$

We call this s the right-action label of the edge.

(**5.3.17**) CLAIM: With this right-action label the graph $G(W,S)$ is essentially the directed ('right') Cayley graph $\Gamma(W,S)$, and s is the 'colour' label.

Proof. ... directed ...

de:Bruhat

(**5.3.18**) Evidently $G(W,S)$ is a rooted acyclic digraph, with root C_0. The partial order so defined is the *Bruhat order* on chambers/elements of W.

(**5.3.19**) CLAIM: Let (W,S) be a Coxeter system and let $\mathbb{H}$ be the set of hyperplanes of W realised as a reflection group on V, so that W acts regularly on chambers, the fundamental chamber C_0 may be identified with $1 \in W$ and the walls of this chamber give S. That is, $(W,S) = (W_{\mathbb{H}}, \mathbb{H}_{C_0})$. Then as directed graphs

$$G(W,S) = D(\mathbb{H})$$

where the root of $D(\mathbb{H})$ is C_0.

Proof. Both graphs have vertex set $W = W_{\mathbb{H}}$ The edge sets also agree: the edge (a,ta) corresponds by definition to a face between these chambers.

claim: $(a, ta = as)$, some s, and corresponds to a face in the W-orbit of the face between 1 and s.

HOW DO WE KNOW THIS? By the argument above. □

(**5.3.20**) Let $v \in C_0$, and let Wv be the W-orbit of v in V. In the same way as above we may associate a graph to this orbit. It will be evident that this graph is isomorphic to $G(W,S)$, for any such v.

5.4 Coxeter/Parabolic systems (W', W) and alcove geometry

ss:CoxeterPara1

Let (W', S') be a Coxeter system containing (W, S) as a parabolic subsystem (i.e. $S \subset S'$ as in (5.2.13)), with both acting on V. With regard to this pairing, the chambers of W' are then called *alcoves*. Thus the alcoves are a further subdivision of the chambers of W. Write $\mathcal{A} = \mathcal{C}_{W'}$ for the set of alcoves, and $\mathcal{C} = \mathcal{C}_W$ for the set of chambers of W.

Recall that for any Coxeter group W we choose a chamber $C_0 \in \mathcal{C}_W$. This serves three purposes. (1) It determines a choice of Coxeter generators S by Lemma 5.3.5. (2) It determines a bijection $\zeta_{C_0} : \mathcal{C}_W \to W$. (3) It defines a length function and a direction on graph $G(W, S)$ (which should more properly be called $G(W, C_0)$).

(**5.4.1**) Consider the choice of C', the prefered alcove of (W', S'), in the W', W setup. We can no longer freely choose a C' in $\mathcal{A}$ to specify S' *and* a C_0 to specify S (or even a specific W), and expect $S \subset S'$. Since C' determines S', while we also require $S \subset S'$, if a specific W is required then C' must be chosen with the inclusion in mind. Alternatively we may choose C' freely, then define C_0 (and hence W) by removing some walls from the wall set $\mathbb{H}_{C'}$ (producing a chamber that contains but is bigger than C' — the Euclidean closure of C' in the complex defined by W). Then $C_0 \supset C'$ is the prefered chamber of (W, S).

(**5.4.2**) Write $\mathcal{A}^+$ for the set of alcoves lying in C_0. Thus $\mathcal{A}^+$ is a representative set for the W-orbits of $\mathcal{A}$. (In this setting we will call any point $v \in C_0$ *dominant*.)

By (5.3.15) the digraph $G(W', S')$ has vertex set $\mathcal{A}$, and (A, B) an edge if $B = sA$ with s simple for A and $l(B) = l(A) + 1$. And $G(W', S')$ is a rooted acyclic digraph, with root C'. The edges of $G(W', S')$ are in correspondence with the set of walls, and may thus be partitioned into W'-orbits, labelled by the walls of C'.

(**5.4.3**) Let $G_a = G_a(W', W)$ denote the full subgraph of $G(W', S')$ with vertex set $\mathcal{A}^+$.

See Figure 5.3 for an example. See Figures 5.4 and 5.5 for more examples (the construction for the latter is described in §5.5).

(**5.4.4**) The subgraph G_a is again rooted with root C'. Thus any alcove $A \in \mathcal{A}^+$ may be reached from C' by a sequence of simple reflections, always remaining in $\mathcal{A}^+$.

de:res Bruhat

(**5.4.5**) We shall denote the poset defined by the acyclic digraph G_a as $(\mathcal{A}^+, <)$. Note that this coincides with the restriction of the Bruhat order (5.3.18).

(**5.4.6**) Here let $\mathbb{H}$ denote the hyperplanes of W' and $\mathbb{H}_+$ the hyperplanes of W. That is, $W' = W_{\mathbb{H}}$ and $W = W_{\mathbb{H}}$. Via $\zeta_{C'} : \mathcal{A} \to W_{\mathbb{H}}$, we have that $\mathcal{A}^+$ is a transversal of right coset space $W_{\mathbb{H}_+} \setminus W_{\mathbb{H}}$. The right action of $\{w \in W_{\mathbb{H}_+}\}$ on $W_{\mathbb{H}_+} \setminus W_{\mathbb{H}}$ thus gives a closed action on $\{a \in \mathcal{A}^+\}$. Denote this action by

$$a \mapsto a\langle w \rangle.$$

This unpacks as follows. We use $\zeta_{C'}$ to consider alcoves and group elements interchangeably. Write $[a]$ for the coset containing a (i.e. $[a] = W_{\mathbb{H}_+} a$, as it were). Then for $w \in W_{\mathbb{H}}$ we have aw for the usual right action on $a \in \mathcal{A}$ (i.e. on $W_{\mathbb{H}}$ itself), and $a\langle w \rangle = b$ where b is the unique element of $[aw] \cap \mathcal{A}^+$. Note that here it can happen that $a = a\langle w \rangle$.

(**5.4.7**) Under certain special circumstances a in $\mathcal{A}^+$ is taken to $aw \in \mathcal{A}^+$ by the ordinary right action (examples shortly). Indeed if $w \in \mathbb{H}_{C'}$ here then (a, aw) is an edge in G_a with right-label

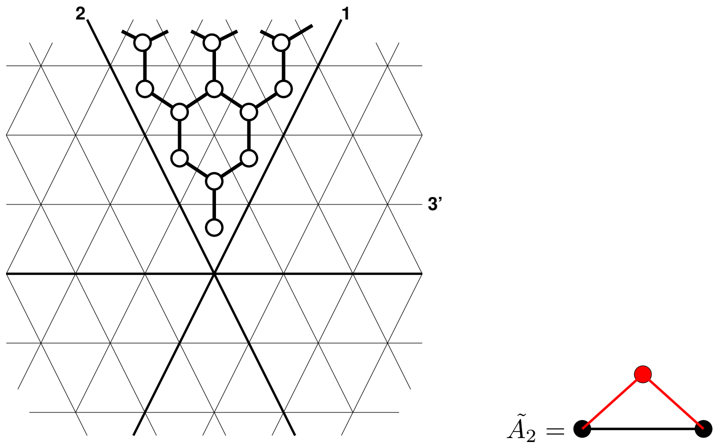


Figure 5.4: Constructing an example of a dominant dual graph G_a: case $G_a(\text{affine-}A_2, A_2)$. `fig:A2dual`

w. More often, though, this action of w does not preserve $\mathcal{A}^+$. Later we will be interested in characterising $a\langle w\rangle$ in case w a reflection element (not necessarily in S), while $a \neq a\langle w\rangle$.

Indeed we will stretch notation slightly by setting $\langle w\rangle a$ to be $a\langle w\rangle$ when $a \neq a\langle w\rangle$; and to be *undefined* otherwise.

5.5 Exercises and examples

`ss:Dex1`

We return to the examples $\mathcal{D}^- \supset \mathcal{D}_+$ from (5.2.18). Our objective here is to make some observations about this case that will be useful later.

(**5.5.1**) We encode elements of $\mathcal{D}^-$ as alcoves via regularity, and then encode alcoves using the regular orbit $\mathcal{D}^- v_-$ where $v_- = (-1, -2, -3, ...)$.

5.5.1 Constructing $G_a(\mathcal{D}^-, \mathcal{D}_+)$ and $G_a(\mathcal{D}, \mathcal{D}_+)$, and beyond

See Fig.5.5 for the beginning of $G_a(\mathcal{D}^-, \mathcal{D}_+)$. The vertex set (of dominant alcoves) is labelled by dominant elements of $\mathcal{D}v_-$ (descending signed perms of $(-1, -2, -3, ...)$), written in $\mathtt{P}(\mathbb{N})$ notation. An edge (a, b) with label 1 means $b = a(1)_-$; otherwise label x means that $b = a(x)$, where these are the *right*-reflection actions of Coxeter generators.

Here we consider a couple of exercises. (0) Produce a version showing the left-action labels.
(1) This graph can be considered as the graph of orbit $\mathcal{D}v_- \cap C_0$. Is there something similar for $\mathcal{D}(0, -2, -3, ...) \cap C_0$ say? This case is not regular so there is no right action, but there is a left action.
(2) How does this compare with $G_a(\mathcal{D}, \mathcal{D}_+)$? In this case we may label with $\mathcal{D}v_-$, which is the subset of descending signed perms of $(-1, -2, -3, ...)$ with an even number of positive terms — and hence with the corresponding subset of $\mathtt{P}(\mathbb{N})$.

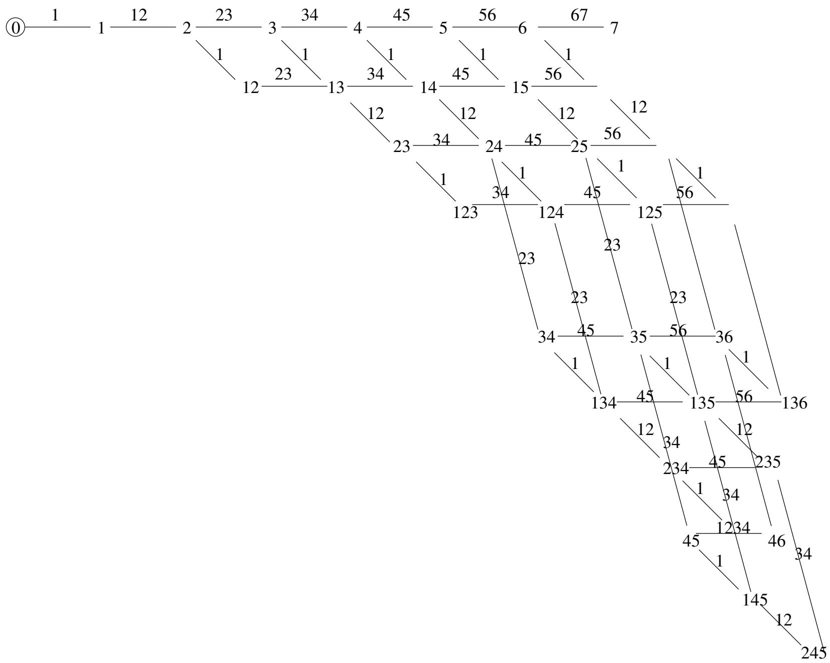


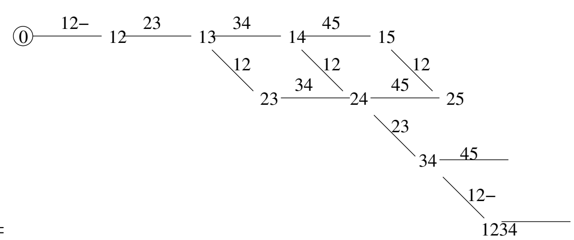
Figure 5.5: Beginning of the type-B/A dominant dual graph (directed from left to right). `fig:KL-HS3`

(3) These are *graphs with extended structure* ('graphes'), in the sense that the edge labels give a partition of the edges. What happens to the comparison when extended structure is noted?

(**5.5.2**) (2): We can describe a vertex set homomorphism h as follows. In what follows a vertex is considered as a string S (an unpunctuated list of natural numbers, so $1S$, say, is a longer string); and given such a string S then $S+1$ denotes the string obtained by adding 1 to each element.

The vertex map is $h(S) = S+1$ if $|S|$ even; and $h(S) = 1(S+1)$ if $|S|$ odd. (Note that this is well-defined.)

$G_a(\mathcal{D}, \mathcal{D}_+) =$



(**5.5.3**) Theorem. *The map $h : G_a(\mathcal{D}^-, \mathcal{D}_+) \to G_a(\mathcal{D}, \mathcal{D}_+)$ is a graph isomorphism; but not a graphe isomorphism.*

Proof. First we need to check that h passes to a graph homomorphism. We do this by checking the 'images' of all possible edge types:

Edges of form $S \xrightarrow{(1)_-} T = 1S$ are taken to $1(S+1) \xrightarrow{(12)} 2(S+1)$ if $|S|$ odd; and to $S+1 \xrightarrow{(12)_-} 12(S+1)$ if $|S|$ even.

Edges of form $S \xrightarrow{(ij)} T$ with $i,j > 1$ are taken to $1(S+1) \xrightarrow{(12)} 1(T+1)$ if $|S|$ odd; and to $S+1 \xrightarrow{(12)_-} T+1$ if $|S|$ even.

Edges of form $1S \xrightarrow{(12)} 2S$ (hence with $1,2 \notin S$) are taken to $2(S+1) \xrightarrow{(23)} 3(S+1)$.

The inverse is as follows: First remove 1 if present; then $S \to S-1$. We check that this is a graph homomorphism. The most interesting cases are:

Edges of form $S \xrightarrow{(12)_-} 12S$ are taken to $(S-1) \xrightarrow{(1)_-} 1(S-1)$.

Edges of form $1S \xrightarrow{(12)} 2S$ are taken to $(S-1) \xrightarrow{(1)_-} 1(S-1)$.

Note that this is injective because the sets S in the two cases necessarily have different parities.

Finally note from this that h is not a graphe isomorphism. Two components in the edge partition in $G_a(\mathcal{D}, \mathcal{D}_+)$ are taken into one. □

(**5.5.4**) (0,1): ...

5.5.2 Right cosets of $\mathcal{D}_+$ in $\mathcal{D}^-$

Consider the right cosets of $\mathcal{D}_+$ in $\mathcal{D}^-$. Every coset contains one strictly descending ('dominant') sequence, and every descending sequence in $\mathcal{D}^- v_-$ arises this way, so we may choose the strictly descending sequences as a set of representatives. A descending sequence in $\mathcal{D}^- v_-$ is determined by its positive elements, and every subset of $\mathbb{N}$ arises, so we may index the cosets by $\mathtt{P}(\mathbb{N})$.

As for any group and subgroup the set of right cosets (the 'coset space') forms a right $\mathcal{D}^-$-set, so we have in this way a right action of $\mathcal{D}^-$ on $\mathtt{P}(\mathbb{N})$. For example:
since

$$(-1,-6,...)(16) = ((26)(-1,-2,...,-6,...))(16) = (26)(16)(-1,-2,...,-6,...)$$
$$= (26)(-6,-2,...,-1,...) = (-6,-1,...)$$

we have $\emptyset(16) = \emptyset$;
since

$$(-1,6,...)(16) = ((26)_-(-1,-2,...,-6,...))(16) = (26)_-(16)(-1,-2,...,-6,...)$$
$$= (26)_-(-6,-2,...,-1,...) = (-6,1,...)$$

we have $\{6\}(16) = \{1\}$;
since $(-1,6,...)(6)_- = (-1,-6,...)$ we have $\{6\}(6)_- = \{\}$;
since $(-1,-6,...)(16)_- = (6,1,...)$ we have $\emptyset(16)_- = \{1,6\}$;
since $(1,-6,...)(16)_- = (-6,1,...)$ we have $\{1\}(16)_- = \{1\}$.

(**5.5.5**) Note the following. Let $\sigma_{ij} : \mathbb{N} \to \mathbb{N}$ denote the i,j-transposition function. The right action of (ij) on $a \in \mathtt{P}(\mathbb{N})$ is given by

$$a(ij) = \sigma_{ij}(a)$$

while $a(i)_-$ is obtained from a by toggling the presence of i.

(**5.5.6**) Let us compare the right action on cosets with the right action restricted to dominant elements as in Fig.5.5.

A small subset of the right actions of $\mathcal{D}^-$ on $\mathcal{D}^- v_-$ take a dominant element to a dominant element (a necessarily different dominant element, of course, unless we act with the identity).

If we restrict attention to the action of reflections ((ij), $(i)_-$ and $(ij)_-$), then a requirement is that the element acting is either of form $(i\,i{+}1)$ or $(1)_-$ (or $(12)_-$) ...

(This is not sufficient. There are also conditions on the dominant element, as we shall see in the proof.)

To see this consider first (ij) acting on a dominant element v. A dominant element is a descending sequence of positive terms then a descending sequence of negative terms, thus precisely one of i, j must be positive. Else both are same sign, and they cannot be descending (as a pair) before *and* after transposition. On the other hand if they are not adjacent then there is $i+1$ (say) between them. Either this positive in v with j before it or i after it; or it is negative with $-i$ before it or $-j$ after. In any case after the transposition, taking account of signs, one of i, j is the wrong side of $i+1$.

Now consider $(i)_-$ on v. Since this changes the sign of i without changing its position, it must either be the last positive or the first negative element, and must be 1.

The argument for $(ij)_-$ is similar. $\square$

If we further require that the action only changes length l by 1 then even $(12)_-$ is excluded in this case.

5.5.3 On connections of reflection groups with representation theory

There are many direct and indirect connections between properties of reflection groups and Coxeter systems and representations theory. Later we will look at Lie groups and algebras; Hecke algebras; and also at the combinatorial structures called Kazhdan–Lusztig polynomials.

5.6 Combinatorics of Kazhdan–Lusztig polynomials

ss:KLP01

KL polynomials can be defined as straightening coefficients between bases for representations of Hecke algebras (of certain Coxeter systems — see §**??** or [86, 68], corresponding to certain parabolic subsystems); or as polynomials encoding Ext data for generalised Verma modules over simple modules for Lie algebras over parabolic subalgebras (see e.g. [11]); or as polynomials whose evaluation at $u = 1$ determines the relationship between Verma characters and simple characters; or indeed in some other ways involving representation theory (see later).

Combinatorially they can be defined as the solutions to certain recursions defined on reflection group alcove geometries. The well-definedness of such definitions may depend on the equivalence with the representation theoretic definitions, but otherwise they are self-contained, and we can look at them without requiring any further machinery. A recursion definition [37] is described next.

5.6.1 The recursion for polynomial array $P(W'/W)$

ss:pKLp def

Let $W' \supset W$ be Coxeter systems as in §5.4 above. Let the set X^+ of dominant alcoves be totally ordered by any order consistent with the Bruhat order (5.4.5). The array $P = P(W'/W)$ is a (generally semiinfinite) lower unitriangular matrix of polynomials in v, with row and column positions indexed by the ordered set X^+, whose remaining entries we describe below.

Write $P = (p_{AB})_{A,B \in X^+}$. It is natural to organise this data into rows p_A (although it is also of interest to organise it into columns). These rows are thus 'finite' (i.e. of finite support), while the columns are not in general.

(**5.6.1**) The recursion for rows of $P(W', W)$ above the root in the poset (acyclic digraph/Bruhat) order may be given as follows (see [142] for equivalent constructions). To compute the row p_A for alcove A, assuming all lower cases known, we first compute another polynomial for each alcove D, p'_{AD}, also denoted $p'_A(D)$ as follows. (Actually $p'_A(D)$ can depend on the choice made next in the computation, but p_A does not and we supress this dependence in notation.)
(**I**) Pick an edge (B, A) in G_a ending at A (so p_B is known). For each alcove D let $\Gamma^{\pm}_D = \Gamma^{\pm}_D(B, A)$ be the set of alcoves D' of G_a such that (D', D) (resp. (D, D')) is an edge in the orbit of the edge (B, A). That is,

$$\Gamma^+_D = \Gamma^+_D(B, A) \;=\; \{D' \mid (D', D) \sim (B, A)\}$$

which says that if Γ^+_D non-empty then it contains a D' obtained by reflecting 'down' from D; and complementarily for Γ^-_D. (By (5.3.15) we can express $(B, A) = (B, Bs)$, $s \in S'$, whereupon any such D' must obey $(D', D) = (D', D's) = (Ds, D)$ (respectively $(D, D') = (D, Ds)$).) Then

$$p'_A(D) = \sum_{D' \in \Gamma^+_D} (v^{-1} p_B(D) + p_B(D')) + \sum_{D' \in \Gamma^-_D} (v p_B(D) + p_B(D')) \tag{5.9}$$

eq:pKL recursion

$$= \begin{cases} v^{-1} p_B(D) + p_B(D') & \exists D' = Ds < D \\ v p_B(D) + p_B(D') & \exists D' = Ds > D \\ 0 & \text{otherwise} \end{cases}$$

(As noted there is at most one edge in the orbit of (B, A) involving any alcove D. Thus at most one of these sums is non-trivial, and that contains only one entry.)
Remark. Another way to read this is that we get p_A from p_B as follows. Keeping in mind that $(B, A) = (B, Bs)$, then whenever there is an entry $p_B(D)$ and a pair D, Ds then there are entries $p_A(D)$ and $p_A(Ds)$, given as follows. If $Ds > D$ (i.e. the s-labelled reflection wall reflects 'up' from D) then $p_A(Ds) = p_B(D)$ and $p_A(D) = v p_B(D)$. If $Ds < D$ (i.e. the s-labelled reflection wall reflects 'down' from D) then $p_A(Ds) = p_B(D)$ and $p_A(D) = v^{-1} p_B(D)$.

Examples: In particular (B, A) is in its own orbit, so $\Gamma^+_A = B$ and $\Gamma^-_A = \emptyset$ and so $p'_A(A) = v^{-1} p_B(A) + p_B(B) = 0 + 1$; and $\Gamma^+_B = \emptyset$ and $\Gamma^-_B = A$ and so $p'_A(B) = v p_B(B) + p_B(A) = v + 0$.

(**II**) To obtain the row of P that we want from p'_A it is then necessary to perform a subtraction in case the evaluation $p'_A(D)(v = 0)$ is non-zero for any $D < A$:

$$p_A = p'_A - \sum_{D<A} p'_A(D)(v = 0)\; p_D \tag{5.10}$$

eq:pKL recurse2

(**5.6.2**) Theorem. *Let W' be a Coxeter system and W a parabolic as above. Then for $A \in \mathcal{A}^+$ row p_A of $P(W'/W)$ does not depend on the choice of B such that $A = Bs$.*

Proof. See [142] and references therein. □

(**5.6.3**) In order to work with this recursion rule in any given alcove geometry it is necessary to be able to manipulate the graph G_a and its edge orbits efficiently. In Section 20.2 we set up the requisite machinery for the case $\mathcal{D}/\mathcal{D}_+$, where $\mathcal{D}_+ = \langle (ij) \rangle_{i,j \in \mathbb{N}}$ (a reflection group on $\mathbb{R}^{\mathbb{N}}$) and $\mathcal{D} = \langle (ij), (ij)_- \rangle_{ij}$.

5.6.2 Example

(**5.6.4**) EXAMPLE. Here we look at the case $\mathcal{D}^-/\mathcal{D}_+$ (what one might call 'type-B'). As already noted, Fig.5.5 gives the graph G_a in this case. We will totally order alcoves by the binary count (of the binary sequence corresponding to the alcove via its $\mathtt{P}(\mathbb{N})$ label), thus 0, 1, 2, 12, 3, 13, 23, 123, 4, ... and so on. We first run the recursion explicitly, and then look at solutions.

(**5.6.5**) We have $p_{00} = 1$. That is, $p_0 = (1, 0, 0, ...)$. We have $(1)_-0 = 0(1)_- = 1$ so

$$p_{11} = 1 \qquad \text{and} \qquad p_{10} = u$$

(Caveat: we get our rows and columns mixed up!) That is

$$p_1 = (u, 1, 0, 0, ...)$$

Comparing with the general recursion we are taking $B = 0$ here, then since it only has one non-zero entry we need only consider edges out of 0. Our next step up is again forced: $1(12) = 2$. Of the alcove labels for non-zero entries in p_1 only 1 has an edge (12), so

$$p_2 = (0, u, 1, 0, 0, ...)$$

From here we can get p_{12} and p_3. For p_{12} we have $2(1)_- = 12$. Two of the alcoves with non-zero entries in p_2 have $(1)_-$-edges. Specifically, fixing $(B, A) = (2, 12)$ we have $\Gamma_0^+ = 1$, since $(1, 0)$ (note the reverse order) is in the orbit of $(2, 12)$ via $12(1)_-(12)(1)_-(12) = 0$ and $2(1)_-(12)(1)_-(12) = 1$. Note that $\Gamma_0^+ = \emptyset$, so $p'_{12}(0) = up_2(0) + p_2(1) = u$. Meanwhile $\Gamma_1^- = 0$, so $p'_{12}(1) = u^{-1}p_2(1) + p_2(0) = 1$. Overall we get

$$p'_{12} = (u, 1, u, 1, 0, 0, ...) \qquad \text{so} \qquad p_{12} = (0, 0, u, 1, 0, 0, ...)$$

Similarly $p_3 = (0, 0, 0, u, 1, 0, 0, ...)$.

For p_{13} we have $12(23) = 13$. Setting $(B, A) = (12, 13)$ we have $(2, 3)$ in the same orbit via $2(1)_- = 12$ and $3(1)_- = 13$. That is, $(2, 2(23)) \sim (12, 12(23))$. Thus $\Gamma_2^+ = \emptyset$, $\Gamma_2^- = 3$, giving

$$p_{13}(2) = up_{12}(2) + p_{12}(3) = u^2 + 0,$$

and $\Gamma_3^+ = 2$, $\Gamma_2^- = \emptyset$, giving $p_{13}(3) = up_{12}(3) + p_{12}(2) = 0 + u$ and so on, giving $p_{13} = (0, 0, u^2, u, u, 1, 0, 0, ...)$. The beginning of the array is in Fig. 5.6.

(**5.6.6**) EXERCISE. Check p_{23} and p_{123}. We have $23 = 13(12)$.

(**5.6.7**) Our exercise in continuing this will be to bring together the existing treatments in Boe [11] (which computes inverse KL polynomials — i.e. the entries in the inverse array to P) and [110], which requires slightly different notation from the former, in order to treat the large n limit. [1] Our characterisation of dominant elements as subsets of $\mathbb{N}$ gives us a binary sequence representation (write 101 if 1 and 3 are positive, etc. — this is equivalent to an '$\alpha\beta$' notation of Lascoux–Schutzenberger [92]).

(**5.6.8**) The *transpose* or reverse of a binary sequence w is the sequence written in reverse order. The *star* of w is w with occurences of 0 and 1 swapped. The *conjugate* of a binary sequence w is

[1] We consider descending sequences in $(-1, -2, ...)$ rather than $(n, n-1, ..., 1)$ as in Boe. We write binary sequences $w = w_1w_2...$ rather than $w = w_nw_{n-1}...w_1$ as in Boe. We write $w_i = 0$ if $-i$ appears in the descending sequence.

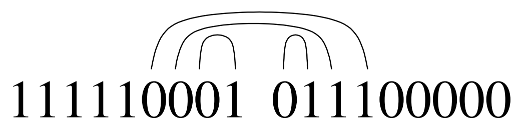
	–	1	2	12	3	13	4	23	14	5	123
	0	1	01	11	001	101	0001	011	1001	00001	111
0	1										
1	u	1									
01		u	1								
11			u	1							
001			u		1						
101			u^2	u	u	1					
0001					u		1				
011		u^2	u			u		1			
1001					u^2	u	u		1		
00001							u			1	
111	u^2	u	u^2	u	u	1		u			1

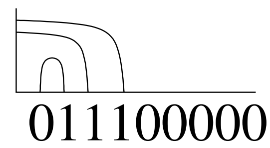
Figure 5.6: Matrix of KL polynomials. fig:KL-HS3matrix

the star-transpose sequence $w^\dagger$. If w is a binary sequence we write w^t for the reverse sequence, so that $w^t w$ is a (possibly doubly infinite) palindromic sequence.

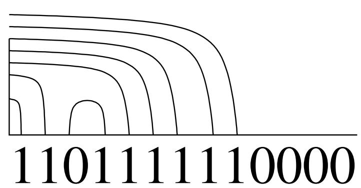
(**5.6.9**) We associate a TL 'half-diagram' $T(w)$ to each binary sequence $w^\dagger w$ (each '*-palindromic' sequence) as follows. We join the elements of each 01 subsequence with an arc (note that every 01 sequence in w begets an 01 sequence in $w^\dagger$, so that this construction is left-right symmetric). Note that if $w_1 = 1$ then in particular we join this with its image 0 in $w^\dagger$. Now ignoring the subsequences covered by arcs so far, we arc each 01 subsequence in (the remainder of) $w^\dagger w$. (Some of these may be parts of matching pairs left and right. Some may arc over *from* left to right.)

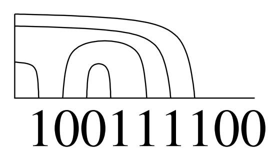
In the infinite finitary case every w has a tail of 0s on the right, so $w^\dagger$ has a tail of 1s on the left. These are not arc'ed.
(**5.6.10**) EXAMPLE.



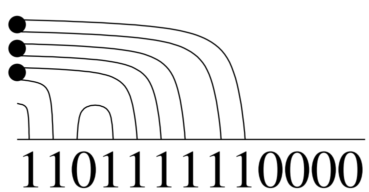


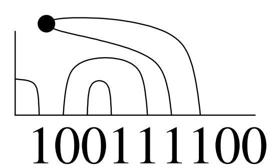
We claim that these TL diagrams are palindromic. Therefore it is enough to draw only the right-hand side, the 'quarter-diagram' as indicated in the figure. Some more examples:





This is Martin's construction in [110]. (Strictly speaking, Martin's construction applies the 'blob' map [116, 112] to symmetric arcs, paired from the left end of the right-half diagram. For our present purposes this is a 'type-D' move, whereas we want a 'type-B' move. So we should leave the first 1 on the left alone, and pair from the left thereafter:





and so on.) To make contact with Boe [11] we should take the dual graph in the sense of (5.1.12), which for TL diagrams is a rooted tree.

(**5.6.11**) A binary sequence is called a *Catalan* binary sequence if the running total of 1s exceeds the running total of 0s. In the case of finitary large n, where every *-palindromic sequence can be considered to begin with many 1s, every sequence trivially has the property that the running total of 1s exceeds the running total of 0s. The *-palindromic sequences are thus a subset of the Catalan sequences. The combinatorics of 'type-A' binary sequences (Catalan binary sequences in general)

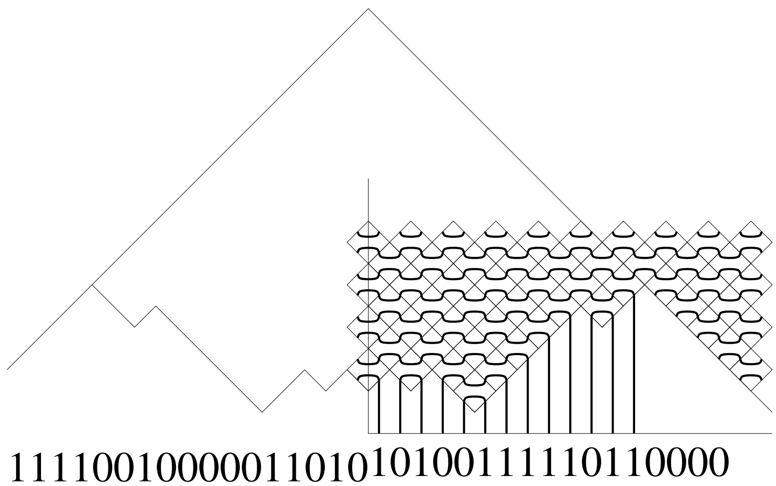


Figure 5.7: *-palindromic binary sequence, Dyck path, and TL diagram via tile map. fig:tile1

covers well-known bijections with many other sets (see Stanley's Catalan notes for example [?], or Marsh–Martin [?]).

Boe uses the correspondence with Dyck paths on $\mathbb{Z} \times \mathbb{Z}$ (Stanley's sequence 17(g), and see below), for example. As Boe notes, this leads to a simple statement of the Bruhat order (**??**) on dominant alcoves ($w > y$ if y can be obtained from w by removing diamond tiles). However it also leads to a simple construction of the associated TL diagrams (Stanley's sequence 17(j)), via the tile map as in Fig.5.7.

Note that the upper envelope Dyck path in the figure is the path for ...111000... which is the *-palindrome of 000..., which is the sequence for the Bruhat-lowest element. Thus we have 0100... > 1000... > 0000... and so on.

(**5.6.12**) Exercise. Recast Boe's construction for inverse Kazhdan–Lusztig polynomials using TL diagrams.

(**5.6.13**) Next we construct hypercubes associated with dominant alcoves (i.e. with sequences) as follows. For each sequence w we define the set of sequences obtained by flipping 01 to 10 across an arc involved in the TL quarter-diagram of $T(w)$. This includes the 'generator' arcs associated to the right actions of $(1)_-$ and $(i\ i{+}1)$ (the lowest arcs), and also any arcs that cover these. Note that the $(1)_-$ arc flip appears to flip an initial 1 to 0 if only the original sequence (the right-hand half) is considered.

What happens to long symmetric arcs (i.e. after the first symmetric arc, which might be in position 1, or might be later if there are 01 pairs to the left in the quarter-diagram)??? Do we have 11 to 00? I guess so. Consider our direct calculation of p_{123} for example.

We perform all possible subsets of flips, so that, if there are r arcs (not counting any odd long-symmetric arc; and only counting paired arcs as one) then it is an r-dimensional hypercube, with 2^r vertices.

We claim that the hypercube of alcove A gives the row p_A in the obvious way (if B occurs in level i then $p_A(B) = u^i$, otherwise it is zero).

Example:

...

The matrix is shown in Fig.5.6.

5.6.3 Alternative constructions: wall-alcove

We may associate arrays of Kazhdan–Lusztig polynomials to other orbits of reflection group actions. In particular we may define an array $P^\omega = P^\omega(W', W)$ of polynomials with rows and columns indexed by dominant walls (the singularity-1 /codimension-1 facets). Each such wall is a wall to two dominant alcoves, and they inherit a corresponding partial order. We can compute the arrays P and P^ω by an interlocking recursion that is in some ways simpler than the original (see e.g. [119]).

Start again from $p_{C'} = (1, 0, 0, 0, ...)$. Moving up the order, we compute row p^ω_w of P^ω, for w a wall above alcove A (say), from p_A as follows. Let s be such that A and As are the alcoves on either side of w. We first compute another row of polynomials $p^{\omega'}_w$. The entry $p^{\omega'}_w(x)$ is non-zero only if x is an s-wall (a wall in the orbit of s), whereupon it is obtained from p_A by $p^{\omega'}_w(x) = p_A(B) + v^{-1}p_A(Bs)$, where $B < Bs$ are the alcoves around x. That is, we 'throw' the polynomial from alcove B up onto the wall, and throw the (modified) polynomial from alcove Bs down onto the wall.

To compute p^ω_w we then make the analogous subtraction to (5.10).

Next, to compute p_{As} we start with p^ω_w. We throw each polynomial in p^ω_w up into the adjacent alcove above, and throw v times each polynomial down into the adjacent alcove below (note that these two throws are never both into the same alcove, since no alcove has more than one wall of a given 'colour').

Again it is not clear that this procedure is well-defined (in a number of regards).

(**5.6.14**) Example. To follow.

5.7 Young graph combinatorics

ss:Young diagram

5.7.1 Young diagrams and the Young lattice

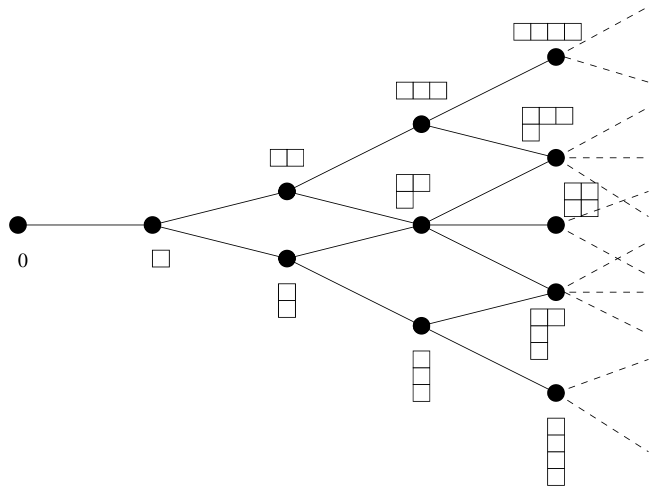
See James–Kerber [77, §1.4,§2.3] for standard terminology for Young diagrams and so on. See also Fulton [49, pp.1-4].

(**5.7.1**) Recall that Λ is the set of all integer partitions and Λ_n the set of integer partitions of degree n. The subpartition order $(\Lambda, \supset)$ is defined by $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all i.

(**5.7.2**) The subpartition order $(\Lambda, \supset)$ is a lattice (the *Young lattice*) — meet is partition intersection and join is union.
In this case λ covers μ (in the sense of (3.4.4)) if the skew λ/μ (see (5.7.8)) is a single box. See Figure 5.8.

de:young graph

(**5.7.3**) The *Young directed graph* $\mathcal{Y}^+$ is the Hasse graph of the Young lattice. The *Young graph* $\mathcal{Y}$ is the underlying (undirected) graph of $\mathcal{Y}^+$.



Younggraph1 Figure 5.8: The Young graph (covering DAG of the Young lattice, increasing from left to right).

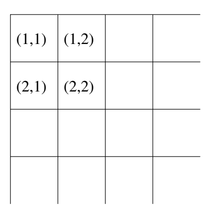
(**5.7.4**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice. We have

$$A(\mathcal{Y}) = A(\mathcal{Y}^+) + A(\mathcal{Y}^+)^t$$

de:yyy (**5.7.5**) Define a map $\mathtt{y} : \Lambda \to \mathtt{P}(\mathbb{N}^2)$ by $\mathtt{y}(\lambda) = \cup_i \{(i,1),(i,2),...,(i,\lambda_i)\}$. Example: $\mathtt{y}(3,1,1) = \{(1,1),(1,2),(1,3),(2,1),(3,1)\}$.

de:lc1 (**5.7.6**) We visualise an integer partition $\lambda = (\lambda_1, \lambda_2, ...) \in \Lambda$ as a *Young diagram* as follows.

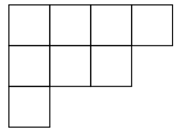
One first pictures the lower right quadrant of the plane partitioned into unit boxes. It is useful to label the boxes by their row and column position in this arrangement — in matrix labelling, as it were; so that the top-left box has label $(1,1)$:



Note that the row-column label only coincides loosely with (rotated) Cartesian coordinates, in which $(0,0)$ would be the top-left corner; and $(1,1)$ would be the bottom-right corner of the box with label $(1,1)$. (We will be vague for now about which box edges are open or closed - in a *diagram* such differences are generally undetectable. For definiteness we might take $(0,1] \times (0,1]$ and so on, noting that the first coordinate increases down; and the second increases to the right.) The set of all these boxes is thus in bijection with $\mathbb{N}^2$. We write $\mathfrak{b}(\mathbb{N}^2)$ for this set; and $\mathfrak{b}(i,j)$ for the corresponding single box. A Young diagram is a certain subset of these boxes, as we now explain.

The Young diagram for partition λ is $\mathfrak{b}(\mathsf{y}(\lambda))$, i.e. it consists of the first λ_i boxes in the i-th row, for each $i = 1, 2, \ldots$. (Hereafter we generally identify λ with its Young diagram.)

Example:

$(4,3,1) \mapsto$ 

(**5.7.7**) We associates to each such box $b \in \mathfrak{b}(\mathbb{N}^2)$ the rectangle of boxes $r(b)$ between it and the top-left box. We define a *light-cone order* on the set of boxes by $b \geq b'$ if $b' \in r(b)$. If $b \geq b'$ we say b *pins* b'.

For B a set of boxes, define

$$r(B) = \bigcup_{b \in B} r(b)$$

A Young diagram is a subset of boxes such that if a box is included, then every box in its rectangle is included. That is, B is a Young diagram if $B = r(B)$.

Note that the identification of a partition λ with the subset of $\mathfrak{b}(\mathbb{N}^2)$ whose i-th row has length λ_i is indeed a Young diagram.

de:skew (**5.7.8**) A *skew* (or skew Young diagram, or skew diagram, or skew shape [49]) of a pair $\lambda \supset \mu \in \Lambda$ is the subset of $\mathfrak{b}(\mathbb{N}^2)$ given by $\lambda \setminus \mu$. This skew is denoted λ/μ.

(**5.7.9**) For any box b there is a minimal $\lambda \in \Lambda$ containing this box, and this coincides with $r(b)$. Given a partition μ and a box, there is a minimal $\lambda \in \Lambda$ containing both. Given a partition μ and a box b, and hence a container λ, the skew λ/μ is called the skew over μ *pinned* by b.

lem:pin1 (**5.7.10**) LEMMA. *Fix a diagram μ. For a set of boxes γ to be a skew λ/μ it must not intersect μ, and must not pin any box outside $\gamma \cup \mu$.*

(**5.7.11**) The *rim* of a diagram λ is the subset of its boxes with the property that the box immediately to the SE is *not* in λ.

Note that the rim is a skew and, in the obvious sense, *connected*.

A *rim hook* of λ is a subset of the rim that is connected. We extend this notion slightly (cf. [77]): A *rim* or *rim hook* in general is any collection of boxes that is a rim hook for some λ.

(**5.7.12**) For each pair of Young diagrams λ, μ there is a skew diagram

$$\lambda \setminus \mu := \lambda/(\lambda \cap \mu)$$

(i.e. such that a box is in $\lambda \setminus \mu$ if it occurs in λ but not in μ); and a skew diagram

$$\lambda \Delta \mu := (\lambda \setminus \mu) \cup (\mu \setminus \lambda) = \lambda \cup \mu/(\lambda \cap \mu)$$

(**5.7.13**) The *dominance order* on Λ_n is the partial order

$$\lambda \leq \mu \iff (\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j \qquad \forall i)$$

5.8 Young graph via alcove geometry on $\mathbb{Z}^{\mathbb{N}}$

ss:YD02

Here we investigate the appearance of the Young graph, and certain subgraphs, within a more homogeneous structure. (This will be useful in representation theory later on.)

5.8.1 Nearest-neighbour graphs on $\mathbb{Z}^n$

(**5.8.1**) We define $\mathbb{Z}^n_{\mathfrak{g}}$ as the graph with vertex set $\mathbb{Z}^n = \mathbb{Z}\{e_1, e_2, ..., e_n\}$ and an edge whenever

$$x - x' = \pm e_i$$

Here $\{e_i\}_i$ could be any set of degree n. If we embed $\mathbb{Z}^n$ in $\mathbb{E}^n$ such that $\langle e_i, e_j\rangle = \delta_{ij}$ then $\mathbb{Z}^n_{\mathfrak{g}}$ is also known as the simple hypercubical lattice in n-dimensions. The edges are between all the nearest neighbour pairs of vertices.

Note that $\mathbb{Z}^n_{\mathfrak{g}}$ is a bipartite and connected graph (in the sense of 3.5.22).

(**5.8.2**) Consider the symmetric group S_n action on $\mathbb{R}^n$ and hence $\mathbb{Z}^n$ given by (5.1) — in which the generator (ij) acts by swapping the corresponding coordinates. This induces an action on $\mathbb{Z}^n_{\mathfrak{g}}$. Note that if there is an edge (x, x') then there is an edge $((ij)x, (ij)x')$. Hence (ij) is a graph automorphism.
(Indeed the set of points in the linear interval between x and x' is taken to the set of points between $(ij)x$ and $(ij)x'$.)

(**5.8.3**) The orbits of the S_n action on $\mathbb{R}^n$ may be represented by the set of (not necessarily strictly) descending sequences.

Alternatively, if we take the euclidean embedding of $\mathbb{Z}^n$ in $\mathbb{E}^n = \mathbb{R}^n$ as above, we may partition $\mathbb{Z}^n$ by the hyperplanes of the reflection group action $\mathcal{D}^n_+ = \langle (ij) \rangle_{ij}$ on $\mathbb{R}^n$ (this is simply S_n realised as a reflection group). The hyperplanes partition $\mathbb{R}^n$ into chambers and facets as in (5.1.4).

(**5.8.4**) The strictly descending sequences of $\mathbb{R}^n$ lie in a single 'dominant' chamber. All descending sequences lie in the closure of that chamber.

For example, with $n = 4$, $(0, 0, 0, 0)$, $(1, 0, 0, 0)$ and $(0, 0, 0, -1)$ lie in the closure of the dominant chamber.

de:Areg

(**5.8.5**) Given the action of $\mathcal{D}^n_+$, a sequence in $\mathbb{Z}^n$ is called *regular* (or A-regular, since $\mathcal{D}_+$ is type-A in the classification of finite crystallographic reflection groups, or *Weyl groups* [68, Ch.2]) if it is not fixed by any group element except the identity.

If a sequence is regular then so is every element of its orbit. The regular orbits may be represented by the set of strictly descending sequences.

For example $(4, 3, -3, -6)$ is a representative element of a regular S_4 orbit in $\mathbb{Z}^4$.

(**5.8.6**) Lemma. *$v \in \mathbb{Z}^n$ is regular if and only if it is singularity-zero.*

(**5.8.7**) We define $\mathbb{Z}^{nA}_{\mathfrak{g}}$ as the following graph. The vertices are the orbits of the S_n action on $\mathbb{Z}^n_{\mathfrak{g}}$ (or equivalently the integer sequences in the closure of the dominant chamber); and there is an edge $([x], [x'])$ in $\mathbb{Z}^{nA}_{\mathfrak{g}}$ if there are representatives x, x' connected by an edge in $\mathbb{Z}^n_{\mathfrak{g}}$.

(**5.8.8**) We define $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$ as the full subgraph of $\mathbb{Z}^{nA}_{\mathfrak{g}}$ on the regular orbits (or equivalently on the integer sequences in the open dominant chamber).

For example $((4,3,-3,-6),(4,3,-2,-6))$ represents an edge in $\mathbb{Z}^{n\,A}_{\mathfrak{g}\,reg}$.

(**5.8.9**) We CLAIM that $\mathbb{Z}^{n\,A}_{\mathfrak{g}\,reg}$ is isomorphic to the full subgraph of $\mathbb{Z}^n_{\mathfrak{g}}$ on vertices in the open dominant chamber.

(**5.8.10**) A walk on $\mathbb{Z}^n_{\mathfrak{g}}$ is A-regular if it visits only A-regular vertices.
For example $(4,3,2,0)-(4,3,2,-1)-(5,3,2,-1)$ is A-regular.

5.8.2 Graphs on $\mathbb{Z}^{\mathbb{N}}$

ss:GrZN

de:RN (**5.8.11**) Define $\mathbb{R}^{\mathbb{N}}$ as the space of real sequences $(x_1, x_2, ...)$. Define $\mathbb{R}^f$ as the subspace of finitary elements. Define $\mathbb{Z}^{\mathbb{N}}$ ('integral points') and $\mathbb{Z}^f$ similarly.
Set $e_i = (0,0,...,0,1,0,...) \in \mathbb{Z}^f$ and

$$w = \frac{-1}{2}(1,1,1,...)$$

which is not finitary or integral (if $x+w$ is integral we say x is *half-integral*). We say an element of $\mathbb{R}^{\mathbb{N}}$ is *dominant* if it is strictly descending.

(**5.8.12**) Define graph $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ with vertices $\mathbb{Z}^{\mathbb{N}}$ and an edge (v,v') whenever $v-v'=\pm e_i$ for some i.

(**5.8.13**) Note that the dot product is not defined for arbitrary pairs in $\mathbb{R}^{\mathbb{N}}$, but is defined for any element of $\mathbb{R}^{\mathbb{N}}$ with any element of $\mathbb{R}^f$. Thus the permutation (ij) may be realised as reflection in a hyperplane; as may the signed permutation $(ij)_-$.

(**5.8.14**) An *integral walk* in $\mathbb{R}^{\mathbb{N}}$ is a sequence of points $x^1, x^2, ... \in \mathbb{R}^{\mathbb{N}}$ such that $x^i - x^{i+1} = \pm e_i$.

(**5.8.15**) Define $\mathfrak{Z}$ as the graph with vertex set $\mathbb{R}^{\mathbb{N}}$ and an edge whenever $x-y=\pm e_i$ for some i. For $x \in \mathbb{R}^{\mathbb{N}}$ define $\mathfrak{Z}(x)$ as the connected component of $\mathfrak{Z}$ containing x. Thus $\mathfrak{Z}(x) \cong \mathfrak{Z}(x')$ and

$$\mathfrak{Z}(0) = \mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}.$$

(**5.8.16**) NB, $y, y' \in \mathfrak{Z}(x)$ implies $y - y' \in \mathbb{Z}^f$.

(**5.8.17**) Here define reflection group actions $\mathcal{D}_+ = \langle \{(ij)\}_{i,j\in\mathbb{N}} \rangle$ and

$$\mathcal{D} = \langle \{(ij), (ij)_-\}_{i,j\in\mathbb{N}} \rangle$$

on $\mathbb{R}^{\mathbb{N}}$. Note that the respective sets $\mathbb{H}_+, \mathbb{H}_\pm$ of hyperplanes corresponding to the given sets of reflections are closed (albeit infinite) in each case.

There follows in (5.8.27) a brief discussion of the hyperfinite and possibly larger reflection groups on $\mathbb{R}^{\mathbb{N}}$. However the characterisation above will be sufficient for our purposes.

(**5.8.18**) CLAIM: If there is an edge (x,x') in $\mathfrak{Z}$ then there is an edge $((ij)x,(ij)x')$, thus $\mathfrak{Z}$ is fixed by $\mathcal{D}_+$. $\mathfrak{Z}(x)$ is fixed by $\mathcal{D}_+$ if and only if every $x_i - x_j \in \mathbb{Z}$.
SOMETHING
$\mathfrak{Z}(0)$ and $\mathfrak{Z}(w)$ are fixed by $\mathcal{D}$.

(**5.8.19**) Define the chamber of $\mathcal{D}_+$ (the chamber of $\mathbb{H}_+$) containing

$$\rho = (0,-1,-2,...)$$

as the *dominant* chamber.

Note that the set of dominant points is the set of points in the dominant chamber.

We call the chambers of $\mathcal{D}$ (the chambers of $\mathbb{H}_\pm$) alcoves. Define the *fundamental* alcove as the chamber of $\mathcal{D}$ containing $(0,-1,-2,...)$.

(**5.8.20**) Here a point $x \in \mathbb{R}^{\mathbb{N}}$ is called *regular* (or A-regular, by analogy with (5.8.5)) if it is not fixed by any element of $\mathcal{D}_+$, i.e. if $x_i = x_j$ implies $i = j$.

Note: The set of points in chambers is the set of regular points. No finitary point is regular.

— Is it possible to step (on an integral walk) from a regular integral point to another in a different chamber? How about from a half-integral point?

(**5.8.21**) Define $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ (similarly to $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$) as the full subgraph of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ on vertices in the dominant chamber.

For v a vertex in the dominant chamber write $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(v)$ for the connected component of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ containing v.

(**5.8.22**) Our other notation: $\mathfrak{Z}_+ = \mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ is the full subgraph of $\mathfrak{Z}$ on vertices in the dominant chamber. For v a vertex in the dominant chamber $\mathfrak{Z}_+(v) = \mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(v)$ is the connected component of $\mathfrak{Z}_+$ containing v.

(**5.8.23**) CLAIM: Define $w, \rho \in \mathbb{R}^{\mathbb{N}}$ by $-2w = (1,1,1,...)$ and $-\rho = (0,1,2,...)$.
(i) Every sequence of form $\rho_\delta = \delta w + \rho$ is A-regular.
(ii) The slowest integral descent (in the obvious sense) from any initial integer γ is $\rho_{-2\gamma}$.
(iii) If $\delta \in 2\mathbb{Z}$ then ρ_δ is a vertex of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$. No two distinct ρ_δs are in the same connected component of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$.
(iv) For each $\delta \in 2\mathbb{Z}$,

$$\mathfrak{Z}^+(\rho_\delta) \cong \mathcal{Y}$$

i.e. $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(\rho_\delta)$ is isomorphic to the Young graph (as defined in (5.7.3)). The isomorphic image of the Young graph is given by $x \mapsto x + \rho_\delta$ for each vertex x in the Young graph.

Proof. (i-iii) are clear. (iv): note that every such $x+\rho_\delta$ is strictly descending and in the connected component. On the other hand every strictly descending element in the component must differ from ρ_δ by a finitary (non-strictly) descending sequence, and hence by an integer partition. □

(**5.8.24**) We define the 'natural' inclusion of $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$ by $(x_1, x_2, ..., x_n) \mapsto (x_1, x_2, ..., x_n, 0)$. We extend this to a natural inclusion of $\mathbb{Z}^n_{\mathfrak{g}}$ in $\mathbb{Z}^{n+1}_{\mathfrak{g}}$.

(**5.8.25**) We define $\mathbb{Z}^f_{\mathfrak{g}}$ as the natural inverse limit of the graphs $\mathbb{Z}^n_{\mathfrak{g}}$.

The 'finitary' graph $\mathbb{Z}^f_{\mathfrak{g}}$ is still bipartite and connected, since for any pair of vertices there is an embedded $\mathbb{Z}^n_{\mathfrak{g}}$ in which they both lie.

(**5.8.26**) We define $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ as the 'infinite' version. For this the vertex set is all integer sequences.

Note that $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ has infinitely many connected components. For example the vertices $(0,0,0,...)$ and $(1,1,1.,,,)$ are not in the same component.

The connected component of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ containing $\mathbb{Z}^f_{\mathfrak{g}}$ is the component containing $(0,0,0,...)$. Note that every other connected component is isomorphic to this one.

rem:infrg

(**5.8.27**) Remark. Consider generalising the partition of $\mathbb{Z}^n$ by orbits/hyperplanes of the S_n action to the case $\mathbb{Z}^{\mathbb{N}}$. Here an example of a descending sequence is $-(0,1,2,3,...)$. A sequence formally in the same orbit is $-(1,0,3,2,...)$ (transpose in pairs). Is $-(1,2,3,...)$ in the same orbit (formally, an infinite cyclic shift)? There are some choices to be made about how one defines the orbits of the reflection group action (one could restrict orbits to sequences related by reflection group elements of finite length, say, so that $-(0,1,2,3,...)$ and $-(1,0,3,2,...)$ are not in the same orbit).

On the other hand, the partition into chambers generalises relatively usefully.

We note that two sequences are only in the same connected component of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ if they are different in finitely many places. This means that if they are in the same connected component *and* in the same orbit then they are necessarily related by a *hyperfinite* group element (i.e. an element in the hyperfinite subgroup of the infinite reflection group — the inductive limit of finite group inclusions). This will be sufficient for our purposes (even though we do not entirely restrict to the hyperfinite graph).

(**5.8.28**) Remark. We might like to try to define $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}}$ similarly to $\mathbb{Z}^{nA}_{\mathfrak{g}}$. Vertices would be integral points in the closure of the dominant chamber.

What about edges?:

The points $(0,0,0,...)$ and $(-1,0,0,0,...)$ are connected in the underlying graph, but $(-1,0,0,0,...)$ does not lie in the closure of the dominant chamber. The image (y, say) of $(-1,0,0,0,...)$ in the closure of the dominant chamber is not a finitary sequence. That is, $(0,0,0,...)-y$ is not polynomial. Further y is not adjacent to $(0,0,0,...)$. That is, there is no $i \in \mathbb{N}$ such that $(0,0,0,...)-y = \pm e_i$.

We shall not need to resolve this obstruction.

Part II

Second Pass

Chapter 6

Basic Category Theory

The good formulation of an exposition of category theory depends somewhat on what is important to the user. So there is not a single 'right' formulation. Even within these notes, categories are being used in multiple radically different ways. A reasonable learning journey is to work through several different expositions and allow one's overview to emerge from this process. (So, in conjunction with these notes, perhaps also read §1.7, some of Adamek [1], Jacobson [72], Mac Lane [97, **?**] and/or Freyd [**?**].)

As for example in Anderson–Fuller [**?**] we here define $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\mathbb{N} = \{1, 2, ...\}$.

6.1 Categories I

ss:cati

There are various equivalent definitions for the notion of a category. Sometimes it is written as a 5-tuple, sometimes as a 3-tuple, and sometimes in other ways. Cf. for example [1, 72, 97] and references therein. In our setting these differences are no more than organisational choices, so we can start with any one of the formulations.

(**6.1.1**) Remark. Here is a place where we engage relatively directly with some fundamental notions in mathematics (that we have largely hidden or taken for granted until now). 'Fundamental' in the sense that they are not defined in terms of other notions. Thus we must deal with various kinds of 'collections' of 'elements'. For our purposes there are essentially two notions: sets and classes. For much of the time we need to work with collections too big to be sets - hence classes. But also for much of the time the distinction is not central to our study here, and we can leave the notion as 'collection'.

For a collection C then $a \in C$ means that a is an element of C, essentially just as if C is a set.

de:cat

(**6.1.2**) A CATEGORY is given by a triple

$$A = (\mathrm{Ob}\, A,\ \hom_A(-,-),\ \bullet)$$

where
(i) $\mathrm{Ob}\, A$ is a collection; [1,2]

[1](The possible failure of this collection to be a set will not be of central concern here [19]. If we wish to emphasise that it *is* a set, we will say that the category is *small*.)

[2]The notation $\mathrm{Ob}\, A$ is used, for example, in [72]; the notation $\hom_A(-,-)$ is used, for example, in [72] and in

Elements of the collection $\mathrm{Ob}\, A$ are called 'objects'.
(ii) for each ordered pair (M,N) of objects then $\hom_A(M,N)$ is a set [3] of 'morphisms' — one sometimes writes $A(-,-)$ for $\hom_A(-,-)$;
(iii) for each ordered triple (L,M,N) of objects then $\bullet$ is a composition

$$\bullet : A(M,N) \times A(L,M) \to A(L,N)$$

(written $\bullet(a,b) = a \bullet b$) such that for each object M there is an 'identity' element $1_M \in A(M,M)$, i.e. $1_M \bullet f = f$, $f \bullet 1_N = f$ whenever defined; and $f \bullet (g \bullet h) = (f \bullet g) \bullet h$ ('associativity') whenever these compositions are defined.

(**6.1.3**) One sometimes writes Ob_A for $\mathrm{Ob}\, A$. Writing A for $\mathrm{Ob}\, A$ should be avoided where possible.

`Set` (**6.1.4**) EXAMPLE. Consider the triple $\mathbf{Set} = (\mathrm{Ob}_{\mathbf{Set}}, \hom_{\mathbf{Set}}(-,-), \circ)$ where $\mathrm{Ob}_{\mathbf{Set}}$ is the collection of all sets; and, for $M,N \in \mathrm{Ob}_{\mathbf{Set}}$, $\hom_{\mathbf{Set}}(M,N) = \mathbf{Set}(M,N)$ is the set of maps from M to N; and $\circ$ is the usual composition of maps. The usual composition of maps is associative and has identities, so $\mathbf{Set} = (\mathrm{Ob}_{\mathbf{Set}}, \mathbf{Set}(-,-), \circ)$ gives a category.

(**6.1.5**) EXAMPLE. Consider $\mathbf{Grp} = (\mathrm{Ob}_{\mathbf{Grp}}, \mathbf{Grp}(-,-), \circ)$ where $\mathrm{Ob}_{\mathbf{Grp}}$ is the class of groups and $\mathbf{Grp}(M,N)$ is the set of group homomorphisms. Then $\mathbf{Grp}$ gives a category.

Given a group G, write $U(G)$ for the underlying set. (We often write just G for the underlying set, but here we will be more careful.) Note that there is a natural inclusion of $\mathbf{Grp}(M,N)$ in $\mathbf{Set}(U(M), U(N))$. But this inclusion depends on the groups, not just the sets.

(**6.1.6**) EXAMPLE. For R a ring R-mod is the category of left R-modules and their homomorphisms.

(**6.1.7**) Let A be a category. Consider a triple A' consisting of (i) any subclass of $\mathrm{Ob}\, A$; (ii) a subset $A'(M,N)$ of $A(M,N)$ for each pair M,N in the subclass, such that $1_M \in A'(M,M)$, and the composition from A closes on these subsets; and (iii) the composition from A. This is a category — a *subcategory* of A.

(**6.1.8**) A subcategory is *full* if every $A'(M,N) = A(M,N)$.

(**6.1.9**) EXAMPLE. Let $\mathbf{Set}^i(M,N)$ denote the restriction of $\mathbf{Set}(M,N)$ to morphisms that are isomorphisms. Thus for example $(\mathbf{Set}^i(\{1,2,...,n\},\{1,2,...,n\}), \circ)$ gives the symmetric group S_n. Let $\mathbf{Set}^i = (\mathrm{Ob}_{\mathbf{Set}}, \mathbf{Set}^i(-,-), \circ)$. This $\mathbf{Set}^i$ gives a subcategory of $\mathbf{Set}$.

`Ab` (**6.1.10**) EXAMPLE. Let $\mathrm{Ob}_{\mathbf{Ab}}$ be the collection of all abelian groups and $\mathbf{Ab}(M,N)$ the set of group homomorphisms from M to N. This gives a full subcategory $\mathbf{Ab}$ of $\mathbf{Grp}$.

(**6.1.11**) EXAMPLE. Let $\mathbf{Set}^f$ denote the full subcategory of $\mathbf{Set}$ with object class consisting of finite sets.

`monoid` (**6.1.12**) PROPOSITION. *Let A be a category and N an object in A. Then the full subcategory of A induced on the single object N consists essentially in the set $A(N,N)$ with its unital associative composition, and hence gives a monoid.*

Conversely any monoid gives a category on (essentially) any single object. □

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[97].

[3](Some workers do not require $A(M,N)$ to be a set either. For them, a category is *locally small* if each $A(M,N)$ is a set.)

(**6.1.13**) Example. Recall $\underline{2} = \{1, 2\}$. The monoid $(\mathbf{Set}(\underline{2}, \underline{2}), \circ) \;= \underline{2}^{\underline{2}}$ is the one studied in Section 4.1.4.

de:genCat1 (**6.1.14**) Let A be a category and S a subset of the class of morphisms. The subcategory *generated* by S is the smallest subcategory of A containing S.

Category A is said to be generated by S if the smallest subcategory containing S is A.

Exercise. Describe the smallest subcategory $\mathcal{C}$ of $\mathbf{Set}^f$ containing all isomorphisms and the inclusion $f : \{1\} \hookrightarrow \{1, 2\}$.
Hints. Note that $\mathcal{C}$ contains all injections of sets of order 1 into sets of order 2.

(**6.1.15**) Remark. As we will see, a category is a very useful notion. But there are some 'handedness' convention choices in the details of the setup above. Thus for example we have the following.

Given a category $A = (\mathrm{Ob}\, A, A(-,-), \bullet)$ one could define a collection of sets $A^r(-,-)$ by $A^r(M, N) = A(N, M)$. Then trivially $\bullet$ can be used to make a composition on $A^r(N, M) \times A^r(M, L)$ to $A^r(N, L)$. That is $(a, b) \mapsto a \bullet b$.

Alternatively one could make from $\bullet$ a composition on $A(L, M) \times A(M, N)$ to $A(L, N)$ by $(a, b) \mapsto b \bullet a$.

With these reorganisations in mind we have the following.

(**6.1.16**) Remark. It is not uncommon to find a 'diagram' variant of (6.1.2)(iii) used instead:

$$\star : A(N, M) \times A(M, L) \to A(N, L)$$

This is ultimately just a matter of organisation — we have reversed the order of writing of all the object pairs. (Reversing the factors in the domain produces an essentially equivalent recasting. Although applying the two changes to a given category does not produce alternative structures with an identical composition. Cf. (6.1.18).)

The first formulation of (6.1.2)(iii) is natural in some settings, and the second formulation in others. In both cases the condition for composibility is a 'matching middle object'. But where the 'middle' naturally appears in composition depends on the context.

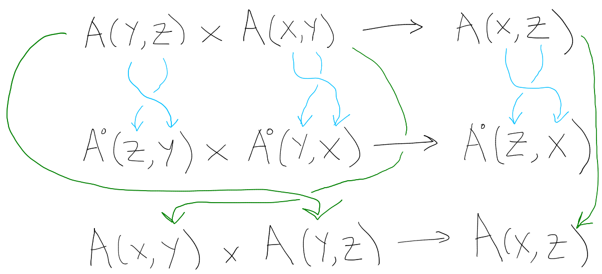
Sometimes we call a category with the second convention a 'diagram category'. But sometimes we simply treat a diagram category as a category.

In the original convention it is natural to think of a morphism as passing from the first object to the second, 'function-wise', with the function symbol written on the left 'acting' on the object on the right... That is, for morphism $f \in \hom(A, B)$ say, i.e. with first object A, we write $f(A)$ for the 'image in B'. In this case composing with a second morphism drawn on the left (as in $(g \circ f)(A)$) we see that g should start in B (and end in C, say). So here the domain of composition is indeed $\hom(B, C) \times \hom(A, B)$ overall.
But in many categories a morphism in $\hom(A, B)$ is not 'passing from' A to B in any meaningful sense (or it may not be natural to write from left to right). For categories of matrices for example (see e.g. (6.1.17)), and for partition categories, the diagram convention is more natural.

de:matcat (**6.1.17**) Example: For R a ring let $\mathbf{Mat}_R$ be the (diagram) category of R-valued matrices. That is $\mathrm{Ob}\, \mathbf{Mat}_R = \mathbb{N}$, and $\hom_{\mathbf{Mat}_R}(m, n)$ is the set of $m \times n$ matrices, and composition is matrix multiplication. See e.g. Adamek [1].

de:opcat (**6.1.18**) Given a category A (with composition $\bullet$, say) there is a Dual category (or opposite category) A^o which is a category with the same objects, and $A^o(M, N) = A(N, M)$, and composition is reversed — that is, if $\star$ denotes composition in A^o then it is given by $f \star g = g \bullet f$.

Notice that if we only reverse the objects (the blue switch below), or only reverse the composition (the green switch), then we turn a category into a diagram category:



While taking the opposite is just a matter of organisation, note that a category and its dual can be very different in local details. For example, $\mathbf{Set}(S, \emptyset)$ is empty unless $S = \emptyset$, while $\mathbf{Set}(\emptyset, S) = \mathbf{Set}^{\circ}(S, \emptyset)$ contains precisely one element for each S (the appropriate empty relation).

(**6.1.19**) Remarks: In a given category we may write

$$M \xrightarrow{\theta} N$$

for $\theta \in A(M, N)$. Then we have 'chains' like

$$M \xrightarrow{\theta} N \xrightarrow{\theta'} P$$

giving

$$M \xrightarrow{\theta' \bullet \theta} P$$

Notice that the θ, θ' are in a sense apparently reversed. (Simply the resolution of a pair of convention choices.)

de:sourcetarget1

Keeping within a fixed category A we may define maps s and t from the class of morphisms to the class of objects such that for $\theta \in A(M, N)$ we have

$$\mathsf{s}(\theta) = M$$

and $\mathsf{t}(\theta) = N$ — the 'source' and 'target' maps.

(**6.1.20**) From the above 'arrow-diagram' perspective, a (small) category is a directed graph with some extra data. To say that a triangle of such homs/arrows

$$\begin{tikzcd} L \arrow[rr, "\mu"'] \arrow[dr, "\phi"'] & & P \\ & M \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes is to say that the long arrow (in the obvious sense) is the composite of the shorter ones. Then associativity says that commutativity of any three of the triangles here:

$$\begin{tikzcd} L \arrow[rrr, "\mu"'] \arrow[dr, "\phi"'] \arrow[drr] & & & P \\ & M \arrow[r, "\theta"'] \arrow[urr] & N \arrow[ur, "\psi"'] & \end{tikzcd}$$

implies commutativity of the fourth.

(**6.1.21**) A morphism $f \in A(M, N)$ is an ISOMORPHISM if there exists $g \in A(N, M)$ such that $gf = 1_M$ and $fg = 1_N$.

(Note that we sometimes write gf for the category composition of g and f, where no ambiguity arises.)
Example: The isomorphisms in $\hom_{\mathbf{Set}}(\underline{n}, \underline{n})$ form a submonoid which is a subgroup — the symmetric group S_n.

brauerexmoBr (**6.1.22**) EXERCISE. Consider

$$\mathbf{Br} \;=\; (\mathbb{N}, \mathbf{Br}(-,-), \circ)$$

where $\mathbf{Br}(m, n) = J_{m,n}$ and $\circ$ is partition composition (both from (3.2.21)). Show that **Br** is a (diagram) category.

(**6.1.23**) A category A is *locally finite* if every $A(M, N)$ is finite. For example **Br** is locally finite.

On set products and product categories

Product categories as defined below are a useful tool. But their constructions contains some choices that can become a little hidden in the process. One of our mottos is to 'control choice', so we proceed by recalling some set theory.

de:prdcat (**6.1.24**) Consider a set I and for each $i \in I$ a set S_i. We note that this notation is already not harmless. From it we have a collection $\{S_i\}_{i \in I}$ of sets — here the sets are not necessarily intrinsically distinct but we may allow the label to distinguish them.

Then $\times_{i \in I} S_i$ denotes the set of I-tuples, i.e. the set of functions from I mapping i to an element of S_i. Thus for example if I has order 2 then the product is a kind of *unordered* Cartesian product, although in practice we might pick a total order on I (for I of any size) and thus write elements of $\times_{i \in I} S_i$ as $(a, b, ...)$.

Notice that the standard notation $S_1 \times S_2$ orders the sets — arbitrarily reading from left to right as we tend to do here! And notice that the obvious isomorphism $S_1 \times S_2 \to S_2 \times S_1$ given by $(a, b) \mapsto (b, a)$ is not the identity even when $S_1 = S_2$ as sets.

'Picking a total order' on I is easier to control if I is finite or countable than otherwise. Confer for example Kamke [**?**]. We will return to the control of this choice later, with regard to its role in the following.

pa:prod cat (**6.1.25**) Now consider a set I and a collection $\{C_i\}_{i \in I}$ of not necessarily distinct categories. The triple, (by an abuse of notation) denoted $\times_i C_i$, consisting of object class $\times_{i \in I} \mathrm{Ob}\, C_i$, corresponding I-tuples of morphisms, and the corresponding pointwise composition, is a category.
(Proof is an exercise.)

As with sets, we write $C_1 \times C_2$ for the ordered version of $\times_i C_i$ in case $I = 1, 2$ with the obvious order.

6.1.1 Functors

ss:functors

(**6.1.26**) CONCRETE CATEGORY. If there is a map $\mathrm{und} : \mathrm{Ob}\, A \to \mathrm{Ob}\, \mathbf{Set}$ from the object collection of category A to **Set**, such that $A(M, N) \subseteq \mathbf{Set}(\mathrm{und}(M), \mathrm{und}(N))$ for each $M, N \in \mathrm{Ob}\, A$, with $1_M = 1_{\mathrm{und}(M)}$, and composition is the usual composition of maps, then A is a *concrete category.*

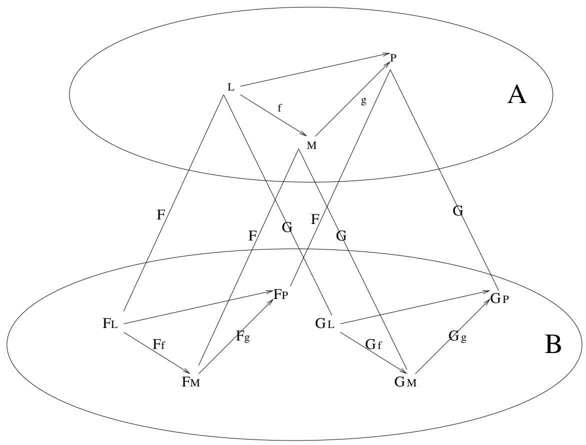


Figure 6.1: Schematic for functors F, G between categories A and B, acting on a commuting triangle in A. A functor preserves commuting triangles (and identities).

:functordiagram1

(**6.1.27**) Examples: Categories **Ab**, **Grp** are concrete categories for which 'und' is simply to 'forget' the extra structure (so a group passes to its *und*erlying set).
Category **Br** is not a concrete category by inclusion (indeed its objects are not directly underlain by sets, although this aspect of the obstruction is easily defeated).

(**6.1.28**) For A, B categories, a (covariant) FUNCTOR $F : A \to B$ is a map on objects together with a map on morphisms which preserves composition and identities. (See Fig.6.1.)
The term 'map' here essentially means 'function' — except only that the domain and codomain may be classes rather than sets; and the 'preservation' means that the map on morphisms is a collection of functions.
Notationally we sometimes write F_0 for the map on objects. Thus formally

$$F_0 : \mathrm{Ob}\, A \to \mathrm{Ob}\, B.$$

And write F_1 for the collection of maps on morphisms:

$$F_1 : A(x, y) \to B(F_0(x), F_0(y))$$

with domain determined by context. But at other times (for example in the Figure) we may simply write F for each and all of the maps, with all details determined by context.

A CONTRAVARIANT FUNCTOR from A to B is a functor from A^o to B (examples later).

pa:func (**6.1.29**) Examples: (I) As noted, **Grp** is concrete. Thus there is an 'und' functor $U : \mathbf{Grp} \to \mathbf{Set}$.
Indeed for each $n \in \mathbb{N}$ there is a 'pointwise' functor $D_n : \mathbf{Grp} \to \mathbf{Set}$ given by

$$D_n(A \xrightarrow{f} B) = A^n \xrightarrow{f^n} B^n$$

where $f^2(a,b) = (fa, fb)$ and so on.

(II) For A any category, the identity object map $1_A(a) = a$ and identity morphism map $1_A(f) = f$ give a functor $1_A : A \to A$.

(III) For A any category, the identity object map and identity morphism map give a contravariant functor $1^o_A : A \to A^o$.

(**6.1.30**) Let A, B, C be categories and $F : A \to B$ and $G : B \to C$ be functors. The composite of two such functors (in the obvious sense) is a functor $GF : A \to C$. That is, for $a \in \mathrm{Ob}\, A$ then $(GF)_0(a) = G_0(F_0(a))$ and so on.

Indeed this composition inherits associativity from the underlying compositions of functions. Thus we may form, for example, a category of (small) categories and functors.

Exercise: The composite of two composable contravariant functors is a functor. The composite of two composable functors exactly one of which is contravariant is a contravariant functor. Thus given a functor $F : A^o \to B$ we can build a contravariant functor $F1^o_A : A \to B$.

p:morexf

(**6.1.31**) More examples: (The following simple examples will come up again later, when we develop the notions of *natural transformation* and of *adjoint pairs* of functors.) Let S be a set. Then there is a functor $F_S : \mathbf{Set} \to \mathbf{Set}$ given by $F_S(T) = T \times S$ and

$$F_S(T \xrightarrow{f} T')(t, s) = (f(t), s)$$

And a functor $F^S : \mathbf{Set} \to \mathbf{Set}$ given by $F^S(T) = \hom(S, T)$ and $F^S(f) : g \mapsto f \circ g$.

(**6.1.32**) Remark. (i) There is a version of F^S for any category C, that is $F^S : C \to \mathbf{Set}$. We take $F^S(X) = C(S, X)$ and $F^S(X \xrightarrow{f} Y) : C(S, X) \to C(S, Y)$ is given by $g \mapsto fg$.
(ii) Some workers write $C(S, -)$ for F^S. It will be clear from (i) that there is a contravariant version of F^S, denoted $C(-, S) : C^o \to \mathbf{Set}$, given by $C(-, S)(X) = C(X, S)$ and f as above taking a map g in $C(X, S)$ to gf in $C(Y, S)$.
(iii) Bifunctor $C(-,-) : C^o \times C \to \mathbf{Set}$ is defined in the obvious way.

exe:preadj

(**6.1.33**) Exercise. Since $F^S(T) = \hom(S, T)$ is a set as well as a hom set, we may consider the hom set $\hom(U, F^S T) = \hom(U, \hom(S, T))$ in $\mathbf{Set}$. Let U', T' be two further sets. A pair of maps $u' : U' \to U$ and $t : T \to T'$ define a map from $\hom(U, F^S T)$ to $\hom(U', F^S T')$ by

$$g \mapsto (u, F^S t)(g) = (F^S t) \circ g \circ u'$$

(note the direction of the map u'!).
Show that this gives rise to a functor

$$\hom(-, F^S -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(cf. (6.1.25)), and construct an analogous functor

$$\hom(F_S -, -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(Remark: Functors from products are sometimes called bifunctors.)
Answer to last part: the map from $\hom(F_S U, T)$ to $\hom(F_S U', T')$ is given by $f \mapsto (F_S u', t)(f) = t \circ f \circ F_S u'$.

(**6.1.34**) A *forgetful functor* is a functor to a category whose objects have some structure (binary operation; inverses; etc) from a category whose objects have this and additional structure. The functor simply forgets the additional structure.

Our 'und' functors are examples of forgetful functors. Another example would be the functor from **Fld** (the category of fields) to the category of integral domains and injective ring maps (call it C), inside the category **Rng** of rings.
(The restriction to injective maps is just because every field homomorphism is injective.)

pa:ind res1 (**6.1.35**) EXAMPLE. Suppose we have an algebra homomorphism $f : A \to B$. Then each B-module can be made an A-module by action $am := f(a)m$. This map lifts to a functor from $B-$ mod to $A-$ mod, called Res$-$ or Res$_f-$.

This means in particular that B itself is a left A-module (while still a right B-module); or a right A-module (while still a left B-module). Each A-module can be made into a B-module by $M \mapsto {}_BB_A \otimes_A M$. (We will say more about tensor products later.) This map also lifts to a functor, called Ind$-$.

6.1.2 Notes and Exercises (optional)

See Figure 6.1 for a schematic representation of functors.

Is every category concrete? No. However:

(**6.1.36**) A *small category* is a category whose object class is a set. Given a small category C, construct an und-functor for C.

Hints: For $x \in$ Ob C define $und(x) = \hom_C(-,x) := \cup_y \hom_C(y,x)$. For $f \in \hom_C(x,y)$ define $und(f) : \hom_C(-,x) \to \hom_C(-,y)$ by $und(f)(h : z \to x) = y \circ h$.
Now check that everything works!

(**6.1.37**) The previous exercise says that every small category is concrete — i.e. it can be equipped with an und-functor. It does not say that there is only one such und-functor.

In each category C the identity element 1_x in each $\hom_C(x,x)$ is uniquely defined. Further any und-functor necessarily obeys $und(1_x) = 1_{und(x)}$ (since all functors do). Can you construct a category with two distinct und-functors?

(**6.1.38**) Suppose we have a triple C which is a candidate to be a category, and Ob C is a set, and the axioms (6.1.2)(iii) are verified except that conditions $f1_x = f$, $1_xf = f$ are only required for $f \in \hom_C(x,x)$. Is this equivalent to the definition of a (small) category?

Hint: consider the triple with Ob $C = \{x,y\}$; $\hom_C(x,y) = \{f,g\}$, $\hom_C(x,x) = \{1_x\}$ and all other hom sets minimal; and $f \circ 1_x = g$.

(**6.1.39**) Recall that a representation of an algebra is, in one sense, an algebra homomorphism from A to some other algebra A'. One good way to study representations is to study the morphisms/intertwiners between them. If we are studying a category C, a functor $F : C \to C'$ to some other category C' is a 'representation' of C. This raises the question of what takes the place of module–morphisms and representation intertwiners for category representations?...

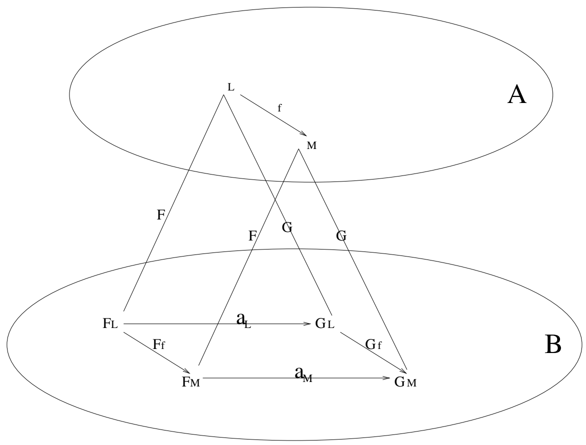


Figure 6.2: Schematic for functors F, G between categories A and B, acting on a map in A. A natural transformation $a : F \to G$ is a collection of maps in B making commuting squares in B as
turaltradiagram1 shown.

6.1.3 Natural transformations

de:natt (**6.1.40**) Let A, B be categories, and T, S be functors from A to B. A 'natural transformation' $a : T \to S$ is a family $a = (a_M)_{M \in A}$ of B–morphisms

$$a_M : TM \to SM$$

such that for each $f \in A(M, N)$ we have $Sfa_M = a_N Tf$. (See also Figure 6.2.)

(**6.1.41**) The functors from A to B are the objects of a category B^A with morphisms the (set of!) natural transformations.

(**6.1.42**) Let $a : T \to S$ be a natural transformation as above, with $T : A \to B$. If the B-morphisms in family a are all isomorphisms then the natural transformation a is a *natural isomorphism*. (We may write $T \cong S$.)

(**6.1.43**) Let $F : A \to B$ and $G : B \to A$ be functors. If $FG \cong 1_B$ and $GF \cong 1_A$ we say that A, B are *category equivalent*.

In §8.10 we will talk about a stronger 'Morita' version of category equivalence, for module categories, in which F, G are also required to be 'additive' (see (6.2.6)).

(**6.1.44**) EXAMPLE. Recall the setup from (6.1.35), which gives us an example of functors back and forth between two categories. Let us abuse notation for a moment and write A and B also for the module categories (of the algebras A and B).

Here (or generally, with two such functors) we could ask if there is a natural transformation between the identity functor $1_B : B \to B$ and $\mathrm{Ind}\,\mathrm{Res} : B \to B$; or between 1_A and $\mathrm{Res}\,\mathrm{Ind}-$.

The existence of a 'natural isomorphism' in either case would have profound implications for the relationship between the module categories.

(**6.1.45**) EXAMPLE. As we have noted, a group is an example of an algebraic system — one with a binary operation. That is, for each group, group multiplication is a function

$$\kappa_G : G \times G \to G$$

The collection of all group multiplications $\kappa = (\kappa_G)$ is thus a candidate to be a natural transformation κ from D_2 to U (the functors **Grp** $\to$ **Set** defined in (6.1.29)):

$$\begin{tikzcd} G \times G \arrow[r, "\kappa_G"'] \arrow[d, "f^2"] & G \arrow[d, "f"'] \\ G' \times G' \arrow[r, "\kappa_{G'}"'] & G' \end{tikzcd}$$

and the commutativity condition $Sfa_M = a_N Tf$ is $Uf\kappa_G(a,b) = \kappa_{G'}(D_2f)(a,b)$ which is simply

$$f(ab) = f(a)f(b)$$

That is, group multiplication (collectively) is a natural transformation.
Other operations in categories of algebraic systems are viewable as natural transformations similarly.

`exa:natiso` (**6.1.46**) EXAMPLE. Recall the functors F_S, F^S from (6.1.31), and $\hom(-, F^S-)$, $\hom(F_S-, -)$ from (6.1.33). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S,U))$ by $(\psi x)(v)(s) = x((v,s)) \in U$. On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v,s) = (y(v))(s)$.

Comparing with (6.1.33) one finds that ψ and ψ' are natural transformations between the functors $\hom(-, F^S-), \hom(F_S-, -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$. For example, for each object (V, U) in $\mathbf{Set}^o \times \mathbf{Set}$ we have $\psi_{U,V}$ such that the diagram

$$\begin{tikzcd} \hom(F_S V, U) \arrow[r, "\psi_{U,V}"] \arrow[d] & \hom(V, F^S U) \arrow[d] \\ \hom(F_S V', U') \arrow[r, "\psi_{U',V'}"] & \hom(V', F^S U') \end{tikzcd}$$

commutes for vertical maps built from any $(f,g) = (V' \xrightarrow{f} V, U \xrightarrow{g} U') \in \hom_{\mathbf{Set}^o \times \mathbf{Set}}((V,U),(V',U')))$.
To see this note that going to the right first we have

$$((\hom(-, F^S-)(f,g))\psi_{U,V})(V \times S \xrightarrow{x} U) = (\hom(-, F^S-)(f,g))(V \xrightarrow{\psi x} \hom(S,U))$$

$$= (V' \xrightarrow{f} V \xrightarrow{\psi x} \hom(S,U) \xrightarrow{F^S g} \hom(S,U'))$$

so this way round the image of x is a map in which $v' \in V'$ is taken to a map which takes s in S to $g(x(f(v'),s))$. The other way round

$$(\psi_{U',V'}(\hom(F_S-,-)(f,g)))(V \times S \xrightarrow{x} U) = (\psi_{U',V'})((V' \times S \xrightarrow{f \otimes 1} V \times S \xrightarrow{x} U \xrightarrow{g} U')$$

which eventually gives the same thing.

(**6.1.47**) As we have said before, we can consider the study of the category B^A to be a very general form of representation theory of category A. A first step in this is to embed A, or at least A^o, in B^A (if B is 'big' enough).

(**6.1.48**) Let C be a category. There is a natural functor

$$F_* : C \to \mathbf{Set}^{C^o},$$

called *Yoneda embedding*, given as follows. Firstly $X \mapsto C(-, X)$. (Here $C(-, X)$ is a functor from C^o to **Set**, and thus an object in $\mathbf{Set}^{C^o}$. It takes an object Y, say, in C^o to $C(Y, X)$ regarded as a set. It takes a morphism $g \in \hom_{C^o}(Y, Z) = C(Z, Y)$, say, to a function from the set $C(Z, X)$ to $C(Y, X)$, mapping $t \in C(Z, X)$, say, to $gt \in C(Y, X)$...) Then $f \in C(X, Y)$ maps to $C(-, f) : C(-, X) \to C(-, Y)$ the natural transformation given by $C(-, f)_T : C(-, X)T \to C(-, Y)T$ (where $C(-, X)T = C(T, X)$) and $g \in C(T, X)$ is mapped to fg.

(**6.1.49**) Exercise. Check that the above *is* a natural transformation.

(**6.1.50**) Exercise. Make an example.

6.2 R-linear and ab-categories

ss:ab

(**6.2.1**) Let R be a commutative ring. An R-linear category is a category in which each hom set is an R-module, and the composition map is bilinear.

A basis for an R-linear category C is a subset $\hom^o_C$ of $\hom_C$ such that

$$\hom^o_C(m, n) = \hom^o_C \cap \hom_C(m, n)$$

is a basis for $\hom_C(m, n)$.

Any category C extends R-linearly to an R-linear category RC.

(**6.2.2**) If C is an R-linear category then each $\hom_C(m, m)$ is an R-algebra.

(**6.2.3**) Remark. A good working aim for this course is to compute the dimensions of the irreducible modules for the $\mathbb{C}$-algebras contained in $\mathbb{C}\mathbf{Br}$ (as defined in Exercise (6.1.22)).

(**6.2.4**) A category C is called an *ab-category* if there is a $+$ operation on each $\hom_C(A, B)$ making it an abelian group; and morphism composition distributes over $+$:

$$f(g + h) = fg + fh \qquad \text{and} \qquad (g + h)f = gf + hf$$

exa:abadd (**6.2.5**) Example: We can define a $+$ for any $\hom_{\mathbf{Ab}}(A, B)$ pointwise:

$$(g + h)(a) = g(a) + h(a)$$

(this defines an element of $\hom_{\mathbf{Set}}(A, B)$, but $(g + h)(a + b) = g(a + b) + h(a + b) = g(a) + g(b) + h(a) + h(b) = (g + h)(a) + (g + h)(b)$, so $(g + h) \in \hom_{\mathbf{Ab}}(A, B)$ as required).
Thus **Ab** is an ab-category.

de:additive (**6.2.6**) A functor $F : A \to B$ between ab-categories is *additive* if for $f, g \in \hom_A(X, Y)$:

$$F(f + g) = F(f) + F(g)$$

6.2.1 Zero object, direct sums, additive categories

(**6.2.7**) If there is an object 0 in a category C such that $|\hom_C(0,A)| = |\hom_C(A,0)| = 1$ for all A then 0 is called a *zero object*.

de:directsum (**6.2.8**) Consider L, M, N objects in an ab-category C. If there are morphisms $a : L \to N$, $a' : N \to L$, $b : M \to N$, $b' : N \to M$ such that $a'a = 1_L$, $b'b = 1_M$ and

$$aa' + bb' = 1_N$$

then we write $N \cong L \oplus M$.
If there is an object $N \cong L \oplus M$ for any two objects L, M we say C *has direct sums*.

pa:ac (**6.2.9**) An *additive category* is an ab-category with direct sums and zero object.

Example: **Ab** with the trivial group as zero object.

(**6.2.10**) Example. Fix a commutative ring K. Consider the (diagram) category $\mathbf{Mat}_K$ of matrices over K, from (6.1.10). Recall that the object set is $\mathbb{N}$. (For now without changing notation) We allow in particular here that 0 is an object — so now the object set is $\mathbb{N}_0$. This object 0 is taken to have the properties of a zero object.

Observe that $\mathbf{Mat}_K$ is an ab-category via matrix addition. For $m \in \mathbb{N}$ we name the singleton element of $\mathbf{Mat}_K(0,m)$ as θ_{0m}. Thus $\theta_{0m} + \theta_{0m} = \theta_{0m}$. We might call these 'improper' matrices — every such matrix has no entries, even though they are formally distinct, with different 'shapes'.

Remark: There is a sense in which we can see the shapes of improper matrices. In §1.1.1 we defined so-called matrix direct sums of 'proper' matrices. We can formally extend this to include the improper matrices. For example

$$1_3 \oplus^{\cdot} \theta_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

— the almost-coincidence of notation (albeit formally resolved by context) will be resolved shortly. Meanwhile

$$M \oplus^{\cdot} \theta_{00} = M = \theta_{00} \oplus^{\cdot} M$$

Next we claim that the *object* needed for $n \cong l \oplus m$ in the sense of (6.2.8) is given by $l+m \cong l \oplus m$. What does this mean here? We are claiming that there are morphisms as follows. Firstly $\mathbf{Mat}_K$ is a diagram category so $a : l \to l+m$ means an element of $\mathbf{Mat}_K(l+m, n)$ and hence we require an $(l+m) \times (l)$-matrix. Then we require $a' : l+m \to l;\ \ b : m \to l+m,\ \ b' : l+m \to m$; such that $aa' + bb' = 1_{l+m}$ where 1_n is the $n \times n$ unit matrix. Altogether, for example with $l = 3$, $m = 2$ we can take:

$$aa' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$bb' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

These choices are far from unique. But it will be clear how to generalise to any l, m.

We see that $\mathbf{Mat}_K$ is additive.

6.2.2 Abelian categories

See for example Freyd's 1964 book [48]. Abelian categories can be regarded as abstractions of the class of module categories, and so are useful in representation theory.

A particular useful notion in a module category is a (short) exact sequence:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

To say that the sequence is exact at A is to say that the image of 0 in A (necessarily the 0 element of the module) is equal to the kernel of f — the subset of elements of A such that $f(\ker f) = \{0\}$. Thus it says that f is an injection of A into B.

These manipulations relied heavily on the nature of a module category. But we can express them in more abstract categorical terms.

(**6.2.11**) Suppose A is an additive category, with category composition denoted $\circ$. Since A is additive it has a zero object (0, say) and hence a zero morphism 0_{MN} in each $\hom_A(M, N)$ — the one that factors through 0.

Example: Fix a commutative ring K and consider **Mat**. This is a diagram category so we have to be careful with composition. Here we have $0_{MN} = \theta_{M0}\theta_{0N}$. This is a product of improper matrices, but since $\theta_{M0} + \theta_{M0} = \theta_{M0}$ we have $0_{MN} + 0_{MN} = 0_{MN}$ and so on, so 0_{MN} is the zero matrix. In $\mathbf{Mat}(2, 3)$ we have

$$0_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(**6.2.12**) For $f \in \hom_A(M, N)$ let K_f denote the set of pairs consisting of an object L and a morphism, $g : L \to M$ say, such that $f \circ g = 0_{LN}$. (Note that this set might be empty.)

eg:Mat22 (**6.2.13**) Example: Consider $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{Mat}(2, 2)$. Putting $L = 2$ we see that $g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{Mat}(2, 2)$ obeys $fg = 0_{22}$. Indeed $f \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = 0_{21}$ for any α. Now consider

$$g' = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \qquad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{pmatrix}$$

Is g' the right way round for K_f? For $fg' = 0$ we need $a = b = c = 0$. For $hf = 0$ we need $h_{11} = h_{21} = h_{31} = 0$.

(**6.2.14**) A *kernel* of $f \in \hom_A(M,N)$ is a pair consisting of an object and a morphism, L, g say, such that $(L,g) \in K_f$ and for every $(L', g') \in K_f$ there is a unique $h : L' \to L$ such that $g \circ h = g'$.

(**6.2.15**) Example. Consider f as in (6.2.13). We have $(1, \begin{pmatrix} 0 \\ \alpha \end{pmatrix}) \in K_f$ for any α; and also $(2,g)$ and $(3,g')$ for any d,e,f. We have

$$\begin{pmatrix} 0 \\ \alpha \end{pmatrix} \begin{pmatrix} \frac{d}{\alpha} & \frac{e}{\alpha} & \frac{f}{\alpha} & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \\ d & e & f & \cdots \end{pmatrix}$$

(for $\alpha \neq 0$) so $(1, \begin{pmatrix} 0 \\ \alpha \end{pmatrix})$ is a kernel, but there are multiple solutions to

$$\begin{pmatrix} 0 & 0 \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

so $(2,g)$ and $(3,g')$ and so on are not.

(**6.2.16**) A *cokernel* is defined by reversing all the arrows in the definition of kernel.

In §6.2.3 we define monomorphisms. Armed with these we will have the following.

(**6.2.17**) An additive category A is an *abelian category* if
(I) every $f \in \hom_A(M,N)$ has a kernel and a cokernel.
(II) every monomorphism is a kernel; every epimorphism is a cokernel.

6.2.3 Monomorphism, Subobjects

ss:mono

(**6.2.18**) Let A be any category. Morphism $f \in \hom_A(M,N)$ is a *monomorphism* if

$$f \circ g = f \circ g' \;\Rightarrow g = g'.$$

(**6.2.19**) An *epimorphism* is defined by reversing all the arrows in the definition of monomorphism.

(**6.2.20**) Let A be any category. Consider the set m_L of monomorphisms $f : M \to L$ with codomain L. We can 'order' these by $f \leq g$ if there is an h such that $g \circ h = f$. This gives an equivalence relation on m_L by $f \equiv g$ if $f \leq g$ and $g \leq f$.

A *subobject* of L is an equivalence class of m_L. The 'order' lifts to a poset structure on the set of subobjects.

6.2.4 Simple objects, Jordan–Holder

(**6.2.21**) Proposition. *If A is abelian then the poset of subobjects of L is a modular lattice.*

(**6.2.22**) From the above ideas we have the idea of a composition chain for L — a chain of subobjects such that each interval is simple. One then has versions of the Schrier Refinement Theorem (§8.3.2) and hence the Jordan–Holder Theorem 1.4.14.

(**6.2.23**) Let A be an abelian category. An object $L \neq 0$ is *simple* if its only subobjects are the class containing $0_L : 0 \to L$ and the class containing $1_L : L \to L$.

6.3 Categories II: On functors, skeletons and adjunctions

ss:catii

(**6.3.1**) A functor $F : A \to B$ is:
full (respectively *faithful*) if all hom set maps

$$F : \hom_A(S,T) \to \hom_B(FS, FT)$$

are surjective (respectively injective);
isomorphism dense if for every object T in B there is an object S in A such that $F(S)$ is isomorphic to T.

(**6.3.2**) A skeleton for a category is a full isomorphism dense subcategory in which no two objects are isomorphic.

(**6.3.3**) Example. The assembly of sets in $\mathbf{Set}^f$ into cardinality classes induces a corresponding set of isomorphisms between hom sets

$$\begin{aligned} f_S : S &\xrightarrow{\sim} S' && (6.1) \\ f : \hom(S,T) &\to \hom(S',T') && (6.2) \\ g &\mapsto f_T \circ g \circ f_S^{-1} && (6.3) \end{aligned}$$

1

Associate a representative element of each class to each cardinality. We may then construct a category $C_{\mathbb{N}}$ whose objects are the set $\mathbb{N}$ of finite cardinalities, and with $\hom_{C_{\mathbb{N}}}(m,n) = \hom(\underline{m}, \underline{n})$. The functor

$$F : C_{\mathbb{N}} \to \mathbf{Set}^f$$

which takes object n to object $\underline{n}$ and identifies the corresponding hom sets is isomorphism dense and full. This $C_{\mathbb{N}}$ is thus a subcategory of $\mathbf{Set}^f$, from which the rest of $\mathbf{Set}^f$ can easily be constructed. We have:

de:CN1

(**6.3.4**) Proposition. *This $C_{\mathbb{N}}$ is a skeleton for $\mathbf{Set}^f$.*

(**6.3.5**) Note that the set of isomorphisms in an end set form a group. The set of isomorphisms in $\hom(\underline{n}, \underline{n})$ form the symmetric group S_n.

(**6.3.6**) A *congruence relation* I on a category C is an equivalence relation on each hom set such that $f' \in [f]_I$ and $g' \in [g]_I$ implies $f'g' \in [fg]_I$ (compositions of morphisms).
The quotient category C/I has the same object class as C but $\hom_{C/I}(F,G) = \hom_C(F,G)/I$, with the obvious composition well-defined by congruence.

6.3.1 Adjunctions

de:adjunction1

(**6.3.7**) An *adjunction* between categories A, B is an ordered pair of functors (G, F) with $F : A \to B$ and $G : B \to A$ such that for all objects (U, V) in $A \times B$ there is a bijection

$$\psi_{U,V} : \hom_A(GV, U) \to \hom_B(V, FU)$$

such that

$$\psi : \hom_A(G-, -) \to \hom_B(-, F-)$$

is a natural isomorphism of bifunctors.
(The bifunctors are $\hom_A(G-,-) : B \times A \to \mathbf{Set}$ and $\hom_B(-,F-) : B \times A \to \mathbf{Set}$.
Remark: For those familiar with induction and restriction functors and Frobenius reciprocity (see later), the order in an adjunction is (ind,res).)

That is, we have

$$\begin{array}{ccc} \hom_A(GV,U) & \xrightarrow{\psi_{U,V}} & \hom_B(V,FU) \\ \downarrow & & \downarrow \\ \hom_A(GV',U') & \xrightarrow{\psi_{U',V'}} & \hom_B(V',FU') \end{array}$$

commutative for each $f \in \hom_{A^o \times B}(V,U)$ (and hence each pair of vertical maps, cf. (6.1.40)).

(**6.3.8**) Example. Recall the functors F_S, F^S from (6.1.31). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S,U))$ by $(\psi x)(v)(s) = x((v,s)) \in U$. On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v,s) = (y(v))(s)$.

Comparing with (6.1.33) one checks that ψ and ψ' are natural transformations (the diagram above commutes for vertical maps built from $\hom_{\mathbf{Set}^o \times \mathbf{Set}}(V,U)$) and hence isomorphisms. Thus (F_S, F^S) is an adjunction.

(**6.3.9**) The left adjoint to a forgetful functor is usually something interesting!

See for example the tensor product, in §8.4.

6.4 Categories III: monoidal categories

See Section 12.10. See also Joyal–Street [83]; and Kassel [84], Reshetikhin–Turaev [135] and references therein.

6.4.1 Tensor/monoidal categories: preliminaries - product functor

ss:tc1

de:tensorprod (**6.4.1**) Let $A = (\mathrm{Ob}\, A, A(-,-), \bullet)$ be a category. Let $A \times A$ be the product category, essentially as in (6.1.24). A functor

$$\mathsf{F} : A \times A \to A$$

is sometimes called a *tensor product*. From such a functor F we have in particular an association of an object $\mathsf{F}(m,n)$ (or $\mathsf{F}_0(m,n)$ if we want to emphasise the object domain notationally) to each pair of objects; and similarly a morphism $\mathsf{F}_1(f,g)$ for each pair of morphisms, hence a collection of binary operations.

eg:Setcp (**6.4.2**) Example. To construct an example here we need an appropriate collection of maps that is formally as in a functor, and then to verify the axioms. This kind of situation is common in construction, so let us call the first step a *formal-functor*.

Consider the formal-functor $\mathsf{F}^\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ given on objects by $\mathsf{F}^\times(A,B) = A \times B$ and on morphisms by $\mathsf{F}^\times(f,g)(a,b) = (f(a), g(b))$. We can check that this gives a functor. To do this we first unpack one of the main conditions in the general case.

(**6.4.3**) Let $A = (\mathrm{Ob}\, A, A(-,-), \mu_A)$ be a category and $\mathsf{F} : A \times A \to A$ a tensor product. Let $ff' = \mu_A(f, f')$ and gg' be well-defined products of morphisms. Recall that the product in $A \times A$ is given by $(f, g)(f', g') = (\mu_A(f, f'), \mu_A(g, g'))$. Note that the functoriality of F implies that

$$\mathsf{F}(\mu_A(f, f'), \mu_A(g, g')) = \mu_A(\mathsf{F}(f, g), \mathsf{F}(f', g')).$$

Notationally we may write $f \otimes g$ for $\mathsf{F}_1(f, g)$, and so on. That is

$$(ff') \otimes (gg') \; = \; (f \otimes g)(f' \otimes g') \tag{6.4}$$ eq:intercha1

(**6.4.4**) Example. In our example above we have $\mathsf{F}^\times(ff', gg')(a, b) = (f(f'(a)), g(g'(b)))$ and $\mathsf{F}^\times(f, g)(\mathsf{F}^\times(f', g')(a, b)) = \mathsf{F}^\times(f, g)(f'(a), g'(b) = (f(f'(a)), g(g'(b)))$ as required.

(**6.4.5**) We continue to think about the example $\mathsf{F}^\times$ in (6.4.2). Note that there is no 'identity' object that would make the formal magma $(\mathrm{Ob}\, \mathbf{Set}, \mathsf{F}_0^\times)$ into a monoid (even a large one!) in this example. Recall here that $\mathsf{F}_0^\times(S, T) = S \times T$. The closest we can get to an identity is to pick an object of order 1, then there are isomorphisms $l_S : \mathsf{F}^\times(\{*\}, S) \to S$ given by $(*, a) \mapsto a$ and $r_S : \mathsf{F}^\times(S, \{*\}) \to S$ similarly.

Also the formal magma $(\mathrm{Ob}\, \mathbf{Set}, \mathsf{F}_0^\times)$ is not associative. But again there are isomorphisms $a_{STU} : (S \times T) \times U \to S \times (T \times U)$ given by $((s, t), u) \mapsto (s, (t, u))$.

Armed with such a collection of isomorphisms, we can ask about 'straightening out' products generally, as in §1.1. Starting with $((ab)c)d$ there are two routes to $a(b(cd))$.

(**6.4.6**) Example. Another possible example is as follows. We use a 'disjoint union' (which always involved choices). Fix $A = \{a < b\}$, an ordered set of two elements, such as $\{0 < 1\}$. We have a formal-functor $\mathsf{F}^\sqcup : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ given by $(S, T) \mapsto S \overset{\sqcup}{\scriptstyle 01} T \; := \; S \times \{0\} \cup T \times \{1\}$ — the corresponding disjoint union; and $(f, f') \mapsto \mathsf{F}^\sqcup(f, f')$ given (for $f : S \to T$ and $f' : S' \to T'$ say, so $\mathsf{F}^\sqcup(f, f') : \; S \overset{\sqcup}{\scriptstyle 01} S' \;\; \longrightarrow \; T \overset{\sqcup}{\scriptstyle 01} T'$) by

$$\mathsf{F}^\sqcup(f, f')(u) = \begin{cases} (f(s), 0) & u = (s, 0) \\ (f'(s), 1) & u = (s, 1) \end{cases}$$

check!

Exercise. What is the smallest submagma of $(\mathrm{Ob}\, \mathbf{Set}, \mathsf{F}_0^\sqcup)$ that contains the single element set $\{*\} \in \mathrm{Ob}\, \mathbf{Set}$?

We have $\mathsf{F}_0^\sqcup(\{*\}, \{*\}) = \{(*, 0), (*, 1)\}$.

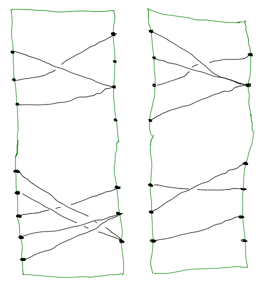
Then

$$\mathsf{F}_0^\sqcup(\{*\}, \{(*, 0), (*, 1)\}) = \mathsf{F}_0^\sqcup(\{*\}, \mathsf{F}_0^\sqcup(\{*\}, \{*\})) \;\; = \{*\} \overset{\sqcup}{\scriptstyle 01} \left(\{*\} \overset{\sqcup}{\scriptstyle 01} \{*\}\right) = \{(*, 0), ((*, 0), 1), ((*, 1), 1)\}$$

and $\mathsf{F}_0^\sqcup(\{(*, 0), (*, 1)\}, \{*\}) = \left(\{*\} \overset{\sqcup}{\scriptstyle 01} \{*\}\right) \overset{\sqcup}{\scriptstyle 01} \{*\} = \{((*, 0), 0), ((*, 1), 0), (*, 1)\}$ and so on.

Are there associator isomorphisms in this case? If S and T are finite ordered sets then $S \overset{\sqcup}{\scriptstyle 01} T$ can be ordered in the obvious way. The size of $S \overset{\sqcup}{\scriptstyle 01} T$ is $|S| + |T|$, so $S \overset{\sqcup}{\scriptstyle 01} \left(T \overset{\sqcup}{\scriptstyle 01} U\right)$ and $\left(S \overset{\sqcup}{\scriptstyle 01} T\right) \overset{\sqcup}{\scriptstyle 01} U$ both have size $|S| + |T| + |U|$. And since they are ordered these orders gives an explicit bijection. The empty set acts as a 'not-quite-identity' here, if required...

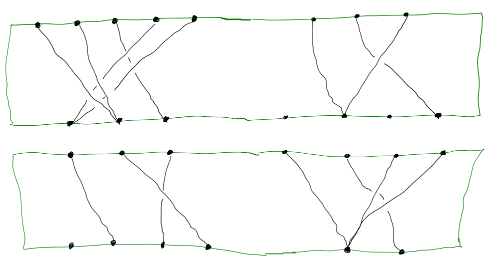
(**6.4.7**) This last example, $\mathsf{F}^{\sqcup}$, has an interesting kind of schematic/diagrammatic incarnation. First we draw morphisms f in **Set** by drawing elements of the domain as vertices on the left, and the codomain on the right, and then edges $(a, f(a))$ left-to-right. An example showing two composable functions, f and g say, is:



Each vertex needs to be labelled by the element it represents. We take this labelling as read for now.

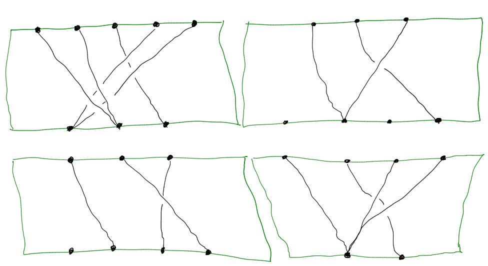
This is a familiar schematic. But there is a lot going on here! The sets have somehow been arranged into their allotted column (a copy of an interval of the real line) in the diagram. This is arbitrary for now, but the arrangement of the codomain of f on the left is consistent with that for the domain of g on the right, so the schematic for composition can be juxtaposition.

The first (trivial) thing we are going to do is rotate the entire schematic to:



This is simply to save space. Now composition is by 'vertical stacking' (here top to bottom).

Now let us draw some more examples, with some different functions 'drawn' side-by-side. The action of drawing side-by-side is arguably philosophically and mathematically rather profound so, while this will look familiar, we need to proceed carefully here. Example:



This becomes a schematic for $\mathsf{F}^{\sqcup}$ if we consider the left box and the right box taken *together* as the disjoint union of the two considered separately. Keep in mind that we are suppressing labels. For example the same label could appear on the left and the right — so although we have drawn the boxes together, we cannot remove the vertical separator lines here.

(**6.4.8**) Questions: What can we say in comparison of $\mathsf{F}^{\sqcup}$ and $\mathsf{F}^{\times}$?
Is $\mathsf{F}^{\sqcup}$ associative? No. But there are some clues here for an 'associative version'.

(**6.4.9**) Claim: Let A be a category. There is a functor $\mathtt{t} : A \times A \to A \times A$ (or $\mathtt{t}_A$ if we want to emphasise the underlying category) given on objects by $\mathtt{t}(a, b) = (b, a)$ and on morphisms by $\mathtt{t}(f, g) = (g, f)$.

If there is a tensor product on A — call it $\otimes$ — then in many cases there is a natural isomorphism $\chi : \otimes \to \otimes\mathtt{t}$ (a collection of isomorphisms all satisfying a commuting square property).

This is true of **Set** with $\mathsf{F}^{\times}$ for example.

In many cases it also happens that this functor and the bracketing of the monoidal composition also 'play well' together. To develop this observation more precisely we first address the bracketing.

Claim (demote): the $\mathtt{t}$ functor itself yields a 'braiding' ...

6.4.2 Strict monoidal category

ss:StMC

(**6.4.10**) Let A be a category and $\mathsf{F} : A \times A \to A$ a functor. If (*unlike* our example in (6.4.2)) there is an object 0 say such that $(\mathrm{Ob}\, A, \mathsf{F}_0, 0)$ is a monoid; and also $(A(-,-), \mathsf{F}_1, id_0)$ is a monoid (i.e. a monoid on all morphisms), then A with F becomes a *strict monoidal category*, denoted (A, F).

(**6.4.11**) Notation: Given such a pair (A, F) we sometimes write $\mathsf{F}(x, y)$ as $x \otimes_A y$, for x, y objects or morphisms, where the symbol $\otimes_A$ here can be declared to be some similar symbol (depending for example on how canonical the choice of F is for underlying category A). And then we sometimes give a strict monoidal category (A, F) as a triple like

$$(A, \mathsf{F}) \;=\; (A, \otimes_A, id_A),$$

giving the underlying category; the symbol for the monoidal product; and the unit of the morphism monoid.

(**6.4.12**) EXAMPLE. Fix a commutative ring K. Consider the category $\mathbf{Mat}_K = (\mathbb{N}, \mathbf{Mat}_K(-,-), .)$ as in (6.1.17), with id_{-} denoting the identity function. (We may write it as a quadruple $\mathbf{Mat}_K = (\mathbb{N}, \mathbf{Mat}_K(-,-), ., id_{-})$.) Define

$$\mathsf{F}_0(m, n) = mn$$

and $\mathsf{F}_1(f, g) = f \otimes g$ the aB-convention Kronecker product, as in §1.1.1. Then $(\mathbf{Mat}_K, \mathsf{F})$ is a strict monoidal category.

The identity object is 1; and the morphism id_1 is the 1×1 unit matrix.

Proof. Exercise. We must verify, for example, the functoriality of F, as made explicit in (6.4).

(**6.4.13**) Alternatively we could define a formal-functor by $\mathsf{F}'(f, g) = g \otimes f$ (the Ab-convention product). This also gives a strict monoidal category, $(\mathbf{Mat}_k, \mathsf{F}')$.

pa:CNprod (**6.4.14**) EXAMPLE. Recall $C_{\mathbb{N}} = (\mathbb{N}_0, \mathbf{Set}^f(\underline{n}, \underline{m}), \circ, id_{-})$ is a skeleton in $\mathbf{Set}^f$. Thus $C_{\mathbb{N}}(0,0) \cong C_{\mathbb{N}}(1,1)$ are both trivial groups. Define $F_0(m, n) = m + n$ and $F_1(f, f') = f \otimes f'$ meaning, for

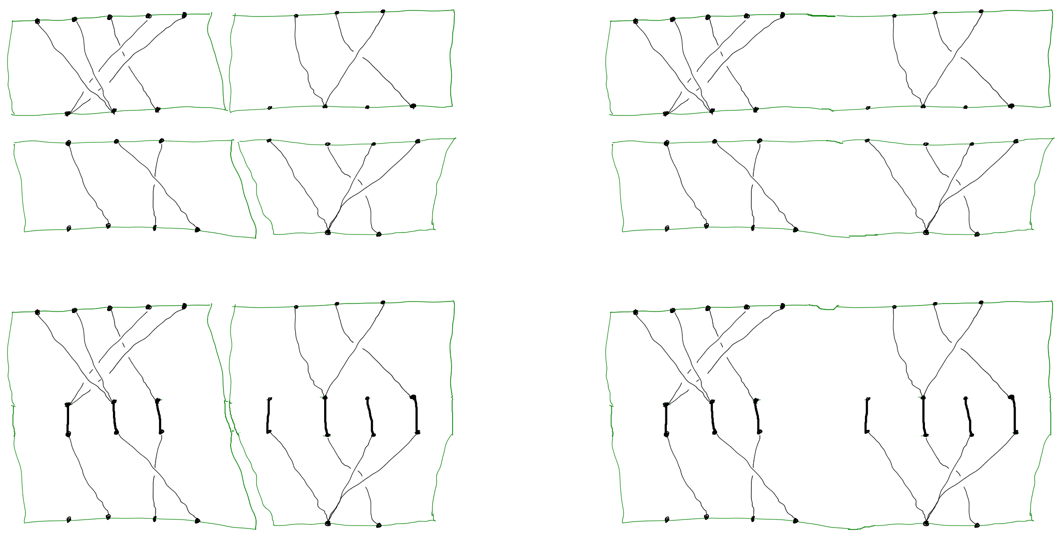


Figure 6.3: Interchange condition for $(C_{\mathbb{N}}, F)$ from (6.4.14), with conventions as given in the text.

fig:interch1

$f \in C_{\mathbb{N}}(m,n)$ and $f' \in C_{\mathbb{N}}(m',n')$ say, the element of $C_{\mathbb{N}}(m+m', n+n')$ obtained by Young's side-by-side concatenation:

$$(f \otimes f')(i) = \begin{cases} f(i) & i \leq m \\ f'(i-m)+n & i > m \end{cases}$$

where, more generally, using s and t for the source and target functions, $m = \mathsf{s}(f)$ and $n = \mathsf{t}(f)$ (see below for an illustration).
Then we claim $(C_{\mathbb{N}}, F)$ is a strict monoidal category. The identity object is 0; and the morphism id_0 is the unique morphism on the empty set.
Proof. We need to show that $\otimes$ is functorial. This is one of many cases in symbiosis with a pictorial realisation. Consider the two sides for the interchange property in an example — in a suitable diagram calculus we have the calculation as in Fig.6.3.

In this figure the vertices are labelled $1, 2, ...$ from left to right. Observe that the top object in the top-left sub-figure is $5 \otimes 3 = 8$ as required. The labelling in order $1, 2, 3, 4, 5||1, 2, 3$ is understood, making use of the available natural order here; and this becomes $1, 2, 3, 4, 5, 6, 7, 8$ in the obvious way.

The upshot is that we arrive at the same bottom-right outcome whether we pass by the down-then-right route or the right-then-down route, so we have functoriality.
It is perhaps worth also dwelling a little on the diagram calculus itself... Here our convention is to draw category composition vertically (unlike earlier, where drawings mimicked in-line functions) — it is in the nature of **Set** that composition is essentially linear, in the sense of sequential, so the choice of orientation is just a convention. Drawing monoidal composition horizontally is more of a

step — it is not clear why such a composition would be linear in general. It plays reasonably with linearity here due to the choice of skeleton...

Exercise: complete the proof. □

(**6.4.15**) We can also make $C_{\mathbb{N}}$ into a monoidal category via $f \overline{\otimes} f' = f' \otimes f$.

de:Scat (**6.4.16**) EXAMPLE. Let $\underline{\Sigma}(n,n)$ denote the symmetric group on $\{1,2,...,n\}$. Thus $\underline{\Sigma}(0,0) \cong \underline{\Sigma}(1,1)$ are both trivial groups. (Recall from (6.3.4) that we sometimes write S_n for the isomorphism class of the group $\underline{\Sigma}(n,n)$. And we could also write $\underline{\Sigma}(n,n) = C^i_{\mathbb{N}}(n,n)$.) Consider the category

$$\underline{\Sigma} = (\mathbb{N}_0, \underline{\Sigma}(-,-), \circ, id_-)$$

(see for example Mac Lane [**?**]). Define $F_0(m,n) = m+n$ and $F_1(f,g) = f \otimes g$ meaning the element of $\underline{\Sigma}(m+n, m+n)$ obtained by Young's side-by-side concatenation:

$$(f \otimes g)(i) = \begin{cases} f(i) & i \leq m \\ g(i-m)+m & i > m \end{cases}$$

Then $(\underline{\Sigma}, F)$ is a strict monoidal category.

The identity object is 0; and the morphism id_0 is the unique morphism on the empty set.

(**6.4.17**) The object monoid of $\underline{\Sigma}$ is $(\mathbb{N}_0, +, 0)$. A *natural* strict monoidal category is a SMC with this property.
Thus for example $(\mathbf{Mat}_K, \mathsf{F})$ is not natural. However see (6.4.18).

pa:MatN (**6.4.18**) EXAMPLE. Now also fix $N \in \mathbb{N}$. The full subcategory $\mathbf{Mat}^N$ of $\mathbf{Mat} = \mathbf{Mat}_K$ on objects of form N^m, with $m \in \mathbb{N}_0$, is a monoidal category with the same F.

We rename the objects from $\mathbb{N}_0$ so the object monoid can be expressed as $(\mathbb{N}_0, +, 0)$, and set $\mathbf{Mat}^N(l,m) = \mathbf{Mat}(N^l, N^m)$.

(**6.4.19**) EXAMPLE. Note that the rows and columns of matrices in $\mathbf{Mat}^N(N^l, N^m)$ are naturally indexed by words of length l and m respectively in the symbols $\{1,2,...,N\}$. Write $\langle w|M|v\rangle$ or M_{wv} for the wv matrix entry. Define the subset $\mathsf{Match}^N(l,m)$ of $\mathbf{Mat}^N(l,m)$ by $M_{wv} = 0$ unless w a perm of v. (We call this 'charge-conserving' property.) Then we have the following.

Proposition. The tuple

$$\mathsf{Match}^N \;=\; ((\mathbb{N}_0, \mathsf{Match}^N(-,-), ., id_-), (+, \otimes)) \;=\; ((\mathbb{N}_0, \mathsf{Match}^N(-,-), ., id_-), (+, \otimes), 0)$$

(sometimes we write a strict monoidal category as a triple including the unit of the object monoid) is a strict monoidal diagonal category.
Proof. We need to verify that Match^N is a category; and then that it is monoidal. For the former consider

$$(M.M')_{uw} = \sum_v M_{uv} M'_{vw} \tag{6.5}$$ eq:Matmult

We need to show that the RHS is nonzero only if u a perm of w. It is a sum of terms $M_{uv}M'_{vw}$ where the first factor is nonzero only if u a perm of v; and the second only if v a perm of w — but a necessary condition here is that u is a perm of w so we are done.

For the monoidal property we first observe that the monoidal unit of $\mathbf{Mat}^N$, which is $id_{N^0} = id_1$, is indeed charge-conserving. Next consider $M \otimes M'$ with $M \in \mathsf{Match}^N(l,l)$ and $M' \in$

$\mathsf{Match}^N(m,m)$. The row and column labels are words in $\{1,2,...,N\}$ of length $l+m$. In particular

$$(M \otimes M')_{vw} = M_{v_1v_2...v_l,w_1w_2...w_l} M'_{v_{l+1}v_{l+2}...v_{l+m},w_{l+1}w_{l+2}...w_{l+m}} \quad (6.6)$$ eq:MatKronecker

Observe now that if $M_{v_1v_2...v_l,w_1w_2...w_l} \neq 0$ then $v_1v_2...v_l$ is a perm of $w_1w_2...w_l$. And similarly if $M'_{v_{l+1}v_{l+2}...v_{l+m},w_{l+1}w_{l+2}...w_{l+m}} \neq 0$ then $v_{l+1}v_{l+2}...v_{l+m}$ is a perm of $w_{l+1}w_{l+2}...w_{l+m}$. So if neither is zero then v is a perm of w, as required. □

Observe that the latter perm is a kind of Young subgroup perm. Exercise: ...(FINISH mE!)

(**6.4.20**) more exercises: make a functor from (some of) Set to Match; compare Match with TL
say something about monoidal subcategories of monoidal categories
say something about diagrams and strictification; about taches; ...

6.4.3 Aside on generators of a strict monoidal category

(**6.4.21**) Fix a strict monoidal category (C,F). Given a subset of morphisms M, we can ask what is the smallest subcategory containing the subset. Let us denote it $\langle M \rangle$.

And we can ask what is a minimal subset of morphisms such that $C = \langle M \rangle$. We call such a subset a generating set.

(**6.4.22**) We can also ask about presentation of strict monoidal categories by generators and relations.

Given a natural strict monoidal category and, say, a single morphism, what morphisms can we build? Obviously given any two morphisms we always have $a \otimes b$. And given two composable morphisms we have ab. We can iterate the process of forming such composites. How does this start?

Let us assume that our object monoid is $(\mathbb{N}_0,+)$. One organisational device is to observe that if we have a generating set M then every morphism can be expressed as a category product of morphisms of the form $id_j \otimes m \otimes id_k$ with $m \in M$. (Such a form is called a whiskered form.)
Proof. Exercise.

Built into the construction we have identity morphisms in every $\hom(j,j)$. We have $id_j \otimes id_k = id_{j+k}$.

(**6.4.23**) Example. The category $\underline{\Sigma}$ is generated by the non-identity morphism σ in $\underline{\Sigma}(2,2)$.

6.4.4 Aside on idempotent subcategories

(**6.4.24**) Given a category C and a fixed collection $(e_X)_{X \in \mathrm{Ob}\, C}$ of idempotent endomorphisms such that $e_X \in C(X,X)$ there is a category C_e given by the same object class and $C_e(X,Y) = e_Y C(X,Y) e_X$. (Note that this automatically gives a closed composition on the subclass of such morphisms.) This is not generally a subcategory because the identity in $C(X,X)$ is e_X. We call such a category an *idempotent subcategory.*

(**6.4.25**) Example. Given a natural strict monoidal category C, with $id_{\_}$ denoting the identity function, and an idempotent $e_1 \in C(1,1)$ then $e_2 = e_1 \otimes e_1 \in C(2,2)$ is an idempotent, and $e_3 = e_1 \times e_1 \otimes e_1 = e_1 \otimes e_2 \in C(3,3)$ (note that we do not need brackets here) and so on. And $e_0 = id_0$ is an idempotent. We claim that the category C_e here inherits the natural strict monoidal

property.
Proof: The object monoid is unchanged from C. The identity map has changed to be given by $id_n = e_n$. The unit of the morphism monoid is id_0, which is unchanged, so it remains to check closure. Consider morphisms f, g in C such that $e_l f e_m$ and $e_n g e_p$ are well-defined and hence in C_e (and note that every morphism in C_e arises this way). By the functoriality of F in C (as in (6.4)) we have $e_l f e_m \otimes e_n g e_p = (e_l \otimes e_n)(f \otimes g)(e_m \otimes e_p) = e_{l+n}(f \otimes g)e_{m+p}$, so the monoidal composition closes in C_e. □

(**6.4.26**) An interesting idempotent subcategory of Match^N is obtained by restricting further, so that $M_{vw} = 0$ if v contains symbol N. For example the element

$$e_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

is in the subset of $\mathsf{Match}^3(1,1)$. To see if this gives a monoidal category we should check the identity and closure conditions as above.
First observe, then, that the identity morphisms are not in this subset. However idempotents like e_1 (and $e_1 \otimes e_1$ and so on) are in. Observe that these act like identities on our subsets.
For the category composition, suppose u in (6.5) contains N. Then each $M_{uv}M'_{vw}$ is zero so $(MM')_{uw} = 0$ as required.
For the monoidal composition, suppose v in (6.6) contains N. If there is an N somewhere in the first l positions then $M_{v_1 v_2 \ldots v_l, w_1 w_2 \ldots w_l} = 0$. If there is an N somewhere in the last m positions then $M'_{v_{l+1} v_{l+2} \ldots v_{l+m}, w_{l+1} w_{l+2} \ldots w_{l+m}} = 0$. Thus the monoidal composition is closed. □

(**6.4.27**) Observe that this category is the idempotent subcategory associated to e_1.

6.4.5 Aside on the category of Match categories

(**6.4.28**) BUT realLY. WHATs going on here. How do we get a good functor from Match^N to Match^{N-1}, say? (And is that what we want/need for representation theory?)

We want such a functor basically to forget all the N stuff. How does this work?? More of a quotient than a sub thing??

pa:RMNN-1 (**6.4.29**) Lemma. For $m, n \in \mathbb{N}$ we define set maps $R : \mathsf{Match}^N(m,n) \to \mathsf{Match}^{N-1}(m,n)$ by restricting to the appropriate subset of rows and columns, that is $R(M)_{vw} = M_{vw}$ (observe that this is necessarily only for $v, w \in \underline{N-1}^m$). This gives a functor $R : \mathsf{Match}^N \to \mathsf{Match}^{N-1}$ with $R_0(n) = n$.

pa:Rfunctor *Proof.* It will be clear that identities are taken to identities. It remains to show that $R(M.M') = R(M).R(M')$. We have $R(M.M') = (R(M.M')_{uw})_{u,w \in \underline{N-1}^m}$ and

$$R(M.M')_{uw} = (M.M')_{uw} = \sum_{v \in \underline{N}^m} M_{uv} M'_{vw} = \sum_{v \in \underline{N-1}^m} M_{uv} M'_{vw}$$

since $u \in \underline{N-1}^m$ implies $M_{uw} = 0$ unless $v \in \underline{N-1}^m$ by the Match^N property (note that this does not hold in general in $\mathbf{Mat}^N$). And $(R(M).R(M'))_{uw} = \sum_{v \in (\underline{N-1})^m} R(M)_{uv} R(M')_{vw} = \sum_{v \in (\underline{N-1})^m} M_{uv} M'_{vw}$, so we are done. □

pa:Fpsi

(**6.4.30**) Observe that there is no specific property of the set $\underline{N}$ used in the construction of Match^N, or indeed $\mathbf{Mat}^N$. In either case we could use any finite set J as the row index set for object 1. Then J^m is the index set for object m. This gives monoidal categories $\mathbf{Mat}^J$ and Match^J say. These have isomorphisms to $\mathbf{Mat}^{|J|}$ and $\mathsf{Match}^{|J|}$ respectively, induced by isomorphisms $\psi : J \to \underline{|J|}$. That is:

LEMMA. Given $\psi \in \mathbf{Set}^i(K,J)$ (and understanding functions in $\mathbf{Set}(K,J)$ to act pointwise on words in K) we have a functor

$$F_\psi : \mathsf{Match}^J \to \mathsf{Match}^K$$

given by $F_\psi(n) = n$ and $F_\psi(M)_{uv} = M_{\psi(u)\psi(v)}$, where $u, v \in K^m$ for the appropriate m.

Proof. Note that provided that ψ is an isomorphism then F_ψ indeed commutes with composition:

$$F_\psi(M.M')_{uw} = (M.M')_{\psi u,\psi w} = \sum_{v\in J^m} M_{\psi u,v} M'_{v,\psi w}$$

where m is the appropriate object, while since $|K| = |J|$,

$$(F_\psi(M).F_\psi(M'))_{uw} = \sum_{v\in K^m} F_\psi(M)_{uv} F_\psi(M')_{vw} = \sum_{v\in K^m} M_{\psi u,\psi v} M'_{\psi v,\psi w}$$

— the same sum written in a different order; and preserves identities. □

Indeed provided that $\psi \in \mathbf{Set}(K,J)$ is an injection a variant of the argument in (6.4.29) continues to work for a functor

$$R_\psi : \mathsf{Match}^J \to \mathsf{Match}^K$$

— but not for $\mathbf{Mat}^J$ unless ψ is an isomorphism.

We can say that R itself restricts to the subset $\underline{N-1}$ of $\underline{N}$. (This is the restriction on the row and column index for object 1, but this generalises simply to $\underline{N-1}^m$ for object m.)

Observe that we can define a version of R that restricts to a different subset of $\underline{N}$. Let us say for $J \subset \underline{N}$ that R_J restricts to J. Thus $R_J(M)_{uv} = M_{uv}$.

Next we introduce functors that commute with the monoidal structure.

Question: can we connect to categorified SW duality and the partition category? Maybe ask this question after mod'ing the monoidal proof below for the iso case...

strictmonfunctor

6.4.6 Strict monoidal functor

A *strict monoidal functor* from strict monoidal category $(C, \otimes_C, I_C)$ to $(D, \otimes_D, I_D)$ is a functor $G : C \to D$ such that

$$G_0(I_C) = I_D$$

$$G_0(a \otimes_C b) = G_0(a) \otimes_D G_0(b)$$

$$G_1(f \otimes_C g) = G_1(f) \otimes_D G_1(g)$$

(Exercise: decide if the last condition is determined by the preceding ones.)

(**6.4.31**) Observe that the identity functor between strict monoidal categories is strict monoidal.

Observe that the composition of composable strict monoidal functors is a strict monoidal functor.

Thus we have a category of strict monoidal categories and strict monoidal functors.

(**6.4.32**) Fixing a strict monoidal category (C, F), we have a monoid of functors from this category to itself. The set/class of these functors becomes the object class of a category whose morphisms are the natural transformations. Sitting inside the monoid of functors is the group of isomorphism functors.

(**6.4.33**) Example. Consider the functor $R : \mathsf{Match}^N \to \mathsf{Match}^{N-1}$ from (6.4.29). This functor is strict monoidal.

Proof. The identity of the object monoid in each case is 0, and $R_0(0) = 0$ as required. Furthermore $R_0(a + b) = a + b = R_0(a) + R_0(b)$. Finally recall, for $M \in \mathsf{Match}^N(l, l)$ and $M' \in \mathsf{Match}^N(m, m)$ say,

$$(R(M \otimes M'))_{vw} = (M \otimes M')_{vw} = M_{v_1 v_2 .. v_l, w_1 w_2 .. w_l} M'_{v_{l+1} v_{l+2} .. v_{l+m}, w_{l+1} w_{l+2} .. w_{l+m}}$$

— with the caveat that $v, w \in \underline{N-1}^{l+m}$. On the other hand, with the same range,

$$(R(M) \otimes R(M'))_{vw} = (R(M))_{v ...., w ...} (R(M'))_{v ..., w ...} = M_{v_1 v_2 .. v_l, w_1 w_2 .. w_l} M'_{v_{l+1} v_{l+2} .. v_{l+m}, w_{l+1} w_{l+2} .. w_{l+m}}$$

Thus R gives a strict monoidal functor. □

Remark. We reiterate that this construction applied to $\mathbf{Mat}^N$ fails even to give a functor.

(**6.4.34**) Example. Consider the functor $F_\psi : \mathsf{Match}^J \to \mathsf{Match}^K$ from (6.4.30). This functor is strict monoidal.

Proof. For $M \in \mathsf{Match}^J(l, l)$ and $M' \in \mathsf{Match}^J(m, m)$ say,

$$(F_\psi(M \otimes M'))_{vw} = (M \otimes M')_{\psi(v)\psi(w)} = M_{\psi(v_1 v_2 .. v_l), \psi(w_1 w_2 .. w_l)} M'_{\psi(v_{l+1} v_{l+2} .. v_{l+m}), \psi(w_{l+1} w_{l+2} .. w_{l+m})}$$

because ψ acts 'diagonally' (i.e. $\psi(v_1 v_2) = \psi(v_1)\psi(v_2)$ and so on). Meanwhile

$$(F_\psi(M) \otimes F_\psi(M'))_{vw} = (F_\psi(M))_{v_1 v_2 ... v_l, w_1 w_2 ... w_l} (F_\psi(M'))_{v_{l+1} ..., w ...}$$

$$= M_{\psi(v_1 v_2 .. v_l), \psi(w_1 w_2 .. w_l)} M'_{\psi(v_{l+1} v_{l+2} .. v_{l+m}), \psi(w_{l+1} w_{l+2} .. w_{l+m})}$$

□

(**6.4.35**) Example/Exercise: consider the two conventions for Kronecker products. Each defines a strict monoidal category as noted above. Is There is a Strict Monoidal Functor $F : (\mathbf{Mat}_K, \mathsf{F}) \to (\mathbf{Mat}_K, \mathsf{F}')$ between them (caveat: symbol F overloaded)? In particular is there an isomorphism - i.e. a pair of functors back and forth that compose to the identity functor?

Any candidate F will have to obey $F_0(n) = n$. (Right? Why? In the isomorphism case at least anything else would require some choices.)

For F_1 we have to give a lot of maps, so it will need to be, in a suitable sense, algorithmically simple. The simplest candidate is $F_1(f) = f$. Let us try this...

We have $F_1(f \otimes g) = ...$

... (FINISH mE!)

(**6.4.36**) A *strong* monoidal functor (as for example in Mac Lane [**?**]) from Strict Monoidal (StM) category $(C, \otimes_C, I_C)$ to StM category $(D, \otimes_D, I_D)$ is a triple (F, f_0, f) where $F : C \to D$ is a functor; f_0 is a morphism from I_D to $F(I_C)$ that is an isomorphism; and $f = f(-,-)$ is a collection of natural isomorphisms

$$f(A,B) : F(A) \otimes F(B) \to F(A \otimes B)$$

(FINISH ME!)

(**6.4.37**) Example. Consider $(\mathbf{Mat}, \mathsf{F})$ and $(\mathbf{Mat}, \mathsf{F}')$ from §6.4.2, and consider the formal triple (F, f_0, f) given by $F_0(m) = m$, $F_1(A) = A$, so in particular $F(id_1) = id_1$, so f_0 ...

6.4.7 Aside on representation theory

ss:asideRT1

Match categories have a helpful rigidity for approaching classification problems in representation theory. Consider first the monoidal category $\underline{\Sigma}$... Then B...

(**6.4.38**) A representation of a strict monoidal category is a monoidal functor to some monoidal category. Suppose we have a strict monoidal category (C, F), generated by a subset M, and that there is a representation $\rho : C \to C'$. Then to give the representation it is sufficient to give the images of the generators. (Here we mean that we have a well-defined representation and we give sufficient data to yield it.) For examples we will make some preparations.

(**6.4.39**) Recall the monoidal category $\underline{\Sigma}$ from (6.4.16). It turns out that we have a monoidal functor $F : \underline{\Sigma} \to \mathbf{Mat}$ given on objects by $1 \mapsto 2^1$ (notice that this gives an injection of the object monoid $(\mathbb{N}_0, +)$ into the object monoid $(\mathbb{N}, \times)$) and on morphisms by

$$\sigma \mapsto \begin{pmatrix} 1 & & & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ & & & 1 \end{pmatrix} \tag{6.7}$$

eq:sigmaf1

One way to verify this is to note that $\underline{\Sigma}$ is isomorphic to a strict monoidal category $\underline{\Sigma}'$ given by a presentation.

The object monoid of $\underline{\Sigma}'$ is $(\mathbb{N}_0, +)$ and there is a single generator $\varsigma \in \underline{\Sigma}'(2,2)$. The relations are

$$\varsigma^2 = 1_2$$

(in the obvious notation) and

$$\varsigma_1\varsigma_2\varsigma_1 = \varsigma_2\varsigma_1\varsigma_2$$

where $\varsigma_1 = \varsigma \otimes 1_1$ and $\varsigma_2 = 1_1 \otimes \varsigma$.

The isomorphism is given by $\sigma \mapsto \varsigma$. To show that $\varsigma \mapsto \sigma$ gives a functor $\psi : \underline{\Sigma}' \to \underline{\Sigma}$ it is sufficient to check that the relations are satisfied. One way to show an isomorphism is to compare orders of hom-sets. We can do this using representations such as (6.7) and a suitable set of generalisations thereof.

The resultant monoidal functor from (6.7) is also a functor. As such we could apply any functor $\mathbf{Mat} \to \mathbf{Mat}$ to get a new functor/representation. But of course not every such functor will result in a composite that is a monoidal functor.

What is an appropriate notion of equivalence of representations in this setting? As noted in (**??**), ordinary representation theory of a group, say, uses some variant of the category Vec as the raw material for the target. Thus representation theory benefits from properties of Vec such as abelianness. Such properties do not survive untouched in the monoidal setting.

(**6.4.40**) In particular given an automorphism a in $\mathsf{Vec}(V,V)$ say, we have an isomorphism functor $F_a : \mathsf{Vec} \to \mathsf{Vec}$ given by $m \mapsto r_m m r'_m$ where r_m is the identity unless it can be a (i.e. $t(a) = V = s(m)$) and r'_m is the identity unless it can be a^{-1}.

Given a representation $\rho : C \to \mathsf{Vec}(V,V)$ ... FINISH ME!

6.4.8 On non-strict monoidal categories

More generally, consider the following.

(**6.4.41**) Let $F : A \times A \to A$ be a functor as in (6.4.1). Consider the category $A \times A \times A$, which we understand to mean ordered triples (a mild abuse of notation). There are isomorphisms with $(A \times A) \times A$ and $A \times (A \times A)$. We have a functor $A \times A \times A \stackrel{F \times id}{\to} A \times A$ with $(F \times id)_0(a,b,c) = (F_0(a,b),c)$ and so on.

(**6.4.42**) We can iterate the use of a tensor product functor as in (6.4.1) in different ways. Given the category $A \times A \times A$ we have two functors $A \times A \times A \stackrel{F \times id}{\to} A \times A \stackrel{F}{\to} A$ and $A \times A \times A \stackrel{id \times F}{\to} A \times A \stackrel{F}{\to} A$. A natural isomorphism between these would take the form

$$a : F \circ (F \times id) \to F \circ (id \times F)$$

and would mean that there are commuting squares involving each combination of morphisms $F(F(f,g),h)$ and $F(f,F(g,h))$.

If the binary operations are associative and unital (so that an object set becomes a monoid and so on as above) then (A,F) is a *strict tensor category*. If the binary operation is associative and unital up to (certain suitable) natural isomorphisms

$$a_{LMN} : F(F(L,M),N) \to F(L,F(M,N))$$
$$l_M : F(1,M) \to M$$
$$r_M : F(M,1) \to M$$

(see later for axioms) then $(A,F) = (A,F,1,a,l,r)$ is a *tensor category*.

Suppose there are additional natural isomorphisms

$$g_{LM} : F(L,M) \to F(M,L)$$

Then we can reorder and move brackets in any expression of form $F(M_1, F(F(M_2, F(M_3, M_4)), M_5))$ by applying suitable a_{LMN} and g_{LM}s. Suppose we associate such a manipulation to an element of the braid group by associating each $g_{M_i M_j}$ to a braiding in that position. If the manipulation morphism depends only on the associated braiding, then the tensor category A together with (the collection) g is a *braided tensor category*.

A natural example is the category of modules of a finite group algebra, where $F(M,N) = M \otimes N$. (Indeed later we will write $F(M,N)$ as $M \otimes N$ quite generally.)

(**6.4.43**) Example. For K a commutative ring the category $\mathbf{Mat}_K$ becomes a (strict) monoidal category when taken with ordinary multiplication on objects (numbers), and the Kronecker product (1.1) on morphisms (matrices).

Note that there is an obvious alternate convention for Kronecker products. Note that this gives rise to a different SMC — although the underlying categories are the same.

6.4.9 Some diagrams for morphisms in monoidal categories

We may write a morphism $f : V \to W$ as $V \xrightarrow{f} W$, and hence as

$$\begin{array}{c} V \\ \uparrow \\ [f] \\ \uparrow \\ W \end{array}$$

(NB the arrows here are perverse; the forward direction of all morphisms is downwards in this convention and the arrows are here to encode something else — see later). Composable morphisms can then be drawn as stacks of such pictures. The identity morphism 1_V is drawn $\uparrow_V$.

Less trivially, tensor products of morphisms in strict monoidal categories are drawn by horizontal concatenation. In we have $V_1 \xrightarrow{f_1} W_1$ and $V_2 \xrightarrow{f_2} W_2$ we have (1) here:

$$\begin{array}{ccc} V_1 & & V_2 \\ \uparrow & & \uparrow \\ [f_1] & \otimes & [f_2] \\ \uparrow & & \uparrow \\ W_1 & & W_2 \end{array} \quad \overset{(1)}{=} \quad \begin{array}{c} V_1 \otimes V_2 \\ \uparrow \\ [f_1][f_2] \\ \uparrow \\ W_1 \otimes W_2 \end{array} \quad \overset{(2)}{=} \quad \begin{array}{cc} V_1 & V_2 \\ \uparrow & \uparrow \\ [f_1] & [f_2] \\ \uparrow & \uparrow \\ W_1 & W_2 \end{array}$$

We go further and replace each (directed) line by a line for each tensor factor, as in (2) above. (Note that iterating the tensor product in a non-strict category would present an issue here.)

What about depiction of morphisms involving the monoidal identity object 1? By strictness we have $1 \otimes V = V$ so

$$\begin{array}{ccc} 1 & & V_2 \\ \uparrow & & \uparrow \\ [f_1] & \otimes & [f_2] \\ \uparrow & & \uparrow \\ 1 & & W_2 \end{array} \quad \overset{(1)}{=} \quad \begin{array}{c} 1 \otimes V_2 \\ \uparrow \\ [f_1][f_2] \\ \uparrow \\ 1 \otimes W_2 \end{array} \quad \overset{(2)}{=} \quad \begin{array}{cc} 1 & V_2 \\ \uparrow & \uparrow \\ [f_1] & [f_2] \\ \uparrow & \uparrow \\ 1 & W_2 \end{array}$$

which we would like to express as a morphism from V_2 to W_2. How do we do this?

...

6.4.10 Duals

(**6.4.44**) A monoidal category with monoidal identity 1 has left duals if for each object A there is an object A^* and morphisms

$$coev_A : 1 \to A \otimes A^*$$

and $ev_A : A^* \otimes A \to 1$ such that

$$(id_A \otimes ev_A)(coev_A \otimes id_A) = id_A \qquad \text{and v.v.} \tag{6.8}$$

`eq:dualax`

If we write A^* in pictures as A with line-direction reversed (note that the direction asigned to lines in pictures has so far been redundant) then the axioms are

...

(**6.4.45**) EXAMPLE. Return to the monoidal category $\mathbf{Mat}_K$ as above. Is there a left dual? We can try setting $n^* = n$ and give $coev_n : 1 \rightarrow n \otimes n$ by, e.g.,

$$coev_4 = (1,0,0,0,\ 0,1,0,0,\ 0,0,1,0,\ 0,0,0,1)^t$$

and $ev_n = coev_n^t$ (transpose). To check the axiom (6.8) we have (case $n = 2$)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1001) = \begin{pmatrix} 10010000 \\ 00001001 \end{pmatrix}, \qquad (1001)^t \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 01 \\ 00 \\ 00 \\ 00 \\ 00 \\ 10 \\ 01 \end{pmatrix}$$

which works!

(**6.4.46**) Next we can use duals to introduce a kind of transpose.

Chapter 7

Rings in representation theory

ch:ring

As noted in Chapter 1, representation theory is, in general, the study of ring homomorphisms. Or even more generally, representation theory is the study of structure preserving maps between algebraic structures. But typically, for a variety of different reasons (both of motivation and tractability), the target ring is an algebra over a commutative ring that is an algebraically closed field. This means that the object ring has the same nominal structure, however such rings are often obtainable by base change from algebras over more general commutative rings, and this perspective can help with their analysis. This means that we are also interested, for example, in commutative rings that help us study this setup.

See §**??** for a short bibliography.

7.1 Rings I

Recall from (3.1.1):

(**7.1.1**) A ring R is a set with two laws of composition such that
$(R, +, 0)$ is a commutative group with identity 0
$(R, ., 1)$ is a monoid with identity 1
$r(s+t) = rs + rt, \qquad (s+t)r = sr + tr$

If we write $r \in R$ we mean r is an element in the underlying set of R. As an algebraic structure a ring can be written $(R, +, ., 0, 1)$. So then we have the odd (but standard) shorthand $R = (R, +, ., 0, 1)$.

(**7.1.2**) For example, $\mathbb{Z} = (\mathbb{Z}, +, ., 0, 1)$ is a ring. Several further examples will be useful to have in mind. See §7.1.1.

(**7.1.3**) More necessary than useful is the *zero ring* $R = \{0\}$. This is the unique ring with $1 = 0$.

Some workers do not require a ring to have a 'multiplicative unit'. See for example [62, p.12].

For ring homomorphisms see §7.2.1. For ring ideals see §7.2. For now we observe some snippets:

There is a notion of *kernel* for ring homomorphisms; as well as a notion of *image*. The image of a homomorphism is a ring, but the kernel is an ideal (a subgroup closed under left and right actions of the ring).

A ring homomorphism h is injective if and only if $\ker(h) = \{0\}$.
A ring R is *simple* if its only ideals are $\{0\}$ and R.
A ring R is simple if and only if every ring homomorphism $h : R \to S$ is injective.
For every ring R there is a unique ring homomorphism from R to the zero ring.

7.1.1 Examples

ss:rexamples

(**7.1.4**) Example. For each abelian group A the set $\hom_{\mathbf{Ab}}(A, A)$ of endomorphisms is of course a monoid. But defining pointwise addition of endomorphisms by

$$(f + g)(a) = f(a) + g(a)$$

(a special case of (6.2.5)) one finds that it is also an abelian group under addition. It is straightforward to check that distributivity holds, so $\hom_{\mathbf{Ab}}(A, A)$ becomes a ring.

de:tri mat

(**7.1.5**) Example. If R is a ring then, as in §1.1.1,

$$M_2(R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R\}$$

with matrix addition and multiplication, is a ring. Similarly $M_n(R)$ ($n \times n$ matrices, as in §1.1.1) and

$$\mathrm{Tri}_2(R) = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R\} \qquad \text{and} \qquad \mathrm{Tri}'_2(R) = \{t_{a,b} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\}$$

de:opp-ring

(**7.1.6**) Example. If $R = (R, +, 0, ., 1)$ is a ring, then R^{op}, the same structure but with multiplication $a * b = b.a$, is a ring. The ring R^{op} is called the *opposite* ring of R.

Note that a commutative ring is equal to its opposite.

(**7.1.7**) Recall from §3.1.1 that we define a *local ring* as a ring R in which the set $R \setminus R^\times$ of non-units is closed under addition (and hence an ideal).

(**7.1.8**) Note that $t_{a,b}t_{c,d} = t_{ac,ad+bc}$ in $\mathrm{Tri}'_2(\mathbb{C})$. Thus $t_{a,b}$ is a unit in $\mathrm{Tri}'_2(\mathbb{C})$ if and only if $a \neq 0$. Thus every nonunit takes the form $t_{0,b}$, and since $t_{0,b} + t_{0,c} = t_{0,b+c}$ we have that $\mathrm{Tri}'_2(\mathbb{C})$ is a local ring (as defined in **??**). On the other hand $\mathrm{Tri}_2(R)$ is not a local ring for any R (since the idempotent elementary matrices are nonunits that sum to 1).

(**7.1.9**) Next we consider examples based broadly on the idea of a pair of rings related by an injective ring homomorphism: $\psi : R \hookrightarrow S$. Then S can be seen as an 'extension' of the image of R. One can, for example, ask for the smallest subring of S that contains R and some $s \in S$.

(**7.1.10**) Example. For R a ring, the set $R[X]$ is the subset of $R^{\mathbb{N}_0}$ of elements $r = (r_0, r_1, r_2, ...)$ such that only finitely many of the r_i are non-zero. This set closes under pointwise addition and 'Cauchy' multiplication, and indeed these operations make this set a ring. A convenient representation of this is

$$r = \sum_{i=0}^{n_r} r_i X^i$$

(where n_r is the index on the last nonzero entry) whereupon the ring operations become 'polynomial' arithmetic, that is, X is treated as if a central extension of R.
For this reason $R[X]$ is often called the ring of polynomials in indeterminate X over R.

Another way to think about this is that X acts as a shift operator $X : R[X] \to R[X]$: $(X(r))_0 = 0$ and otherwise

$$(X(r))_i = r_{i-1}$$

Most of the time we shall be interested in the case where R is commutative, so that $R[X]$ is also commutative. See for example (7.1.12).

(**7.1.11**) For R a ring, the ring $R[X_1, ..., X_n] := R[X_1, ..., X_{n-1}][X_n]$. Similarly to the above, this is thought of as the ring of polynomials in n (commuting) indeterminates. By this construction each X_i is a central extension, and in particular all the X_is commute with each other. (One could also consider extensions that do not commute with each other. For example one could consider combining two extensions, each separately central, of a commutative ring, but where the extensions do not commute. This non-commutation could be 'free'; or determined, such as in extensions by X and d/dX. However we shall not introduce notation for this here.)

pa:comm p ring (**7.1.12**) For S commutative and $x_1, ..., x_n \in S \supseteq R$ we write $R[x_1, ..., x_n]$ for the ring of polynomials in $x_1, ..., x_n$.

7.1.2 Properties of elements of a ring

de: elem mat (**7.1.13**) Fix a ring R. By $\epsilon_{ij} \in M_n(R)$ we mean the *elementary matrix* which is zero in every position except $(\epsilon_{ij})_{ij} = 1$.

(**7.1.14**) IDEMPOTENTS. An element $e \in R$ is an *idempotent* if $ee = e$.

For example $\epsilon_{11}\epsilon_{11} = \epsilon_{11}$ in $M_n(R)$ (any R). Indeed in $M_2(R)$ we have

$$\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} \qquad \text{(any } r \in R\text{)}$$

$$\begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix} \qquad \text{(any } r \in R\text{)}$$

(**7.1.15**) $x \in R$ is:
a *unit* if it has an inverse. (Write $R^\times$ for the set of units - a group.)
irreducible if R is commutative and x has no proper factorisation into non-units.

(**7.1.16**) Let R be commutative, then $x \in R \supset S$ is:
algebraic if there exists $p(X) \in S[X]^*$ such that $p(x) = 0$.
integral if there exists p as above, monic.

(**7.1.17**) LEMMA. *Let R be commutative. Ring $R[x] \cong R[X]$ if x is* transcendental *(i.e. not algebraic).*

(**7.1.18**) Let us unpack the notion of integrality a little. Let $A \subset B$ be rings and $x \in B$. If there exists a subset $\{a_0, ..., a_n\}$ of A such that

$$x^{n+1} = \sum_i a_i x^i$$

we say x is *integral over* A. This is to say, there is a monic polynomial $f = X^{n+1} - \sum_i a_i X^i$ in the polynomial ring $A[X]$ such that the evaluation image $f(x) = 0$ in B.
If there is a not-necessarily monic polynomial $g \in A[X]$ such that $g(x) = 0$ we say x is *algebraic* over A. (If x is not algebraic it is *transcendental*.)
Example: $a \in A$ is integral over A, since $a = aa^0$.
Example: $\sqrt{2} \in \mathbb{R}$ is integral over $\mathbb{Z}$ since $\sqrt{2}^2 = 2\sqrt{2}^0$.
Example: with $n, m \in \mathbb{N}$, then $\sqrt{n} + m \in \mathbb{R}$ is integral over $\mathbb{Z}$ since $(\sqrt{n} + m)^2 = 2m(\sqrt{n} + m)^1 + (n - m^2)(\sqrt{n} + m)^0$.

pa:int A mod

(**7.1.19**) Suppose x is integral over A, via polynomial f as above. Then $\{x^0, x, ..., x^n\}$ contains a basis of $A[x]$, so the ring $A[x]$ is a finite A-module.
In fact the converse holds (see e.g. [149, V,1,255]): if $A[x]$ is a finite A-module then x is integral over A.

(**7.1.20**) Suppose r, s integral over A. Then there are appropriately vanishing monic polynomials, f_r, f_s say. Using (7.1.19) we can show that $r + s$ is also integral over A.

The set of elements of B that are integral over A is a ring, called the *integral closure* of A in B. If every $b \in B$ is integral over A then say B is *integrally dependent* on A.

(**7.1.21**) Remark. See [35, §1A] for the case of B an A-algebra (so A is commutative but B is not necessarily so).

7.2 Ideals and homomorphisms

ss:rideal

Here R is a ring.

(**7.2.1**) A subgroup I of R is an *ideal* if $rI \subseteq I$ and $Ir \subseteq I$ for all $r \in R$.

More generally, a (left/right) ideal I of a ring R is an additive subgroup closed under (left/right) multiplication by R.

For example, RrR is an ideal for each $r \in R$. Further

$$RrsR \subseteq RrR \subseteq R$$

$(r, s \in R)$ is a nest of ideals. In particular $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$, and indeed mZ is an ideal for each $m \in Z$.

Note that the left ideal Rr can only be proper if r a nonunit (else $Rr \supseteq Rr^{-1}r = R1 = R$), and that every element of a proper left ideal is a nonunit. However the nonunits do not form an ideal in general, since the sum of two nonunits may be a unit. Since this does not happen in a local ring, the set of nonunits is a proper left ideal in a local ring. This left ideal is clearly maximal (the only remaining elements of the ring are units) and indeed contains every other left ideal. Thus a local ring has a unique maximal left ideal. (Or right ideal, by the same argument.)

(**7.2.2**) Note that

$$U_2(R) = \{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R\}$$

is a ring without 1. It is an ideal of $\mathrm{Tri}_2(R)$ contained in $\mathrm{Tri}'_2(R)$, and hence also an ideal of $\mathrm{Tri}'_2(R)$.

Neither $\mathrm{Tri}_2(R)$ nor $U_2(R)$ is an ideal of $M_2(R)$. The set

$$C_2(R) = \{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in R\}$$

is a left ideal of $M_2(R)$ but not an ideal or right ideal.

(**7.2.3**) Lemma. *For R a ring, the only ideals of $M_n(R)$ are the subsets $M_n(I)$ for I an ideal of R.*

ss:rhomo

7.2.1 Ring homomorphisms

(**7.2.4**) A ring homomorphism is a map

$$\phi : R \to S$$

such that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$.

(**7.2.5**) Lemma. *For $\phi : R \to S$ a ring homomorphism, $\ker(\phi)$ is an ideal of R.*

(**7.2.6**) Let I be an ideal of R. Then the set R/I of cosets of I in R form a ring, and

$$\phi_I : x \mapsto I + x$$

is a ring epimorphism

$$\phi_I : R \to R/I$$

with $\ker(\phi_I) = I$.

(**7.2.7**) Example. We have

$$\mathrm{Tri}_2(\mathbb{Q})/U_2(\mathbb{Q}) \cong D_2(\mathbb{Q})$$

the ring of diagonal matrices.

7.2.2 Posets revisited

See also §3.4. Recall:

(**7.2.8**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

7.2.3 Properties of ideals: Artinian and Noetherian rings

(**7.2.9**) A proper ideal I of a ring R is *maximal* if there is no proper ideal $J \supset I$.

For example, $p\mathbb{Z}$ is maximal iff p is prime.
Similarly $U_2(\mathbb{C})$ is a maximal ideal of $T_2'(\mathbb{C})$. (Any extension of $U_2(\mathbb{C})$ would have to be by an element with non-zero diagonal, but this would then be the whole of $T_2'(\mathbb{C})$.) Indeed $U_2(\mathbb{C})$ is the unique maximal ideal of $T_2'(\mathbb{C})$.

(**7.2.10**) Lemma. *Every commutative ring with $1 \neq 0$ has a maximal ideal.*

Proof. Consider the poset P of proper ideals in R. Consider a chain T in P, and let I be the union of all ideals in T. This is (evidently) an ideal, and does not contain 1, so $I \in P$. *Zorn's Lemma* states that a poset in which every chain has an upper bound ($u \in P$ such that $u \geq t$ for all $t \in T$) has a maximal element ($m \in P$ such that $\nexists x \in P,\ x > m$). Thus our P has a maximal element.

(**7.2.11**) A ring is *left-Noetherian* if the poset of left ideals satisfies ACC.
(Similarly right-Noetherian / Noetherian.)
A ring is *left-Artinian* if the poset of left ideals satisfies DCC. (Similarly right-Artinian / Artinian.)

Thus any field is Noetherian; $\mathbb{Z}$ is Neotherian; $\mathbb{Z}[x]$ is Neotherian.
$\mathbb{Z}[x_1, x_2, ...]$ is not Neotherian, since $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset ...$
The set F of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a ring by pointwise operations. For each subset $S \subseteq \mathbb{R}$ the subset F_S of functions with $\ker f = S$ is an ideal. Each chain $S_1 \supset S_2 \supset S_3 \supset ...$ induces a chain of ideals $F_{S_1} \subset F_{S_2} \subset F_{S_3} \subset ...$ (ascending), so F is not Noetherian.

(**7.2.12**) THEOREM. *[Hopkins-Levitzki] A left(right)-Artinian ring R is left(right)-Noetherian. (Indeed if R is Artinian then any finitely generated R-module, as in §8, is Artinian and Noetherian.)*

7.2.4 Properties of ideals: Integral and Dedekind domains

(**7.2.13**) An ideal I in R is *prime* if $xRy \subset I$ implies that at least one of x, y lies in I.
This is the same as to say that an ideal I is prime if, whenever it contains a product of ideals HJ then at least one of H, J is contained in I.

Example: (i) $p\mathbb{Z}$ is prime iff p is prime.
(ii) $I = 2x\mathbb{Z}[x] = 2\mathbb{Z}[x]x$ is not prime, since neither of $2, x \in I$.

(**7.2.14**) The *field of fractions* of an integral domain R (as in (3.1.1)) is the quotient of $R \times (R \setminus \{0\})$ by $(r, s) \sim (t, u)$ if $ru = st$. Addition is represented by $(r, s) + (t, u) = (ru + ts, su)$ and multiplication by $(r, s)(t, u) = (rt, su)$.

Example: The field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

pa:inv id (**7.2.15**) Suppose I is an ideal of integral domain R, and $\mathbb{Q}_R$ the field of fractions. Embedding $R \hookrightarrow \mathbb{Q}_R$ by $r \mapsto (r, 1)$, we write

$$I^{-1} := \{q \in \mathbb{Q}_R \mid qI \subseteq R\}$$

Example: $(p\mathbb{Z})^{-1} = \{(r, p^i) \mid r \in R;\ i \in \{0, 1\}\}$
We say ideal I is *invertible* if $II^{-1} = R \subset \mathbb{Q}_R$.

(**7.2.16**) Let R be an integral domain and I a prime ideal. Then the subset of the field of fractions with representatives of form (r, s) with $s \notin I$ is a subring.
This subring, denoted R_I, is the *localisation* of R at I.

(**7.2.17**) An integral domain in which every ideal can be written as a product of prime ideals (with those prime factors unique up to ordering) is a *Dedekind domain*.

(**7.2.18**) PROPOSITION. *An integral domain is a Dedekind domain iff every ideal is invertible.*

(**7.2.19**) PROPOSITION. *An integral domain is a Dedekind domain iff it is noetherian, integrally closed in its field of fractions, and each prime ideal is maximal.*

(**7.2.20**) An integral domain in which every non-unit has a finite irreducible factorisation, that is unique up to order and units, is a UFD.
An integral domain with a division algorithm is a Euclidean domain.

(**7.2.21**) We can summarize the commutative part of the story so far:
commutative rings $\supset$ integral domains $\supset$ UFDs/Dedekind domains $\supset$ PIDs $\supset$ Euclidean domains $\supset$ fields.

7.3 Rings II: rings for algebra

In this Section we want to construct rings with certain special properties that will be useful later. We start with the notion of valuation on a ring, from which we build absolute value. This gives a metric, and hence a notion of Cauchy sequences, and hence of completion (generalising the completion of $\mathbb{Q}$ to $\mathbb{R}$).

7.3.1 Order and valuation

See e.g. MacLane–Birkoff [**?**, Ch.VIII].

Let $(S, \geq)$ be a poset. A LOWER BOUND of $T \subset S$ is an element $b \in S$ such that $b \leq t\ \forall t \in T$. A GREATEST LOWER BOUND of $T \subset S$ is an element $b \in S$ such that b is a lower bound and for each lower bound c we have $b \geq c$.

de:ordered ring (**7.3.1**) ORDERED RING (respectively DOMAIN, FIELD) R, R a ring, $0 \neq 1$, $\exists$ nonempty $P \subset R$ such that $a, b \in P$ implies $a + b, ab \in P$, $0 \notin P$, $\forall a \in R$ either $a \in P$ or $(-a) \in P$.
In an ordered ring we write $a > b$ if $a + (-b) \in P$ ('positive').

complete1 (**7.3.2**) COMPLETE ORDERED DOMAIN D, D an ordered domain, every nonempty $S \subset P$ has GLB in D.
(Counter)Example: $\mathbb{Q}$ is an ordered field with $x \in P$ if $x > 0$, but $\mathbb{Q}$ is not complete. (Consider any sequence of rational approximations to $\sqrt{2}$ which approaches from above — any *rational* $r \leq \sqrt{2}$ would be a LB. But a GLB g would have to be rational and equal to $\sqrt{2}$ — a contradiction! — else a rational $O < \epsilon < \sqrt{2} - g$ could be added to g contradicting 'greatest'.)

(**7.3.3**) VALUATION. For D an integral domain and $(\Gamma, +, 0)$ a totally ordered group a *valuation* on D is a map $v : D \to \Gamma \cup \{\infty\}$ (here $\infty + g = \infty$ and $\infty > g$ for all $g \in \Gamma$) such that $v(0) = \infty$, $v(a) < \infty$ if $a \neq 0$, $v(ab) = v(a) + v(b)$, $v(a + b) \geq \min(v(a), v(b))$.

(**7.3.4**) The set $R = \{x \in D | v(x) \geq 0\}$ is a subring of D called the VALUATION RING.
Note in particular that $v(1) = v(1.1) = v(1) + v(1)$, so $v(1) = 0$.
Also, if $d \in D$ is invertible then $v(d) + v(d^{-1}) = v(d.d^{-1}) = v(1) = 0$, so either $v(d) = v(d^{-1}) = 0$ or precisely *one* of d, d^{-1} lies in the valuation ring.

(**7.3.5**) Examples: (i) $\mathbb{Q} \ni x = \pm p^{\mu} \frac{r}{s}$ with p, r, s coprime, p prime, $\mu(0) = \infty$, defines $\mu = \mu(x)$ uniquely. This is the p–ADIC VALUATION on $\mathbb{Q}$.
Check: Consider $p = 5$, so $\mu(99/100) = -2$ and $\mu(15/27) = 1$. We have $\mu(\frac{m}{n} + \frac{r}{s}) = \mu(\frac{ms+nr}{ns})$ and with $m = 99$, $n = 100$, $r = 15$, $s = 27$ we have $\mu(\frac{4173}{2700}) = -2$ which obeys $-2 \geq \min(-2, 1)$ ˇ.
(N.B., The feature illustrated in this example is generally true: if $\mu(a) \neq \mu(b)$ then $\mu(a + b) = \min(\mu(a), \mu(b))$.)

(ii) Let K be a field and $K(X)$ the field of fractions of the polynomial ring $K[X]$. Then $K(X) \ni x = p(X)^{\mu} \frac{f(X)}{g(X)}$ with f, g, p coprime and p irreducible, and $\mu(0) = \infty$, defines $\mu = \mu(x)$ uniquely.
(iii) (More generally) Let K be the field of fractions of a Dedekind domain R, and P a maximal ideal of R. For $x \in K$ there is a factorisation of Rx into a product of prime ideal powers (some possibly negative in the sense of (7.2.15); some possibly repeated). Let $\mu(x)$ be the power to which P appears, and $\mu(0) = \infty$.

valuation ring

(**7.3.6**) Proposition. *A subring R of a field F is a valuation ring in F iff $x \in F \setminus \{0\}$ implies [either $x \in R$ or $x^{-1} \in R$].*

In particular a field F with a valuation v is an ordered field (as in (7.3.1)) with set P equated to the valuation ring.
If $\Gamma = \mathbb{R}$ a valuation is REAL VALUED, and if $\Gamma = \mathbb{Z}$ a valuation is PRINCIPAL or DISCRETE RANK 1.

de:naav1

(**7.3.7**) Non-Archimedean absolute value. For R a commutative ring a non–archimedian absolute value on R is a real–valued function $x \mapsto |x|$ such that $|x+y| \leq \max(|x|, |y|)$ (ultrametric inequality), $|xy| = |x|.|y|$ and $|x| \geq 0$ (saturated iff $x = 0$).
Example: Let K be a field with real–valued valuation v then

$$|x| = 2^{-v(x)}$$

defines a non–archimedean absolute value on K.
Check: $|xy| = 2^{-v(xy)} = 2^{-(v(x)+v(y))} = 2^{-v(x)} 2^{-v(y)} = |x||y|$;
$|x+y| = 2^{-v(x+y)} = 2^{\leq -\min(v(x),v(y))} \leq \max(2^{-v(x)}, 2^{-v(y)})$.

(**7.3.8**) An *Archimedean* absolute value (or simply an absolute value) obeys the weaker relation $|x+y| \leq |x| + |y|$ in place of the ultrametric inequality.

(**7.3.9**) Let R be a ring with (non–archimedean) absolute value $||$. Then $d(x, y) = |x - y|$ is a metric on R. The ring operations are continuous so R becomes a TOPOLOGICAL RING with the metric topology.

(**7.3.10**) Suppose that K is some non-archimedean valued field. For $r > 0 \in \mathbb{R}$ and $a \in K$ we set $B_a(r) = \{x \in K : |x-a| \leq r\}$ and $B_a(r^-) = \{x \in K : |x-a| < r\}$. The sets $B_0(1)$ and $B_0(1^-)$ are called the CLOSED (respectively OPEN) UNIT DISCS. Note that each is both open and closed however!

Now $B_0(1)$ is a subring of K, and $B_0(1^-)$ is an ideal in $B_0(1)$. Indeed since every non-zero $d \in K$ has an inverse, and every d with $|d| = 1$ is in $B_0(1)$, every d with $|d| = 1$ has an inverse in $B_0(1)$; thus $B_0(1^-)$ is a maximal ideal, since any larger ideal contains some d with $|d| = 1$ and this d generates everything: $B_0(1)d^{-1}d = B_0(1)$.

It is clear that every non-zero element of the quotient ring $B_0(1)/B_0(1^-)$ takes the form $d + B_0(1^-)$, with $|d| = 1$, and has an inverse. Thus $B_0(1)/B_0(1^-)$ is a field, which we call the residue class field k of K.

(**7.3.11**) Now suppose that there is a discrete valuation μ on K. In this case we have $B_0(1) = \{x \in K : \mu(x) \geq 0\}$. An element $p \in B_0(1)$ is a *primal* element if $\mu(p)$ generates the additive group $\mu(K \setminus \{0\})$. We might as well assume that $\mu(K \setminus \{0\}) = (\mathbb{Z}, +)$ and that $\mu(p) = 1$. Note that $\mu(p^n) = n$.

Suppose I is a proper ideal of $B_0(1)$. Since μ is discrete, there is some minimum $\mu(a)$ among the $a \in I$, call it n_I. Let $a \in I$ such that $\mu(a) = n_I$. If $\mu(b) \geq \mu(a)$ then b/a (computed in K) lies in $B_0(1)$, so $b = a(b/a)$ lies in I. Thus in particular I contains p^{n_I} and indeed every element of K of this or greater valuation. We have $p^{n_I} B_0(1) \subset I \subset p^{n_I} B_0(1)$. Thus every ideal I of $B_0(1)$ is a principal ideal.

7.3.2 Complete discrete valuation ring

(**7.3.12**) Let R be a ring with absolute value $||$ as in (7.3.7). For $\{c_i\} \in R^{\mathbb{N}}$, write $\lim_{i\to\infty} |c_i| = c$ (or $|c_i| \to c$ as $i \to \infty$) if given any real $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|c_i - c| < \epsilon \; \forall i \geq N$.

A sequence $\{c_i\}$ in R *converges* to $c \in R$ if $|c_i - c| \to 0$ as $i \to \infty$.
A CAUCHY SEQUENCE is a sequence such that $|c_i - c_j| \to 0$ as $i, j \to \infty$.
In particular this does *not* require that $\{c_i\}$ converges to any $c \in R$.

(**7.3.13**) We say a ring R is COMPLETE with respect to $||$ if every Cauchy sequence is convergent. This is equivalent to (7.3.2).

complete_ring (**7.3.14**) PROPOSITION. *For R a commutative ring with absolute value $||$ there exists a complete ring $\check{R}$ unique up to metric isomorphism (?) and a dense embedding $R \to \check{R}$ preserving $||$.*

Proof: Firstly we construct a candidate for $\check{R}$. Take the set C of all Cauchy sequences over R and endow with componentwise addition and multiplication. The set of sequences of the form $\{c\} = \{c_i = c \forall i\}$ form a subring isomorphic to R (identify this with R in C). Then $(C, +, ., \{0\}, \{1\})$ is a commutative ring. Sequences convergent to 0 form an ideal $\mathfrak{n}$ in C (need that $\{|c_i|\}$ is bounded for any Cauchy sequence). Note that $R \cap \mathfrak{n} = \{\{0\}\}$. Take $\check{R} = C/\mathfrak{n}$. We can again identify R with a subring of $\check{R}$, namely by $a \mapsto \{a\} + \mathfrak{n}$.

For $\{c_i\} \in C$ then $\{|c_i|\}$ is a Cauchy sequence of real numbers, so $\lim |c_i| \in \mathbb{R}$ by completeness of $\mathbb{R}$ (!). If $\{c'_i\} \in \{c_i\} + \mathfrak{n}$ then $\lim |c'_i| = \lim |c_i|$ so we can define

$$|\{c_i\} + \mathfrak{n}| = \lim |c_i|$$

giving a map of $\check{R}$ into $\mathbb{R}$ (indeed into $\mathbb{R}_{\geq 0}$). Eventually (!) we can verify that this $||$ is an absolute value on $\check{R}$. In particular

$$|(\{c_i\} + \mathfrak{n})(\{d_i\} + \mathfrak{n})| = |(\{c_i d_i\} + \mathfrak{n})| = \lim |c_i d_i| = \lim |c_i||d_i|$$

$$= \lim |c_i| \lim |d_i| = |(\{c_i\} + \mathfrak{n})||(\{d_i\} + \mathfrak{n})|$$

Denseness requires to show that any element a' of $\check{R}$ is a limit of a sequence of elements of R. But if $a' = \{c_i\} + \mathfrak{n}$ then $\lim_{i\to\infty} c_i = a'$ (for $\{c_i\}$ is a Cauchy sequence...) so we are done.

Let $A = \{a'_j\}$ be a Cauchy sequence of elements of $\check{R}$. This is a sequence of sequences, since $a' = \{a_i\}$ (we may write $A = \{\{a_{i_j}\}\}$, neglecting $\mathfrak{n}$). That is to say,

$$\lim_{k\to\infty, j\to\infty} |a'_k - a'_j| = \lim_{k,j} |\{a_{i_k}\} - \{a_{i_j}\}| = \lim_{k,j} |\{a_{i_k} - a_{i_j}\}| = \lim_{k,j} \lim_i |a_{i_k} - a_{i_j}| = 0. \tag{7.1}$$ lim0

Since this must hold however $i, j, k \to \infty$ it must hold if $i = k$, hence

$$\lim_j \lim_i |a_{i_i} - a_{i_j}| = 0. \tag{7.2}$$ lim1

Since each a'_j is a Cauchy sequence in $(R, ||)$ we also have

$$\lim_{i,k} |a_{i_j} - a_{k_j}| = 0 \qquad (7.3) \quad \boxed{\texttt{lim2}}$$

for each j. In particular this holds in lim_j, and then if limits are taken with $k = j$, giving

$$\lim_{i,j} |a_{i_j} - a_{j_j}| = 0 \qquad (7.4) \quad \boxed{\texttt{lim3}}$$

We require to show that the sequence A converges to an element of $\check{R}$, and we will consider the element $\{c_i = a_{i_i}\}$ (i.e. the i^{th} element is the i^{th} element of a'_i). Why is this a Cauchy sequence? — RTS

$$\lim_{i,j} |a_{i_i} - a_{j_j}| = 0.$$

By the ultrametic inequality we have

$$|\{a_{i_i} - a_{j_j}\}| \leq |\{a_{i_i} - a_{i_j}\}| + |\{a_{i_j} - a_{j_j}\}|$$

but on applying $\lim_{ij}$ the RHS becomes zero using equation (7.2) and equation (7.4).

Thus we RTS

$$\lim_{k\to\infty} |\{a'_k - \{a_{i_i}\}\}| = \lim_k |\{a_{i_k} - a_{i_i}\}| = \lim_k \lim_i |a_{i_k} - a_{i_i}| = 0$$

Now note that

$$|\{a_{i_k} - a_{i_i}\}| \leq |\{a_{i_k} - a_{i_j}\}| + |\{a_{i_i} - a_{i_j}\}|$$

by the ultrametric inequality. This holds for any j, and hence in $\lim_j$ on the RHS. Applying $\lim_{k,i}$ the first term on the RHS then becomes $\lim_{ijk} |\{a_{i_k} - a_{i_j}\}|$ which vanishes by equation (7.1), and the last term vanishes by equation (7.2). (Still have to show denseness and uniqueness.) $\Box$

7.3.3 p–adic numbers

(**7.3.15**) Fix a prime p. The set of p-ADIC INTEGERS $\mathbb{Z}_p$ is the set of all infinite sequences $\cdots, a_2, a_1, a_0$ where each $a_i \in \{0, \ldots, p-1\}$. Given $n \in \mathbb{N}$, we can write $n = \sum a_i p^i$ with each $a_i \in \{0, \ldots, p-1\}$. In this way we identify $\mathbb{N}$ with the subset of $\mathbb{Z}_p$ where almost every a_i is zero.

Addition and multiplication are defined in $\mathbb{Z}_p$ so as to extend the usual definition on $\mathbb{N}$. However, every element also has an additive inverse. For example in $\mathbb{Z}_2$:

$$(...1111) + (...0001) = (...000) = 0$$

(where we use ellipsis when a pattern is established sufficient to determine all terms). Further

$$(...1111)(...00011) = (...1111)((...00010) + (...0001)) = (...11110) + (...1111) = (...11101)$$

and

$$(...10101011)(...00011) = (...101010110) + (...10101011) = (...0001) = 1$$

Indeed, $\mathbb{Z}_p$ is a commutative ring with subring $\mathbb{Z}$.

It is clear that, beside our earlier identification of $\mathbb{N}$, the set $-\mathbb{N}$ can be identified with the subset of $\mathbb{Z}_p$ where almost every a_i is $p-1$.

So far we have not use the primality of p. However, if p is composite then there exist $x, y \neq 0$ such that $xy = 0$. For p prime the ring $\mathbb{Z}_p$ is an integral domain, and an element of $\mathbb{Z}_p$ has an inverse if and only if $a_0 \neq 0$.

(**7.3.16**) Given a non-zero p-adic integer a, we let $v(a)$ be the least m for which a_m is non-zero. It is easy to see that for a non-zero element a of $\mathbb{Z}_p$, we have $a = p^{v(a)}b$ where b is invertible in $\mathbb{Z}_p$. Thus, to find a field containing $\mathbb{Z}_p$, it is enough to find an inverse for p.

(**7.3.17**) Inspired by decimal notation, we define the p-ADIC NUMBERS $\mathbb{Q}_p$ to be the set of all infinite sequences $\cdots, a_2, a_1, a_0, a_{-1}, a_{-2}, \cdots$ where each $a_i \in \{0, \ldots, p-1\}$ and $a_{-n} = 0$ for all $n >> 0$. We identify $\mathbb{Z}_p$ with the subset of sequences where $a_{-n} = 0$ for all $n > 0$. Addition and multiplication are extended from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ in the obvious way.

It is easy to show that $\mathbb{Q}_p$ is a field containing $\mathbb{Q}$ as a subfield and $\mathbb{Z}_p$ as a subring. In fact, $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$. Under our identifications, the elements of $\mathbb{Q}$ in $\mathbb{Q}_p$ are precisely those sequences which are periodic — that is those for which there exists $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a_i = a_{i+n}$ for all $i \geq m$.

(**7.3.18**) For $0 \neq a \in \mathbb{Q}_p$ we define (as before) $v(a)$ to be the least m for which a_m is non-zero. Then the p-ADIC VALUE of a is

$$|a|_p = \begin{cases} p^{-v(a)} & a \neq 0 \\ 0 & a = 0 \end{cases}$$

A routine exercise shows that v is a valuation on $\mathbb{Q}_p$, and that the closed unit disc in Q_p with respect to $|.|_p$ is $\mathbb{Z}_p$.

(**7.3.19**) For convenience, we now summarise some of the main topological properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$. First, $\mathbb{Z}_p$ is compact (and hence complete), containing $\mathbb{Z}$ as a dense subset.

Similarly, $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, which is locally compact (and hence complete and separable). Thus we could also have defined $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|.|_p$.

(**7.3.20**) Note that $\mathbb{Q}_p$ is very different from $\mathbb{R}$; for example, $\mathbb{Q}_p$ is totally disconnected. However, like $\mathbb{R}$ it is complete but not algebraically closed. We denote the completion of $\mathbb{Q}_p$ by $\mathbb{C}_p$.

Proposition 7.1. *The field $\mathbb{C}_p$ is algebraically closed. As a $\mathbb{Q}_p$-vector space it is infinite dimensional. It is separable, but not locally compact. The residue class field of $\mathbb{C}_p$ is the algebraic closure of $\mathbb{F}_p$ (the field of p elements).*

(**7.3.21**) Perhaps the most surprising result concerning $\mathbb{C}_p$ (and the one that shall be the key in our applications) is

Theorem 7.2. *As fields we have $\mathbb{C}_p \cong \mathbb{C}$.*

(**7.3.22**) SUMMARY.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{Z} & \hookrightarrow & \mathbb{Q} & \hookrightarrow & \mathbb{C} \\
 & & | & & \downarrow & & \downarrow \\
 & & | & & \mathbb{Q}_p & \overset{c}{\hookrightarrow} & \mathbb{C}_p \\
 & & \downarrow & \overset{a}{\nearrow} & & & \downarrow \\
\mathbb{N} & \hookrightarrow & \mathbb{Z}_p & & & & 0 \\
 & & & \searrow & & & \\
 & & & & \mathbb{F}_p & & \\
n & \mapsto & \sum_{i\geq 0} n_i p^i & \mapsto & n_0 & &
\end{array}
$$

vertical sequences exact, a is an inclusion.

7.3.4 Idempotents over the p–adics

ss:porder1

(**7.3.23**) Why might we be interested in the p-adic numbers, from a representation theory perspective? The reason is that they provide a concrete example of the following general setup. (See [34, 7] for more details.)

Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K of characteristic zero, maximal ideal $\mathfrak{p}$, and quotient field $k = \mathcal{O}/\mathfrak{p}$. Then if $p \in \mathcal{O}$ generates $\mathfrak{p}$ we call the triple $(K, \mathcal{O}, k)$ a p-MODULAR SYSTEM.

By the various results collected together in the previous section, one can check that $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ is an example of such a system. (The only fact we have not explicitly seen is the identification of k with $\mathbb{F}_p$ in this case — but this follows from the description of the invertible elements in $\mathbb{Z}_p$.)

(**7.3.24**) Let Λ be an algebra over $\mathcal{O}$ that as an $\mathcal{O}$-module is free of finite rank. We set $\bar{\Lambda} = k \otimes_{\mathcal{O}} \Lambda$, and extend this notation to elements of the tensor product in the obvious way. The importance of p-modular systems is that we can lift idempotents from algebras defined over the quotient field by the following theorem.

(**7.3.25**) THEOREM. *(i) Let e be an idempotent in $\bar{\Lambda}$. Then there is an idempotent f in Λ with $e = \bar{f}$. If e_1 is conjugate to e_2 in $\bar{\Lambda}$, with $\bar{f}_1 = e_1$ and $\bar{f}_2 = e_2$, then f_1 is conjugate to f_2 in Λ.*

(ii) By this lifting process we may lift a decomposition of 1 into a sum of primitive orthogonal idempotents from $\bar{\Lambda}$ to Λ.

(iii) Suppose that reduction modulo $\mathfrak{p}$ is a surjective map between the respective centres of Λ and $\bar{\Lambda}$. Then we may lift a decomposition of 1 into a sum of primitive central idempotents from $\bar{\Lambda}$ to Λ.

We shall see later that these results on idempotents provide a paradigm for passing representation theoretic information between algebras constructed by different base changes.

(**7.3.26**) Very briefly, this is exemplified as follows: In the group algebra $\mathbb{F}_2 S_n$ $(n > 2)$ we have $e = (1 + (123) + (321))$ obeying

$$(1 + (123) + (321))^2 = (1 + (123) + (321)) \qquad (mod.2)$$

so we should be able to lift this to $\mathbb{Z}_2 S_n$. Considering $f = (\ldots 10101011)(1 + (123) + (321)) \in \mathbb{Z}_2 S_n$ we see that this reduces to e mod.2; and that

$$(1 + (123) + (321))^2 = (\ldots 0011)(1 + (123) + (321))$$

and $(\ldots 0011)(\ldots 10101011) = (\ldots 001)$, so

$$f^2 = (\ldots 10101011)^2(\ldots 0011)(1 + (123) + (321)) = f$$

Of course the inclusion of $\mathbb{N}$, and the inverses of odd numbers, in $\mathbb{Z}_2$ means that $(\ldots 0011) = 3$ and $(\ldots 10101011) = \frac{1}{3}$. Since $\mathbb{Q}$ (which includes in $\mathbb{Q}_2$) is a splitting field for S_n we anticipate that we can decompose this idempotent further over $\mathbb{Q}S_n$. But any further decomposition requires an inverse of 2, which we do not have *here* over $\mathbb{Z}_2$. Over $\mathbb{Q}$ we have

$$f = e_{(3)} + e_{(1^3)}$$

(each term on the RHS requiring coefficients of form $\frac{1}{6}$).

Chapter 8

Ring–modules

ch:ringmod

8.1 Ring–modules

Here R will be a ring.

(**8.1.1**) (Left) R–Module M: M an abelian group with map $R \times M \to M$ (written $(r, m) \mapsto rm$) such that $r(x + y) = rx + ry$, $(r + s)x = rx + sx$,

$$(rs)x = r(sx) \qquad \text{(`left-action condition')},$$

$1x = x$ ($r \in R$, $x, y \in M$).

Right modules defined similarly (but instead $(rs)x = s(rx)$; or equivalently we can change the notation to $(r, m) \mapsto mr$ giving $x(rs) = (xr)s$).

8.1.1 Examples

(**8.1.2**) Example. Ring R is both a left and a right R–module by the ring multiplication (on the left and on the right respectively).
Every left R–module is a right R^{op} module, where R^{op} is the opposite ring.

(**8.1.3**) Example. Let M be a left module and $m \in M$, then

$$Rm = \{rm \mid r \in R\}$$

ex:CRmod1 is a submodule.

In particular consider $\mathbb{C}$ as an $\mathbb{R}$-module. Then $\mathbb{R}1 = \mathbb{R}$ is a submodule — the real line; and $\mathbb{R}i$ is a submodule — the imaginary line; and so on. Note here that there are infinitely many such submodules, but the sum of any two of them is $\mathbb{C}$.

(**8.1.4**) Example. The set $R \times R$ is an abelian group by $(a, b) + (c, d) = (a + c, b + d)$ and an R-module by the action $r(a, b) = (ra, rb)$.

(**8.1.5**) Example. Let k be a field and V a k-space. Then the set $\mathrm{End}(V)$ of k-linear maps on V is a ring (and a k-algebra). Thus V is a module over this ring.

In the k-algebra setting, given a subset $S \in \mathrm{End}(V)$ we may write $\langle S \rangle_k$ for the smallest subalgebra containing S (the subalgebra *generated* by S). Thus V is also a $\langle S \rangle_k$-module.

(**8.1.6**) In general a k-space V as above comes with various possible choices of bilinear form defined upon it (Cf. §10.1.5). (If $k = \mathbb{R}$ or $\mathbb{C}$ we even have an *inner product* — a (conjugate) symmetric (sesqui/)bilinear form $(-,-)$ with associated positive definite quadratic form $q(v) = (v,v)$ (*positive definite*: $v \neq 0$ implies $q(v)$ real, positive) — in the sense of (**??**).)

Fixing such a product $(-,-)$, then each subspace $M \subset V$ has an *orthogonal complement* — the subset $M' \subseteq V$ such that $(m', m) = 0$ for $m' \in M'$ and all $m \in M$.

Now suppose that M is a submodule of V as a module over one of the above subalgebras of $\mathrm{End}(V)$. We can investigate the conditions under which M' is also a submodule. (We consider this further in §10.)

In case $k = \mathbb{C}$, a striking examples of an inner product is the *Hermitian product*

$$(a, b) = \sum_i (a_i)^* b_i$$

where $*$ denotes complex conjugate, and a_i is the coefficient in the standard basis. In this case evidently $M \cap M' = \{0\}$. Now consider a set S of matrices realising a linear action on V (in the standard basis) that fix M. That is, consider S such that M is an $\langle S \rangle_{\mathbb{C}}$-submodule of V. It follows (exercise) that the set of matrices obtained from S by hermitian conjugate fixes M'. And hence if S is fixed setwise by hermtitian conjugation, then it follows that $V = M \oplus M'$ (see (8.2.9)) as a module for the corresponding algebra.

8.1.2 The lattice of submodules of a module

(**8.1.7**) For any module M the set of submodules partially ordered by inclusion forms a lattice. (Meet and join are intersection and sum respectively.)

Example: Consider the R-module $R \times R$ defined above, and suppose for simplicity that R is a field. Then the submodule lattice looks, in part, like

$$\begin{tikzcd}
 & & R \times R \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{r(1,0) \mid r \in R\} \arrow[drr, dash] & \{r(0,1) \mid r \in R\} \arrow[dr, dash] & \{r(1,1) \mid r \in R\} \arrow[d, dash] & \ldots \arrow[dl, dash] \\
 & & \{(0,0)\} &
\end{tikzcd}$$

Setting $R_{ab} = \{r(a,b) \mid r \in R\}$ note that

$$(R_{10} + R_{01}) \cap R_{11} = R_{11}$$

$$(R_{10} \cap R_{11}) + (R_{01} \cap R_{11}) = R_{00}$$

so the distributive law does *not* hold in this lattice.
(Later we will consider finite dimensional modules for algebras over fields. There we will see that the distributive law holds iff the module has no section isomorphic to a direct sum of isomorphic simple modules. Note that our example illustrates the only-if part of this.)

8.2 R-homomorphisms and the category R-mod

ss:R-hom

(**8.2.1**) For M, N left R-modules, an R-module homomorphism, or R-homomorphism, is an element

$$f : M \to N$$

of $\hom_{\mathbf{Ab}}(M, N)$ such that $f(rm) = rf(m)$ $(r \in R,\ m \in M)$.

Note that such an f is, in particular, an abelian group homomorphism, so that $\ker f$ is defined.

We write $\mathrm{Hom}_R(M, N)$ for the set of these homomorphisms. We write $\mathrm{End}_R(M)$ for $\mathrm{Hom}_R(M, M)$.

(**8.2.2**) The class $R-$mod of left R–modules is a category, with morphisms the R–module homomorphisms. (mod $- R$ defined similarly for right modules.)

8.2.1 quotients

(**8.2.3**) If M' is a submodule of $M \in R-$mod then there is an action of R on the cosets of M' in M given by

$$r(m + M') = rm + M'$$

making the collection of cosets M/M' a left R-module.

For example, in (8.3.2) we have $I_d/I_{d-1} \cong I_2$ for all $d > 1$.

(**8.2.4**) If $f \in \mathrm{Hom}_R(M, N)$ then $\ker f$ is an R-submodule of M; $f(M) = \mathrm{im}\, f$ is an R-submodule of N; and $f(M)$ and $M/(\ker f)$ are isomorphic R-modules.

(**8.2.5**) Let $f \in \mathrm{Hom}_R(M, N)$. For $n \in N$ define $f^{-1}n$ (as for any map $f : M \to N$) as the set of elements m of M such that $f(m) = n$. For $S \subset N$ then $f^{-1}S$ is the subset of M whose images lie in S.

Note that f^{-1} is not an R-homomorphism (it is not even a set map). However:

pr:isRmod (**8.2.6**) Proposition. *If N' is a submodule of N then $f^{-1}N'$ is a submodule of M. If M' is a submodule of M then fM' is a submodule of N.*

Proof. If $a, b \in f^{-1}N'$ then $f(a), f(b) \in N'$ so $f(a) + f(b) \in N'$ so $f(a) + f(b) = f(a + b) \in N'$, so $a + b \in f^{-1}N'$; and if $r \in R$ then $f(ra) = rf(a) \in N'$ so $ra \in f^{-1}N'$. The proof of the other claim is similar. □

(**8.2.7**) Remark. For more on quotients see for example Zariski–Samuel [150, §III.3].

8.2.2 Direct sums and simple modules

ss:DSandSimple

(**8.2.8**) If M, N are R-modules then the external direct sum $M \dotplus N$ is $M \times N$ with componentwise addition and

$$r(m, n) = (rm, rn)$$

This is an R-module. Further

$$M' = \{(m, 0) \mid m \in M\}$$

is a submodule (as is N' defined similarly).

de:sum mod **(8.2.9)** If M_1, M_2 submodules of R-module M then we write $M_1 + M_2$ for the obvious subset of M. This is another submodule.

(8.2.10) If M_1, M_2 submodules of R-module M we write

$$M_1 + M_2 = M_1 \oplus M_2$$

if $m_1 + m_2 = 0$ ($m_i \in M_i$) implies that each $m_i = 0$.

The module $M_1 \oplus M_2$ is the *(internal) direct sum* of M_1 and M_2. This extends to $\oplus_i M_i$.

Refering back to the external direct sum we have:

$$M \dotplus N = M' \oplus N'$$

p:irred **(8.2.11)** A left R-module M is *irreducible* (or *simple*) if $M' \subset M$ implies $M' = \{0\}$.

pr:schur **(8.2.12)** PROPOSITION. [SCHUR'S LEMMA] *Let S be a simple R–module. Then $End_R(S)$ is a division ring.*

Proof. Let $f \in \mathrm{End}_R(S)$ be non–zero. The kernel of f is a submodule of S, so it is empty. Thus f is an injection. Similarly the image of f is S, so f is a surjection and hence a bijection, and so has an inverse. □

e:semisimple mod **(8.2.13)** SEMISIMPLE MODULE M, M is a module which is a direct sum of simple modules.

(8.2.14) A non-zero left R-module M is *indecomposable* if it cannot be expressed as a direct sum of two non-zero submodules.

Example: The ring $T_2'(\mathbb{C})$ (from (7.1.5)) is indecomposable as a left-module for itself. It is not irreducible, since $U_2(\mathbb{C})$ is a submodule, but the only other nonzero submodule is $T_2'(\mathbb{C})$ itself, so there is evidently no direct sum decomposition.

(8.2.15) A diagram $L \xrightarrow{f} N \xrightarrow{g} M$ in R-mod is *exact at N* if $im(f) = ker(g)$.
A finite sequence of maps in R-mod is an *exact sequence* if it is exact at every step.
An exact sequence of form

$$0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$$

is called a *short exact sequence*.
If such a sequence has a reverse (there is an $f' : N \to L$ with $f'f = 1_L$), it is *split*.

For example, the natural sequence

$$0 \to L \to L \oplus M \to M \to 0$$

is split.

pa:splito **(8.2.16)** Note that R-mod is an additive category (as in 6.2.9), with the category direct sum given by module direct sum. In particular, for every split short exact sequence

$$0 \rightleftarrows L \underset{a'}{\overset{a}{\rightleftarrows}} N \underset{b}{\overset{b'}{\rightleftarrows}} M \rightleftarrows 0$$

there is an idempotent decomposition of $1_N \in \mathrm{End}_R(N)$:

$$aa' + bb' = 1_N$$

ss:free modules

8.2.3 Free modules

(**8.2.17**) A set $\{m_i\}$ of elements of an R-module M is called *R-free* if the only solution to

$$\sum_i r_i m_i = 0$$

is $r_i = 0$ for all i.

(**8.2.18**) A set $\{m_i\}$ of elements of R-module M is called a *set of generators* of M if every $m \in M$ can be expressed in the form

$$m \;=\; \sum_i r_i(m)\, m_i \qquad r_i(m) \in R$$

an R-linear combination of a finite number of the $\{m_i\}$.

de:basis

(**8.2.19**) A set of generators of M that is R-free is called a basis.
A (left) R-module with a basis is called a free (left) R-module.

(**8.2.20**) Free module M, M is a module with a basis.

(**8.2.21**) For S any set we may write RS for 'the free (left) R–module with basis S'. (Caveat: this notation is not entirely safe, but useful. A module M could have a subset S that is a basis. But in that case we do not *start* with S. For example $\{(x, y)|x, y \in \mathbb{R}\}$ is an $\mathbb{R}$-module with basis $\{(1, 0), (0, 1)\}$, while $\mathbb{R}\{0, 1\}$ and $\mathbb{R}\{(1, 0), (0, 1)\}$ are isomorphic modules generated by some dangerously named formal symbols! Later we will meet modules such as (in this notation) $\mathbb{CR}$ and $\mathbb{RR}$, so care will be needed.) This RS is the set of formal sums of the form

$$m = \sum_{s \in S} r_s s, \qquad r_s \in R \tag{8.1}$$ eq:freem1

where only finitely many of the r_s are non-zero ('finite support'); and $s \in S$ is identified with the element given by $r_s = 1_R$ and $r_t = 0_R$ otherwise. Addition and multiplication are given in the obvious way.

Note that S might be uncountably infinite (so, at least without the finite support condition, the notations like $\sum_i$ above are somewhat stretched). The formal sum can also be seen as the subset of $\hom(S, R)$ of elements f with $f(s) = 0$ for all but finitely many $s \in S$.

Note that given $m \in RS$ the coefficients r_s in the sum (8.1) are uniquely determined.

(**8.2.22**) Examples. See e.g. §8.4.

(**8.2.23**) There is a forgetful functor (as in §6.1.1, 6.1.35) from $R-$mod to **Set**. One can consider the construction of a left adjoint (as in 6.3.7) to this functor.

free module

(**8.2.24**) Proposition. *For any R–module M there is a short exact sequence*

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is free.

(**8.2.25**) Suppose (R commutative and) that $\rho : A \to M_n(R)$ is a representation of an R-algebra A. Let $\{b_1, .., b_n\}$ be a set of symbols, and let M be the free R-module with this set as basis. Then the action of A on M given by $ab_i = \sum_j \rho(a)_{ij} b_j$ makes M an A-module.
Note however that this M is not a free A-module in general.

8.2.4 Matrices over R and free module basis change

ss:fmbc

Here we shall take R to be commutative.

(**8.2.26**) A matrix $Y \in M_n(R)$ is *unimodular* if there exists Y' such that $YY' = Y'Y = 1_n$. Equivalently Y is unimodular if $\det(Y)$ is a unit in R.

(**8.2.27**) Matrices $S, T \in M_{m,n}(R)$ are *equivalent* if $S = Y_1 T Y_2$ with Y_i unimodular. We write $S \sim T$.

(**8.2.28**) Let M be a free R-module with ordered basis $m = (m_1, m_2, .., m_n)$. Let $Y \in M_n(R)$. Then $m' = mY$ is a basis iff Y unimodular.

(**8.2.29**) Example. From (4.1.5) *et seq*, for the monoid algebra of $\mathbf{Set}(\underline{2},\underline{2})$ over $\mathbb{Z}$, say, we have

$$(-11+12+21-22,\ 11,\ 11-22,\ 12-21) \ = \ (11,12,21,22)\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and $\det\ Y = -(-1+(-1)) = 2$.

(**8.2.30**) Example. From (2.1.2) *et seq* consider the usual basis of the $n = 4$ Temperley–Lieb standard module $\Delta_n(0)$, given in terms of partitions of $\underline{4}$, thus $\{(12)(34),\ (14)(23)\}$. We have the corresponding cv-form gram matrix $\begin{pmatrix} \delta^2 & \delta \\ \delta & \delta^2 \end{pmatrix} = \delta \begin{pmatrix} \delta & 1 \\ 1 & \delta \end{pmatrix}$. Working over $\mathbb{C}[\delta]$, say, we have

$$\underbrace{\begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix}}_{Y_1} \underbrace{\delta \begin{pmatrix} \delta & 1 \\ 1 & \delta \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} -\delta & 1 \\ -1 & 0 \end{pmatrix}}_{Y_2} = \delta \begin{pmatrix} \delta^2-1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that we changed the basis differently on each side here.

For $\Delta_4(2)$ we have basis $\{(12)(3)(4), (1)(23)(4), (1)(2)(34)\}$ giving gram matrix T as shown below:

$$\underbrace{\begin{pmatrix} 1 & -\delta & \delta^2-1 \\ 0 & 1 & -\delta \\ 0 & 0 & 1 \end{pmatrix}}_{Y_1} \underbrace{\begin{pmatrix} \delta & 1 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} \delta^2-1 & 1 & 0 \\ -\delta & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{Y_2} = \begin{pmatrix} \delta(\delta^2-2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

-homework: check this!

Recall that $\mathbb{Z}$ is a principle ideal domain (PID); that $\mathbb{Z}[x]$ is not a PID; but that $k[x]$ is a PID if k is a field.

(**8.2.31**) Theorem. *Suppose R a PID. Let $S \in M_{m,n}(R)$. Then*

$$S \sim diag(d_1, d_2, .., d_l, 0, 0, .., 0)$$

where $d_i | d_{i+1} \in R$. The elements $\{d_i\}$ are the *invariant factors* of S (defined up to units in R). The diagonal matrix is the *Smith normal form*.
See e.g. [32] for a proof.

th:RPIDsubfree (**8.2.32**) THEOREM. *Let R be a PID. If M is a free R-module of rank n and N is a submodule, then N is free of rank $n' \leq n$. (See e.g. [32, §16.1].) Further, there is an ordered basis $(b_1, b_2, ..., b_n)$ of M, and elements $r_1|r_2|...|r_{n'} \in R^*$ such that $(r_1b_1, r_2b_2, ..., r_{n'}b_{n'})$ is an ordered basis for N. The r_i are uniquely determined up to units. (See e.g. [32, §16.8].)*

8.3 Finiteness issues

ss:finite1

(**8.3.1**) A module M is *finitely generated* (fg) if there is a finite set of generators.

Example: $R = R1$ is finitely generated. (Next we construct a non-fg module.)

de:egDCC (**8.3.2**) Fix a prime p. Consider the abelian additive group I of rational numbers of form $\frac{q}{p^l}$, modulo the integers. Thus writing simply x for $x + \mathbb{Z}$:

$$\frac{80}{3^4} + \frac{79}{3^4} \equiv \frac{26}{3^3}$$

Regard I as a $\mathbb{Z}$-module. The subset $I_d = \{0, \frac{q}{p^l} \mid d > l > 0\}$ is a finite subgroup and a submodule for any $d \in \mathbb{N}$. For example, $I_3 = \{\frac{q}{p^2} \mid q = 0, 1, 2, ..., p^2 - 1\}$.

We have

$$\{0\} = I_1 \subset I_2 \subset I_3 ...$$

I_2 is generated by $\frac{1}{p}$, and I_d by $\frac{1}{p^{d-1}}$, but I is not finitely generated as a $\mathbb{Z}$-module.

(**8.3.3**) The submodules of an R-module M satisfy the *ascending chain condition* (ACC) if every chain

$$M_1 \subset M_2 \subset M_3 \subset ...$$

terminates (i.e. there is an index i such that the chain cannot be extended on the right beyond M_i, except by $M_{i+1} = M_i$).

DCC means, analogously, that every descending chain terminates.

th:accfg (**8.3.4**) THEOREM. *TFAE:*
(i) The submodules of R-module M satisfy ACC
(ii) Every submodule of M is finitely generated
(iii) Every collection of submodules of M contains a maximal element.

(**8.3.5**) Our example (8.3.2) above satisfies DCC but not ACC.

(**8.3.6**) THEOREM. *Submodules of left R-module M satisfy DCC iff every set $\{M_i \mid M_i \subset M\}$ has a minimal element.*

(**8.3.7**) If R is a commutative integral domain in which every ideal is principle (PID), then the left ideals of R satisfy ACC.

(**8.3.8**) A ring whose left ideals satisfy DCC (i.e. a left-Artinian ring) is sometimes called a *ring with (left) minimum condition* (MC).

Examples: All finite dimensional algebras over fields are rings with MC. On the other hand the ring $\mathbb{Z}/p_1p_2\mathbb{Z}$ is finite, with p_1p_2 elements, and so has MC, but is not an algebra over a field (since this field would be finite, and any vector space over it would be finite and have prime power elements).

Ring $\mathbb{Z}$ itself does not satisfy MC.

(**8.3.9**) The *socle* of a left R-module M is the sum of its irreducible submodules.
Of course R may not have any irreducible submodules, if it does not have MC.

8.3.1 Radicals and semisimple rings

(**8.3.10**) The *radical* of a module is the intersection of its maximal submodules. (Set $\mathbf{rad} M = M$ if there are no maximal submodules.)
By Theorem (8.3.4) a nonzero fg R-module has

$$\mathbf{rad}\ M \subset M$$

(**8.3.11**) If module M satisfies DCC then $\mathbf{rad}(M) = 0$ iff M is a finite direct sum of irreducible modules.

(**8.3.12**) The *Jacobson radical* J of a ring R is the radical of R as a left-module for itself.

That is, J is the intersection of the maximal left ideals of R. (One can check that this J is a two-sided ideal; and the same as the intersection of maximal right ideals.)
Example: $\mathbf{rad}\ \mathbb{Z} = \{0\}$.
If R is a ring with Jacobson radical J then

$$\mathbf{rad}\ M_n(R) = M_n(J)$$

Let $\mathrm{Tri}_n(R)$ denote the ring of upper-triangular $n \times n$ matrices over R, and $U_n(R)$ the ideal of strictly upper-triangular matrices (0s on the diagonal). Then $\mathbf{rad}\ \mathrm{Tri}_n(K) = U_n(K)$ for any field K.

(**8.3.13**) Nil ideal I of ring R is an ideal such that for each $x \in I$ there is a natural number n such that $x^n = 0$.

(**8.3.14**) Note that every nil ideal of R is contained in the radical J.

(**8.3.15**) Nilpotent ideal I of ring R is an ideal such that there is a natural number n such that $\prod_{i=1}^n x_i = 0$ for every n-tuple $x \in I^{\times n}$.

(**8.3.16**) Let R be a ring with radical J. Note that every nilpotent ideal is a nil ideal — specifically $r \in N$ such that $N^n = 0$ implies $r^n = 0$. (In a (left) artinian ring every nil ideal is nilpotent. [1]) Thus every nilpotent ideal is contained in the radical. If R is (left) artinian then J is nilpotent.

(**8.3.17**) A ring is said to be *semisimple* if the Jacobson radical $J = 0$.

(**8.3.18**) Cf. the definition 8.2.13 of semisimple module. Some workers define a '(left) semisimple ring' to be a ring whose left regular module is semisimple. The two definitions do not coincide in general, but they do for artinian rings:

[1] On the other hand consider the ring $\mathbb{Z}[x_1, x_2, ...]/\{x_i^i = 0\}$. The set of polynomials with vanishing constant term form an ideal I (non fg, indeed generated by $\{x_2, x_3, ...\}$ say, so the ring is not artinian) and each individual polynomial is evidently nilpotent, so the ideal is nil. However there is no n such that $I^n = 0$, since in particular $x_2 x_3 ... x_{n+2} \in I^n$ is not zero for any n.

(**8.3.19**) THEOREM. *A ring whose left regular module is semisimple is left (and right) artinian and has radical $J = 0$.*

A left artinian ring with radical $J = 0$ has semisimple left regular module.

de:mc rad (**8.3.20**) THEOREM. *For a ring R with MC the Jacobson radical coincides with the (two-sided) ideal which is the sum of all nilpotent left ideals.*

(**8.3.21**) LEMMA. *If R has MC and is semisimple then every R-module is a direct sum of irreducible modules.*

Conversely, if R has MC and is a direct sum of irreducible left modules (as a left module for itself) then it is a semisimple ring.

th:wedda (**8.3.22**) THEOREM. [Wedderburn-Artin] *If ring R is semisimple and has MC then it is isomorphic to a direct sum of a uniquely determined set of matrix rings over division rings $\{M_{d_i}(D_i) \mid i = 1, ..., r\}$. This index set $i = 1, ..., r$ also indexes the isomorphism classes of simple modules.*

8.3.2 Composition series

ss:compos

de:filtration2 (**8.3.23**) Let Γ be a set of R-modules. An R-module M has a Γ-*filtration* if there is a chain of modules $M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$ such that every factor M_k/M_{k+1} is isomorphic to some element of Γ.

(**8.3.24**) A chain of modules

$$M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$$

is a *composition series* for M if the factors M_k/M_{k+1} are irreducible.

(**8.3.25**) A left R-module M has a composition series iff it satisfies ACC and DCC.

(**8.3.26**) THEOREM. *[Jordan-Holder] Any two composition series for a left R-module are equivalent (i.e. of the same length, and the sequence of factors is the same up to order and R-isomorphism).*

Proof. Outline: Let $M \supset A_1$ and $M \supset B_1 \supset B_2$ be chains of submodules. Then

$$B_2 + (A_1 \cap B_1) = (B_2 + A_1) \cap B_1$$

(just show an inclusion both ways). Now suppose $A_1 \supset A_2$. Then one can show

$$\frac{A_1 \cap B_1}{(B_1 \cap A_2) + (A_1 \cap B_2)} \cong \frac{(A_1 + B_2) \cap B_1}{(A_2 + B_2) \cap B_1}$$

Let $M \supset ... \supset M_i \supset M_{i-1}$ and $M \supset ... \supset M'_j \supset M'_{j-1}$ be series of submodules. We can refine the first by

$$M_i \supseteq ... \supseteq (M'_j + M_{i-1}) \cap M_i \supseteq (M'_{j-1} + M_{i-1}) \cap M_i \supseteq ... \supseteq M_{i-1}$$

and the second analogously. This gives us series of equal length. There may be many equalities, in different positions, in each. But the isomorphism above allows us to match the sections up in pairs. □

(**8.3.27**) If S is an irreducible R-module we write $[M : S]$ for the multiplicity of S as a composition factor in M up to isomorphism.

(**8.3.28**) THEOREM. *If submodules of a left R-module M satisfy DCC then M can be expressed as a direct sum of finitely many indecomposable modules.*

(**8.3.29**) THEOREM. [Krull-Schmidt] *If M is a left R-module satisfying ACC and DCC then any two decompositions into a direct sum of indecomposables have the same length, and an ordering bringing the summands into pairwise isomorphism.*

(**8.3.30**) In summary, we have seen that modules satisfying ACC and DCC are characterised in large part by the list of their simple factors, together with the possible orderings of these factors. We are interested in fd algebras over fields, for which modules satisfying ACC and DCC are available. We will see that the possible orderings are determined by the radical. So interest turns naturally to the construction and 'detection' of simple modules.

8.3.3 More on chains of modules and composition series

Suppose we have a chain of R-modules $M = M_0 \supset M_1 \supset \ldots \supset M_l = 0$ and not every section is simple. In particular, suppose $M_i/M_{i+1} = N = N_0 \supset N_1 \supset N_2 = 0$. Can we refine the first chain using the second? I.e. can we insert M'_i in $M_i \supset M_{i+1}$ so that $M_i/M'_i \cong N_0/N_1$ and $M'_i/M_{i+1} \cong N_1/N_2 = N_1$?

pr:chain refine

(**8.3.31**) PROPOSITION. *(Cf. [150, §III.4 Th.4]) Let $M_0 \supset M_1$ be R-modules, and $f : M_0 \to M_0/M_1$ be the quotient map; i.e. there is a short exact sequence*

$$0 \to M_1 \to M_0 \xrightarrow{f} M_0/M_1 \to 0$$

Every submodule $L' \subset M_0/M_1$ can be expressed as M'_0/M_1 where R-module $M'_0 = f^{-1}L'$ obeys $M_0 \supset M'_0 \supset M_1$.

Proof. Note that $f^{-1}L' \subset M_0$ (it is the set of elements of M_0 taken to elements of L' by f). We need to show (i) that it is an R-module; and (ii) that $M'_0/M_1 \cong L'$. For (i) we note Proposition 8.2.6. For (ii) we note that $f(f^{-1}L') = fM'_0 = L'$. □

(**8.3.32**) COROLLARY: Any R-module chain that is not a composition series can be refined.

PROOF: Consider $M_i \supset M_{i+1}$ with M_i/M_{i+1} not simple. Then there is a proper submodule $L' \subset M_i/M_{i+1}$. Then for $f : M_i \to M_i/M_{i+1}$ we have $f^{-1}L' \subset M_i$, refining the chain between M_i and M_{i+1}.

8.4 Tensor product of ring-modules

ss:tensor1

See also Th.8.9.1 et seq.

One can take the view that restriction is a fairly natural way of getting a 'new' module from an existing module — one is keeping the module as a set, and almost simply forgetting some of the properties of the module. Here 'natural' means that we do not need to make choices. One can then approach tensor products by asking if there is an adjoint, a left adjoint in the sense of (6.3.7), to this restriction construction. (As is often the case with trying to construct adjunctions, there are then questions of existence and uniqueness.)

de:rest funct

(**8.4.1**) Given a pair of rings $R \subset S$ we have a functor

$$\mathrm{Res}_R^S : S\!-\!\mathrm{mod} \;\; \to \;\; R\!-\!\mathrm{mod}$$

where $\mathrm{Res}_R^S M = M$, an R-module via inclusion in the S-action ($r \in R \subset S$).

Indeed, given any ring homomorphism $\psi : R \to S$ each $M \in S-\mathrm{mod}$ becomes an R-module via ψ; and this extends to a functor Res_ψ. (This functor is exact — see e.g. Anderson–Fuller [3, §16 Ex.1]; and cf. §8.5.2.)

(**8.4.2**) We may write ${}_SM$ to indicate that M is an S-module above. Then we may write simply ${}_RM$ for $\mathrm{Res}_R^S M$.

(**8.4.3**) The *restriction functor* Res_R^S- introduced in (8.4.1) will be very useful (see e.g. §8.9.1). It would also be useful to have a 'paired' functor going the other way (see also e.g. §8.5.2). One of the ways to do this requires a new technical device — the *tensor product*.

The idea, mechanistically speaking, is that if we have an R, S-bimodule ${}_RM_S$ and a left S-module ${}_SN$ then we should be able to 'glue' these together at their 'dual' S-module structures to make a new left R-module in a natural way. To think about what this means in practice, the first thing is to forget about the R-module structure for a moment, and just think about how to make a suitably 'balanced gluing' of the S-module structures.

What is the minimum property that this glued object should have? Later it is going to be an R-module, but we are not looking at R for now, so what property is common to all modules for all rings? Certainly they are all sets, so we must glue M and N to make a set — such as $M \times N$ — but also they are all abelian groups. The Cartesian product is not of itself a group. It can be given an internal additive structure but (since we will see later that this does not give an adjunction) we can start with something 'free-er'.

(**8.4.4**) The free abelian group on a set A is the free $\mathbb{Z}$-module with basis A (cf. §8.2.3), denoted $\mathbb{Z}A$.
Remark. Here note that A may be uncountable. The module is the subset of $\hom(A, \mathbb{Z})$ of elements f of finite support, i.e. for which $f(a) = 0$ for all but finitely many $a \in A$. We define $f + f'$ by $(f + f')(a) = f(a) + f'(a)$ (NB a different +!). Note that this is indeed a group. We can write 0 for the identity. We can consider a set map $\iota : A \to \mathbb{Z}A$ by $a \mapsto f_a$ where $f_a(a) = 1$ and $f_a(b) = 0$ for $b \neq a$. The elements f_a generate $\mathbb{Z}A$. (If A is, say, uncountable, this is certainly not finite generation!) Where it is safe to do so, we might write simply a for $f_a \in \mathbb{Z}A$. Then we have elements like $a + b$ and $a + a$ and $a - b$, and $a - a = 0$.

(**8.4.5**) So now we have in mind $\mathbb{Z}(M \times N)$. On the other hand $\mathbb{Z}(M \times N)$ is rather big, and takes no account of the S-module structures inside (neither their group property nor the ring actions). We have elements in $\mathbb{Z}(M \times N)$ such as $f_{(m,n)}$ and, with our notation above, $(m, n) + (m', n')$, $(m, n) - (m', n')$ and so on. In particular we have elements like $(m + m', n)$ and $(m + m', n) - (m, n) - (m', n)$, where we have used the additive structure in M. The set of all elements of the latter form (as m, m', n vary) generate a subgroup, H_M say. The quotient group by this subgroup obeys $(m + m', n) + H_M = (m, n) + (m', n) + H_M$ — we can say it is left-additive. Observe that H_M is the smallest subgroup we could quotient by to achieve this.

Similarly we can use the additive structure in N, and we can use both additive structures, thus generating a subgroup H_{MN}. This is then the smallest we could quotient by to achieve left- and right-additivity.

Now consider (mr, n) with $r \in R$. For the same m, n, r we have also (m, rn), and we are now bridging between the two module structures. In particular consider $(mr, n) - (m, rn)$, and the group S_{MN} generated by all elements of this form, together with the generators above. The quotient $\mathbb{Z}(M \times N)/S_{MN}$ has components in its construction making use of all aspects of the two modules. So far it remains rather ad hoc that we use them in these particular ways. (But let us keep going! Later we will have the question of adjunction to test the construction against.)

de:balanced (**8.4.6**) Let R be a ring; M a right R-module; N a left R-module; and Q an additive abelian group. A *balanced map* is a set map

$$g : M \times N \to Q$$

that is (B1) a biadditive map, i.e.
(B1l) left additive: $g(m + m', n) = g(m, n) + g(m' + n)$, and
(B1r) right additive;
such that
(B2) $g(m, rn) = g(mr, n)$.

rem:tp universal (**8.4.7**) Remark. Given R, M, N as above, a balanced map $\psi : M \times N \to T$ is 'universal' if for every balanced map $\phi : M \times N \to Q$ there is a morphism of abelian groups $f : T \to Q$ such that $f \circ \psi = \phi$ (that is, if every ϕ factors through T).

(**8.4.8**) Let R be a ring; M a right R-module; and N a left R-module. Let S_{MN} be the subgroup of $\mathbb{Z}(M \times N)$ generated by all formal sums of form $(m + m', n) - (m, n) - (m', n)$, $(m, n + n') - (m, n) - (m, n')$, $(m, rn) - (mr, n)$. Let $z : M \times N \to \mathbb{Z}(M \times N)/S_{MN}$ via

$$z(m, n) = (m, n) + S_{MN}$$

Then z is a balanced map ((B1) is ensured by the first two types of sums appearing in S_{MN}; and (B2) by the third type).

(**8.4.9**) We define the tensor product

$$M \otimes_R N = \mathbb{Z}(M \times N)/S_{MN}$$

We write $m \otimes n$ for the image under z of (m, n).

(**8.4.10**) Remark. A general element of $M \otimes_R N$ is of form $\sum_i m_i \otimes n_i$. As noted above this is shorthand for the class $f + S_{MN}$ of an almost-always-zero function $f \in \hom(M \times N, \mathbb{Z})$. It is not uncommon to find this further abbreviated to $a \otimes b$ (i.e. a sum, and a suitable unpacking of a and b, is understood).

(**8.4.11**) Remark. In §8.5.2 we shall see that the object map $M_R \otimes_R - : R - \text{mod} \to \mathbf{Ab}$ is a functor.

rem:tp univ (**8.4.12**) Remark. The tensor product has the universal property. (*Outline Proof*: Any balanced map $g : M \times N \to Q$ lifts to a homomorphism g' of $\mathbb{Z}(M \times N)$ to Q in the obvious way. In particular $g'(S_{MN}) = 0$. Therefore the map $g'' : M \otimes_R N \to P$ given by $g''((m, n) + S_{MN}) = g(m, n)$ is well-defined. That is, $g''(z(m, n)) = g(m, n)$. □)

(**8.4.13**) Several further properties of tensor products are discussed in Th.8.9.4 et seq.

8.4.1 Notes and Examples of tensor products

(**8.4.14**) For any ring R and R-module N we have the isomorphism of abelian groups

$$R_R \otimes_R {}_R N \;\cong N, \tag{8.2}$$ eq:RxNN

since $r \otimes n \equiv 1 \otimes rn$.

ex:BRC1 (**8.4.15**) EXAMPLE. Consider the ring $R = \mathbf{Mat}_2(\mathbb{C})$ and the right R-module B of 2-component row matrices (with the natural right action of R); and the left R-module C of 2-component column matrices. We claim that $B \otimes_R C \;\cong \mathbb{C}$ as an abelian group.

Proof: First note that $R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C$ (C is simple). Thus every class in $B \otimes_R C$ has an element of form $v \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (the generic element $\xi \;= \sum_i m_i \otimes n_i = \;\sum_i m_i \otimes r_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some set of r_is in R, and this $\xi \;\equiv\; \sum_i m_i r_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\equiv\; (\sum_i m_i r_i) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by construction). Indeed every class has an element of form $(x, 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some $x \in \mathbb{C}$, since $v \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv v\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_1, 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Clearly every such representative (every $v_1 \in \mathbb{C}$) occurs, so it remains to show that no two distinct such representatives are in the same class. (In Re.8.4.16 we give a quick way to do this here. For now we continue with a general strategy.) So far we have shown a surjection $\mathbb{C} \to B \otimes_R C \to 0$ as an abelian group.

But the map $B \times C \to \mathbb{C}$ given by $(b, v) \mapsto b.c$ is a *balanced map* in the sense of (8.4.6). Now use a universal property as in (8.4.7,8.4.12) [**?**]. □

rem:BRC2 (**8.4.16**) REMARK. In the example (8.4.15) above it is perverse to consider $B \otimes_R C$ as merely an abelian group. The right module B is also a 'left' $\mathbb{C}$-module (and C is a 'right' $\mathbb{C}$-module) and one can see (exercise) that this property survives the construction, to make $B \otimes_R C$ a 'left' $\mathbb{C}$-module.

(**8.4.17**) EXAMPLE. With definitions as in (8.4.15) above, noting the subring $\mathbb{C} \hookrightarrow R$, what can we say about the abelian group $B \otimes_{\mathbb{C}} C$?

In this case we claim that the Kronecker product is a balanced map. This is a 4-dimensional $\mathbb{C}$-space. Noting that the tensor product is at most 4-dimensional as a $\mathbb{C}$-space, we conclude that the Kronecker product is the tensor product.

exa:odd2 (**8.4.18**) EXAMPLE. Consider $\mathbb{Q} \supset \mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$, since $x \otimes y = xpp^{-1} \otimes y = xp^{-1} \otimes py = 0$. Meanwhile $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$.

exa:S3S2 (**8.4.19**) EXERCISE. Regard $\mathbb{Z}S_3$ as a right $\mathbb{Z}S_2$-module by restriction. Compute the abelian group $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0$ where M_0 is the trivial $\mathbb{Z}S_2$-module.
Hints: Consider

$$S_3 = \;\langle \sigma_1 = (12),\; \sigma_2 = (23) \rangle \quad \supset \quad S_2 = \;\langle \sigma_1 \rangle.$$

We have $\mathbb{Z}S_3 = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. Now $\sigma_1 \otimes m = 1 * \sigma_1 \otimes m = 1 \otimes \sigma_1 m = 1 \otimes m$ and so on, so $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0 = \mathbb{Z}\{1 \otimes m, \sigma_2 \otimes m, \sigma_1\sigma_2 \otimes m\}$.

(**8.4.20**) Suppose M' a submodule of M as above. Then another balanced map $z' : M' \times N \to M' \otimes_R N$ arises immediately. In this way there are two possible meanings for $m' \otimes n$. Unfortunately it is not generally possible to embed $M' \otimes_R N$ in $M \otimes N$ (as comparison of the two examples in (8.4.18) shows), so care is needed with this notation.

(**8.4.21**) THEOREM. *Let $f \in Hom(M, M')_R$ (right modules) and $f' \in Hom_R(N, N')$. Then the map $(m, n) \mapsto f(m) \otimes f'(n)$ is balanced; and there is a unique map*

$$f \otimes f' : M \otimes_R N \to M' \otimes_R N'$$

such that

$$(f \otimes f')(m \otimes n) = f(m) \otimes f'(n)$$

(**8.4.22**) EXERCISE. Consider the case in which the ring is a field K, and M, N finite dimensional vector spaces, with bases B_M and B_N respectively. Construct a basis for $M \otimes_K N$, regarded as a vector space via the obvious left action of K on M.
Elevate this whole picture to describe homomorphisms of based tensor spaces (matrices) constructed from homomorphisms of their tensor factor spaces (again realised as matrices).

(**8.4.23**) MORE TO GO HERE! See e.g. §13.4.1 for more examples.

8.4.2 R-lattices etc

(**8.4.24**) For R a Dedekind domain (e.g. a PID) the torsion submodule of a module M is $\tau(M) = \{m \mid rm = 0, \ \text{ some } r \in R^*\}$.

(**8.4.25**) An R-lattice for a Dedekind domain R is a f.g. torsion-free R-module.
Let R^0 be the field of fractions of R. Then $V = R^0 \otimes_R M$ is a vector space. The dimension of this space is the R-rank of M. (See [35, §4D].)

8.5 Functors on categories of modules

(**8.5.1**) A, B–BIMODULE $_AM_B$ is a left A–module and right B–module such that $a(mb) = (am)b$ for all $a \in A$, $b \in B$, $m \in M$.

(**8.5.2**) REMARK. Jacobson [70, §9], for example, uses the term *left A, right B-module* for such a bimodule.

(**8.5.3**) A *left A– left B–bimodule* $M = {}_{A,B}M$ is a left A–module and a right B–module such that $a(bm) = b(am)$ for all $a \in A$, $b \in B$, $m \in M$.
(Again Jacobson omits the 'bi'.)

map by r mult (**8.5.4**) Let $_AM_B$ be a left A– right B–bimodule. Then for each $b \in B$ we may define an element $b' \in \mathrm{Hom}_A({}_AM_B, {}_AM_B)$ by $b'(m) = mb$. Note that

$$(bc)'(m) = m(bc) = (mb)c = c'(b'(m)). \qquad (8.3)$$ maps by r mult

ss:Hom

8.5.1 Hom functors

(**8.5.5**) For each $M \in R-$mod there is a (covariant additive) functor

$$\hom_R(M,-):\ R\text{−mod} \ \to\ \mathbf{Ab}$$

with object map given by $X \mapsto \hom_R(M,X)$. The action on maps is $f \in \hom_R(X,Y)$ goes to $f_* : \hom_R(M,X) \to \hom_R(M,Y)$ given by $f_* a = fa$, $a \in \hom_R(M,X)$.

pa:hom left exact

(**8.5.6**) Fix M as above and consider any $A' \xrightarrow{f} A \ \in\ \hom_R(A',A)$. What can we say about $\ker f_*$? Unpacking we have

$$\hom_R(M,A') \overset{\hom_R(M,f)}{\to} \hom_R(M,A)$$

$$\begin{tikzcd} M \xrightarrow{g} A' \arrow[r, phantom, "\mapsto"] & M \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & A \\ & & A' \arrow[ur, "f"'] & \end{tikzcd}$$

We have $\ker(\hom_R(M,f)) = \{M \xrightarrow{g} A' \mid f\circ g(M) = 0\} = \{M \xrightarrow{g} A' \mid g(M) \in \ker f\}$ so

$$\ker(\hom_R(M,f)) = \Psi_{A'}(\hom_R(M,\ker f))$$

— the isomorphic image of $\hom_R(M,\ker f\} \hookrightarrow \hom_R(M,A')$ got by simply enlarging the codomain to A'. Thus for $0 \to A' \xrightarrow{f} A \xrightarrow{h} A''$ exact ($\ker f = 0$; im $f = \ker h$) we have $\ker f_* = 0$ and $\ker h_* = \Psi_A(\hom_R(M,\ker h)) = \Psi_A(\hom_R(M,\mathrm{im}\ f)) = \Psi_A(\hom_R(M,f(A'))) = \Psi_A(\hom_R(M,A'))$ by the injectivity of f. On the other hand im $f_* =$ im $(\hom_R(M,f)) \cong \hom_R(M,A')$. Thus

$$\ker h_* = \mathrm{im}\ f_*$$

(Note that this is not true in general for the image of a sequence exact at A — we have used the short-exactness on the left.)

de:cvfhom

(**8.5.7**) There is similarly a contravariant functor $\hom_R(-,M)$. It is contravariant because the construction takes $g \in \hom_R(Y,X)$ and builds an element g^* in $\hom_{\mathbf{Ab}}(\ \hom_R(X,M),\ \hom_R(Y,M))$ mapping $a \in \hom_R(X,M)$ to $g^*(a) = a \circ g \in \hom_R(Y,M)$.

(**8.5.8**) We may go further. Taking $M = R$, the right action of R on R (by the ring multiplication), commutes with the left action we are using, and hence survives to equip

$$X^* \ :=\ \hom_R(X,R)$$

with the property of right R–module. The functor is then from $R-$mod to mod$-R$ (and is called *duality*). In particular the image of a sequence of modules under duality is a sequence in the other direction:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & 0 \\ 0 & (M')^* \arrow[l] & M^* \arrow[l] & (M'')^* \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

pa:hom funct

(**8.5.9**) Next we elevate the target of the $\hom_R(M,-)$ functor from $\mathbf{Ab}$ to a module category, in case M a bimodule.

We can make $\mathrm{Hom}_A({}_AM_B, {}_AL)$ an element of $B-\mathrm{mod}$ (not $\mathrm{mod}-B$, note), as follows. Applying $\mathrm{Hom}_A(-, L)$ to $b' \in \mathrm{End}_A(M)$ (from (8.5.4)) we obtain

$$\begin{array}{rcl} b'^* : \mathrm{Hom}_A(M,L) & \to & \mathrm{Hom}_A(M,L) \\ \beta & \mapsto & b'^*\beta \end{array} \tag{8.4}$$ bmapp

given by

$$b'^*\beta(m) = \beta(b'(m)).$$

Then we define the (left) action of B on $\mathrm{Hom}_A({}_AM_B, {}_AL)$ by

$$b.\beta(m) = b'^*\beta(m) \tag{8.5}$$ eq:lact

Finally we verify for $b, c \in B$, $\beta \in \mathrm{Hom}_A(M, L)$ that $(bc).\beta = b.(c.\beta)$ (i.e. that it *is* a left action):

$$((bc).\beta)(m) = ((bc)'^*\beta)(m) = (\{c'b'\}^*\beta)(m) = \beta(c'b'(m)) = (b'^*(c'^*\beta))(m) = (b.\{c.\beta\})(m).$$

(We leave it as an exercise to check that image maps are also maps in $B - \mathrm{mod}$.)

ex:isoshock (**8.5.10**) Example. By the action in (8.5) $\mathrm{Hom}_R(R, M)$ is, in particular, a left R-module, for any ring R and R-module M. In fact, the map:

$$\psi : M \to \mathrm{Hom}_R(R, M) \tag{8.6}$$ eq:nasi

$$m \mapsto r \overset{\psi(m)}{\mapsto} rm \tag{8.7}$$

is an isomorphism of left R-modules.

$$\mathrm{Hom}_R({}_RR_R, {}_RM) \cong {}_RM$$

Proof: First note that each $\psi(m)$ is indeed in $\mathrm{Hom}_R(R, M)$ (and not just some arbitrary set map). Secondly note that ψ is R-linear. Since $\psi(m) = 0$ implies $m = \psi(m)(1) = 0$, ψ is an injection. Finally, for any $\gamma \in \mathrm{Hom}_R(R, M)$ choosing $m = \gamma(1)$ gives $\psi(m)(r) = \psi(\gamma(1))(r) = r\gamma(1)$. We have $\gamma(r) = r\gamma(1)$, so ψ is surjective. □

(**8.5.11**) Of course this says that $\mathrm{Hom}_R(R, M)$ is nonzero for any nonzero M. Thus there is a nonzero map from R to M for any such M, and in particular a surjection from R to M for any simple M. For a ring with MC, this says that every simple module appears as a composition factor[2] of R (regarded as a left module for itself); and indeed that any simple module may be chosen as the last composition factor.

dual basis (**8.5.12**) Proposition. *Let $\{m_i\}$ a basis for $N \in R - mod$. A set $\{\phi_j\} \subset N^*$ is a basis iff matrix $(\phi_j(m_i))_{ij}$ is full rank.*

Proof: Define $\theta_j \in N^* = \mathrm{Hom}_R(N, R)$ by $\theta_j(m_i) = \delta_{ij}$. Note that $\{\theta_j\}$ is a basis of N^*. Now use linearity. □

[2]Hopkins Theorem says that a ring with MC has a composition series, as a left module. (See for example [34, (54.1)].)

8.5.2 Tensor functors and tensor-hom adjointness

ss:tf

(**8.5.13**) For ${}_RM \in R-\text{mod}$ define (covariant) functor

$$- \otimes_R {}_RM : \text{mod} - R \to \mathbf{Ab}$$

by $- \otimes_R {}_RM : X_R \mapsto X_R \otimes_R {}_RM$ and for $a \in \hom_R(X, Y)$, $- \otimes_R {}_RM(a) = a \otimes 1$.

(**8.5.14**) For ${}_SM_R$ a bimodule as indicated, define (covariant) functor

$${}_SM_R \otimes_R - : R-\text{mod} \;\to\; S-\text{mod}$$

by ${}_SM_R \otimes_R - : {}_RX \mapsto {}_SM_R \otimes_R {}_RX$ and for $a \in \hom_R(X, Y)$, $({}_SM_R \otimes_R -)(a) = 1 \otimes a$.

(**8.5.15**) Adjointness.
Let R, S be rings with modules ${}_RL, {}_SM_R, {}_SN$. There is an isomorphism of additive groups

$$\gamma : \hom_R(L, \hom_S(M, N)) \cong \hom_S(M \otimes_R L, N) \tag{8.8}$$

eq:adj isom

given by

$$(\gamma f)(m \otimes l) = f_l(m)$$

where $f_l \in \hom_S(M, N)$ is the image of l under f.
That is to say, the pair of functors $(M \otimes_R -,\ \hom_S(M, -))$ form an adjunction, in the sense of (6.3.7), between categories $R-\text{mod}$ and $S-\text{mod}$.

$$R-\text{mod} \underset{\hom_S(M,-)}{\overset{M\otimes_R -}{\rightleftarrows}} S-\text{mod}$$

Outline Proof: $\hom_S(M, N)$ is a left R–module, and $M \otimes_R L \in\ S-\text{mod}$, and γf is well defined. It may be shown that γ has an inverse μ defined as follows. For each $g \in \hom_S(M \otimes_R L, N)$ let $\mu g \in \hom_R(L, \hom_S(M, N))$ be given by $\{(\mu g)_l\}m = g(m \otimes l)$, where $(\mu g)_l$ is the image of l under μg. Done.

We have that $M \otimes_R -$ is *left adjoint* to $\hom_S(M, -)$.

pa:frob rep

(**8.5.16**) The canonical example of an adjunction of functors on module categories is Frobenius reciprocity. This is where we take $M = S$ and R a subring of S:

$$\hom_R({}_RL, {}_RN) \cong \hom_S({}_SS_R \otimes_R {}_RL, {}_SN)$$

Here ${}_RN = \text{Res}^S_R N$ is the obvious *restriction* to R, as is S_R, and we have used $\text{Hom}_S(S, N) \cong N$ from (8.5.10).

The functor ${}_SS_R \otimes_R -$ in this case is called *induction* from R to S.

(**8.5.17**) In particular suppose that S is an algebra over a field k, take $R = k$, and consider that $L = k$ and N is a simple S-module. Then this Frobenius reciprocity is

$$\hom_k(k, {}_kN) \cong \hom_S({}_SS, {}_SN)$$

an isomorphism of k-vector spaces of dimension $\dim N$. This tells us that there are $\dim N$ copies of ${}_SN$ in the head of the regular module ${}_SS$. We will see in (1) that this tells us that the number of copies of the indecomposable projective cover of ${}_SN$ in ${}_SS$ is $\dim N$.

8.5.3 Exact functors

(**8.5.18**) A functor F between module categories is EXACT if it takes an exact sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

to an exact sequence

$$F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N).$$

A functor F between module categories is LEFT EXACT (respectively right exact) if it takes a short exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0 \qquad (8.9)$$ sesy

to a sequence

$$0 \longrightarrow F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N) \longrightarrow 0$$

that is exact at $F(L)$ and at $F(M)$ (respectively at $F(M)$ and at $F(N)$).

A functor which is left and right exact is exact.

th:adju (**8.5.19**) THEOREM. *If functors F, G form an adjunction (F, G) between module categories*

$$\begin{tikzcd} C_R \arrow[r, shift left, "F"] & C_S \arrow[l, shift left, "G"] \end{tikzcd}$$

then the left adjoint F is right exact and the right adjoint G is left exact.

Proof. Applying the adjunction isomorphism to a short exact sequence

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

in the 'N position', and any L, we get

$$\begin{tikzcd} 0 \arrow[r] & \hom_R(L, GA') \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA'') \arrow[d, "\cong"'] \\ 0 \arrow[r] & \hom_S(FL, A') \arrow[r] & \hom_S(FL, A) \arrow[r] & \hom_S(FL, A'') \end{tikzcd}$$

[3] The bottom row is exact since the functor $\hom_S(FL, -)$ is left exact (see (8.5.6)). Therefore the top row is exact. Note that the top row is the image under $\hom(L, -)$ of another sequence — the image of the original sequence under G. Since L can be chosen freely, if the preimage (the image

[3] In the hom/tensor case (as in (8.8)) this is:

$$\begin{tikzcd} 0 \arrow[r] & \hom_R(L, \hom_S(M, A')) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, \hom_S(M, A)) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, \hom_S(M, A'')) \arrow[d, "\cong"'] \\ 0 \arrow[r] & \hom_S(M \otimes_R L, A') \arrow[r] & \hom_S(M \otimes_R L, A) \arrow[r] & \hom_S(M \otimes_R L, A'') \end{tikzcd}$$

under G) were not exact it would pass to an inexact sequence for some choice (exercise), so it is exact. Thus G is left exact. A similar argument with all the arrows reversed shows F right exact. $\square$

In particular:

t left exact etc

(**8.5.20**) Proposition. *Let $_AV_B$ be a left $A-$ right $B-$ bimodule. The hom functor $Hom_A({}_AV_B, -)$ from $A-mod$ to $B-mod$ is left exact.*
The tensor functor F_V given by $_AV_B \otimes_B -$ from $B-mod$ to $A-mod$ is right exact. That is, if (8.9) is a short exact sequence in $B-mod$ then

$$_AV_B \otimes_B L \xrightarrow{F_V(\lambda)} {}_AV_B \otimes_B M \xrightarrow{F_V(\mu)} {}_AV_B \otimes_B N \longrightarrow 0$$

is exact in $A-mod$. $\square$

To address the question of when such functors are properly exact, it is useful to consider *projective modules*, which we do in §8.6.

(**8.5.21**) Note that we have shown that restriction is left exact (in fact it is exact); and that induction is right exact.

8.6 Simple modules, idempotents and projective modules

ss:proj21

8.6.1 Idempotents

ss:id21

Note that the only unit idempotent in a ring is 1, since if $a^2 = a$ and a has an inverse then $a = aaa^{-1} = aa^{-1} = 1$. Thus if $e^2 = e \in R$ is not 1 then $1 = e + (1-e)$ is a sum of nonunits and R is not a local ring (as defined in §3.1.1). That is, the only idempotents in a local ring are 1 and 0.

(**8.6.1**) Two idempotents e_1, e_2 in a ring R are *orthogonal* if $e_1e_2 = e_2e_1 = 0$.
For example the idempotent elementary matrices ϵ_{ii} in $M_n(\mathbb{C})$ (see (7.1.13)) obey $\epsilon_{11}\epsilon_{22} = 0$; and

$$\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -r \\ 0 & 1 \end{pmatrix} = 0.$$

Indeed, if idempotent $e \in R$ is not 1 then $1 = e + (1-e)$ is a decomposition of 1 into orthogonal idempotents.

(**8.6.2**) An idempotent $e \in R$ is *primitive* if it has no proper decomposition $e = e_1 + e_2$ into orthogonal idempotents.
(Note that if e, e_1, e_2 are idempotent and $e = e_1 + e_2$ so that $e_2 = e - e_1$ then $e_1e_2 = e_1(e - e_1) = e_1e - e_1$, but this is not necessarily an orthogonal decomposition.)
(Note from our example above that while it may be possible to decompose an idempotent into primitive idempotents, such a decomposition is not unique in general. However given a decomposition $1 = \sum_i e_i$ in R, every decomposition is of form $\sum_i ae_ia^{-1}$ for some unit $a \in R$.)

(**8.6.3**) Proposition. *If R is a ring then an idempotent $e \in R$ is primitive if and only if Re is an indecomposable left module.*

Proof. (Only if:) Suppose for a contradiction that $Re = M \oplus M'$. Then $e = x + x'$ with $x = xe \in M$ and $x' = x'e \in M'$ nonzero. We have $M \ni (1-x)x = xe - x^2 = x(x+x') - x^2 = xx' \in M'$, so $xx' = 0$ and $x^2 = x$, an idempotent, contradicting primitivity of e.

(If:) Exercise. □

(**8.6.4**) If $1 = \sum_i e_i$ is a decomposition into orthogonal idempotents in R then

$$R = \bigoplus_i Re_i$$

as a left module. Here Re_i is indecomposable iff e_i is primitive.

e exact

(**8.6.5**) Suppose $M \in R-$mod and $e^2 = e \in R$. Then $eM \subset M$ is the abelian group $\{em \mid m \in M\}$. (Note that eM is also an eRe-module.) Note that an R-module morphism $\psi : M' \to M$ defines an abelian group/eRe-module morphism $eM' \to eM$, by restriction ($em \mapsto \psi(em) = e\psi(em) \in eM$). Thus $M \mapsto eM$ defines a functor from R-mod to **Ab** (or to eRe-mod).

Now consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. We *claim* that $0 \to eM' \to eM \to eM'' \to 0$ is short exact (i.e. that the functor is exact).

Proof. Certainly $eM' \hookrightarrow eM$. We have $eM/eM' = \{em + eM' \mid m \in M\}$ and $e(M/M') = \{e(n + M') \mid n \in M\}$. But $e(n + M') = en + eM'$. □

de:id lifting

(**8.6.6**) Let R be a ring and I an ideal. We say idempotents can be *lifted* mod.I if for every $a \in R$ such that $a^2 - a \in I$ (i.e. every a that passes to an idempotent in the quotient ring R/I) there is an idempotent $e \in R$ such that the images of a and e in R/I coincide. (See also §7.3.4, (9.3.3) et seq..)

For example, if I is a nil ideal (such as a nilpotent ideal) of R then idempotents can be lifted. (Exercise, and see (8.6.22).)

de:semiperfect

(**8.6.7**) A ring R is *semiperfect* if (i) idempotents can be lifted mod. the radical J; (ii) R/J is completely reducible.

For example, any left or right artinian ring is semiperfect, since in this case the radical is nilpotent. (If R is not artinian then the radical is not necessarily nilpotent and the requirement of idempotents lifting mod the radical is not automatically satisfied.)

pr:Re M

(**8.6.8**) PROPOSITION. *If M is an R-module and e an idempotent then*

$$\begin{aligned} f : eM &\to Hom_R(Re, M) && (8.10)\\ em &\mapsto f(em) : ae \mapsto aem && (8.11) \end{aligned}$$

is an isomorphism of abelian groups, with inverse $g(\gamma) = \gamma(e)$.

Proof. $f(em)(e) = em$ and $f(\gamma(e))(ae) = f(e\gamma(e))(ae) = ae\gamma(e) = \gamma(ae)$. □

pr:HomRe-

(**8.6.9**) PROPOSITION. *If $e^2 = e \in R$ then the functor $Hom_R(Re, -)$ is exact.*

Proof. Consider a short exact sequence in R-mod: $0 \to M' \to M \to M'' \to 0$. Applying the functor $\mathrm{Hom}_R(Re, -)$ we obtain a sequence of abelian groups. We *claim* that this sequence is also exact (i.e. that the functor is exact).

(Since hom-functors are only left-exact in general (see (8.5.20)), there is something to show here.)

Now the sequence $0 \to eM' \to eM \to eM'' \to 0$ is exact (by (8.6.5)). Applying (8.6.8) to this sequence we see that the $\mathrm{Hom}_R(Re, -)$ image of the original sequence is exact. □

p:feef (**8.6.10**) Let $e \in R$ be an idempotent and consider $f \in \mathrm{End}_R(Re)$. We have $f(x) = f(xe) = xf(e)$, so f may be realised by right multiplication by $f(e)$.

eRe (**8.6.11**) PROPOSITION. *$End_R(Re) \cong (eRe)^{op}$*

Proof. Right multiplication by $ef(e)e$ would also realise $f \in \mathrm{End}_R(Re)$, as in (8.6.10), so there is a map, and $f(e) = 0$ only if $f = 0$, so it is an injection. Indeed the map $x \to xeae$ is in $\mathrm{End}_R(Re)$ for any a, so the map is surjective.

(**8.6.12**) PROPOSITION. *A left R–module M is indecomposable if and only if $End_R(M)$ has no idempotents except 1.*

For example, if $eRe \cong \mathbb{Z}$ then Re is indecomposable and e is primitive. N.B. the converse does not hold in general.

eRe=divr (**8.6.13**) PROPOSITION. *Let R be a ring and $e \in R$ an idempotent.*
(I) Ring eRe a local ring (e.g. a division ring or a field) implies Re indecomposable and e primitive.
(II) Ring R semiperfect (as in (8.6.7); e.g. artinian) and $e^2 = e \in R$ primitive implies eRe local.

Proof. (I) For a contradiction suppose Re has two non–zero direct summands, and that p_1 is the projection map onto the first. Then $p_1 \in \mathrm{End}_R(Re)$ and not invertible, indeed idempotent and not the identity map. Thus $\mathrm{End}_R(Re)$ is not a local ring, and by proposition 8.6.11 neither is eRe.
(II) Exercise. □

(**8.6.14**) PROPOSITION. *A left ideal J of R is a direct summand of R (as a left-module for itself) iff*

$$J = Re$$

for some idempotent $e \in R$; whereupon

$$R = Re \oplus R(1 - e)$$

as a left-module.

8.6.2 Projective modules

There are a number of equivalent conditions for an R-module to be projective. Which one is the 'definition' depends on ones perspective. For us (for now, cf. e.g. 1.4.73):

de:proj1 (**8.6.15**) PROJECTIVE MODULE P, P is an R-module and the functor $\hom_R({}_RP_S, -) : R-\mathrm{mod} \to S-\mathrm{mod}$ (as in (8.5.9)) is exact for each bi-module structure ${}_RP_S$ on P.

Example: If $e \in R$ is idempotent then Re is projective, by Prop.8.6.9.

projective equiv (**8.6.16**) PROPOSITION. *The following are equivalent:*

i 1. *(P1: Exactness) Module P is projective (as defined in (8.6.15));*

2. *(P2: Lifting) For every R-module surjection $M \xrightarrow{f} M'' \to 0$ and homomorphism $P \xrightarrow{a} M''$ there is a homomorphism $P \xrightarrow{a'} M$ such that $fa' = a$.*

iii 3. *(P3: Summand) Module P is a direct summand of a free module;*

iiii 4. *(P4: Splitting) Every short exact sequence of the form*

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} P \longrightarrow 0 \qquad (8.12)$$ short exact

splits. (That is, $Ext^1_R(P, L) = 0$ for all L — see later; and cf. e.g. [7, 2.6.2].)

Proof. (1) implies (4) since given equation (8.12) (1) says its image under $\hom_R(P, -)$ is exact, but $1_P \in \hom_R(P, P)$ and so in particular there is a ν in $\hom_R(P, M)$ such that $\nu\mu = 1_P$, splitting equation (8.12).

Now, (4) implies (3) since by proposition 8.2.24 there is an F free such that

$$0 \longrightarrow \ker \mu \longrightarrow F \xrightarrow{\mu} P \longrightarrow 0$$

and (4) says this splits.

Next we set off to prove that (3) implies (1).

(**8.6.17**) Proposition. *Let $\{M_i\}$ be a set of right R-modules. Then for any left R-module N*

$$(\oplus_i M_i) \otimes_R N \xrightarrow{\sim} \oplus_i (M_i \otimes_R N)$$

(See for example Jacobson[72, p.154] for a proof.)

(**8.6.18**) A right R-module M is *flat* if $M \otimes_R -$ is exact.

lem:flat1 (**8.6.19**) Lemma. *A right R-module $M = \oplus_i M_i$ is flat iff each M_i is flat.*

pr:tenprpflat1 (**8.6.20**) Proposition. *If M_R is projective in sense (3) of (8.6.16) then the functor $M_R \otimes_R -$ is exact.*

Proof. $R \otimes_R -$ takes any sequence to an isomorphic sequence, so is flat. Thus by Lemma (8.6.19) any free R-module F is flat. For any projective P, for some such F we have $F = P \oplus P'$, by Prop. (8.6.16)(3). Thus $P \otimes_R -$ is exact by Lemma (8.6.19) again. □
(See also Hilton–Stambach[64, p.111], Anderson–Fuller[3, p.227].)

(**8.6.21**) Exercise. Show that (3) implies (1) in Prop. (8.6.16).

We omit (2) from the loop for now. □

8.6.3 Idempotent refinement

We see that idempotents are important structural tools in ring theory. We also see that if I is a nilpotent ideal in ring R then it contains no idempotent. Thus the idempotents of R and R/I are related.

Before we start, note that ideal $I^2 \subseteq I$ in R (indeed $I^2 \subset I$ if I is nilpotent). Suppose $e \in R/I$. Then $e = r_e + I$ for some $r_e \in R$. Similarly $f \in R/I^2$ is $f = r_f + I^2$ for some $r_f \in R$. Thus $f + I = r_f + I^2 + I = r_f + I \in R/I$. In other words $f + I$ makes sense, because I is, roughly speaking, a 'cruder', bigger thing than I^2.

lem:id ref (**8.6.22**) LEMMA. *If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R/I^2$ such that $e = f + I$.*

Proof. Let $r \in R/I^2$ such that $e = r + I$. Then

$$0 = e(e-1) = (r+I)(r-1+I) = r(r-1) + rI + I(r-1+I)$$

so $r(r-1) \in I$ and so $r^2(r-1)^2 = 0$ in R/I^2. Note that $e_2 := (1+2(1-r))r^2$ obeys $e_2 = r + I$ and

$$e_2(e_2-1) = (1+2(1-r))r^2((1+2(1-r))r^2 - 1) = (1+2(1-r))r^2(-(1+2r))(r-1)^2 = 0$$

Thus we can take $f = e_2$. □

th:id ref (**8.6.23**) THEOREM. [Idempotent refinement] *(i) If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R$ such that $e = f + I$.*
(ii) If $1 = \sum_i e_i$ is a primitive orthogonal idempotent decomposition in R/I then there is a corresponding decomposition $1 = \sum_i f_i$ in R with $f_i + I = e_i$. Further if $(R/I)e_i \cong (R/I)e_j$ then $Rf_i \cong Rf_j$.

Proof. (i) Since I is nilpotent, $I \supset I^2 \supset ... \supset I^i \supset I^{i+1} \supset ...$ until some $I^n = 0$. Let $f' \in R/I^2$ be idempotent passing to e as in Lemma 8.6.22. Of course $I^4 = (I^2)^2$ so there is an idempotent $f'' \in R/I^4$ that passes to f' by the same Lemma. Iterating we shall eventually reach an idempotent f in R/I^m with $m \geq n$, so that $R/I^m = R$.
(ii) Exercise.

8.7 Structure of an Artinian ring

ss:artin2

Hereafter let us suppose that R is a ring with MC.

(**8.7.1**) For an Artinian ring the set of nil ideals coincides with the set of nilpotent ideals.

(**8.7.2**) RADICAL J of Artinian ring R: J is the maximal nilpotent ideal of R.
(Recall from (8.3.20) that the Jacobson radical of an Artinian ring is the sum of all nilpotent left ideals, and so coincides with J.)

(**8.7.3**) RADICAL FILTRATION of an R–module M.
The HEAD (or TOP) of a module is M/JM, a semisimple module. We have $M \supset JM \supset J^2M \supset \cdots \supset 0$, and each section is semisimple. (The term *head* is used, for example, by Benson [7].)

(**8.7.4**) SOCLE FILTRATION of an R–module M.
Tail The SOCLE (TAIL) of a module M is the maximal semisimple submodule $\mathbf{Soc}(M)$, i.e. the sum of all simple submodules. There is a sequence of submodules $M \supset M_1 \supset M_2 \supset \cdots \supset \mathbf{Soc}(M)$ unique up to isomorphism such that each section M_i/M_{i+1} is a maximal semisimple submodule of M/M_{i+1}.

(**8.7.5**) If a module M is a direct sum of d copies of a module N we may write simply

$$M = dN$$

Since ring R has MC here, and $R/J(R)$ is semisimple and has MC, then by the Artin-Wedderburn Theorem (8.3.22) we have

$$R/J(R) \cong \bigoplus_i M_{d_i}(D_i) \cong \bigoplus_i d_i L_i$$

(where the first isomorphism is as a ring and the second is) as a left module, where L_i is the simple module. (That is, the multiplicity of a given simple module in the left regular module for a semisimple ring is given by the dimension of that simple (over the opposite of the associated division ring).) Let $1 = \sum_i e_i$ be the corresponding orthogonal idempotent decomposition in $R/J(R)$; and $1 = \sum_i f_i$ the associated decomposition in R (as in Theorem (8.6.23)). Then

$$R \cong \bigoplus_i d_i P_i$$

as a left module. That is, the multiplicity of an indecomposable projective module in R (as a left module for itself) is given by the dimension (in the same sense as before) of the corresponding simple module $L_i = P_i/(J(R)P_i)$.

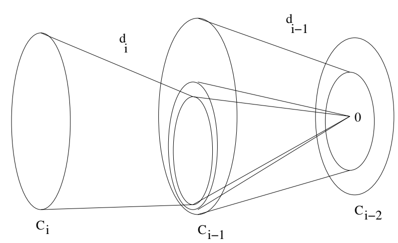
8.8 Homology, complexes and derived functors

See for example Jacobson II [72].

(**8.8.1**) A chain of R-module (or indeed abelian group) homomorphisms $d_i : C_i \to C_{i-1}$ ($i \in \mathbb{Z}$) is a *complex* (C, d) if $d_i d_{i+1} = 0$ for all i. That is, in a complex (C, d) we have

$$C_i \stackrel{d_i}{\to} C_{i-1} \stackrel{d_{i-1}}{\to} C_{i-2} \qquad (i \in \mathbb{Z})$$

and $d_{i-1}(d_i(c)) = 0$ for all $c \in C_i$ (for all i):



(**8.8.2**) A *chain homomorphism* $a : (C, d) \to (C', d')$ is a set of homomorphisms $a_i : C_i \to C'_i$ such that $a_{i-1}d_i = d'_i a_i$ for all i.

(**8.8.3**) If (C, d) is a complex and $C_i = 0$ for all $i < 0$ then (C, d) is a *positive complex*.

(**8.8.4**) Example. (I) For M an R module then $C_i = M$ and $d_i = 0 : M \to 0$ is a complex.
(II) A short exact sequence

$$0 \to M' \stackrel{\alpha}{\to} M \stackrel{\alpha'}{\to} M'' \to 0$$

defines a complex with $C_3, C_2, C_1 = M', M, M''$ and all other $C_i = 0$ and $d_2, d_3 = \alpha', \alpha$ and all other $d_i = 0$.

(**8.8.5**) Let (C, d) be a complex and note that $\ker d_i$ is a submodule of C_i; and that the image $d_{i+1}C_{i+1}$ is a submodule of $\ker d_i$. Define the i-the *homology module* of (C, d) as

$$H_i(C) = \ker d_i / d_{i+1}C_{i+1}$$

Note that the complex is exact at C_i iff $H_i(C) = 0$.

(**8.8.6**) For M an R-module, a *chain over* M is a positive complex (C, d) together with a homomorphism $e : C_0 \to M$ such that $ed_1 = 0$.

A complex over M is a *resolution* of M if the extended chain including

$$\ldots \overset{d_2}{\to} C_1 \to C_0 \to M \to 0$$

is exact. It is a *projective* complex if every C_i is projective.

(**8.8.7**) Theorem. *If (C, e) is a projective complex over M and (C', e') is a resolution of M', and $u : M \to M'$ a homomorphism, then there is a chain homomorphism $a : C \to C'$ such that $ue = e'a_0$.*

(**8.8.8**) Let M be an R-module and

$$\ldots \overset{d_2}{\to} C_1 \overset{d_1}{\to} C_0 \overset{e}{\to} M \to 0$$

a projective resolution of M. Now suppose we apply some functor G (from R-mod to **Ab**):

$$\ldots \overset{Gd_2}{\to} GC_1 \overset{Gd_1}{\to} GC_0 \overset{Ge}{\to} GM \to G0$$

If G is multiplicative then $(GC, \{Gd, Ge\})$ is a positive complex over GM. If G is exact then the image sequence is exact, but this is not true in general, so $H_i(GC)$ may not vanish. Define

$$L_iGM \ := \ H_i(GC)$$

This object map may be extended (for each i) to another functor from R-mod to **Ab**, called the i-th *left derived functor* of G. Note that it depends on the choice of resolution, but that the notation omits explicit reference to this (and in fact the dependence is, in a suitable sense, almost negligible — see e.g. [72, §6.6]).

(**8.8.9**) What is handy about L_iG is that if G is, say, right exact but not exact we can use it to develop exact sequences from the G-image of short exact sequences:

$$0 \to A \to B \to C \to 0$$

$$\ldots \to L_1GC \to L_0GA \to L_0GB \to L_0GC \to 0$$

(**8.8.10**) Exercise. Prove it.

8.9 More on tensor products

Recall from the construction of the tensor product in (8.4) (or see for example Curtis–Reiner I [33, §12]):

bal fac (**8.9.1**) THEOREM. *For left R-module M and right R-module L, and abelian group A, every balanced map $\mu_0 : L \times M \to A$ factors through $L_R \otimes_R M$.* That is, every balanced map μ_0 induces a map $\mu : l \otimes m \mapsto \mu_0(l, m)$.

mult map2 (**8.9.2**) Suppose that left R-module M and right R-module L are in fact a principle left and right ideal respectively: $M = Rm_0$, $L = l_0R$. Note that the restriction of the ring multiplication to $L \times M$, i.e. $(l, m) \mapsto lm$, is balanced. Then by Theorem (8.9.1), associated to the tensor product $L_R \otimes_R M$ is a *multiplication map* given by

$$\mu : L_R \otimes_R M \to l_0Rm_0$$

$$l \otimes m \mapsto lm$$

(Actually we don't need the restriction to principle ideals — the main point is that multiplication makes sense.) Every element of l_0Rm_0 is clearly hit; but is this also an injection? Suppose that one of the generators is idempotent (say l_0, WLOG), then define

$$\nu : l_0Rm_0 \to L_R \otimes_R M$$

$$r \mapsto l_0 \otimes r$$

Note that in this case ν is inverse to μ.

(**8.9.3**) EXAMPLE. We return to Example (8.4.19). The module M_0 is isomorphic to an ideal in $\mathbb{Z}S_2$: $M_0 \cong \mathbb{Z}(1 + \sigma_1)$. This allows us to include $M_0 \subset \mathbb{Z}S_2 \subset \mathbb{Z}S_3$, and thus for M_0 to act on $\mathbb{Z}S_3$ directly by the mutliplication in this algebra. Thus we have a balanced map $\mu_0 : \mathbb{Z}S_3 \times M_0 \to \mathbb{Z}S_3$ (given by $(a, b) \mapsto ab$); and a multiplication map μ taking $\mathbb{Z}S_3 \otimes M_0 \to \mathbb{Z}S_3(1 + \sigma_1)$. The image is $\mathbb{Z}\{e', \sigma_2e', \sigma_1\sigma_2e'\}$ where $e' = 1 + \sigma_1$ (it is easy to see that these elements span; and they are linearly independent in $\mathbb{Z}S_3$). That is, the image is free of rank 3.

Preimages of the basis elements are, for example $\{1 \otimes e', \sigma_2 \otimes e', \sigma_1\sigma_2 \otimes e'\}$. We have already noted that these elements span the tensor product. We now see that they must also be linearly independent in the tensor product, since if some combination $\sum_i c_ia_i \otimes e' = 0$ then $\mu(\sum_i c_ia_i \otimes e') = \sum_i \mu(c_ia_i \otimes e') = \sum_i c_ia_ie' = 0$ so their images would be linearly dependent.

th:assoc (**8.9.4**) THEOREM. *Tensor product is associative up to isomorphism, i.e. for L_R, ${}_RM_S$, ${}_SN$ modules as indicated:*

$$L_R \otimes_R ({}_RM_S \otimes_S {}_SN) \cong (L_R \otimes_R {}_RM_S) \otimes_S {}_SN$$

as abelian groups.

Outline Proof. One idea here is to show that there is a map given by

$$l \otimes (m \otimes n) \mapsto (l \otimes m) \otimes n. \tag{8.13}$$ eq:lmnlmn

We should unpack this to check well-definedness and so on. Firstly $l \in L$, $m \in M$ and $n \in N$ is to be understood here. From this we have that $M \otimes N$ is spanned, i.e. generated as an abelan/additive group, by elements of form $m \otimes n$ (albeit not in general freely). It follows that $L \otimes (M \otimes N)$

is generated by elements of form $l \otimes (m \otimes n)$. From this it follows that any group hom from $L \otimes (M \otimes N)$ is given by its image on these elements. We need to check that the formal set map indicated by (8.13) is well-defined. Firstly, the indicated image elements lie in $(L \otimes M) \otimes N$ by a similar argument to above. But also we need to check since different triples (l, m, n) can produce the same element. For example, $l \otimes (mr \otimes n) = l \otimes (m \otimes rn)$ for all $r \in S$, so we require $(l \otimes mr) \otimes n$ to equal $(l \otimes m) \otimes rn$. We have $(l \otimes m) \otimes rn = (l \otimes m)r \otimes n$ and $(l \otimes m)r = (l \otimes mr)$, so the equality holds. Other such checks work similarly. ...

th:tp dist

(**8.9.5**) Theorem. *Distributivity:*

$$(L \oplus M)_R \otimes_R N \cong (L_R \otimes_R N) \dotplus (M_R \otimes_R N)$$

ss:indres

8.9.1 Induction and restriction functors

We now return to one of our original motivations for introducing tensor products — the construction of an (left) adjoint to the restriction functor.

Recall that for a pair of rings with a homomorphism $\phi : R \to S$ we have a functor $\mathrm{Res}_\phi : S-\mathrm{mod} \to R-\mathrm{mod}$ given on objects by $\mathrm{Res}_\phi M = M$ and $rm = \phi(r)m$.

We already considered the case where ϕ is injective. See (8.5.16).

8.9.2 Globalisation and localisation functors

(**8.9.6**) Let ${}_SM_R$ and ${}_RN_S$ be bimodules as indicated. Suppose

$${}_SM_R \otimes_R {}_RN_S \cong S$$

as S-bimodule. Then the functor $G = {}_RN_S \otimes_S -$ is called a *globalisation*; and the functor $F = {}_SM_R \otimes_R -$ is called a localisation. We have

$$F(G(A)) = {}_SM_R \otimes_R {}_RN_S \otimes_S A \;\cong S \otimes_S A \cong A$$

so that F is a kind of left inverse to G.

(**8.9.7**) We return to consider such functors for rings that are k-algebras in §9.4.
[4]

ss:morita

8.10 Morita equivalence

See also Chapter 6 of Anderson–Fuller [3]; Section 2.2 of Benson [7]; and Chapter 3 of Jacobson II [72].

[4]This DOES NOT WORK!!! Need something more like $S = eRe$...

(**8.9.8**) With this setup, supposing also that F is exact, if L a simple S-module and B a proper submodule of $G(L)$, then $F(B) = 0$.

Proof. $F(B) \subseteq L$ by construction, but L is simple, so either $F(B) = 0$ or $F(B) = L$.

Categories A, B (not necessarily module categories) are *category equivalent* if there are a pair of functors $G : A \to B$ and $F : B \to A$ such that there are natural isomorphisms $GF \cong 1_A$ and $FG \cong 1_B$.

(**8.10.1**) Suppose S, T are module categories. Recall from (6.2.6) that a functor $F : S \to T$ is additive if, for $f, f' : L \to M$ in S, we have $F(f + f') = F(f) + F(f')$.

Rings A, B are *Morita equivalent*, denoted $A \approx B$, if there are a pair of additive functors $G : A-\text{mod} \to B-\text{mod}$ and $F : B-\text{mod} \to A-\text{mod}$ such that there are natural isomorphisms as above.

(**8.10.2**) Proposition. *Let A be a ring and $e = e^2$ in A such that $AeA = A$. Then*

$$A \approx eAe$$

Proof. ...

First note that for each $M \in A-\text{mod}$ there is a morphism $X_M : Ae \otimes_{eAe} eA \otimes_A M \to M$ given by $ae \otimes eb \otimes m \mapsto aebm$. Consider

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ Ae \otimes_{eAe} eA \otimes_A M \arrow[u, "X_M"] \arrow[r] & Ae \otimes_{eAe} eA \otimes_A N \arrow[u] \end{tikzcd}$$

We note that the two composite morphisms coincide: Through M we have $ae \otimes eb \otimes m \mapsto f(aebm)$. Through $Ae \otimes_{eAe} eA \otimes_A N$ we have $ae \otimes eb \otimes m \mapsto ae \otimes eb \otimes f(m) \mapsto aebf(m) = f(aebm)$. That is, $X_- : (ae \otimes eb \otimes -) \to 1_{A\text{–mod}}$ is a natural homomorphism.

If $AeA = A$ then we claim X_- is even a natural isomorphism.

...

pr:MEconseq1 (**8.10.3**) Proposition. *If S, T are rings and $F : S - \text{mod} \to T - \text{mod}$ is a Morita equivalence functor then the lattice of submodules of a module ${}_SM$ is isomorphic to the lattice of submodules of $F({}_SM)$.*

Proof. See e.g. Anderson-Fuller §21.

(**8.10.4**) It follows from Prop.8.10.3 for example that $F(M)$ is indecomposable iff M is.

Chapter 9

Algebras

ch:alg

9.1 Algebras and A-modules

ss:alg1

Here R is a commutative ring. We start by recalling the (second) definition from (1.2.17).

de:algebra

(**9.1.1**) R–ALGEBRA A: $(A, +, ., 1, 0)$ a ring (as for example in 3.1.1) and A an R–module such that $ax.y = x.ay = a(x.y)$ $\forall x, y \in A$, and $a \in R$.
Examples:
(i) Any ring K is a $\mathbb{Z}$–algebra with $na = a + a + \cdots + a$ (n summands).
(ii) Let G be a finite group or monoid, R a commutative ring and RG the free R–module with basis G. Then R–linear extension of the group multiplication equips RG with the property of R–algebra.
(iii) Let A' be the free abelian monoid generated by 1 and a symbol x, and let RA' be the monoid algebra as above. Let RA be the quotient of this algebra by the ideal generated by $x^2 - 2$. If $R = \mathbb{Q}$ then $\{1, x\}$ is a basis for RA.

Observe from the definition that $r1 \in A$ for all $r \in R$, so this induces a ring homomorphism from R to A, with image lying in the centre (cf. the first definition of algebra in (1.2.17)).

(**9.1.2**) REMARK. Let A be an R-algebra. It is interesting to recast the definition in terms of commutative diagrams. To this end let us first unpack the multiplication notation, writing it as $\mu : A \times A \to A$ — that is, $a.b = \mu(a, b)$. Then associativity: $a.(b.c) = (a.b).c$ for all $a, b, c \in A$ becomes $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ for all $a, b, c \in A$. If we abuse notation and write $A \times A \times A$ for the set of ordered triples then we have commutativity of

$$\begin{tikzcd}
A \times A \times A \arrow[rrr, "\mu\otimes id", "(a,b,c)\mapsto(\mu(a,b),c)"'] \arrow[dd, "id\otimes\mu"'] & & & A \times A \arrow[dd, "\mu"] \\
& \begin{array}{c}(a,b,c)\\ \mapsto \\ (a,\mu(b,c))\end{array} & \begin{array}{c}(\mu(a,b),c)\\ \mapsto \\ \mu(\mu(a,b),c)\end{array} & \\
A \times A \arrow[rrr, "(a,\mu(b,c))\mapsto\mu(a,\mu(b,c))", "\mu"'] & & & A
\end{tikzcd}$$

The ring operations of A itself obey $(x+y).z = x.z + y.z$ and $z.(x+y) - z.x - z.y = 0$ of course, and (supposing that we can identify R with its image in the centre) the centrality of R and associativity of . means that the interaction with the R-module structure can be written $(ax).y = (a.x).y = (x.a).y = x.(a.y) = x.(ay)$ $(a \in R)$. Thus the multiplication factors through a balanced map of R-modules (compare (8.4.6)), and may be considered as an R-linear map $\nabla : A \otimes_R A \to A$. Specifically recall $A \otimes_R A$ is spanned by elements of form $(a,b) + S_{AA}$ with $(a,b) \in A \times A$ (in general elements are finite sums $\sum_i (a_i, b_i)$); and $\nabla(a,b) = \mu(a,b) = a.b$ and $\nabla(\sum_i (a_i, b_i)) = \sum_i \mu(a_i, b_i)$.

Similarly the multiplicative identity induces an R-linear map $\eta : R \to A$ taking $1 \mapsto 1$.

In these terms we claim associativity of ∇ means $\nabla(\nabla(a,b),c) = \nabla(a, \nabla(b,c))$ for all $(a,b,c) \in A^{\times 3}$ as before, together with linear extensions. We have $(A \otimes A) \otimes A$ with elements of form $\sum_i ((a_i), d_i)$ where $a_i = \sum_j (b_{ij}, c_{ij})$ and all $b_{ij}, c_{ij}, d_i \in A$. Using linearity we have $\sum_{ij} (b_{ij}, c_{ij}, d_i)$ ...UNFINISHED! This becomes commutativity of

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

where we use the already asserted associativity of tensor product (Theorem 8.9.4) to give the two interpretations of $A \otimes A \otimes A$ needed. Meanwhile

$$\begin{tikzcd} A \otimes R \cong A \cong R \otimes A \arrow[r, "\eta \otimes id"] \arrow[d, "id \otimes \eta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "\nabla"'] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

also commutes.

(**9.1.3**) If an R-algebra A is free of finite rank as an R-module then that rank is its *rank* as an algebra. In particular if R is a field and A is a finite dimensional R-vector space, then that is its *dimension* as an algebra over the field.

xAx1 (**9.1.4**) Suppose A is an algebra over a commutative ring $\mathcal{Z}$. For k a ring that is a $\mathcal{Z}$-algebra define k-algebra

$$A^k = k_{\mathcal{Z}} \otimes_{\mathcal{Z}} A$$

If the map $A \to A^k$ given by $a \mapsto 1 \otimes a$ is an injection we may write simply a for $1 \otimes a$.

(**9.1.5**) Suppose A as above, $x \in A$ obeys $x^2 \neq 0$ and

$$xAx \subseteq \mathcal{Z}x$$

Then $x^2 = c_x x$ for some $c_x \in \mathcal{Z}$. Suppose further that k is such that $1 \otimes c_x \neq 0$. Then $(1 \otimes x)A^k(1 \otimes x) = k(1 \otimes x)$. This implies $(1 \otimes x)$ is proportional to an idempotent e_x (say) in A^k, and hence that $e_x A^k e_x$ is a field. Thus by Prop.(8.6.13) we have that $A^k(1 \otimes x)$ is indecomposable. Since the functor $k_{\mathcal{Z}} \otimes_{\mathbb{Z}} -$ preserves direct sums this implies that Ax is indecomposable.

Now suppose there is a k' such that $1 \otimes_{k'} c_x = 0$. Is $A^{k'}(1 \otimes_{k'} x)$ indecomposable?

9.2 Finite dimensional algebras over fields

In the remainder of this section k is a field, and A a finite dimensional algebra over k.

go (**9.2.1**) THEOREM. [Burnside's Theorem] *Let A be a f.d. algebra over k and M a simple left A-module. If $End_A(M) = k.1_M$ then every k-endomorphism of M is realised by left by multiplication by some element of A.*

9.2.1 Dependence on the field

Let M be a simple left A-module. Schur's Lemma ((1.4.32) or (**??**)) says $\mathrm{End}_A(M)$ is a division ring.

go2 (**9.2.2**) PROPOSITION. *Let M be a simple left A-module. If k is algebraically closed then $End_A(M) = k.1_M$.*

(**9.2.3**) ABSOLUTELY SIMPLE MODULE. A left A-module M is absolutely simple if $k' \otimes_k M$ is simple for every extension field $k' \supseteq k$.

(**9.2.4**) Example: The $\mathbb{Q}$-algebra generated by x obeying $x^2 = 2$ defined in 9.1.1(iii) is simple as a left module for itself, but not absolutely simple. To see this we note that if there is a submodule then it must be spanned by an element of the form $\alpha 1+\beta x$. We have $x(\alpha 1+\beta x) = 2\beta 1+\alpha x = c(\alpha 1+\beta x)$ for some scalar c. Elimination gives $c^2 = 2$ and $\frac{\alpha}{\beta} = c$. This has no solution in $\mathbb{Q}$, so there is no submodule. However it has a solution over $\mathbb{Q}[\sqrt{2}]$ (indeed $x + \sqrt{2}.1$ spans a submodule), so the algebra is not absolutely simple as a module for itself.

go3 (**9.2.5**) THEOREM. *A simple left A-module obeys $End_A(M) = k.1_M$ iff it is absolutely simple.*

In our example consider the map $\phi \in \mathrm{End}_A(M = A)$ given by $\phi : m \mapsto xm$, which takes $1 \mapsto x$ and $x \mapsto 2$. Evidently $\phi \notin k.1_M$. On the other hand $\phi^2 = 2.1_M$ so ϕ is invertible over $\mathbb{Q}$. One easily checks that 1_M and ϕ generate a division ring.

(**9.2.6**) SPLITTING FIELD. For A a f.d. algebra over k, an extension $k' \supseteq k$ is a splitting field for A over k if every simple left $k' \otimes_k A$-module is absolutely simple.

Note that every algebraically closed field containing k is splitting for A over k.

9.2.2 Representation theory preliminaries

(**9.2.7**) Suppose R is a field and A an R-algebra. The dual R–module $X^* = \hom_R(X, R)$ of $X \in A-\mathrm{mod}$ is equipped with the property of right A–module via: $\psi \in X^*$, $a \in A$,

$$(\psi a)(x) = \psi(ax).$$

To see that X^* is a *right* module, we need to check that $m(ab) = (ma)b$ ($m \in X^*$, $a, b \in A$):

$$(\psi(ab))(x) \;=\; \psi(abx) \;=\; (\psi a)(bx) \;=\; ((\psi a)b)(x).$$

(**9.2.8**) An algebra A is *primitive* if it has a faithful irreducible representation and *semi–primitive* if there is a faithful direct sum of irreducibles.

The radical is the intersection of the kernels of the irreducible representations of an algebra A.

(**9.2.9**) An algebra A which is finite dimensional over a field k obeys the structure and representation theory of an *Artinian ring* (see [73, §4.5]), as described in §8.7. Then **rad** A is a nilpotent ideal containing every nilpotent one sided ideal of A.

rad

If k is algebraically closed, or even a splitting field for A, then each division ring D_i appearing in the Artin-Wedderburn Theorem for $A/\mathbf{rad}\, A$ is in fact k.

(**9.2.10**) A free module for an algebra is a direct sum of copies of the algebra as a module for itself.

9.2.3 Structure of a finite dimensional algebra over a field

Let A be a finite dimensional algebra over a splitting field. Then every simple module appears as a composition factor of the left regular module. Thus there are finitely many isomorphism classes of simple modules. The enumeration of these isomorphism classes, or more generally the construction of an indexing set, is thus a fundamental problem in representation theory.

(**9.2.11**) The regular module will not split up into a direct sum of simple modules in general (unless the algebra is semisimple); but the indecomposable summands of the regular module will be projective modules P_i:

$$A \cong \bigoplus_i d_i P_i$$

as left modules. Each such indecomposable projective module has a simple head, and the collection of these heads contains a complete set of simple modules (indeed two such indecomposable projective modules are isomorphic if and only if they have isomorphic heads, and the set of heads of a complete set of class representatives is a complete set of simple modules). Another fundamental problem is thus to compute the multiplicity of each simple module L_j as a composition factor in each P_i.

However, in general neither the simple nor projective modules are amenable to direct construction. An intermediate problem is to compute the blocks of A.

(**9.2.12**) The *blocks* of algebra A are the parts in the partition of the index set for simples given by the closure of the relation $i \sim j$ if L_j appears as a composition factor in P_i.

Matters are somewhat more straightforward under certain special circumstances, some examples of which we collect in the next few sections.

9.3 Cartan invariants (Draft)

ss:CartanInv01

Here we discuss further the methods outlined in §1.8.

(**9.3.1**) We will now encounter several algebras that are defined integrally (i.e. over a ring $\mathcal{Z}$ amenable to multiple distinct localisations), and that are free of finite rank as $\mathcal{Z}$-modules. For each of these, A say, we will want to study the representation theory of various f.d. algebras over fields obtained from it by base change:

$$A_k = k_{\mathcal{Z}} \otimes_{\mathcal{Z}} A$$

Among these may be semisimple cases, and others more complicated. Our next task is to consider how we might pass information (on the structure of modules) *between* these various cases, using the integral case as a conduit.

Essentially we consider two kinds of field under $\mathcal{Z}$: extensions (sometimes written K), such as the field of fractions of $\mathcal{Z}$ an integral domain; and quotients (written k), by a maximal ideal. (These may turn out to be splitting for A or not, but we will largely be able to assume that they are splitting.) The main issue will be the different methods for passing between A-modules and A_K-modules (integral basis for special modules), and between A-modules and A_k-modules (idempotent lifting).

(**9.3.2**) Let $\mathcal{Z}$ be a complete rank 1 discrete valuation ring. Let K be its field of fractions, I a maximal ideal, and $k = \mathcal{Z}/I$ the quotient field.

(For example see §7.3.4.)

pa:129 (**9.3.3**) Under suitable circumstances (for example, that k and $\mathcal{Z}$ are as above, but see also later) there is, for each idempotent in A_k an idempotent in A which reduces to it. Indeed a primitive orthogonal idempotent decomposition of 1 corresponding to the decompostion of A_k (as a left module for itself) into indecomposable projectives lifts to a primitive orthogonal idempotent decomposition of 1 in A, and hence to a certain indecomposable projective decomposition of A. We write Π_i for the A-module in this decomposition corresponding to k-projective P_i. That is

$$P_i = k \otimes \Pi_i.$$

We will write S_i for the simple head of P_i.

9.3.1 Examples

ss:idex8

(**9.3.4**) We start with some examples from the group algebras of symmetric groups. The field $\mathbb{Q}$ is a splitting field in every case here (see later), and it is known how to construct at least one primitive idempotent decomposition of 1 in $\mathbb{Q}S_n$ for each n (see e.g. [78]). Working with $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ for various choices of prime number p, the field $\mathbb{Q}_p$ of p-adic numbers always contains $\mathbb{Q}$, so the idempotents in $\mathbb{Q}S_n$ provide useful hints for constructing idempotents over the other rings. (We do not claim a systematic procedure here — just some useful low rank hints.)

(**9.3.5**) In $\mathbb{Q}S_2$, set $e'_{(2)} = 1 + (12)$ and $e_{(2)} = \frac{1}{2}e'_{(2)}$. Then $1 = e_{(2)} + (1 - e_{(2)})$ is a primitive idempotent decomposition (in this case, the only one). A corresponding decomposition will be possible over some given ring if and only if (the image of) 2 has an inverse.

In $\mathbb{F}_3S_n$ then, setting $e = 2(1 + (12))$ we have $ee = e$:

$$2(1 + (12)) \,.\, 2(1 + (12)) \;=\; 2(1 + (12)) \qquad mod.3$$

In $\mathbb{Z}_3$ we have (...0002)(...1112)=(...0001) so in $\mathbb{Z}_3S_n$

$$f \;=\; (...1112)(1 + (12))$$

obeys $f.f = f$ and reduces to e over $\mathbb{F}_3$.

In $\mathbb{F}_2S_n$ no such decomposition is possible, i.e. the idempotent 1 itself remains primitive.

(**9.3.6**) In $\mathbb{Q}S_3$ the primitive decomposition is not uniquely defined. Set $e'_{(3)} = 1 + (12) + (23) + (13) + (123) + (132)$; $e'_{(1^3)} = 1 - (12) - (23) - (13) + (123) + (132)$; and $f' = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (123)$. Then $e_{(3)} = \frac{1}{6}e'_{(3)}$, $e_{(1^3)} = \frac{1}{6}e'_{(1^3)}$, and $f = \frac{1}{3}f'$ are idempotents obeying $fe_{(3)} = fe_{(1^3)} = e_{(3)}e_{(1^3)} = 0$. Setting $g = 1 - f - e_{(1^3)} - e_{(3)}$ then

$$1 = f + g + e_{(1^3)} + e_{(1^3)}$$

is a primitive idempotent decomposition.

Alternatively, $1 = e_{(2)} + (1 - e_{(2)})$ remains an orthogonal (but not primitive) decomposition. We have

$$1 \;=\; e_{(3)} \;+\; (e_{(2)} - e_{(3)}) \;+\; e_{(1^3)} \;+\; (e_{(1^2)} - e_{(1^3)})$$

as a primitive refinement of this. Indeed since $e_{(3)}$ and $e_{(1^3)}$ must appear in every primitive decomposition, this is the only decomposition refining the S_2 decomposition.

Neither decomposition above can be mirrored over another ring unless 2 and 3 have inverses. If 2 or 3 a nonunit then we can try to find a partial decomposition.

Over $\mathbb{F}_2$ we define idempotent $f_2 = (1+(12))(1-(13))$ from f, and $e_2 = 1+(123)+(132)$ from $e_{(3)} + e_{(1^3)}$; and obtain decomposition

$$1 = f_2 + e_2 + (1 - f_2 - e_2)$$

as primitive idempotent decomposition. It is easy to see that f_2 and $1 - f_2 - e_2$ are conjugate, so $\mathbb{F}_2S_3f_2$ and $\mathbb{F}_2S_3e_2$ are a complete set of inequivalent indecomposable projectives. We computed the lift of e_2 earlier: $e_I \;=\; (...10101011)(1 + (123) + (132))$. The lift of f_2 is

$$f_I \;=\; (...10101011)(1 + (12))(1 - (13)).$$

Thus we claim $\mathbb{Z}_2S_3f_I$ and $\mathbb{Z}_2S_3e_I$ are indecomposable projectives. Finally we can base-change these modules to $\mathbb{Q}_p$, where they will be projective (indeed trivially so, since the algebra is semisimple) but not necessarily indecomposable.

On the other hand, over $\mathbb{F}_3$ we can proceed at least as far as the S_2 decomposition, but then no further. Thus we get a different pair of indecomposables over $\mathbb{F}_3$, with different 'lifts' to $\mathbb{Z}_3$, and then different base-changes to $\mathbb{Q}_3$ (and hence to $\mathbb{Q}$, since the extension of $\mathbb{Q}$ to $\mathbb{Q}_3$ is not relevant to the module structure).

9.3.2 Idempotent lifting revisited

ss:p-mod22

What are some other examples of $(K, \mathcal{Z}, k = \mathcal{Z}/I)$-systems where we can build connections between A and A_k as above? We shall be interested later in algebras that can be defined over $\mathbb{Z}[\delta]$, where we want to do representation theory over the field $\mathbb{C}$ (made a $\mathbb{Z}[\delta]$-algebra by mapping δ to some $\delta_c \in \mathbb{C}$). We can think of this as follows.

First embed $\mathbb{Z}[\delta] \subset \mathbb{C}[\delta]$ (this adds in inverses to all primes, and hence *precludes* passing to a field of finite chractersitic — see later for this). Then consider ideal $I = \mathbb{C}[\delta](\delta - \delta_c)$.

(**9.3.7**) Exercise. Show that $\mathbb{C}[\delta]$ is a PID and hence a Dedekind domain.

(**9.3.8**) Recall from (**??**)(ii) that there is a valuation on $\mathbb{C}(\delta)$, and hence an absolute value. Thus by Prop.**??** there is a complete ring containing it.

Indeed from (**??**)(iii) *et seq.* there is such a construction for any Dedekind domain.

(**9.3.9**) EXERCISE. Prove the idemotent lifting claim from (9.3.3).

We start with the idempotent refinement theorem, (8.6.23). Note that I^{i-1} is nilpotent in $\mathcal{Z}/I^i$, so $I^{i-1}A$ is nilpotent in A/I^iA. Thus we can associate idempotents in A/I^i with idempotents in A/I^{i-1}. The idea is to take a kind of limit (of idempotents in A/I^n as n gets large). We need to consider the 'suitable circumstances' provided by the ring $\mathcal{Z}$, and convince ourselves that we can reach an idempotent in A by this kind of limit.

Let us have in mind the example of $I = (x-3)\mathcal{Z}$, with $\mathcal{Z} \supseteq \mathbb{Z}[x]$ (as well as, say, $I = 3\mathbb{Z}$).

...

9.3.3 Brauer reciprocity

(**9.3.10**) Continuing from (9.3.3), we next suppose that there are A-modules $\{\Delta_i\}_i$, free of finite rank over $\mathcal{Z}$, such that $\{K \otimes \Delta_i\}_i$ are a complete set of simple modules for the split semisimple algebra A_K. In other words

$$A_K = \bigoplus_i d_i(K \otimes \Delta_i)$$

(d_i will be the rank of Δ_i, by the Artin-Wedderburn Theorem (8.3.22)), and there is a corresponding primitive orthogonal idempotent decomposition of 1 in A_K. Of course *this* decomposition will not have a lift to A in general.
(For example the idempotent $e_{(n)} = (1/n!)\sum_{w \in S_n} w \ \in \mathbb{Q}S_n$ has no lift to $\mathbb{Z}S_n$.)
However this just says that Δ_i is not projective in A. We can still use Δ_i to investigate the structure of projectives in A_k.

de:spechtdecmat (**9.3.11**) We define

$$D_{ij} = [k \otimes \Delta_i : S_j]$$

— the multiplicity of simple A_k-module S_j as a composition factor in $k \otimes \Delta_i$.

This means that there is a composition series

$$k \otimes \Delta_i = M_0 \supset M_1 \supset ... \supset M_l = 0$$

and that there are D_{ij} instances of $M_a/M_{a+1} \cong S_j$ (as a varies).

de:spechtdecmat2 (**9.3.12**) Consider the image under the exact functor $\mathrm{Hom}_{A_k}(P_j, -)$ of the short exact sequence (in the a-th position in the above, or indeed any, composition series) $0 \to M_{a+1} \to M_a \to S_{i(a)} \to 0$:

$$0 \to \mathrm{Hom}_{A_k}(P_j, M_{a+1}) \to \mathrm{Hom}_{A_k}(P_j, M_a) \to \mathrm{Hom}_{A_k}(P_j, S_{i(a)}) \to 0$$

Note that every time $i(a) = j$, the last term had dimension $+1$, and so the dimension of the middle term is $+1$ greater than that of the first term. Thus

$$\dim_k \mathrm{Hom}_{A_k}(P_j, M) \ = \ [M : S_j]$$

(**9.3.13**) PROPOSITION. *Under the conditions of this section (in particular that there is a projective Π_j such that $k \otimes \Pi_j = P_j$)*

$$D_{ij} = [K \otimes \Pi_j : K \otimes \Delta_i]$$

(Note that since A_K is semisimple this just counts the number of direct summands in $K \otimes \Pi_j$ isomorphic to simple module $K \otimes \Delta_i$.

Aside: This data 'lifts' to a statement about characters over A; and hence passes to one over A_k. If $\{k \otimes \Delta_i\}_i$ is a basis for the Grothendieck group of $A_k - mod$ and P_j has a filtration by the Δ-modules, then this data would determine the filtration multiplicities.)

Proof. (Outline) Since S_i lies uniquely in the head of $P_i = k \otimes \Pi_j$ and P_i is projective, we have $[M : S_j] = \dim_k \hom_{A_k}(k \otimes \Pi_j, M)$ for any A_k-module M, and in particular

$$[k \otimes \Delta_i : S_j] = \dim_k \hom_{A_k}(k \otimes \Pi_j, k \otimes \Delta_i) \tag{9.1}$$ eq:cart1

Since Π_i is idempotently generated it is projective, so $\hom_A(\Pi_j, \Delta_i)$ has a basis. Any such basis passes to a basis for $\hom_{A_k}(k \otimes \Pi_j, k \otimes \Delta_i)$, and so must have order $[k \otimes \Delta_i : S_j]$ by (9.1). It also works as a basis for $\hom_{A_K}(K \otimes \Pi_j, K \otimes \Delta_i)$. But the natural basis for this is the set of projections onto each of the copies of $K \otimes \Delta_i$ in $K \otimes \Pi_j$, of which there are $[K \otimes \Pi_j : K \otimes \Delta_i]$. Thus $[K \otimes \Pi_j : K \otimes \Delta_i] = [k \otimes \Delta_i : S_j]$. □

We have no particular interest in (and quite possibly no way of gaining access to) $K \otimes \Pi_j$. Rather we are interested in $k \otimes \Pi_j$. The point is that if P_i has a filtration by $\{k \otimes \Delta_i\}_i$, and this set is a basis for the Grothendieck group, then the filtration multiplicities will be given by $[K \otimes \Pi_j : K \otimes \Delta_i]$, so that

$$[P_i, S_j] = \sum_k D_{kj} D_{ki}$$

In fact this holds true regardless of the filtration and basis properties, simply by considering simple characters throughout.

This says that we can completely determine these fundamental invariants of the representation theory of A_k by studying either the simple content of $\{k \otimes \Delta_i\}_i$, or (if the filtration and basis conditions *do* hold) the $\{k \otimes \Delta_i\}_i$ content, as it were, of the indecomposable projectives $\{P_i\}_i$.

(**9.3.14**) Note that, for fixed k, the index sets for $\{P_i\}_i$ and $\{S_i\}_i$ are the same, but are not necessarily the same as for $\{\Delta_i\}_i$.

We will give some rather detailed examples of the use of this machinery later.

9.4 Globalisation functors

ss:glob4alg

NB This section is a brief summary. The material is revisited in §13.4, using the TL algebra as an example.

Here A is an algebra over a field.

e trick

Lemma 9.1. *If $e \in A$ idempotent and $L \in A - mod$ simple then EITHER*

$$[A/AeA : L] \neq 0$$

OR

$$[head\ (Ae) : L] \neq 0.$$

Proof. The first inequality implies that L is not killed by the quotient, so that $eL = 0$. Every L obeying $eL = 0$ will appear in A/AeA, so the condition is also sufficient for this multiplicity to be nonzero. On the other hand, the second inequality implies $eL \neq 0$. Let e_L be an idempotent such that Ae_L is the indecomposable projective with head L. If $eL \neq 0$ then $eAe_L \neq 0$ and e_L is a primitive component of e, and Ae_L is isomorphic to a summand of Ae, so L occurs in its head. □

9.4.1 Globalisation functors and projective modules

ss:jump1

Here A is an algebra over a field k and $e^2 = e$ in A. Define functors between $A-$mod and $eAe-$mod by $G = Ae \otimes_{eAe} -$ (forward) and $F = e- \cong eA \otimes_A -$ (back). Then F is exact, and (G, F) is an adjunction. Note also that $G(A \oplus B) \cong G(A) \oplus G(B)$ by Theorem 8.9.5.

Note that $FG- = 1_{eAe}-$. Consider the special case of the adjoint isomorphism, $\mathrm{Hom}(M, FGM = M) \cong \mathrm{Hom}_A(GM, GM)$. This is $\mathrm{End}(M) \cong \mathrm{End}_A(GM)$. Thus functor G takes indecomposables to indecomposables.

Let P be a projective eAe-module. Then for any surjection $M \to M''$ in $A-$mod we have a surjection $FM \to FM''$ in $eAe-$mod (since F is exact). Since P is projective, $\mathrm{Hom}_{eAe}(P, eM) \to \mathrm{Hom}_{eAe}(P, eM'')$, that is $\mathrm{Hom}(P, FM) \to \mathrm{Hom}(P, FM'')$, is also a surjection. But then so is $\mathrm{Hom}_A(GP, M) \to \mathrm{Hom}_A(GP, M'')$, by using the adjoint isomorphism (on both sides). Hence the left exact functor $\mathrm{Hom}(GP, -)$ is also right exact and hence exact, and hence GP is projective.

Let index set Λ^e be such that $\{P^e_\lambda\}_{\lambda \in \Lambda^e}$ is a complete set of indecomposable projectives for eAe, distinct up to isomorphism (and so the set of heads, $\{L^e_\lambda\}_{\lambda \in \Lambda^e}$ say, is a complete set of simples). Then *add* $:= \{GP^e_\lambda\}_{\lambda \in \Lambda^e}$ is a subset of a complete set for A. Of course $FGP^e_\lambda = P^e_\lambda$ so F takes any projective in *add* back to the corresponding projective in $eAe-$mod.

(**9.4.1**) Let $C(A)$ denote the Cartan decomposition matrix (for any algebra A). Consider the universal property of projective modules: for each map $a : P_\mu \to M/M'$ there is a map f such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M/M' \arrow[r] & 0 \\ & & P_\mu \arrow[u, "f"] \arrow[ur, "a"'] & & \end{tikzcd}$$

From this, noting that there is such an a for each simple factor μ of M, we have

$$C(eAe)_{\lambda\mu} \;=\; [P^e_\mu, L^e_\lambda] \;=\; \dim \mathrm{Hom}(P^e_\lambda, P^e_\mu)$$

(See also (9.3.12).) Combining this with the above we have the following.
LEMMA. The Λ^e part of $C(A)$ coincides with $C(eAe)$. □

$$C(A) = \left(\begin{array}{c|c} C(eAe) & \\ \hline & \end{array} \right)$$

(**9.4.2**) The next question is: 1. How to characterise the projectives of A missing from this set? (How to characterise their simple heads?)

(**9.4.3**) We write $\{P_\lambda\}_{\lambda \in \Lambda^0}$ for the missing projectives; and let $\Lambda = \Lambda^e \sqcup \Lambda^0$ index a complete set.

(**9.4.4**) LEMMA. *The A-modules $G(P^e_\lambda)$ are the indecomposable direct summands of Ae (up to multiplicity).*

Proof. Every P^e_λ is a summand of eAe. But $G(eAe) = Ae \otimes_{eAe} eAe \cong Ae$. □

pa:e trick cor

(**9.4.5**) By Lemma 9.1 we see that the simples of A coinciding with simples of A/AeA (which are the simples obeying $eL = 0$) are disjoint from those corresponding to the heads of the indecomposable projectives in Ae.

(**9.4.6**) Lemma. (i) The projectives $\{P_\lambda\}_{\lambda\in\Lambda^0}$ are precisely those whose heads obey $eL_\lambda = 0$. (ii) For any $M \in eAe-$mod the head of $G(M)$ does not contain any of the $eL = 0$ simples. (iii) Every indecomposable projective of A that arises in $G(eAe-\text{mod})$ arises in the form $G(P^e_\lambda)$.

Proof. (i) Follows from (9.4.5). (ii) Recall that for every algebra module M there is an exact sequence $P' \to P \to M \to 0$ where P, P' are projective. Thus every module $G(M)$ in $G(eAe-\text{mod})$ has an exact sequence $G(P') \to G(P) \to G(M) \to 0$ (this is exact at $G(P)$ since G is right exact). In particular the head of $G(M)$ does not contain any of the $eL = 0$ simples. (iii) Follows immediately. □

(**9.4.7**) Next we attempt to say something about the missing parts of $C(A)$. Our approach is to combine this picture with the D-matrices of the Brauer-modular-system formalism, roughly as per Green [57, §6.6].

More questions:
2. How bring Δ-decomposition matrices into this picture? (And what are they?)
3. What happens to projectives under the involutive antiautomorphism, when there is one? (And what happens to simples? And does this lead to a symmetric Cartan matrix?)
4. What happens to projectives of A *not* in *add* under F?

9.4.2 Brauer-modules in a Brauer-modular-system for A

(**9.4.8**) Let $\{\Delta^e_a\}_{a\in I}$ be a set of eAe-modules. Let $(D^{e,I}_{a\lambda})_{a\in I,\lambda\in\Lambda^e}$ be the corresponding simple decomposition matrix. Then this is also *part of* the simple decomposition matrix $(D^I_{a\lambda})_{a\in I,\lambda\in\Lambda}$ for the set of images $G(\Delta^e_a)$. Is it possible for $G(\Delta^e_a)$ to have *other* composition factors?

$$D^I = \begin{pmatrix} D^{e,I} \\ \hline \end{pmatrix}$$

(**9.4.9**) Suppose now that A/AeA is semisimple. What do we get?

The simple modules of A/AeA are also simples of A, and obey $eL = 0$. Can they occur in Ae? (In the socle? As composition factors?)

What can we say about the composition factors of the corresponding indecomposable projectives in $A-\text{mod}$?
CLAIM: they lie in Λ^e (in the obvious sense).

(**9.4.10**) Suppose A is the 'modular' part of a π-modular system, and that e is integral; and that eAe is also modular.

Suppose that the Δ-modules are *Brauer-modules* for eAe, i.e. modular images of 'lifts' of 'ordinary' simple modules (note that the lifts are not uniquely defined, and neither are their modular images — it seems reasonable, for example, that if M is such a module, and there is an involutive antiautomorphism, then M's contravariant dual is also a candidate — these modules are not isomorphic in general ...although they do have the same composition factors). What can we say about the relationship between the G-images of these modules and the corresponding (set of possible) modules from $A-\text{mod}$? Are the G-images *candidates* to be Brauer-modules for A?

9.5 On Quasi-heredity — an axiomatic framework

ss:qh

Here A is an algebra over a field, and J its radical. We first give the conditions for A to be a quasi-hereditary algebra. We then discuss properties of quasi-hereditary algebras, and their *standard modules.*

Quasi-hereditary algebras appear in Scott [**?**]. They also appear in Cline-Parshall-Scott [25] (here is a discussion of recollement and highest weight categories — the category of A-modules for a qh algebra is highest weight; NB the authors cite Green's treatement [57, §6] of the Schur algebra for motivation), Parshall-Scott [131] (derived categories), and references therein. They are further studied by Dlab-Ringel in [40, 42]; by Ringel in [137]. There is also a review in Donkin [45]. Some useful practical machinery appears in Dlab-Ringel [43] (dimension and chain refinement); Dlab-Ringel [44] (module theoretic perspective); Ringel [136] (Ringel duality); Irving [**?**] (BGG reciprocity); Donkin-Reiten [**?**] (Schur algebras); Martin [107, Appendix] (partition algebras and Jones basic construction). Qh algebras with a duality: Xi [**?**]. See §9.6 for more references.

hered ideal

Definition 9.2. (See e.g. Dlab-Ringel [40, §4].) An ideal S of algebra A is a *heredity ideal* if $S^2 = S$, $SJS = 0$, and S is a projective left A-module.

(**9.5.1**) Example. A itself is a heredity ideal iff $J = 0$, i.e. if A is semisimple.

(**9.5.2**) Example. Consider the complex TL diagram algebra $T = T_4(\delta)$ and the ideal TU_1U_3T. In case $\delta \neq 0$ we can see (1) that $TU_1U_3TTU_1U_3T = TU_1U_3T$; (2) that $U_1U_3TU_1U_3 = kU_1U_3$, which is a semisimple algebra, so $U_1U_3JU_1U_3 = 0$ ($U_1U_3JU_1U_3$ lies in J and hence is nilpotent); (3) that $TU_1U_3T \cong \oplus_i TU_1U_3$ as a left-module. That is, $S = TU_1U_3T$ is a heredity ideal in T.

On the other hand if $\delta = 0$ then $S^2 = (TU_1U_3T)^2 = 0$ (every element contains at least one loop) and this ideal is not heredity.

Next consider the quotient algebra $T^1 := T/(TU_1U_3T)$ (with radical denoted J^1 say), and the ideal $S^1 = T^1U_1T^1$. For $\delta \neq 0$ we have $U_1T^1U_1 = kU_1$ so (1) $S^1S^1 = S^1$ and (2) $U_1J^1U_1 = 0$. Still taking care of the quotient we have (3) $T^1U_1T^1 \cong \oplus_i T^1U_1$ as a left module. Thus S^1 is a heredity ideal in $T^1 = T/S$.

Finally consider the quotient $T^2 := T/TU_1T \cong k$, which is simple and hence semisimple, and hence a heredity ideal for itself.

(**9.5.3**) Proposition. *(See e.g. [42, Stat.2].) (I) $S^2 = S$ iff $Hom_A(S_A, X_A) = 0$ for any A/S-module X.*
(II) If S_A projective then $S^2 = S$ iff $Hom_A(S_A, (A/S)_A) = 0$.

Proof. See [42, Stat.2]. □

(**9.5.4**) Scott's original definition of qh of algebra A: requires an ideal S that is projective and obeys $\mathrm{Hom}_A(S_A, (A/S)_A) = 0$ and $SJS = 0$; and that if $A \neq S$ then A/S also has such an ideal.

By the above we can replace the Hom condition with $S^2 = S$.

Next we recast the recursive ideal conditions firstly in terms of a chain of ideals, and then in terms of idempotents, in the spirit of Dlab-Ringel.

9.5.1 Basic Lemmas

ss:qhbasic

By semisimplicity, every ideal of A/J is generated by an idempotent. This is of the form $f = e+J$, where $e \in A$ idempotent. Thus the image of any A-ideal S is $S+J = AeA+J$ for some e. Note that $(AeA+J)^2 = AeAeA + AeAJ + JAeA + J^2 = AeA + J^2$ for any idempotent e. Meanwhile, supposing $S^2 = S$, we have $(S+J)^2 = S^2 + SJ + JS + J^2 = S + J^2$. Thus $(S+J)^r = S + J^r$, while $(AeA+J)^r = AeA + J^r$, for $r = 1, 2, \ldots$. For some such r we have $J^r = 0$, so

S=AeA **Lemma 9.3.** *Every idempotent ideal can be written in the form $S = AeA$ for some idempotent $e \in A$.* □

Now suppose that $S^2 = S$ is an ideal, and let e be such that $S = AeA$. Clearly $eJe = 0$ if and only if $SJS = AeAJAeA = AeJeA = 0$.

AeA proj **Lemma 9.4.** *(See e.g. Dlab-Ringel [42, Stat.7].) Let $e \in A$ be idempotent, and m the multiplication map $Ae \otimes_{eAe} eA \xrightarrow{m} AeA$.*
(I) If $(AeA)_A$ is projective then m is bijective;
(II) if $eJe = 0$ and m is bijective then $(AeA)_A$ is projective.

Proof. (I) Exercise. (Hint: Recall from (1.4.75) that one of the equivalent conditions for projectivity of a module P is that every surjection $t : M \to P$ splits. Thus here in particular m splits as a map of right-A-modules. That is, the ses

$$0 \to X_A \to Ae \otimes_{eAe} eA_A \xrightarrow{m} AeA_A \to 0$$

splits, where $X = \ker(m)$. Right multiplying by e is an exact functor F to mod $-$ eAe:

$$0 \to Xe \to Ae \otimes_{eAe} eAe \xrightarrow{Fm} AeAe_{eAe} \to 0$$

but Fm is an isomorphism, so $Xe = 0$ (so far regardless of the splitting property). OTOH $Ae \otimes_{eAe} eAeA = Ae \otimes_{eAe} eA$ so e does not kill any direct summand of $Ae \otimes_{eAe} eA$ (regarded as a right-A-module), so $X = 0$. Thus it is sufficient to prove the general property of projectives that we used.) (II) Note that $eJe = 0$ implies *every* eAe module is projective. Since $(Ae)_{eAe}$ and $(eA)_A$ are projective, $(Ae \otimes_{eAe} eA)_A$ is projective. □

(**9.5.5**) Example. Consider the TL algebra T_4 (which 'looks the same' from left or right). Take $e = U_1U_3$ (let us take $\delta = 1$ for simplicity). (I) Here $eT_4e = ke \cong k$, so $eJe = 0$. (II) dim(AeA) = dim(Ae).dim(eA) so m is bijective 'by comparing bases'. (III) Left-ideal T_4e is evidently left-projective, and one sees that ${}_{T_4}T_4eT_4$ is a direct sum of isomorphic copies.

Unsurprisingly given the results in this Section, we have the following definitions.

9.5.2 Definitions

hered e **Definition 9.5.** An idempotent $e \in A$ is a *heredity idempotent* if
(I) eAe is a semisimple algebra
(II) the multiplication map

$$Ae \otimes_{eAe} eA \to AeA$$

is a bijection.

(**9.5.6**) The point, from above, is that if e is heredity then the ideal AeA is heredity in the sense of Def.9.2. Yet another way to look at this is that an ideal of form $S = AeA$, where $e^2 = e$ is such that eAe is semisimple, (necessarily obeys $SS = S$ and $SJS = 0$ and hence) is heredity if it is left-projective.

hered ch **Definition 9.6.** A sequence of idempotents in A, $(e^1, \ldots, e^l)$, is a *heredity (idempotent) chain* if
(I) $A = Ae^1A \supset Ae^2A \supset \ldots \supset Ae^lA$ (all proper inclusions);
(II) each e^i is a heredity idempotent modulo $Ae^{i+1}A$.

ideal chain **Definition 9.7.** (See e.g. Dlab-Ringel [40, §4].) Algebra A is *quasi-hereditary* if there exists a chain of ideals

$$A = S_1 \supset S_2 \supset \ldots \supset S_l \supset S_{l+1} = 0$$

such that S_r/S_{r+1} is an heredity ideal of A/S_{r+1}. (Such a chain of ideals is again a *heredity chain.*)

Remark. In particular S_l is a heredity ideal of A.

(**9.5.7**) Note that an algebra is quasi-hereditary if and only if it has a heredity idempotent chain.

9.5.3 Consequences of quasi-heredity of A for $A-$mod

Let A have heredity chain $e^- = (e^1, \ldots, e^l)$ and put $A^i = A/Ae^{i+1}A$. Now consider

$$A^ie^i \;=\; Ae^i/(Ae^{i+1}A)$$

which is projective as an A^i-module and a quotient of a projective A-module (not necessarily indecomposable). We have the following.
(1) Since $e^iA^ie^i$ is semisimple by heredity (with identity e^i) we can decompose e^i into a sum of primitive idempotents of this algebra:

$$e^i = \sum_{j\in\Lambda(e^iA^ie^i)} \sum_k e^i_{j,k}$$

where $\Lambda(e^iA^ie^i)$ is an index set for simples; and $\sum_k$ is over a primitive decomposition of the primitive-central idempotent e^i_j. Hence:

$$A^ie^i \;\cong\; \bigoplus_{j\in\Lambda(e^iA^ie^i)} \bigoplus_k A^ie^i_{j,k}$$

NOW COMPARE with standard idempotent properties...
(2) For each simple L_x there is a unique i such that $[\mathrm{top}(A^ie^i) : L_x] \neq 0$. For this i write Δ_x for the projective cover of L_x as an A^i-module (an indecomposable summand of A^ie^i.

de:deltaqhx (**9.5.8**) Claim (see below): We may take the label x from $\Lambda(e^iA^ie^i)$, and have

$$\Delta_x \;\cong\; A^ie^i_{x,k} \tag{9.2}$$ eq:delthx

for any k.

(**9.5.9**) Note that this association of each L_x to an e^i associates a partial order to the complete set of simple modules.

Since Δ_x is indecomposable projective (as A^i-module) it has simple head (as A-module). Indeed it has many beautiful properties (derived by following the same line).

(**9.5.10**) To see (9.5.8): We claim that the heredity idempotent chain e^- can be refined to a *maximal* chain, for which the 'sections' $e^x A^x e^x$ (say) are simple algebras. The refinement inserts idempotents into the original chain at point i, corresponding to the structure, and in particular the primitive idempotent decomposition of 1, of $e^i A^i e^i$. The next idempotent after e^{i-1} (or previous to e^{i+1} ??) can be $e^i e_{j,k}$ where $e_{j,k}$ is any primitive idempotent of $e^i A^i e^i$. The next id after that is slightly more complicated (to accommodate the quotient), but since the order of choosing the first one here does not matter we get (9.2).

(**9.5.11**) Exercise. Derive some properties of Δ_x. Hint: What can we say about $[\Delta_x : L_y]$, where L_y is associated to e^j, say?

9.5.4 Quasi-heredity defined module-theoretically

...

9.5.5 Examples

(**9.5.12**) Exercise. Show that the TL algebra and the partition algebra are quasi-hereditary for $\delta \neq 0$.

9.6 Notes and References

ss:qhnr

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott, to deal with highest weight categories (see [25], and later). The notion has turned out to be quite deep, and their definition has since been cast in a variety of ways, depending on the intended use. Here we merely review the cast(s) most useful in the context of diagram algebras. Useful sources include Dlab and Ringel [40, 41].

9.7 More axiomatic frameworks

9.7.1 Summary of 'Donkin's Appendix' on finite dimensional algebras

We summarize from Cline–Parshall-Scott [24], Dlab–Ringel [41, 42, 44] and Donkin [45, Appendix].

First we assemble some generalities about finite dimensional algebras. Later we extend these in the quasi-hereditary case. Here: k is a field and

A a finite dimensional k-algebra.
$\{L(\lambda) \mid \lambda \in \Lambda^+\}$ is a complete set of simple A-modules up to isomorphism.
$P(\lambda)$ and $I(\lambda)$ are the corresponding projective cover and injective envelope of $L(\lambda)$.

We shall routinely notationally confuse $L(\lambda)$ with λ (where no ambiguity can arise). For π any subset of Λ^+, we say A-module V belongs to π if $[V : L(\lambda)] \neq 0$ implies $\lambda \in \pi$.

(**9.7.1**) Fixing π, there is a unique maximal submodule of $V \in A-\mathrm{mod}$, denoted $O_\pi(V)$, such that $O_\pi(V)$ belongs to π (since if two submodules belong to π then so does their sum); and a unique minimal submodule $O^\pi(V)$ such that $V/O^\pi(V)$ belongs to π.

(**9.7.2**) $A(\pi) := A/O^\pi(A)$

(**9.7.3**) $\{L(\lambda) \; : \; \lambda \in \pi\}$ is a complete set of simple $A(\pi)$-modules.

Fixing a partial order $\leq$ on Λ^+ we define $\pi(\lambda)$ as the $\leq$-ideal below λ. Define
$M(\lambda)$ as the unique maximal submodule of $P(\lambda)$;
$K(\lambda) = O^{\pi(\lambda)}(M(\lambda))$ and

$$\Delta^{\leq}(\lambda) = P(\lambda)/K(\lambda).$$

Note that whatever the partial order, $\Delta^{\leq}(\lambda)$ has simple head $L(\lambda)$, and $L(\lambda)$ is *not* a composition factor of the maximal submodule.

(**9.7.4**) Lemma. *For all $\lambda \in \Lambda^+$, $End_A(\Delta^{\leq}(\lambda)) = k$.*

Let us try to prove the more general assertion:

(**9.7.5**) Lemma. *CONSIDER CLAIM: Let module M have simple head L and no other composition factor L. Then $End(M) = k$.*

(**9.7.6**) Let $e \in A$ be idempotent. Let

$$A_e = eAe.$$

The Schur functor $f_e : A-\text{mod} \to A_e-\text{mod}$ is given by

$$f_e M = eM$$

(**9.7.7**) f_e is exact.

(**9.7.8**) $\Lambda^+_e := \{\lambda \in \Lambda^+ \mid eL(\lambda) \neq 0\}$

(**9.7.9**) Let M be an A-module and X a set of A-modules. We write $M \in F(X)$ if M has a filtration by modules from X.

(**9.7.10**) (Donkin [45, p.189]) $A-$mod is a *high weight category wrt* Λ^+ if
$I(\lambda)/\nabla^{\leq}(\lambda) \in F(\nabla^{\leq})$
if $(I(\lambda)/\nabla^{\leq}(\lambda) : \nabla^{\leq}(\mu)) \neq 0$ then $\mu > \lambda$.

(**9.7.11**) Example: Let $k = \mathbb{C}$. Then T_n-mod is high weight wrt propagating number (cf. **??**) if $\delta \neq 0$.

(**9.7.12**) Subset $\pi \subseteq \Lambda^+$ is *saturated* if $\lambda < \mu$ and $\mu \in \pi$ implies $\lambda \in \pi$.

(**9.7.13**) Example: ...

(**9.7.14**) Prop. If π saturated then $A(\pi)$-mod is a hwc.

(**9.7.15**) ... IDEMPOTENTS! ...

(**9.7.16**) Prop. Consider a 'dual' setup to that for a hwc wrt Λ^+ above: Datum (A, Λ^+) gives a hwc iff, with $\Delta^{\leq}(\lambda) = P(\lambda)/K(\lambda)$ as before, (i) $K(\lambda) \in F(\Delta^{\leq})$; (ii) $(K(\lambda) : \Delta^{\leq}(\mu)) \neq 0$ implies $\mu > \lambda$.

Proof. Exercise ...

(**9.7.17**) Prop. Algebra A yields a hwc iff it has a quasihereditary structure.

Proof. [45, Appendix] If: Exercise.
Only if: Recall A is qh if there is a chain of ideals $A = H_0 > H_1 > ... > H_n = 0$ with $S = H_i/H_{i+1}$ hereditary in A/H_{i+1} (i.e. $S^2 = S$, $SJS = 0$ and S is projective). Let Λ^+ label simples; and let $\Lambda^+(i)$ be the subset appearing in A/H_i. Let $\leq$ be the obvious partial order on Λ^+. We aim to show that we get a hwc with respect to this order.

Note (I) that if $n = 1$ then $A = H_0$ is itself hereditary so $J_A = 0$ and $(A, \textit{any order})$ gives a hwc. Also, (II) if A is qh then so is each A/H_i. So let us have an inductive hypothesis that the claim holds for heredity chains of length $\leq n-1$. The observation (I) is the base of induction. Thus we require to show that A gives a hwc given the inductive hypothesis.

Consider $H = H_{n-1}$ (e.g. in even n TL case this is the ideal with no propagating lines ... here high in the order means low number of propagating lines), which is projective. In general we can show that each indecomposable summand here is a $\Delta^{\leq}(\mu)$ for μ 'high' in the order, i.e. in $\Lambda^+ \setminus \Lambda^+(n-1)$. I.e. $P(\mu) = \Delta^{\leq}(\mu)$. ...

For $\lambda \in \Lambda^+(n-1)$ consider the module $HP(\lambda)$. Then $P_0 = P(\lambda)/HP(\lambda)$ is the projective cover of $L(\lambda)$ in A/H. Define M_0 as the unique maximal submodule: $0 \to M_0 \to P_0 \to L(\lambda) \to 0$ and $K_0 = O^{\pi(\lambda)}(M_0)$, and $\Delta_0 = P_0/K_0$. CLAIM: since H is projective, $HP(\lambda)$ is projective with summands all from 'high'. ...

9.7.2 Quasi-hereditary algebras (take III?)

See Dlab–Ringel [39], Cline–Parshall–Scott [24], Koenig, Koenig–Xi, or ...

9.7.3 Cellular algebras

See Graham—Lehrer [56].

Chapter 10

Forms, module morphisms and Gram matrices

ss:Forms

10.1 Forms, module morphisms and Gram matrices (Draft)

Here we consider some representation-theoretically useful forms on modules for algebras that are isomorphic to their opposites. (Many interesting algebras are isomorphic to their opposites, such as partition algebras. For interesting examples that are *not*, consider for example the full transformation semigroups from §4.1.4.)

Aside: A k-algebra A isomorphic to its opposite via an involutive set map has an involutive antiautomorphism. (For any algebra A let $op : A \to A^{op}$ be the identity map on A as a set; and hence an algebra antihomomorphism. Since the image of op is A as a set, we can apply op again - i.e. to A^{op}: $op \circ op = 1$. Let $i : A \to A^{op}$ be an algebra isomorphism. Certainly i is a bijective set map from A to itself (not the identity map, unless A is commutative, but possibly an involution). Then $(op \circ i) : A \to A$ is an antiautomorphism. Note that as a set map this is just the map i. For N in A-mod we have N^i in A-mod given by $N^i = \hom_k(N, k)$ as a set.)

Overview:
The basic idea is this... Let A be a k-algebra. Firstly, *any* A-module morphism $\psi : M \to N$ gives us information about M and N. The kernel is a submodule of M for example.

If there is an algebra antiautomorphism for A then module morphisms $\psi : M \to N^i$ are in bijection with contravariant forms $\langle\rangle : M \times N \to k$ by $\langle m, n\rangle = \psi(m)(n)$ — thus contravariant forms become a useful source of morphisms.

Next, for certain special classes of modules, only morphisms from them (or to them) of special kinds are possible in principle; explicit morphisms are then particularly revealing. We give a small indicative example in (10.1.2).

(**10.1.1**) ... the ideas discussed in this section have been used historically to study the symmetric group in particular (see e.g. [78]). See also [57]. Outside the classical group representation theory setting, it has proved very useful for 'diagram algebras' such as the partition algebra (see [114, 116], [115] for some early examples).

An outline of the section is this... We start with a brief review of basics designed to make the section as self-contained as possible. In particular it will be useful to recall properties of ordinary left-right duality. Then we discuss bilinear forms; and then contravariant forms; then examples from both the group and the diagram algebra settings; ...

Indicative (meta) Example

exa:eAeeke2 (**10.1.2**) Suppose A is a k-algebra, $e^2 = e \in A$ and $A^/$ is a quotient of A. Let $\bar{a} \in A^/$ denote the natural image of $a \in A$ (when unambiguous, we simply write a for the image of a in $A^/$). Suppose now that

$$\bar{e}A^/\bar{e} = k\bar{e}. \tag{10.1}$$ eq:eAe=ke2

Then by (1.6.7): (the image of) e is primitive; $A^/\bar{e}$ is an indecomposable projective left-$A^/$-module (and an indecomposable A-module with simple head); and $\bar{e}A^/$ is an indecomposable projective right-module. Furthermore for any $a \in A$ we have a scalar $\langle a \rangle_e \in k$ given by $\bar{e}\bar{a}\bar{e} = \langle a \rangle_e \bar{e}$.

(**10.1.3**) Given (10.1) we have a bilinear map

$$\langle\rangle : A^/\bar{e} \ \times \ \bar{e}A^/ \ \to k$$

given by $(a\bar{e}, \bar{e}b) \mapsto \langle a\bar{e}, \bar{e}b \rangle$ where $\langle a\bar{e}, \bar{e}b \rangle$ is given by $\bar{e}ba\bar{e} = \langle a\bar{e}, \bar{e}b \rangle \bar{e}$. That is $\langle a\bar{e}, \bar{e}b \rangle = \langle \bar{e}ba\bar{e} \rangle_e$. This map induces a linear map

$$\psi : A^/\bar{e} \to (\bar{e}A^/)^*$$

where $(\bar{e}A^/)^* = \hom_k(\bar{e}A^/, k)$, by $ae \mapsto (eb \mapsto \langle ae, eb \rangle)$. Both the domain and the codomain now have a left A-module structure (the dual to the right projective is left injective, with simple socle). One checks if this linear map is a left A-module morphism (comparing $\psi(c(ae))$ with $c\psi(ae)$ for $c \in A$)

$$\psi(c(ae)) \ = \ (eb \mapsto \langle cae, eb \rangle) \ = \ (eb \mapsto \langle eb(cae) \rangle_e)$$
$$c\psi(ae) \ = \ c(eb \mapsto \langle ebae \rangle_e) \ = \ (eb \mapsto \psi(ae)((eb)c) = \langle (ebc)ae \rangle_e)$$

Recall (see e.g. Prop.8.6.8) that for any $A^/$-module M there is an isomorphism of abelian groups $\bar{e}M \cong \mathrm{Hom}_{A^/}(A^/\bar{e}, M)$. So in particular $\bar{e}A^/\bar{e} \cong \mathrm{Hom}_{A^/}(A^/\bar{e}, A^/\bar{e})$ (this can even be made into a ring anti-isomorphism). Also recall that (if k is algebraically closed) then composition multiplicities in any module M are given by

$$\dim_k \mathrm{Hom}_{A^/}(P_i, M) \ = \ [M : S_i] \tag{10.2}$$ eq:PiMSi

(see e.g. (9.3.12)) for any indecomposable projective P_i with simple head S_i. Thus in case $\bar{e}A^/\bar{e} = k\bar{e}$ we see that the dimension is 1 and so the head simple in indecomposable left projective $A^/\bar{e}$ occurs *only* in the head. (Caveat: expressed this way it is essentially a tautology. The hom space in (10.2) can be thought of as spanned by morphisms taking the head simple S_i in P_i to each of the various 'copies' in M.) Here the 1-d space of morphisms is spanned by the identity morphism taking head to head.

Hereafter, we are more interested in the 1-d space of morphisms from the projective to the 'corresponding' injective (necessarily taking the head to the socle, as we will see). Here the rank of the morphism is not full in general - it corresponds to the dimension of the head/socle.

In general (e.g. for an algebra not isomorphic to its opposite) the structure (the dimension; the character and so on) of the right projective $\bar{e}A^/$ can be quite different to the left projective. See for example (4.5). However the quotient by the radical reduces both modules to simples, and yields a semisimple (and hence opposite-isomorphic) quotient algebra, so that the head simples are paired.

It follows - see also e.g. **??** - that the simple module S' in the socle of the indecomposable left injective $(\bar{e}A^/)^*$ above occurs only in the socle.

...We'd like to consider when the map from projective to injective is head S to socle S', i.e. when $S \cong S'$. To investigate this we consider $\bar{e}(\bar{e}A^/)^*$. The element $f : \bar{e}m \mapsto f(\bar{e}m)$ is taken to $(\bar{e}f) : \bar{e}m \mapsto f(\bar{e}m\bar{e})$. Since ... One should compare with the quasi-heredity framework as in §9.5. And see also §2.3 for extensive examples.

Note that the dual of a projective right-module is not necessarily a projective left-module.

10.1.1 Basic preliminaries recalled: ordinary left-right duality

ss:vs dual2

(**10.1.4**) Recall the convention (e.g. from Ch.8) that we write the action of ring or algebra A on left A-module M 'on the left':

$$A \times M \rightarrow M \tag{10.3}$$
$$(a, m) \mapsto am \tag{10.4}$$

so that $b(am) = (ba)m$; and the action of $a \in A$ on a right A-module on the right: ma.
(If we write ma for a left action we get $(ma)b = m(ba)$ which just looks odd.)

de:dualM

(**10.1.5**) Recall that if A is an R-algebra and $M = {}_AM$ a left A-module then the *dual right module* is

$$M^* \;=\; M^*_A \;:=\; \mathrm{Hom}_R({}_AM, R)$$

Thus elements of M^* are maps

$$\mu : M \rightarrow R \tag{10.5}$$
$$m \mapsto \mu(m) \tag{10.6}$$

It is a *right* A-module by the action of $a \in A$ on any μ as above, to give μa as follows:

$$\mu a : m \mapsto \mu(am)$$

We check the right action property by comparing $(\mu a)b$ with $\mu(ab)$:

$$(\mu a)b : m \mapsto \mu a(bm) = \mu(a(bm))$$

$$\mu(ab) : m \mapsto \mu((ab)m)$$

(**10.1.6**) Does this $*$ lift to a functor from $A-$mod to mod$-A$? Consider the possible image of a map in $A-$mod (or indeed a sequence in $A-$mod):

$$\begin{tikzcd}
M' \arrow[r, "f"] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[d, "\hom_R(-,R)"'] \\
(M')^* & M^* \arrow[l, "?"] & (M'')^* \arrow[l, "?"]
\end{tikzcd}$$

Given a map $f \in \hom_R(M', M)$, then for each $a \in \hom_R(M, R)$ we can form $f^*(a) \in \hom_R(M', R)$ by $f^*(a) = a \circ f$. That is

$$\begin{tikzcd} M' \arrow[rr, "f^*(a)"] \arrow[dr, "f"'] & & R \\ & M \arrow[ur, "a"'] & \end{tikzcd}$$

Thus $f^* \in \hom_R(M^*, (M')^*)$ and we have

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[d, "\hom_R(-,R)"'] \\ (M')^* & M^* \arrow[l, "f^*"] & (M'')^* \arrow[l, "g^*"] \end{tikzcd}$$

In other words $\hom_R(-, R) : A-\text{mod} \;\to\; \text{mod}-A$ defined in this way is a *contravariant* functor. Indeed this $H- = \hom_R(-, R)$ is a left-exact contravariant functor, meaning that an exact sequence $M' \to M \to M'' \to 0$ passes to an exact sequence $0 \to HM'' \to HM \to HM'$.

(**10.1.7**) Exercise. (I) Suppose an A-module M is simple. What can we say about M^*?
(II) Suppose we have a Jordan–Holder series (see e.g. §8.3.2) for an A-module M. What can we say about M^*?

(**10.1.8**) If in particular A is a finite dimensional algebra over a *field* R then every finitely-generated A-module M is a finite-dimensional R-vector space, and M^* has the same dimension, and we have the following (see e.g. [3, §23]).

(I) M and M^{**} are isomorphic A-modules.

(II) M is (semi)simple if and only if M^* is (semi)simple.

(III) A sequence $0 \to M' \to M \to M'' \to 0$ is (split) exact if and only if the dual sequence is (split) exact.

(IV) $\text{Soc}\, M^* \cong (M/\text{Rad}\ M)^*$.

10.1.2 Contravariant duality

ss:contra dual2

de:opM0

(**10.1.9**) Let R be a commutative ring and let A be any R-algebra. We may regard $M = {}_AM$ as a right A^{op}-module $M = M_{A^{op}}$ by defining a right action as follows:

$$ma \;:=\; am$$

(Check: recall that for $a, b \in A^{op}$, with multiplication denoted $*$ say, then $a * b = ba$ as computed in A; thus $(ma)b = b(ma) = b(am) = (ba)m = m(a * b)$). Similarly each right module gives a left module.

A left A-module homomorphism $M' \xrightarrow{f} M$ becomes a right A^{op}-module homomorphism, so this construction $\Phi : A-\text{mod} \to \; \text{mod}-A^{op}$ (say) is a (covariant) functor. We also use Φ for the right-to-left version.

avariant duality

(**10.1.10**) A group algebra over a commutative ring R is isomorphic to its opposite (defined as for opposite ring) since $g \mapsto g^{-1}$ defines a group antiautomorphism (an isomorphism $G \cong G^{op}$); and this extends to RG via: $\sum_i r_i g_i \mapsto \sum_i r_i g_i^{-1}$.

There may be other isomorphisms. For example, a suitable group of matrices may be mapped to its opposite by $g \mapsto g^{tr}$ (transpose matrix). Here, when considering any algebra isomorphic to its opposite, we will generally fix a given involutive antiautomorphism.

de:opM

(**10.1.11**) Let R be a commutative ring and let A be any R-algebra with an involutive antiautomorphism (generally denoted $g \mapsto g^t$, or $g \mapsto g^\tau$). We may regard $M = {}_AM$ as a right A-module by

$$ma \;:=\; a^t m$$

(Check: $(ma)b = b^t(ma) = b^ta^tm = (ab)^tm = m(ab)$); and similarly each right module gives a left module.

(**10.1.12**) It follows from (10.1.5) and (10.1.11) that for each $M \in A-\mathrm{mod}$ there is another left module M^o obtained from the dual right module M^* by applying Φ:

$$M^o \;:=\; \Phi(M^*)$$

(i.e. via the opposite isomorphism, regarding M^* as a left module for the opposite). This construction has the property that $R-\mathrm{mod}$ is invariant under taking to its dual combined with taking all $M \mapsto M^o$. (I.e. if defined, $(\mathrm{Head}\ M)^o \cong \mathrm{Soc}\ M^o$, and so on.)

We will call the map $M \mapsto M^o$ *contravariant duality* (see e.g. [57]). We have $(M^o)^o = M$.

(**10.1.13**) Exercise. Let G be a finite group and R a field. The 'contragredient' of a projective RG-module is projective (claim (10.29) in Curtis–Reiner [35]). Prove this. Give a counter-example for general finite-dimensional R-algebra A with t.
(For the symmetric group, and indeed any finite G, we will see in (10.1.35) that the regular module is contravariant self-dual for any R. Thus the collection of indecomposable summands over R a field must be fixed under duality, which verifies the claim in this case. However the regular module is not always self-dual for an algebra A with t (we shall have an example from the Temperley–Lieb algebras shortly).)

(**10.1.14**) Exercise. (Optional) Why are duals of lattices done differently in Curtis–Reiner [35] p.89 cf. p.245?

(**10.1.15**) Exercise. (Optional) Claim: Suppose A is in fact a finite group algebra over R. Let $x \in A$ be mapped to x^o by the opposite isomorphism (and regard x^o as an element of A). Then $M = Ax$ implies $M^o = Ax^o$.
Prove this, or provide a counter-example.

10.1.3 On 'Λ-standard' modules and duals of simples

pa:dualsimple

(**10.1.16**) Given an algebra A with an involutive antiautomorphism t as above, when is a simple A-module $L = L^o$?

(**10.1.17**) Let $\Lambda = (\Lambda, \leq)$ be a poset whose underlying set indexes the simple modules of algebra A up to isomorphism. A *set* Δ of A-modules M_λ is a Λ-*standard set* if it is indexed by the poset $(\Lambda, \leq)$, with the property that (i) the set of heads is a complete set of simple modules of A; and (ii) the decomposition matrix of this set is lower unitriangular with respect to the partial order.

(**10.1.18**) Let L be the head of $M \in \Delta$ as above. The unitriangular property means that if simple L' occurs below the head in M then L will not occur below the head in the standard M' with head L'.

Note that the set of simples of A is a Λ-standard set for any order.

(**10.1.19**) Lemma. *Let A be an algebra with involutive antiautomorphism. Suppose Δ is a Λ-standard set for A; and that there is a nonzero contravariant form on each $M_\lambda \in \Delta$. Then $L \cong L^o$ for all simples L.*

Proof. Since there is a nonzero contravariant form on each $M = M_\lambda$ there is a nonzero module map from M to M^o (Prop.10.1.34). This means that there must be a copy of head L in M^o. But we know the socle of M^o is L^o, so both L and L^o are in the image of the module map from M. Thus either $L \cong L^o$ and the image is just L, or L^o lies in M below the head. But now consider the standard module M' for which L^o is the head. If $L \ncong L^o$ then M, M' are nonisomorphic and M' must contain $L = L^{oo}$ and L^o by an analogous argument, contradicting the unitriangular property. Thus here $L \cong L^o$. □

10.1.4 On a Schur Lemma for 'standard' modules

prepre (**10.1.20**) Proposition. *Suppose that R is a field, and A is an R-algebra with a given involutive antiautomorphism. If left A-module M has a unique maximal proper submodule (call it M_o) and hence simple head $L = M/M_o$, and this composition factor L has multiplicity one in M, and $L \cong L^o$, then* $\dim Hom_A(M, M^o) = 1$*, and* $\psi \in Hom_A(M, M^o)$ *has rank* $\dim(L)$.

Proof: NB, every simple factor in M^o is extended by L below it. There *is* a map $\psi \in \mathrm{Hom}_A(M, M^o)$ — that which kills the unique maximal proper submodule M_o and so makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M_o \arrow[r] & M \arrow[r] \arrow[d, "\psi"'] & L \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 & (M_o)^o \arrow[l] & M^o \arrow[l] & L^o \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

(or any scalar multiple thereof). No reduction is possible in the kernel, since this would require factors appearing in the image below L, which M^o does not have, as already noted. No enlargement of the image is possible since this, correspondingly, requires factors above L in M. □
(NB, the converse does not hold in general.)

10.1.5 Bilinear forms

ss:bilinear

See §1.4.9 and (1.4.56) for examples, connections and applications.

de:bilinearform (**10.1.21**) Bilinear form: For R a field (or commutative ring), a bilinear form on $M, N \in R{-}\mathrm{mod}$ is an R-bilinear function

$$<,>: M \times N \to R$$

(cf. Perlis [132], Maclane-Birkoff [97, §X.1]).

(**10.1.22**) Suppose M, N free R-modules. Let $B_M = \{b_1, b_2, \ldots, b_k\}$ be an ordered basis of M and $B_N = \{c_1, c_2, \ldots, c_l\}$ of N. Then a matrix $B(M, N)$ of form

$$(B(M,N))_{ij} \ = < b_i, c_j >$$

determines $<,>$.

(**10.1.23**) In an obvious notation,

$$B(M,N) = \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \left(\ |c_1\rangle, \ |c_2\rangle, \cdots, \ |c_l\rangle \ \right)$$

By linearity, the effect of basis changes $B'_M = B_M U$, $B'_N = B_N W$ (B_M arranged as a row vector $(b_1, b_2, ...)$; U, W unimodular matrices, as in §8.2.4) is

$$B'(M,N) = U^t \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \left(\ |c_1\rangle, \ |c_2\rangle, \cdots, \ |c_l\rangle \ \right) W \ = \ U^t B(M,N) W$$

Recall (e.g. from §8.2.4) that if R is a PID then among the possible such basis changes would be a pair that bring $B'(M,N)$ into Smith normal form.

Remark. The reason for keeping the ground ring R at the level of generality of commutative ring (despite the fact that our eventual objects of study are typically algebras over fields) will become apparent shortly.

`pa:form-map` (**10.1.24**) Keep $M, N \in R-\text{mod}$ as before. Recall that the dual $N^* = \mathrm{Hom}_R(N,R)$ has the structure of R-module (since R is commutative, 'left' and 'right' modules are the same here). Note that form $<,>$ defines an R-module homomorphism

$$\psi : M \to N^*$$

by

$$\psi(m)(n) \ = < m, n > .$$

From this perspective we can think of $B(M,N)$ (or $B'(M,N)$) as characterising the image of M in N^*.

If R is a field, then the rank of matrix $B(M,N)$ is independent of the specific choice of bases (to see this note that it is unchanged on replacing b_1 by a linear combination with nonzero component of b_1). Accordingly

$$\text{rank} \ \ <,> \ \ := \ \ \text{rank} \ (B(M,N)).$$

(**10.1.25**) For I an ideal of R define $L^I_{<,>}$ as the subset of M such that $< m, n > \in \ I$ for every $m \in L^I_{<,>}$ and $n \in N$. By linearity $L^I_{<,>}$ is a submodule.

The Left radical $L_{<,>} = L^0_{<,>}$ of $<,>$ is the submodule M' of M such that $< m, n > = \ 0$ for every $m \in M'$ and $n \in N$.

If R is a field then

$$\text{rank} \ \ <,> \ \ = \ \ \dim M - \dim L_{<,>}.$$

(**10.1.26**) Note that $\psi(b_j)$ is the map such that $(\psi(b_j))(c_i) = \ < b_j, c_i >$. Our basis of N^* is $\{f_i\}$ such that $f_i c_j = \delta_{i,j}$ therefore

$$\psi(b_j) = \sum_i < b_j, c_i > f_i$$

(check: we have $(\sum_i \alpha_i f_i)c_j = \alpha_j$ so $(\sum_i < b_j, c_i > f_i)c_k \; = \; < b_j, c_k >$ as required).

In an example, this says that, expressing the basis of M as $(1,0,0), (0,1,0), (0,0,1)$ (say); and of N^* as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can express ψ acting on M by $B(M,N)$ acting on the right:

$$(1,0,0)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix} = (B_{11}, B_{12}) = B_{11}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T + B_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$$

or by $B(M,N)^T$ acting on the left. In this formulation the inner product can be written, say,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (B_{11})$$

ss:contravf 10.1.6 Contravariant forms on A-modules

In (10.1.24) we gave a correspondence between bilinear forms and maps from modules to dual-modules over a commutative ring R. Next we lift this to modules over an R-algebra A (NB, suitably changing dual right modules to left modules for a *non*-commutative algebra A requires an antiautomorphism).

This will not work for an arbitrary form $<,>: M \to N$, or corresponding $\psi : M \to N^*$, since we have the A-module morphism condition $\psi(am) = a\psi(m)$ for all $a \in A, m \in M$ to satisfy.

de:intertwine (**10.1.27**) We can think of this condition directly or over specific bases for M and N, where a acts as a matrix $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$ respectively and we must check $B\,\rho_{B_M}(a) \;=\; \rho_{B_N}(a)\,B$ for all $a \in A$ — i.e. that $B = B(M,N)$ 'intertwines' representations $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$. It turns out that this condition can be expressed quite neatly.

(**10.1.28**) Contravariant form.
Let A be an R–algebra with involutive antiautomorphism t as above. For $M, N \in A-$mod, an R–bilinear form $<,>: M \times N \to R$ is *contravariant* if $< am, n > = < m, a^t n >$ for all $a \in A$, $m \in M$, $n \in N$.

(**10.1.29**) Note that there is a bilinear form $<,>$ for every choice of matrix $B(N,M)$. The requirement of satisfying the constraints of *contravariant form* on A-modules, however, will in general be very restrictive on possible choices of $B(M,N)$.

de:permf (**10.1.30**) Example. Let R be a commutative ring and G a finite group. Let $M = R\{m_1, m_2, ..., m_l\}$ be an RG-module that is free as an R-module, with the given basis. Define a bilinear form on M (i.e. on the pair $(M,N) = (M,M)$) by setting $B(M,M)$ to the identity matrix, that is, by

$$< m_i, m_j > = \; \delta_{m_i,m_j}.$$

We have $gm_i = m_j \iff m_i = g^{-1}m_j$. Thus *if G acts on M by permuting basis elements* we have $< gm_i, m_j > = < m_i, g^{-1}m_j >$, and hence $< am_i, m_j > = < m_i, a^t m_j >$ for $a \in RG$. Thus this $<,>$ is a contravariant form.

In particular if $M = RG$ is the left regular module then G indeed acts by permuting basis elements, so this $<,>$ is contravariant in this case. (We will give a more explicit example in (10.1.35).)

(**10.1.31**) Exercise. (Optional) Construct an example as above but where G does not act on M by permutation, and where the $<,>$ above is indeed not contravariant.

contraformperp (**10.1.32**) Proposition. *Let A be an R-algebra with involutive antiautomorphism t, and $M, N \in A-mod$. If $<,>: M \times N \to R$ is a contravariant form and I an ideal in R then the subset $S = L^I_{<,>} \subseteq M$ (such that $< s, N > \in I$ for all $s \in S$) is an A-submodule of M.*

Proof. For $a \in A$, $s \in S$, $n \in N$ we have $< as, n >=< s, a^t n > \in I$ since $a^t n \in N$, so $as \in S$. □

(**10.1.33**) Remark. Note that if $I' \subset I$ then $L^{I'}_{<,>} \subseteq L^I_{<,>}$. Of course if R is a field then the only possibility is $I = 0$.
Note that if we start with R a commutative ring and compute $S = L^0_{<,>}$; then base change to some $A^k = k \otimes_R A$ and write S^k for the corresponding submodule computed here, then S^k may be bigger than $k \otimes_R S$.

contraformx (**10.1.34**) Proposition. *Let R be a commutative ring and $A = RG$ for some group G (or else an R-algebra with involutive antiautomorphism).*
(I) To each contravariant form $<,>: M \times N \to R$ we may associate an element

$$\psi \in \hom_A(M, N^o)$$

given by $\psi(m)(n) = <m,n>$.
(II) This association defines a bijective correspondence between such forms and morphisms.
(III) If R is a field and $M = N$ satisfies the assumptions in Proposition (10.1.20) then there is a unique form up to scalars, and the form is non–singular iff the associated ψ is an isomorphism (in particular it is non–singular if $M = N$ is simple). (cf. [57, §2.7].)

Proof. (I) We first need to show that ψ defined in this way is a homomorphism of left A–modules, i.e. that $a\psi(m) = \psi(am)$. Putting aside the way A acts on it for a moment we have $N^o = N^* = \mathrm{Hom}_R(N, R)$, so $\psi(m) \in \mathrm{Hom}_R(N, R)$. By construction we have that $\psi(am) \in \mathrm{Hom}_R(N, R)$ is given by:

$$\psi(am)(n) \; = < am, n > = < m, a^t n > = \; \psi(m)(a^t n)$$

Meanwhile for $a\psi(m)(n)$ the action of a on the left is achieved by the action of a^t on the right of N^*, which we recall is given by $(\phi a^t)(n) = \phi(a^t n)$ for any $\phi \in N^*$. Thus $(a \circ \psi(m))(n)) = (\psi(m)a^t)(n) = \psi(m)(a^t n)$ as required.

(II) Note that for given $\psi \in \hom_A(M, N^o)$ we can define a form by $<m,n>_\psi \; = \; \psi(m)(n)$.

(III) Finally observe (cf. proposition 8.5.12, noting that the difference between N^o and N^* is not relevant, since the algebra action will not be used) that the rank of the image under ψ is rank $<,>$. □

10.1.7 Examples: contravariant forms on S_n modules

ss:S_n form

See also §11.4 for generalisations of the following example.

ex:S_3 form (**10.1.35**) Example. The symmetric group S_3 acts on the set of sequences $T_{2,1} = \{211, 121, 112\}$ by place permutation:

$$g_1 211 = 121, \qquad\qquad g_1 112 = 112$$

and so on (g_i denotes elementary transposition $(i\; i{+}1) \in S_n$). This is a left-action:

$$g_2(g_1 211) = g_2 121 = 112 = (g_2 g_1)211$$

These sequences thus form a basis for a left $\mathbb{Z}S_3$-module, $M = \mathbb{Z}T_{2,1}$. (Or similarly over any ground ring.)

The bilinear form on M given on $T_{2,1}$ by

$$< t, t' > \; = \; \delta_{t,t'}$$

is contravariant. We can see this by (10.1.30), or as follows. If $gt = t'$ for some $g \in S_3$, then $g^{-1}t' = t$ so $< gt, t' > = < t, g^{-1}t' >$.

Note that the rank of a form depends, in general, on the ground field. However in our case there is clearly no such dependence. Since this form is of full rank it defines an isomorphism between M and M^o. (Indeed since the intertwiner, as in (10.1.27), is the identity matrix, it is a rather 'uninteresting looking' isomorphism. Of course M does not satisfy the conditions of proposition 10.1.20. Next we construct a submodule which does.)

(**10.1.36**) We can restrict our form above to a form on a submodule S. For example, consider the element of $\mathbb{Z}S_3$ given by $V_{13} = 1 - (13)$ and the submodule of M generated by

$$e_{112} \; := \; V_{13}112 \; = 112 - 211$$

This is spanned by e_{112} and $e_{121} = 121 - 211$. (Remark: This submodule will turn out to satisfy the conditions of proposition 10.1.20.)

Just for completeness we note that the antirepresentation corresponding to this basis is given as follows:

$$g_1 \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix} = g_1 \begin{pmatrix} 112-211 \\ 121-211 \end{pmatrix} = \begin{pmatrix} 112-121 \\ 211-121 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}}_{\rho'(g_1)} \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix}$$

$$g_2 \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix} = g_2 \begin{pmatrix} 112-211 \\ 121-211 \end{pmatrix} = \begin{pmatrix} 121-211 \\ 112-211 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\rho'(g_2)} \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix}$$

A couple more identities:

$$g_1(e_{112} + e_{121}) = e_{112} - 2e_{121}, \qquad\qquad g_2(e_{112} + e_{121}) = e_{112} + e_{121} \tag{10.7}$$ linalgxxx

(**10.1.37**) Writing $E = \{e_{112}, e_{121}\}$ for the ordered basis of S here, the restricted form has Gram matrix

$$B(E,E) \; = \; \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \left(\; |e_{112}\rangle, \; |e_{121}\rangle \; \right) \; = \; \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(in the obvious notation).

Via *different* changes to the left and right basis, realised here by unimodular transformations, we get that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{B(E,E)} \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right) = \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{B(E',E'')} \tag{10.8}$$ eq:unimexa

is an equivalent (and in Smith form). This clearly gives Gram determinant 3 ('up to units').

We see from this that the Gram matrix has full rank over $\mathbb{Q}$, but not so over certain other fields. Specifically over a field of char.3 then, the rank is not full.

(**10.1.38**) What can we deduce from this? The form tells us that there is a module map out of S with a kernel $K \subset S$ of dimension equal to the corank. The kernel is a submodule. Thus S is not simple in the char.3 case.

The dimension of the quotient module S/K is equal to the gram matrix rank. But we cannot yet deduce directly from this that the quotient by the kernel is simple (of course in char.3 here the rank is 1 and we are done, but this would not be the case in general). Indeed, if 3 is a unit in the field then the rank is full. But this is *not* enough, of itself, to deduce that S is then simple (as the case of module M itself illustrates). For this we need to investigate the specifics of S a little further.

As we shall see, over a field F this submodule S of M has a simple head (which may or may not be the whole thing, depending on F). Note that if it is not the whole thing then the submodule is not a direct summand of the original module M (since this is contravariant self-dual).

We note that the unimodular transformations applied in (10.8) give

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |e_{112}\rangle, & |e_{121}\rangle \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_{112}+e_{121}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |2e_{112}-e_{121}\rangle, & |-e_{112}+e_{121}\rangle \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

From (10.7), or directly, we see that the first new basis vector on the left

$$a_1 \;:=\; e_{112}+e_{121}$$

(and separately $2e_{112}-e_{121}$ on the right) spans a submodule in char.3. Indeed this is a copy of the trivial module (from which we may deduce that the quotient is a copy of the alternating module).

Exercise: check that this vector lies in the kernel of the induced module map in char.3.

(**10.1.39**) Examples of this specific type — Specht modules S in permutation modules M — are very important in S_n representation theory. We finish here with some manipulations specific to this setting (see §11.4 for more details). We investigate S by returning to look at other submodules of M .

Fixing a field F, let S' be the set of vectors in M orthogonal to the above submodule S with respect to the original form ($112+121+211 \in S'$ for example, over any field). Note that S' is another submodule of M. Note that, depending on F, this new submodule is either non-intersecting of S, in which case the restricted form is of full rank, or intersects S in a submodule of dimension given by the discrepancy between the full rank and the actual rank of the form on S. (In our case $S \ni 2e_{112}-e_{121} = 2.112-121-211 \equiv -(112+121+211)$ over a field of characteristic 3, so the new module, over such a field, intersects S.) We will see that the quotient of S by $S \cap S'$ is simple (indeed absolutely irreducible). This is because of the following key result.
Evidently $V_{13}121 = 0$, so for any $m \in M$ we have $V_{13}m \in Fe_{112}$.
Suppose m is in some submodule T of M, so either $V_{13}m = 0$ or $V_{13}m \neq 0$. In the latter case $e_{112} \in T$ so $S \hookrightarrow T$. In the former case

$$0 = < V_{13}m, 112 > = < m, V_{13}^t 112 > = < m, V_{13}112 > = < m, e_{112} >$$

so $T \hookrightarrow S'$. Now suppose in particular that T is a submodule of S in M. Then either it is the whole of S, or it is also in S', and hence in $S \cap S'$. This shows that $S/S \cap S'$ is irreducible.
This is the same as to say that its Gram matrix is non-singular over F.

(**10.1.40**) There are several further related exampled later. See (11.4.5), (**??**) ...

10.1.8 Examples: T_n modules (Temperley–Lieb)

ss:Tncov01

(**10.1.41**) Recall the T_n-modules $D_n^{\text{TL}}(l)$ constructed in §2.2, and the c-v form from (2.3.14). In particular

$$\mathrm{Gram}_n(n-2) \;=\; \begin{pmatrix} [2] & 1 & 0 & & \\ 1 & [2] & 1 & 0 & \\ 0 & 1 & [2] & 1 & \\ & & & \ddots & \\ 0 & \dots & 0 & 1 & [2] \end{pmatrix} \tag{10.9}$$

eq:TLgram0002

$$|\mathrm{Gram}_n(n-2)| = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{10.10}$$

xTLgramx

(**10.1.42**) Remark. We can view this gram matrix as an intertwiner for the map $\psi : D_n^{\text{TL}}(n-2) \to D_n^{\text{TL}}(n-2)^o$ corresponding to the c-v form. To see this let us first construct the representations $\rho_{n,n-2}$ and $\rho^o_{n,n-2}$ from the diagram bases. We have for example

$$\rho_{5,3}(U_1) = \begin{pmatrix} [2] & 1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad \rho^o_{5,3}(U_1) = \begin{pmatrix} [2] & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

and so on, whereupon we check

$$\begin{pmatrix} [2] & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} [2] & 1 & 0 & \\ 1 & [2] & 1 & 0 \\ 0 & 1 & [2] & 1 \\ 0 & & 1 & [2] \end{pmatrix} = \begin{pmatrix} [2] & 1 & 0 & \\ 1 & [2] & 1 & 0 \\ 0 & 1 & [2] & 1 \\ 0 & & 1 & [2] \end{pmatrix} \begin{pmatrix} [2] & 1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

...

Observe that this makes sense over the 'integral' ground ring, where the gram matrix is not invertible; and over any specialisation, including cases where the gram matrix has maximal rank or various different ranks.

(**10.1.43**) One easily checks that there is a Smith normal form for $\mathrm{Gram}_n(n-2)$:

$$\mathfrak{SN}_n(n-2) \;=\; \begin{pmatrix} [n] & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

(Indeed this comes via unimodular matrices over $\mathbb{Z}[q, q^{-1}]$, even though this is not a PID.)

(**10.1.44**) For the algebras T_n one does not need to compute any more gram matrices to determine the structure of the algebra. However, as an exercise, we can consider $\mathrm{Gram}_n(n-4)$. We have for example

$$\mathrm{Gram}_n(n-4) \quad =$$

and in general

$$|\mathrm{Gram}_n(n-4)| \quad = \quad \frac{[n-1]}{[2]}[n-2]^{n-1}$$

...

(**10.1.45**) JOBS: FIX AND FINISH THE ABOVE!

10.1.9 Examples: To Do! Full transformation semigroup

...

Part III

Examples

Chapter 11

Basic representation theory of the symmetric group

ch:Sn

Recall that the symmetric group S_n is the group of permutations of a set of n objects. I.e. the group of self-isomorphisms and their compositions as functions. It is also a Coxeter group, as per §5.2 (with $M = M_{A_n}$).

Background references for this Chapter are discussed at the end. Particularly important contributions include Young [**?**], Schur [140] and James [74]. Books on the subject include Hamermesh [60], Boerner [12] and Robinson [36].

11.1 Introduction

11.1.1 Conventions

We take it here that the set permuted by S_n is $\underline{n} = \{1, 2, ..., n\}$ (as in §3.2 for example). Note that $\underline{n}$ is an ordered set, in the natural way. Let us further assume that n is single digit (or otherwise that each element of $\underline{n}$ somehow has a symbol that is connected). Then *'string notation'* for $w \in S_n$ is the string $w(1)w(2)...w(n)$, a string-perm of $12...n$. In particular $12...n$ is the notation for the identity element in S_n.

By convention then, for the composition $w' \circ w$, we have for example

$$213 \circ 132 = 231 \tag{11.1}$$ eq:213x

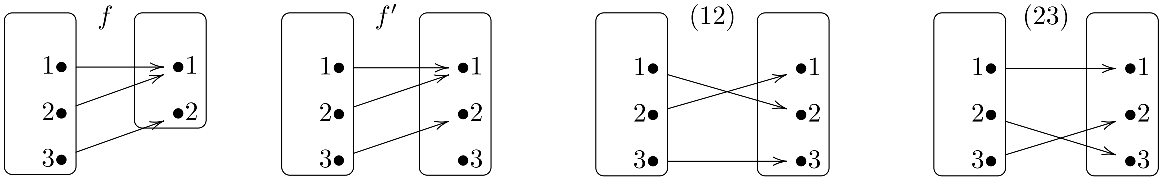
In particular here, $132(1) = 1$ because $w(1)w(2)w(3) = 132$ yields $w(1) = 1$; while $w'(1)w'(2)w'(3) = 213$ so $w'(1) = 2$, so $(w' \circ w)(1) = w'(w(1)) = 2$.

The group S_n is isomorphic to the group with the opposite multiplication, but it is not commutative, so the order in the convention matters 'internally'. Indeed

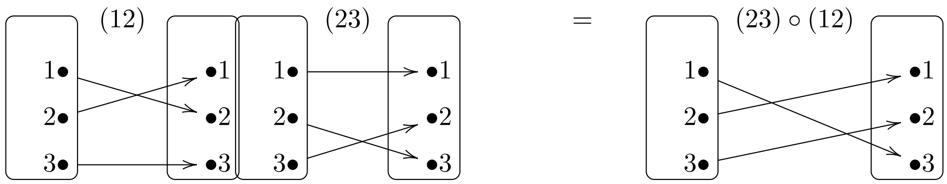
$$132 \circ 213 = 312 \tag{11.2}$$ eq:312

here. Note that the convention with composition given by $f * g = g \circ f$ is also in fairly common use. Indeed in cases where S_n arises as a subalgebra of another structure the latter composition may be natural.

Here are some function diagrams in the spirit of §3.2:



The function $f \in \underline{2}^{\underline{3}}$. Its string notation is $f = f(1)f(2)f(3) = 112$. Meanwhile $f', (12), (23) \in \underline{3}^{\underline{3}}$, with the last two invertible and hence in S_3 — see below for the general form of the (12) notation. For composition here we have a concatenation in a suitable order:



This verifies $(23) \circ (12) = 312$ as in (11.2).

(**11.1.1**) If n is fixed, we write (ij) for the elementary transposition in the symmetric group S_n. That is

$$(12) = 21345...n \in S_n$$

and so on. Note that:

Lemma. Group S_n is generated by $\{(i\; i{+}1) \mid i = 1, 2, ..., n-1\}$. □

(**11.1.2**) Theorem. *The symmetric group S_n is isomorphic to the group with presentation*

$$\langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \rangle / \sim$$

where the relations are $\sigma_i^2 = 1$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \neq 1$. The isomorphism is given by $\sigma_i \mapsto (i\; i{+}1)$.

(**11.1.3**) The *trivial representation* of S_n is given by $R_{(n)}((12)) = 1$; and the *alternating representation* by $R_{(1^n)}((12)) = -1$.

(**11.1.4**) We write $(i_1, i_2, ..., i_k) \in S_n$ for the cyclic permutation of the listed distinct elements of $\{1, 2, ..., n\}$. Note that $(i_1, i_2, ..., i_k) = (i_k, i_1, i_2, ..., i_{k-1})$. We declare a canonical expression for this element to be the one in which the lowest number appears first.

Any two such cyclic elements of S_n commute if they have no list element in common. Thus a set partition of $\{1, 2, ..., n\}$ and an ordering of the elements in each part defines an element w of S_n. Furthermore, any element $w \in S_n$ can be built this way.

The *cycle structure* of $w \in S_n$ is the integer partition associated to the set partition as above. For example the cycle structure of $(12)(34) \in S_7$ is $22111 = 2^2 1^3$.

Two elements of S_n are in the same class if and only if they have the same cycle structure. We have shown the following.

`th:classn` (**11.1.5**) Theorem. *The classes of S_n may be indexed by the integer partitions of n.*

11.1.2 Preamble

(**11.1.6**) The study of the representation theory of S_n is built in large part, of course, on general machinery for the representation theory of finite groups (character theory, Maschke's Theorem, Brauer–modular systems and so on). The case specific aspects, however, involve various historically distinct approaches.

Theorem 11.1.5 can be seen as giving rise to one of the main 'combinatorial' approaches to S_n representation theory (perhaps largely attributable to Young). We have *Young diagrams* as pictures of integer partitions, and *Young tableaux*, which are various combinatorial embellishments. These involve Young subgroups, the Robinson-Schensted correspondence, and various iterative (n-varying) approaches.

A different approach is through the 'dual' study of the 'combinatorial' representation theory of the general linear groups, including 'Schur–Weyl duality'. (Yet another is Springer theory, which uses geometrical properties of general linear groups.)

One also has an approach (specifying the general machinery of character theory) involving symmetric polynomials. And there is a collection of approaches triggered by the so-called Jucys–Murphy elements of the group algebra. There are approaches which could be characterised as 'probablistic' (via the natural connection between combinatorics and probability), and others perhaps 'asymptotic'.

From a contemporary perspective some of these approaches manifest commonalities between them beyond the fact that they all speak to the symmetric group.

There are also contemporary approaches which, while not producing new results, shed interesting new light on the subject and its remaining open problems. An example here is the study of **Set** as a symmetric group module.

11.1.3 Integer partitions, Young diagrams and the Young lattice

In light of Th.11.1.5, integer partitions and their Young diagrams (see e.g. Fig.11.1) play a useful role in S_n representation theory. See §5.7 for some basic properties. The details we shall need are given in §11.4.

As we shall see, the symmetric group algebras are split semisimple over the rationals. The irreducible representations of S_n over the rationals (and hence over the containing complex field) are indexed by the integer partitions of n.

(**11.1.7**) Note that there is an inclusion of S_{n-1} in S_n given in string notation by

$$w \mapsto wn$$

- where wn is understood as the concatenation of the singleton string n onto the end of string w.

(**11.1.8**) The corresponding restriction rule for irreducible representations over the splitting field is that the representation indexed by integer partition λ restricts to the direct sum of irreducible representations indexed by integer partitions of $n-1$ that are sub-partitions of λ (i.e. their Young diagrams are sub-diagrams of the diagram for λ).

(**11.1.9**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice (as in Figure 5.8 - reproduced here in Fig.11.1).

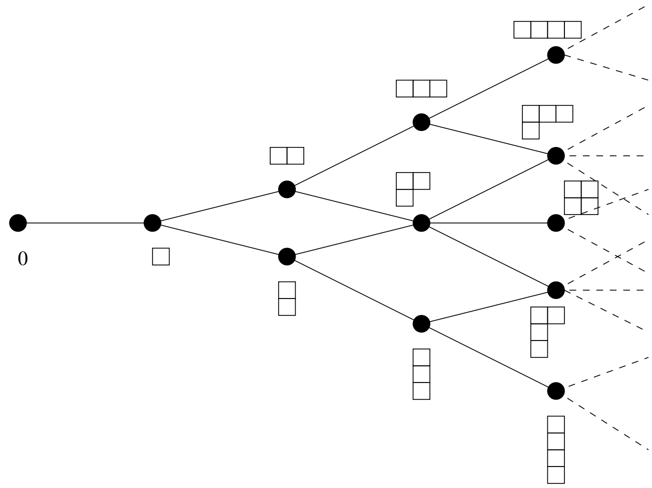


Figure 11.1: The start of the Young graph (covering DAG of the Young lattice, increasing from left to right).

Younggraph2

11.1.4 Realisation of S_n as a reflection group

ss:Snrefl

Consider S_n acting on the standard ordered basis of $\mathbb{R}^n$ by permutation: $we_i = e_{w(i)}$. This lifts to an action on $\mathbb{R}^n$. In particular, setting $\langle e_i, e_j \rangle = \delta_{i,j}$, then perm (ij) is a reflection fixing the hyperplane $H_{ij} = \{x = (x_1, x_2, ..., x_i, ..., x_j = x_i, ..., x_n) \mid x \in \mathbb{R}^n\}$. See also §5.2.

Note that reflection in a *different* hyperplane does not fix H_{ij} in general, but does take it to a hyperplane. In fact the set H of hyperplanes of this form closes under the group generated by the corresponding reflections. (Note that these are all the reflections in the group, but they are not all the involutions in general. We also have (12)(34) for example.)

A single chamber of $\mathbb{R}^n \setminus H$ has open facets touching only a subset of H. For example there are chambers touching only the hyperplanes $H_{i\ i+1}$ (the chamber of strictly ascending sequences and the chamber of strictly descending sequences).

The chamber of ascending sequences a_+ contains vector $v_+ := (1, 2, ..., n)$. If we act with (ij) this is taken to the chamber containing $(1, 2, ..., j, ..., i, ..., n)$, and so on. In this way we may place the chambers of $\mathbb{R}^n \setminus H$ in bijective correspondence with the elements of S_n. That is, writing v for the chamber containing v, $(1, 2, ..., n) \overset{\beta}{\rightsquigarrow} 12...n = 1 \in S_n$. Then writing the reflection action of (ij) on the left we have $(ij)(1, 2, ..., n) \overset{\beta}{\rightsquigarrow} (ij)12...n = (ij)1$, and so on (note that we have to be careful with notation here).

(**11.1.10**) Remark. Note that we have chosen a representative element of the chamber a_+. There is another action of S_n permuting the entries regarded as distinct symbols (instead of the positions of the entries).

(**11.1.11**) Note that the vectors $S_n v_+$ define a convex polytope P_n embedded in $\mathbb{R}^n$. Note that P_n is codimension-1, since the sum of entries in a vertex vector is fixed.

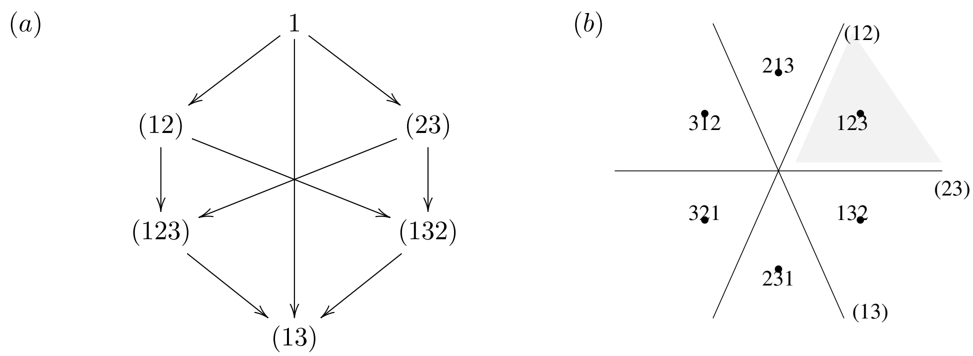


Figure 11.2: (a) Bruhat relation. (b) hyperplanes. `fig:bruhat`

The *permutahedron of order n* (see e.g. Ziegler [151]) is the polytope P_n.

The vertices of P_n (the faces of dimension zero) are precisely the set $S_n v_+$. The vertices and edges (faces of dimension one) of P_n define a graph (directed, by choice of a root vertex, such as v_+, away from which paths then flow, until converging on the image $(n, n-1, ..., 1)$ of the longest element of S_n), also denoted P_n. For example, P_3 is a hexagon; P_4 is a polyhedron with 4- and 6-sided faces.

(**11.1.12**) Let G be a group and $S \subset G$ a set of generators. The *Cayley graph* $\Gamma(G, S)$ of (G, S) is the digraph with vertex set G and an edge (a, b) whenever $b = as$ for some $s \in S$.

The *colour Cayley graph* is the Cayley graph together with edge labels from S — if $b = as$ then the edge label on (a, b) is $s = a^{-1}b$. This partitions the set of edges into parts labelled by S.

LEMMA. Let $S = \{(i\ i{+}1) \mid i = 1, 2, ..., n-1\}$ be the set of Coxeter generators of S_n. Then

$$\Gamma(S_n, S) \cong P_n.$$

Proof. One needs to show that the faces of dimension one in P_n join vertices related by adjacent transpositions. See Ziegler [151]. $\square$

Note that the edge label s on (a, as) is not in general the same as the elementary transposition w that takes a to $wa = as$. Note however that $w = asa^{-1}$ so that any w, s in this 'left-versus-right-action' relationship to each other are in the same class, and in particular here are both reflections. Although w is not generally a simple reflection. See (5.3.15).

(**11.1.13**) Considering $G(S_n, a_+ = (1, 2, ..., n))$ as defined in §5.2 we have $G(S_n, a_+ = (1, 2, ..., n)) \cong \Gamma(S_n, S)$.

The undirected version $G(S_n)$ of $G(S_n, a_+ = (1, 2, 3))$ as defined in §5.2 is the geometric dual graph to the chamber geometry in Fig.11.2 (b) seen as a complex (one considers the alcoves and their walls from the complex — these become the vertices and edges in the dual).

Note that the automorphism group of graph $G(S_n)$ is not S_n, but none-the-less acts transitively (i.e. fixes no proper subset of vertices), so that each single vertex is of equal 'standing'. (This corresponds to the freedom to choose a fundamental chamber.)

de:bruhat1 (**11.1.14**) The *Bruhat order* $(S_n, >)$ is a partial order defined as follows. We first define a relation $(S_n, \rightarrow)$ by $w \rightarrow w'$ if the chambers of w, w' are images either side of a hyperplane (not necessarily 'adjacent'), and w is on the same side of the hyperplane as $(1, 2, ..., n)$. The order $(S_n, >)$ is the transitive closure of the $\rightarrow$ relation (i.e. $w < w'$ if $w \rightarrow w'$ and so on).

Example: Fig.11.2 (a) digraph of the $\rightarrow$ relation for S_3; (b) hyperplanes and chambers in $\mathbb{R}^3$ viewed down the $(1, 1, 1)$ line.

(**11.1.15**) NB graph (a) in Fig.11.2 is not the Hasse graph of the Bruhat order — for this we should omit the long vertical line, even though it is direct in the $\rightarrow$ relation.

Note: The digraph $G(S_3, a_+ = (1, 2, 3))$ from §5.2 is a directed hexagon. Its transitive closure, the *cone order*, is not equal to the Bruhat order since the diagonal lines in Fig.11.2 (a) are not present in the cone order.

NB the Bruhat order is not the same as the *weak order*, the order generated by Coxeter reflections (reflections in walls 'touching' the fundamental chamber), since that order contains none of the vertical lines in (a) (the long one is not needed for Bruhat, but the shorter ones are).

11.2 Representations of S_n from the category Set

Recall the category **Set** from §6.1. For convenience define $\mathbf{Set}(m, n) = \mathbf{Set}(\underline{m}, \underline{n})$. Recall that the category composition equips $\mathbf{Set}(m, m)$ with the property of monoid; and $\mathbf{Set}(m, n)$ with the property of left $\mathbf{Set}(m, m)$-set; and right $\mathbf{Set}(n, n)$-set. For any commutative ring R these sets extend R-linearly to modules. Since $S_m \subset \mathbf{Set}(m, m)$ we can also build a left S_m-module by restriction (respectively a right S_n-module).

What does $\mathbf{Set}(3, 2)$ look like as a left $\mathbf{Set}(3, 3)$- or left S_3-module? In our notation the (ordered) basis is $\{111, 112, 121, 122, 211, 212, 221, 222\}$ and we have

$$R_{3,2}((12)) = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 0 && 1 &&& \\ &&& 0 && 1 && \\ && 1 && 0 &&& \\ &&& 1 && 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$$

We return to consider the general case in §11.2.3.

11.2.1 Connection with Schur's work and Schur functors

In this subsection we follow Green[57, §3,§6] closely. (Note that §6.2 of Green is also focal to §13.4.)

(**11.2.1**) Note that Green uses $I(n, r)$ for $\mathbf{Set}(r, n)$, and writes $i \in I(n, r)$, that is $i : \underline{r} \rightarrow \underline{n}$, as a vector or multi-index: $i = (i_1, ..., i_r)$.

(**11.2.2**) Let $G = GL_n(K)$ be the group of invertible matrices over an infinite field K. We regard $\mathbf{Set}(G, K)$ as a commutative K-algebra by $ab(g) = a(g)b(g)$; $(a + b)(g) = a(g) + b(g)$.

The identity element is given by $1(g) = 1_K$ for all g.

For each $s \in G$ we define $L_s \in \mathrm{End}_K(\mathbf{Set}(G,K))$ by $L_s f(g) = f(sg)$ (and define $R_s f(g) = f(gs)$ similarly).

PROPOSITION. The map $R(s) = R_s$ is a representation of G (and L gives an antirepresentation). Thus $\mathbf{Set}(G,K)$ is a left KG-module:

$$sf = R_s f \qquad\qquad \text{(similarly } fs = L_s f\text{)}$$

The two actions commute.

(**11.2.3**) Let $c_{ij} : G \to K$ be given by $c_{ij}(g) = g_{ij}$. Write A for the K-subalgebra of $\mathbf{Set}(G,K)$ generated by all the c_{ij}s.

For each r we write $A(n,r)$ for the subspace of A of polynomials homogeneous of degree r in the c_{ij}s. For example $A(n,1) = K\{c_{11}, c_{12}, ..., c_{1n}, c_{21}, c_{22}, ..., c_{2n}, ..., c_{nn}\} = K\{c_{ij}\}_{ij}$ (a K-space of dimension n^2). Thus A has grading

$$A = \sum_r A(n,r)$$

as a K-algebra.

For $i, j \in \mathbf{Set}(r,n)$

$$c_{ij} \;:=\; c_{i_1 j_1} ... c_{i_r j_r} \tag{11.3}$$ eq:codet1

Then $A(n,r)$ is spanned by these monomials. Note that the RHS of (11.3) does not determine i, j.

(**11.2.4**) Define

$$S_K(n,r) = \hom_K(A(n,r), K)$$

This has basis $\{\xi_{ij} \mid i,j \in \mathbf{Set}(r,n)\}$ with ξ_{ij} given by

$$\xi_{ij}(c_{kl}) = \begin{cases} 1 & if \quad (i,j) \sim (k,l) \\ 0 & o/w \end{cases}$$

where $\sim$ means the orbit of the diagonal action of S_r.

Algebra $A(n,r)$ is a coalgebra, so the dual $S_K(n,r)$ is an associative algebra. We have (see Schur [140], Green [57, p.21], or Martin–Woodcock [120])

$$\xi_{ij}\xi_{kl} \;=\; \sum_{p,q} Z(i,j,k,l,p,q).1_K \; \xi_{pq} \tag{11.4}$$ eq:Zorder2

where the sum is over a transversal of $\sim$; and

$$Z(i,j,k,l,p,q) = |\{s \in \mathbf{Set}(r,n) \mid (i,j) \sim (p,s),\; (k,l) \sim (s,q)\}|$$

(**11.2.5**) For example $\xi_{ii}^2 = \xi_{ii}$ and $\xi_{ii}\xi_{jj} = 0$ if i,j not in the same orbit of the S_r-action (i.e. if $\xi_{ii} \neq \xi_{jj}$). Indeed

$$1_{S_K(n,r)} = \sum_i \xi_{ii}$$

where the sum is over the distinct elements.

(**11.2.6**) Note from (11.4) that the $\mathbb{Z}$-submodule $S_{\mathbb{Z}}(n,r)$ of $S_{\mathbb{Q}}(n,r)$ generated by the ξ_{ij} is multiplicatively closed. That is, it is a $\mathbb{Z}$-order in $S_{\mathbb{Q}}(n,r)$. For any field K there is an isomorphism of K-algebras $S_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} K \cong S_K(n,r)$. In this sense, for fixed n,r, the 'scheme' or family of algebras $S_K(n,r)$ is 'defined over $\mathbb{Z}$'.

(**11.2.7**) Let $\Lambda(n,r)$ be the set of S_r-orbits in $\mathbf{Set}(r,n)$ (the set of 'weights'). For example, for $r \leq n$ there exist functions $i \in \mathbf{Set}(r,n)$ of form $i = (s(1), s(2), ..., s(r))$, where $s \in S_r$. The weight w of any such i is

$$w = (1,1,...,1,0,0,...,0) \in \mathbb{Z}^n$$

(r nonzero entries).

Consider the commuting S_n action on $\mathbf{Set}(r,n)$. Each S_n-orbit contains one dominant weight. Write $\Lambda^+(n,r)$ for the set of dominant weights.

(**11.2.8**) If $i \in \mathbf{Set}(r,n)$ belongs to $a \in \Lambda(n,r)$ we may write ξ_a for ξ_{ii}.

(**11.2.9**) Proposition. There is an isomorphism of K-algebras

$$\xi_w S_K(n,r) \xi_w \;\cong\; KS_r$$

which takes $\xi_{us,u}$ to s for all $s \in S_r$.

(**11.2.10**) This allows us to construct a 'Schur' functor relating the representation theory of the Schur algebra $S_K(n,r)$ (and hence part of the representation theory of the general linear group) to the symmetric group S_r:

$$F : S_K(n,r) - \text{mod} \;\to\; KS_r - \text{mod} \tag{11.5}$$ eq:schur functor

where $FM = \xi_w M$.

(**11.2.11**) (TO CONTINUE we should summarize [57, §3.2].)

(**11.2.12**) A closely related idea is that, CLAIM:

$$S_K(n,r) \cong \text{End}_{KS_r}((K^n)^{\otimes r})$$

(**11.2.13**) Schur–Weyl duality. Let $V = k\{e_1, e_2, ..., e_N\}$ for some field k. For physics $k = \mathbb{C}$. Now see Fig.11.3.

11.2.2 Idempotents and other elements in $\mathbb{Z}S_n$

(**11.2.14**) For $x \subset \underline{n}$ we write S_x for the subgroup of S_n in which only the elements in x may be permuted nontrivially. If $p = \{p_1, p_2, ...\}$ is a set partition of $\underline{n}$ then each S_{p_i} is a subgroup; these subgroups commute pairwise; and we write S_p for the Young subgroup

$$S_p = S_{p_1} S_{p_2} ...$$

(Note that the given part names in p suggest an ordering of parts, but this is not intrinsic to a set partition in general. In our case it is easy to give an ordering rule - for example 'children first' - but note that in any case S_p will not depend on this order.)

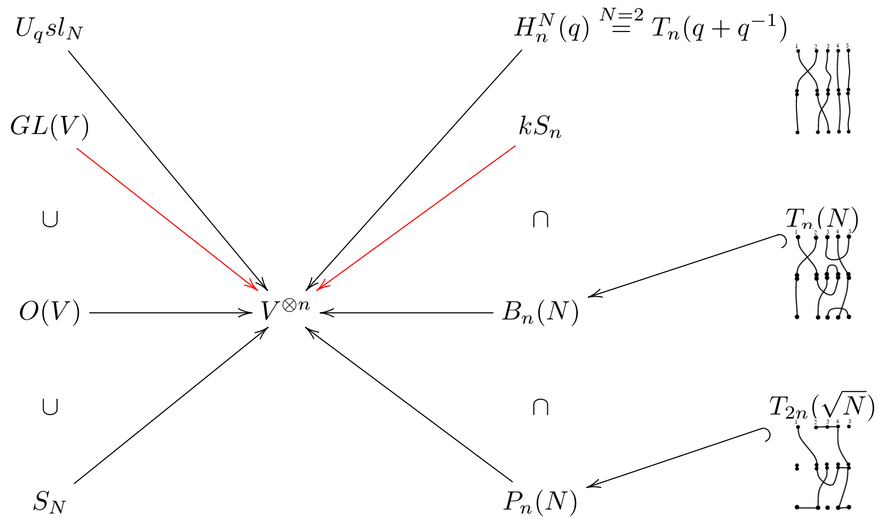


Figure 11.3: Schur–Weyl duality schematic. For N-state Potts on n-site wide lattice. fig:SW01

(**11.2.15**) A *composition* of n (into m parts) is a sequence of elements of $\mathbb{N}_0$ (of length m) that sums to n. We may associate a composition to each ordered set partition of $\underline{n}$ (or indeed of any finite set), $p = \{p_1, p_2, ...\}$, by

$$\lambda_i = |p_i|$$

It will be evident that

$$S_p \cong \times_i S_{\lambda_i}$$

For each integer partition $\lambda \vdash n$ we associate a set partition by

$$p(\lambda) = \{\{1, 2, .., \lambda_1\}, \{\lambda_1 + 1, .., \lambda_1 + \lambda_2\}, ...\}$$

We write $S_\lambda = S_{p(\lambda)}$. When the factor groups are simply arranged side-by-side in this way we write $\otimes_i w_i$ for the image of $(w_1, w_2, ...) \in \times_i S_{\lambda_i}$ in S_n.

(**11.2.16**) Define elements of $\mathbb{Z}S_n$ by

$$e'_{(n)} = \sum_{w \in S_n} R_{(n)}(w)\, w$$

$$e'_{(1^n)} = \sum_{w \in S_n} R_{(1^n)}(w)\, w$$

Note that

$$e'_{(n)} e'_{(1^m)} = 0 \qquad \text{for } m, n > 1 \tag{11.6}$$ eq:orth1

Using the side-by-side notation define

$$e'_\lambda = \otimes_i e'_{(\lambda_i)}$$

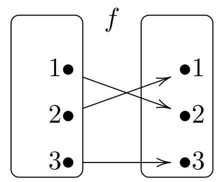


Figure 11.4: Mapping diagram of $(12) \in S_3$. fig:map diag

$$f'_\lambda = \otimes_i e'_{(1^{\lambda'_i})}$$

where λ' is the transpose partition to λ.

(**11.2.17**) Define a map C_L from S_n to set partitions of $\underline{n}$ as follows. Draw the mapping diagram of $w \in S_n$ (as illustrated in Figure 11.4), and put i, j $(i < j)$ in the same part of $C_L(w)$ if the lines from i, j cross, i.e. if $i < j$ and $w(i) > w(j)$. Define $C_R(w) = C_L(w^{-1})$.

Example: $C_L((12)) = \{\{1,2\},\{3\},...\}$.

(**11.2.18**) For any set partition $p \in P_{\underline{n}}$ we call a $w \in S_n$ left p-noncrossing if $i \sim^p j$ implies $i \not\sim^{C_L(w)} j$.
The idea here is that if we look at any of the parts of p regarded as a subset of $\underline{n}$, we will find that the corresponding strings (as labelled on the left) do not cross each other in w.
For example, if p is the partition into singletons then every $w \in S_n$ is p-noncrossing. If $p = \{\underline{n}\}$ then only $1 \in S_n$ is p-noncrossing.

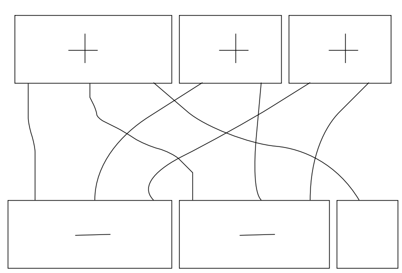
Define right p-noncrossing similarly. Write $S_n(p,q)$ for the set of left p-noncrossing right q-noncrossing elements of S_n. Write $S'_n(p,q)$ for the subset of elements such that no two strings in the same part at p are in the same part at q.

pr:pig1 (**11.2.19**) PROPOSITION. *There is a $w_\lambda \in S_n$ such that $e'_\lambda w_\lambda f'_\lambda \neq 0$ and*

$$e'_\lambda w f'_\lambda \;=\; \epsilon_\lambda(w)\, e'_\lambda w_\lambda f'_\lambda$$

with $\epsilon_\lambda(w) \in \{0, \pm 1\}$, for all $w \in S_n$.

Proof. A conceptual/diagramatic proof is effective here. Draw a box on the left for each factor in e'_λ and one on the right for each factor in f'_λ. For example (rotating so that boxes 'on the left' are on the top, just to save space) the case $\lambda = (3,2,2)$ is:



By (11.6) if w connects any box on the left to any box on the right by more than one line, then $e'_\lambda w f'_\lambda = 0$. Also, if two lines coming out of any box cross this crossing can be removed (at cost a factor -1 if it is a box on the right). Since λ'_i is the number of boxes on the left with at least i lines, one sees that there is precisely one w producing a non-vanishing product and with no removable crossings. This is w_λ. One readily confirms that $e'_\lambda w_\lambda f'_\lambda = w_\lambda + ...$ is non-vanishing. □

(**11.2.20**) Note from the proof that there is more than one choice for w_λ above in general, but there is a unique choice of least length.

(**11.2.21**) Set $E_\lambda = e'_\lambda w_\lambda f'_\lambda$. Evidently $E_\lambda x E_\lambda = m_x E_\lambda$ for all $x \in kS_n$ for some scalar m_x. Thus the left ideal $I = kS_n e'_\lambda w_\lambda f'_\lambda$ obeys $E_\lambda I \subseteq kE_\lambda$.

(**11.2.22**) Suppose we work over a field k of char.0. Then $I^2 \neq 0$, so $E_\lambda g E_\lambda \neq 0$ for some $g \in S_n$. Thus there exists a g for which $E_\lambda g E_\lambda g = m_g E_\lambda g \neq 0$, so that $E_\lambda g$ is an unnormalised idempotent.

(**11.2.23**) Suppose J is a subideal of I. Then either $E_\lambda J = kE_\lambda$ or $E_\lambda J = 0$. In the first case $J = kS_n E_\lambda J = I$; in the second $J^2 = \{0\}$ so if $k \supseteq \mathbb{Q}$ then kS_n is semisimple and $J = 0$. Thus:

(**11.2.24**) Lemma. *If $k \supseteq \mathbb{Q}$ then left ideal $I = kS_n E_\lambda$ is minimal.* □

(**11.2.25**) Set $s_\lambda = e'_\lambda(w_\lambda f'_\lambda w_\lambda^{-1})$. We claim this is the unnormalised idempotent in the char.0 case.

In this case $s^2 = m_w s \neq 0$. Write simply $s = ef$ for a moment. Then $s^2 = efef \neq 0$, so $efe \neq 0$. Note that efe is fixed under the standard involutive antiautomorphism map $g \mapsto g^{-1}$. We claim this efe is then a fixed (unnormalised) primitive idempotent conjugate to s.

11.2.3 Young modules

ss:youngm1

Recall from §6.2 and 6.3 that for k a ring, $k\mathbf{Set}^f$ is the k-linear category over category $\mathbf{Set}^f$, and $kC_\mathbb{N}$ is a corresponding skeleton. We may write $\hom(m,n)$ for $\mathbf{Set}(m,n)$, a basis for $k\mathbf{Set}^f(m,n)$.

As noted in §6.3, S_n is the group of isomorphisms in $\hom(n,n)$, so $k\hom(n,m)$ is a left kS_n right kS_m bimodule by restriction. Recall that the basis $\hom(n,m)$ of functions $f : \underline{n} \to \underline{m}$ may be written as words $f(1)f(2)...f(n)$ in $\underline{m}$. Then the action of S_n is to permute the n entries in this list; while the action of S_m is to permute the set of symbols. For example

$$(23)1244 = 1424$$

$$1244(23) = 1344$$

(**11.2.26**) We may associate a composition λ of n to each function f in $\hom(n,m)$ by

$$\lambda_i = \#\{j \mid f(j) = i\}$$

For example $\lambda(12444) = (1,1,0,3)$.
(Note that the notation $12444 \in \hom(n,m)$ tells us that $n = 5$, but only that $m > 3$. In the above we have assumed $12444 \in \hom(5,4)$. If we regard $12444 \in \hom(5,6)$ then we have $\lambda(12444) = (1,1,0,3,0,0)$.)

For $\lambda \in \mathbb{N}_0^m$, we write $\hom(n,\lambda)$ for the subset of $\hom(n,m)$ of functions of fixed λ. For example

$$\hom(4,(3,1)) = \{1112, 1121, 1211, 2111\}$$

(For the moment we leave this simply as an abuse of notation, as far as the categorical context is concerned.)

Note that the left action of S_n on $\hom(n,m)$ fixes this composition of n, so we may decompose $k\hom(n,m)$ as a direct sum of left modules indexed by compositions:

$$k\hom(n,m) \cong \bigoplus_{\lambda \in \mathbb{N}_0^m} k\hom(n,\lambda)$$

Example:

$$k\hom(3,2) \cong k\{111\} \oplus k\{112,121,211\} \oplus k\{122,212,221\} \oplus k\{222\}$$

Any two such left submodules are isomorphic if their compositions are related by a reordering of the terms in the sequence λ (since the invertible right action by S_m achieves all such reorderings). We will take integer partitions to be representative elements of the orbits of compositions under this action.

(**11.2.27**) We now consider the integral, or ring independent, decomposition of the regular left $\hom(n,n)$-module $k\hom(n,n)$. We are interested in this as a left kS_n-module by restriction.

We have seen that formally ignoring the difference in codomain gives an inclusion of sets $\hom(n,m) \hookrightarrow \hom(n,m+1)$. This gives an example of an injection $k\hom(n,m) \hookrightarrow k\hom(n,m+1)$, and indeed this injection is split. The sections contain sums of the $k\hom(n,\lambda)$'s with precicely $m+1$ parts.

It follows that we can find all the simple left modules for kS_n by looking in the $k\hom(n,\lambda)$'s with $\lambda \vdash n$.
(From now on we shall mean $\lambda \vdash n$ by λ, unless otherwise stated.)
(Indeed the claim follows directly on noting that $k\hom(n,(1^n))$ is isomorphic to kS_n as a left module. But we shall make use of the others too.)

(**11.2.28**) In particular if k is a field, all simple left kS_n-modules will appear as composition factors for this collection of modules — $\{k\hom(n,\lambda) \mid \lambda \vdash n\}$ — which we shall call the *Young modules* of kS_n. However it will be evident that these modules are not themselves simple in general. For example

$$e'_{(n)} k\hom(n,\lambda) \neq \{0\}$$

for any λ, so $e'_{(n)} k\hom(n,\lambda)$ is a proper submodule of $k\hom(n,\lambda)$ for any $\lambda \neq (n)$.

On the other hand

$$e'_{(1^n)} k\hom(n,\lambda) = \{0\} \qquad \text{for any } \lambda \neq (1^n).$$

This gives us a clue as to how to extract useful submodules from the Young modules more systematically.

pr:yyoung1 (**11.2.29**) Proposition. *(I) Let $\lambda \vdash n$. The left kS_n module $kS_n e'_\lambda$ is k-free with basis the elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor $e'_{(\lambda_i)}$ in e'_λ. (II) There is a bijection between this basis and* $\hom(n,\lambda)$ *obtained by modifying w to an element of* $\hom(n,\lambda'_1)$ *by making each of these subsets of λ_i noncrossing lines meet at a point i on the target side. (III) This bijection extends k-linearly to an isomorphism of left modules:*

$$kS_n e'_\lambda \cong k\hom(n,\lambda)$$

Proof. (I) A spanning set for the LHS is elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor in e'_λ. One can check that we'_λ contains the group element w with coefficient 1, and no other such group element, so the set is k-free and a basis. (II,III) The map described is evidently a set bijection and hence an isomorphism of free k-modules, but it also commutes with the S_n action (indeed the computation of this action is essentially the same computation on each side). □

11.2.4 Specht modules

ss:specht1

See for examples James' Lecture Notes [74]. Here we give a quick summary; with more details in the next section.

specht is ideal

(**11.2.30**) Comparing Prop. (11.2.29) with Prop. (11.2.19) it follows that $f'_\lambda k\,\hom(n,\lambda)$ is a rank-1 k-module.

QUESTION/CAVEAT: How do we know that the non-vanishing claim in Prop. (11.2.19) does not fail in finite characteristic?

(**11.2.31**) The kS_n-modules of form

$$\mathcal{S}(\lambda) \;:=\; S_n f'_\lambda k\,\hom(n,\lambda) \;\;\cong S_n f'_\lambda k S_n e'_\lambda$$

are called *Specht* modules after [144].

These are free modules of finite rank over $\mathbb{Z}$, and hence the rank is not affected by base change to k. (For this reason the dependence of $\mathcal{S}(\lambda)$ on k is often left implicit. However some properties do depend on k.)

(**11.2.32**) Proposition. [James] For field k of characteristic $p \neq 2$ this $\mathcal{S}(\lambda)$ is an indecomposable kS_n submodule of $k\,\hom(n,\lambda)$ (cf. (9.1.4)).

However for $p = 2$ the Specht module with $\lambda = (5,1,1)$ is decomposable (and Murphy shows that there are infinitely many others such).

(**11.2.33**) An integer partition λ is *p-regular* if no part is repeated p or more times.

(**11.2.34**) Let k be a field of characteristic $p > 0$. James has shown that if λ is p-regular then $\mathcal{S}(\lambda)$ has simple head over k, and that

$$\{L^k(\lambda) = \text{head}\,{}^k\mathcal{S}(\lambda) \;\mid\; \lambda \;\; p\text{-regular}\}$$

is a complete set of kS_n-modules.

(**11.2.35**) Basis / restriction rules — see below.

The notation $\mathcal{S}(\lambda)^\perp$ for the next Theorem is explained in the following section (where the module $S^\lambda \cong \mathcal{S}(\lambda)$).

(**11.2.36**) Let $\triangleleft$ denote the dominance order. Note that dictionary order is a total order refining the dominance order.

For k be a field of char. p, let

$$D^p_n = [\mathcal{S}(\lambda) : L^k(\mu)]_{\lambda,\mu}$$

be the S_n Specht decomposition matrix over k; where for the rows p-regular partitions and then other partitions (and for the columns, p-regular partitions) are written out in the dictionary order.

(**11.2.37**) THEOREM. *[James78 12.2] Fix k a field of char. p, and any n. Then* $L^k(\lambda) = \mathcal{S}(\lambda)/(\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp)$ *if λ is p-regular; and* $[\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp \ : L^k(\mu)] \neq 0$ *implies* $\mu \rhd \lambda$ *for all λ; and* $[\mathcal{S}(\lambda) \ : L^k(\mu)] \neq 0$ *implies* $\mu \rhd \lambda$ *for λ non-p-regular.*

That is, D_n^p is lower unitriangular.

pa:specht filt (**11.2.38**) In e.g. Hemmer [**?**] (see also Green [57, §6.3]) it is asserted that the Schur functor from (11.5) takes Weyl and coWeyl modules to Specht and dual Specht modules respectively; and takes projective modules to Young permutation modules.

The Schur functor is exact, and one then has (again from Hemmer) that *Young permutation modules have Specht and dual-Specht filtrations.* In particular the regular module has such filtrations. Distinct such filtrations do not necessarily have the same multiplicities, unless the characteristic is $p > 3$.

Note that filtration of the regular module does not necessarily imply that an indecomposable projective module has a filtration, since a Specht module may not be indecomposable. But this is only an issue for $p = 2$.

11.3 Characteristic p, Nakayama and the James abacus

See for example James–Kerber [78].

(**11.3.1**) A *rim-hook* of a Young diagram d is a skew-subdiagram of d whose dual graph is a chain. Define *the rim* of d as the maximal rim-hook.
Note that a rim-hook is not necessarily a hook

For $p \in \mathbb{N}$, a *p-hook* of diagram d is a rim-hook of length p. We define a partial order on Young diagrams by $d' \overset{p}{<} d$ if $d' \subset d$ and the skew is a p-hook. A *p-core* of d is a minimal element in any chain containing d.

FACT: All p-cores of d coincide. (This follows from (11.3.5) below.)

th:Nakayama (**11.3.2**) THEOREM. *[Nakayama conjecture] Let k be a field of prime characteristic p. Then Specht module $\mathcal{S}(\lambda)$ lies in the same block as $\mathcal{S}(\lambda')$ iff their diagrams have the same p-core.*

Proof. See e.g. James–Kerber [78, p.245].

(**11.3.3**) REMARK. Note that we have not shown that the block of $\mathcal{S}(\lambda)$ is well-defined if $p = 2$.

(**11.3.4**) Abacus: see James–Kerber [78, p.77-78]. A q-abacus is an abacus with q (vertical) runners, with the upper frame fixed and the lower frame very far away. The abacus may be considered to be filled with equal sized beads, almost all of which are 'empty' beads. The bead positions are numbered from 0 in reading order (i.e. left to right then top to bottom). A *bead configuration* records the position of the non-empty beads.

To each bead configuration there corresponds a Young diagram as follows. Associate to each non-empty bead the number of empty beads encountered in reading to that point. This gives a non-decreasing sequence from $\mathbb{N}_0$. Thus writing the sequence in reverse order and ignoring any 0s we get a non-increasing sequence in $\mathbb{N}$, and hence a Young diagram.

A useful fact is the following.

de:pcores2 (**11.3.5**) CLAIM: Suppose that removing a rim q-hook from d gives d'. Then replacing any bead configuration for d with a bead configuration in which one bead has been moved one space up, gives a bead configuration for d'.

(**11.3.6**) Example. (Omitting the vertical runners etc for reasons of laziness)

$$\begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \end{pmatrix} \begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \end{pmatrix}$$

The first abacus gives 00011233334 and hence $43^4 21^2$. The second gives 00011112234 and hence $432^2 1^4$. The difference is a rim 5-hook:

$$\begin{pmatrix} x & x & x & x \\ x & x & x & \\ x & x & x' & \\ x & x & x' & \\ x & x' & x' & \\ x & x' & & \\ x & & & \\ x & & & \end{pmatrix}$$

11.4 James–Murphy theory

JamesMurphy

This section is a summary of standard symmetric group results, cast in a form following [78, Ch.7]. See also §10.1.7. The objective is to give a definite (if not canonical) construction for symmetric group Specht module Gram matrices.

We use various notations for S_n elements. If $f \in S_n$ then $(f(1), f(2), ..., f(n))$ is a permutation. On the other hand in *cycle* notation, if $S = \{i, j, ..., k\} \subseteq \underline{n}$ then by $(ij..k) \in S_n$ we mean $f(i) = j$,..., $f(k) = i$ and $f(l) = l$ for $l \notin S$. Thus

$$(13).(12) = (123)$$

is an example of group multiplication.
(Then again, the common *diagram algebra* notation effectively composes permutations backwards, i.e. as in the opposite group. This is not a major issue, since the groups are isomorphic.)

(**11.4.1**) A *tableau* is an ordering of the boxes in a Young diagram, usually given by writing the counting numbers in the boxes. A β-tableau is a tableau for diagram β. If $\beta \vdash n$ then $\pi \in S_n$ acts on tableau in the obvious way:

$$(12)\, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We have

$$(13)(12)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = (13)\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} = (123)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

so this is a 'left' action, as written.

(**11.4.2**) A β-tabloid is an equivalence class of tableaux under permutations within rows. (Abusing notation somewhat) One writes $\{t\}$ for the equivalence class of t. Thus for example

$$\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\} = \{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\}$$

Note that $\pi\{t\} = \{\pi t\}$ is well-defined.

There is a sequence notation for tabloids, in which one writes $s(t)_i = j$ if the number i appears in row j in a tableau $u \in \{t\}$. (Note that this does not depend on the choice of class representative u.) Thus for example

$$s(t) = 112$$

for the tabloid above.

(**11.4.3**) Let F be an arbitrary field. Define S_n module

$$M^\beta = F\{\{t\}\}_{t \in \beta\text{-tableau}}$$

Note that $M^\beta = FS_n\{t\}$ for any suitable t. Examples: $M^{(2,1)}$ has basis

$$\{\{\ytableaushort{12,3}\},\ \{\ytableaushort{13,2}\},\ \{\ytableaushort{23,1}\}\}$$

$M^{(1^3)}$ has basis

$$\left\{\{\ytableaushort{1,2,3}\},\ \{\ytableaushort{1,3,2}\},\ \{\ytableaushort{2,1,3}\},\ \{\ytableaushort{2,3,1}\},\ \{\ytableaushort{3,1,2}\},\ \{\ytableaushort{3,2,1}\}\right\}$$

Note that the different treatment of rows and colums is a source of non-canonicalness. This is unavoidable.

Let $V(t) \in \mathbb{Z}S_n$ be the unnormalised column antisymmetriser associated to tableau t. Examples:

$$V(\ytableaushort{125,43}) = (1-(14))(1-(23))$$

and

$$V(\ytableaushort{1,2,3}) = 1-(12)-(23)-(13)+(123)+(321)$$

Note that for all $\pi \in S_n$

$$\pi V(t) = V(\pi t)\pi.$$

Then

$$e_t \ := \ V(t)\{t\}$$

is a β-polytabloid in M^β.
Examples:

$$e_{\ytableaushort{12,3}} = (1-(13))\{\ytableaushort{12,3}\} = \{\ytableaushort{12,3}\} - \{\ytableaushort{23,1}\}$$

$$e_{\ytableaushort{13,2}} = (1-(12))\{\ytableaushort{13,2}\} = \{\ytableaushort{13,2}\} - \{\ytableaushort{23,1}\}$$

$$e_{\ytableaushort{23,1}} = (1-(12))\{\ytableaushort{23,1}\} = \{\ytableaushort{23,1}\} - \{\ytableaushort{13,2}\}$$

de:S_n specht (**11.4.4**) The Specht module S^β is the subspace of M^β spanned by β-polytabloids:

$$\pi e_t = \pi V(t)\{t\} = V(\pi t)\pi\{t\} = V(\pi t)\{\pi t\} = e_{\pi t}.$$

Cf. §11.2.4.

11.4.1 Gram matrices

de:S_n Gram (**11.4.5**) Define a bilinear form (as in §10.1.5) on the FS_N-module M^β by

$$\Phi(\{t\},\{t'\}) = \begin{cases} 1 & \{t\}=\{t'\} \\ 0 & o/w \end{cases}$$

This form is a contravariant form with respect to the $g \mapsto g^{-1}$ antiautomorphism (since $g\{t\} = \{t'\}$ iff $g^{-1}\{t'\} = \{t\}$ — see e.g. §10.1.7).

(**11.4.6**) Define $Gram(\beta)$ as the matrix with entries $\Phi(e_t, e_{t'})$, with t, t' varying over standard β-tabloids in the lexicographic total order.
Examples: $Gram((1^3)) = (6)$; $Gram((3)) = (1)$;

$$Gram((2,1)) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(**11.4.7**) Define $S^{\beta\perp}$ as the submodule of M^β spanned by elements orthogonal to β-polytabloids.
oonee **Theorem 11.1.** *Let F be any field. Suppose $A \hookrightarrow M^\beta$ is an inclusion of FS_n-modules. Then either $S^\beta \hookrightarrow A$ or $A \hookrightarrow S^{\beta\perp}$.*
Proof: Suppose $a \in A$, and t a β-tabloid. Then it is easy to see that $V(t)a \in Fe_t$. Thus if $V(t)a \neq 0$ then $e_t \in A$, so $S^\beta = FS_n e_t \hookrightarrow A$. If $V(t)a = 0$ then

$$0 = \Phi(V(t)a, \{t\}) = \Phi(a, V(t)\{t\}) = \Phi(a, e)$$

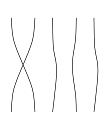
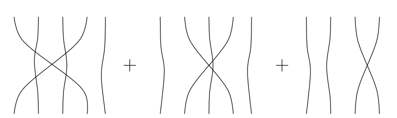
so $A \hookrightarrow S^{\beta\perp}$. □
Theorem 11.2.*Let F be any field. Then $S^\beta/S^\beta \cap S^{\beta\perp}$ is either an absolutely irreducible FS_n-module (i.e. remains irreducible under any field extension) or zero.*
Proof: If A is a submodule of S^β (and hence of M^β) then by Theorem 11.1 it is either S^β or else in $S^\beta \cap S^{\beta\perp}$. Thus $S^\beta/S^\beta \cap S^{\beta\perp}$ has no submodule, so it is irreducible. Absolute irreducibility follows from a consideration of the rank of the Gram matrix over the prime subfield. □

11.4.2 Murphy elements

For $(i,j) \in S_n$ the pair permutation define Murphy elements of S_n:

$$\mathcal{M}_m = \sum_{i=1}^{m-1} (i,m)$$

ss:Murphy1 Example: In $\mathbb{Z}S_5$ we have

$\mathcal{M}_2 =$  , $\mathcal{M}_4 =$ 

Since $(ij)(im)(ij) = (jm)$ we have $(ij)\mathcal{M}_m(ij) = \mathcal{M}_m$ whenever $i,j < m$ or $i,j > m$. Thus

$$[\mathcal{M}_m, S_{m-1}] = 0$$

and so for all m, m':

$$[\mathcal{M}_m, \mathcal{M}_{m'}] = 0$$

That is, the Murphy elements of $\mathbb{Z}S_n$ form a commutative subalgebra. Also

$$\sigma_m \mathcal{M}_m \sigma_m \; = \; \mathcal{M}_{m+1} - \sigma_m \qquad \text{so} \qquad \sigma_m \mathcal{M}_{m+1} \sigma_m \; = \; \mathcal{M}_m + \sigma_m$$

so

$$\sigma_m (\mathcal{M}_m + \mathcal{M}_{m+1}) \sigma_m \; = \; \mathcal{M}_m + \mathcal{M}_{m+1}$$

pa:Murphy1SNF From this one can show that symmetric polynomials in $\mathcal{M}_2, ..., \mathcal{M}_n$ are central in $\mathbb{Z}S_n$.

Murphy [127] (see also, e.g. Green–Diaconis [38]) computed the representation matrices for these elements in Young's seminormal form (see §11.5) of each Specht module $\mathcal{S}_\lambda$:

$$\rho_\lambda(\mathcal{M}_i) = \; diag(c_1(i), c_2(i), ...)$$

where $c_l(i)$ is the content of the box containing i in the l-th standard Young tableau of shape λ (in some chosen total order of tableau, which will not be important to us).

Since $\sum_i \mathcal{M}_i$ is central it acts like a scalar, so we only need the first diagonal entry to determine this scalar. This is, then, the sum of the contents of all the boxes in the 'first' (or indeed any) standard tableau of shape λ. That is

$$\rho_\lambda(\sum_i \mathcal{M}_i) = \; (\sum_{b \in \lambda} c(b)) I$$

where the sum is simply over the contents of boxes of λ; and I is the identity matrix.

For (a rather trivial) example, $\rho_{(2)}(\sum_i \mathcal{M}_i) = \rho_{(2)}(\mathcal{M}_2) = 1$ while $\rho_{(1^2)}(\sum_i \mathcal{M}_i) = -1$. We deduce that these representations are in different blocks unless $1 = -1$ in the ground field k. (Cf. the Nakayama conjecture (11.3.2).)

We return to these elements in §19.1.1.

11.5 Young forms for S_n irreducible representations

ss:SNF

Young obtained explicit constructions of the irreducible representations of S_n over $\mathbb{C}$ satisfying various specific properties as matrices (of importance in physical applications, for example). See e.g. Boerner [12].

(**11.5.1**) Young's Orthogonal form, normal form and SNF all require the notion of hook length.

11.5.1 Hooks, diamond pairs and the Young Forms for S_n

ss:diamond1

de:diamondpair (**11.5.2**) Fix a digraph G that is a multiplicity-free Bratteli diagram for a tower of algebras starting from the ground field (hence a single root vertex in G). It follows immediately that the set of walks from the root to $\lambda \in G$ is a basis for the corresponding simple module.

Following Martin[104, §7.4], a pair of partial walks of length 2 on G is called a *diamond pair* if they start at the same label and finish at the same label. That is, supposing we start in layer $m-2$ (some m), we may write

$$S \; = \; (s(m-2), \; s(m-1), \; s(m)),$$

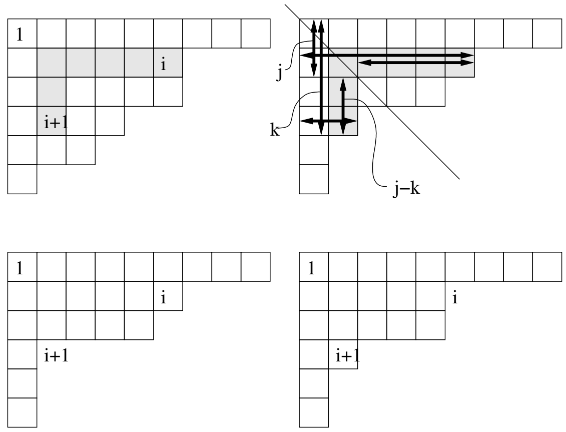


Figure 11.5: Schematic for (a) a tableau t and a hook length; and, (b,c,d) associated diagrams. fig:hookcontent1

and $T = (s(m-2), t(m-1), s(m))$. We say that an algebra element u acts *locally* on the basis of walks if it mixes between two walks only if they differ at a single vertex. That is, they agree except for forming a single diamond pair. (We say such a pair of full walks form a diamond pair at the vertex in question.)

(**11.5.3**) The Bratteli diagram for the tower of symmetric group algebras over $\mathbb{C}$ is the directed Young graph. Walks on this are encoded by standard tableaux.

The matrix for a representation of σ_i in Young's SNF, or OF, (see e.g. Boerner [12]) mixes between two tableaux only if they differ, *as walks*, at a single point — i.e. in two steps. (Thus each tableau is mixed by σ_i with at most one other.)

(**11.5.4**) The mixing is determined by the *tableau hook distance* between i and $i+1$ in the first tableau t say (or equivalently the second, u), denoted $h_t(i, i+1)$. See Fig.11.5.

The *tableau hook length* between i and $i+1$ (shaded in Fig.) is

$$h_t(i, i+1) = (\nu_j - j) - (\nu_k - k) \;=\; c_t(i) - c_t(i+1)$$

where $c_t(i)$ means the content of the box determined by the position of symbol i in tableau t.

(**11.5.5**) The *box hook length* $h(i,j) = h_\lambda(i,j)$ for a box (i,j) in a partition λ is the length of the hook in λ with axis box (i,j).

Tableau hook length is not to be confused with this.

(**11.5.6**) Recall (see e.g. Hamermesh [60], or Yamanouchi [148]) that the S_n orthogonal action on a standard tableau t as above is then:

$$\sigma_i t \;=\; \frac{1}{h_t(i,i+1)} t \;+\; \sqrt{\frac{h_t(i,i+1)^2 - 1}{h_t(i,i+1)^2}}\, \sigma_i(t) \tag{11.7}$$ ortho1

where $\sigma_i(t)$ denotes the (not necessarily standard) image of t under swapping i and $i+1$.

Note that if $\sigma_i(t)$ nonstandard then $h_t(i, i+1) \in \{1, -1\}$ so nonstandard tableau can be ignored.

11.5.2 Asides on geometry

(**11.5.7**) REMARK. It is useful to think about Young forms geometrically. To this end, one can extract a certain foursome of Young diagrams from the t, u tableaux mixing setup as follows:
(1) the diagram containing $1, 2., , , .i-1$ in t;
(2_t) the diagram containing $1, 2., , , .i$ in t;
(2_u) the diagram containing $1, 2., , , .i$ in u;
(3) the diagram containing $1, 2., , , .i+1$ in t.

The figure on the top left in Figure 11.5 illustrates a tableau t, of shape λ say. The figure on the top right is the diagram $\nu \subset \lambda$ containing $1, 2., , , .i+1$. The diagrams in the second row in the figure are cases 2_t and 2_u respectively.

(**11.5.8**) Chat: Consider whether 2_t and 2_u are taken in to each other by reflection in a suitable affine wall? What does this mean?
(1) We must embed the diagrams in a Euclidean space;
(2) We must associate a reflection group to this space.

There are at least two ways of embedding diagrams in $\mathbb{R}^{\mathbb{N}}$, since we can embed diagrams or their transposes, by regarding either as an integral vector in the obvious way. It is not obvious how to *combine* these! Reflection usually involves row permutation by convention, since the vector associated to rows seems marginally more natural than that associated to columns. Thus any row perm take a partition to a composition at worst. Column perms do not preserve row compositions in general, so they do not play together well with row actions. Perhaps the structure preserved by both is that subsets of the boxes in the 'matrix' quadrant are take into each other.

Note however that *some* column operations on *some* diagrams *do* preserve compositions, partitions even,... Consider our example above.

Also in the S_n case we are focussing on certain kinds of diagram 'progressions' — forward walks on the Young graph. If we do not require this, then maybe more freedom arises... ...indeeed we can try to relate to the direct product rep construction, or variants thereof (semidirect products say).

11.6 Outer product and related representations of S_n

ss:outer1

See e.g. Robinson [36], Hamermesh [60], Hoefsmit [66], Martin–Woodcock–Levy [122]. There are several aspects to this topic. Later we look at the classical construction (modules induced from representations of Young and other special subgroups), but there is also a geometric aspect related to affine Hecke algebras and also to the invariant theory, and the theory of highest weight and Verma modules, of sl_n. (This leads to connections with Yang–Baxter equations and elsewhere.)

11.6.1 Multipartitions and their tableaux

ss:multipart

We start with some combinatorial preliminaries.

Recall from §5.7 that Λ is the set of integer partitions. For $d \in \mathbb{N}$ define the set of *d-component multipartitions*

$$\Gamma^d = \Lambda^{\times d}$$

and Γ^d_n as the subset of multipartitions of total degree n (we take this notation from Martin–Woodcock–Levy [122]).

We write

$$\mu = (\mu^1, \mu^2, ..., \mu^d) \; \in \; \Gamma_n^d$$

The *shape* of μ is the composition of n whose i-th component is $|\mu^i|$.

(**11.6.1**) The natural inclusion of $\Gamma^d \hookrightarrow \Gamma^{d+1}$ is given by appending an empty component. We define Γ^{fin} as the inverse limit of these inclusions.

(**11.6.2**) We define a digraph $\mathcal{Y}_*$ on vertex set Γ^{fin} such that there is an edge from μ to ν if, regarded as multi(-Young)diagrams, they differ by the addition of a single box.

Note that this digraph is rooted, simple, loop-free and non-tree. However there are infinitely many edges out of every vertex, since there are infinitely many component partitions in the multitude (most of them the empty partition) to which a box may be added.

(**11.6.3**) A *tableau* of shape $\mu \in \Gamma_n^d$ is an arrangement of the symbols $1, 2, ..., n$ in the n boxes of μ regarded as a multi-Young diagram. As a *set* we may regard μ as a certain set of n boxes. Thus the set of tableaux is $\mathbf{Set}^{\sim}(\mu, \underline{n})$ (here $\sim$ means to take the subset of bijective maps).

(**11.6.4**) A tableau of shape μ is *standard* if each component tableau μ^i is standard. We write T^μ for the set of standard tableau of shape μ. Assuming a given total order on this set, we write

$$T^\mu = \{T_1^\mu, T_2^\mu, ..., T_t^\mu\}$$

A tableau in T^μ determines a walk on the digraph $\mathcal{Y}_*$ from $\emptyset$ to μ.

11.6.2 Actions of S_n on tableaux

de:nat act (**11.6.5**) We define a 'natural' (or 'regular' or 'Cayley') action of S_n on tableaux of shape $\mu \vdash n$ so that $\sigma_i(T_p^\mu)$ (and indeed $\sigma_i(T)$ for any tableau $T \in \mathbf{Set}^{\sim}(\mu, \underline{n})$) is obtained by interchanging symbols i and $i+1$ in T_p^μ.

Note that this action does not necessarily take standard to standard, depending on μ (NB, in the example $\mu = ((1), (1), ..., (1))$ all tableaux are standard). Over all (not necessarily standard) tableau, however, this action evidently makes $\mathbf{Set}^{\sim}(\mu, \underline{n})$ a basis for the regular module for any μ.

(**11.6.6**) Note that this can be seen as a Cayley representation (i.e. a (left) regular permutation representation). Consider the reflection group action $\sigma_i \mapsto (i\ i{+}1)$ of S_n on $\mathbb{R}^n$, with reflection hyperplanes H_{ij} (as in §**??**), associated chambers of $\mathbb{R}^n \setminus \cup_{ij} H_{ij}$, 'dominant' chamber containing $(1, 2, ..., n)$, and so on. Recall S_n acts simply transitively on the set of chambers (which may be indexed by the perms of $(1, 2, ..., n)$). We associate $1 \in S_n$ to $(1, 2, ..., n)$ and $w \in S_n$ to $w(1, 2, ..., n)$. (The 'left' action.) Taking $T_1^\mu = (1, 2, ..., n)$ (in the obvious notation for tableau of this shape) we get the required correspondence. Note that the reflection action of S_n also closes on the 'integral box' $\underline{n}^{\times n} \subset \mathbb{R}^n$ containing all our chamber representative points and some 'singular' points. Here the action is isomorphic (indeed identical) to the 'tensor space' action as for example on $\mathbf{Set}(3, 3)$ in §**??**.

We want to describe various other actions of S_n on $\mathbf{Set}^{\sim}(\mu, \underline{n})$ to $\mathbb{C}\mathbf{Set}^{\sim}(\mu, \underline{n})$, some of which will make certain submodules *manifest*.

(**11.6.7**) For the moment suppose that $\mu = ((1), (1), ..., (1))$ (that is, an n-tuple of single boxes), and let $p(i)$ denote the position in the tuple in which i appears in T_p^μ. Let x be an n-tuple of complex numbers. Define $\sigma_i T^\mu_p$ by

$$\sigma_i \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix} = \frac{1}{h} \begin{pmatrix} -1 & -(h-1) \\ -(h+1) & 1 \end{pmatrix} \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix} \qquad (11.8)$$ eq:YBx1

where $h = x_{p(i)} - x_{p(i+1)}$.

In this way σ_i acts by a linear transformation on the vector space with tableaux basis, and hence as a particular matrix on the tableau basis. Let us write R^x for this map from Coxeter generators to matrices (for x indeterminate, and otherwise when x such that this map is well-defined per se).

(**11.6.8**) Note that x may be chosen so that all h are large magnitude (complex). In the large magnitude limit, then action σ_i- coincides with the regular (as in Cayley, as in simply transitive) representation $\sigma_i(-)$.

pa:rep x (**11.6.9**) We claim that R^x extends to a representation. To verify this we need to show:

(1) $\sigma_i\sigma_i - 1 \overset{R^x}{=} 0$:

This is clear since every matrix $R(\sigma_i)$ falls into 2x2 blocks as above, each of which is traceless and has det$= -1$.

(2) $\sigma_i\sigma_{i+1}\sigma_i - \sigma_{i+1}\sigma_i\sigma_{i+1} \overset{R^x}{=} 0$:

The action of $\{\sigma_i, \sigma_{i+1}\}$ evidently falls into blocks of tableaux of size six. The identity can therefore be checked by looking at σ_i and σ_{i+1} on a typical block of tableaux. Here $h = x_{p(i)} - x_{p(i+1)}$, $h_2 = x_{p(i)} - x_{p(i+2)}$ and $h_1 = x_{p(i+1)} - x_{p(i+2)}$:

$$\sigma_i \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \\ \sigma_{i+1}(T^\mu_p) \\ \sigma_i(\sigma_{i+1}(T^\mu_p)) \\ \sigma_{i+1}(\sigma_i(T^\mu_p)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T^\mu_p))) \end{pmatrix} = \begin{pmatrix} \frac{-1}{h} & \frac{1-h}{h} & 0 & & & \\ \frac{-h-1}{h} & \frac{1}{h} & 0 & & & \\ 0 & 0 & \frac{-1}{h_2} & \frac{1-h_2}{h_2} & 0 & 0 \\ 0 & 0 & \frac{-h_2-1}{h_2} & \frac{1}{h_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{h_1} & \frac{1-h_1}{h_1} \\ 0 & 0 & 0 & 0 & \frac{-h_1-1}{h_1} & \frac{1}{h_1} \end{pmatrix} \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \\ \sigma_{i+1}(T^\mu_p) \\ \sigma_i(\sigma_{i+1}(T^\mu_p)) \\ \sigma_{i+1}(\sigma_i(T^\mu_p)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T^\mu_p))) \end{pmatrix}$$

On the same part of the basis we have:

$$R_{part}(\sigma_{i+1}) = \begin{pmatrix} \frac{-1}{h_1} & 0 & \frac{1-h_1}{h_1} & 0 & & \\ 0 & \frac{-1}{h_2} & 0 & 0 & \frac{1-h_2}{h_2} & 0 \\ \frac{-h_1-1}{h_1} & 0 & \frac{1}{h_1} & 0 & & \\ 0 & 0 & 0 & \frac{-1}{h} & 0 & \frac{1-h}{h} \\ 0 & \frac{-h_2-1}{h_2} & 0 & 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & 0 & \frac{-h-1}{h} & 0 & \frac{1}{h} \end{pmatrix}$$

One can then verify the relation by brute force[1].

(3) distant commutation...

(**11.6.10**) It follows from (11.6.9) that R^x is a representation of S_n over $\mathbb{C}$ for almost every $x \in \mathbb{C}^n$. Thus there are a continuum of such representations. Thus these representations are all isomorphic.

[1] For example, *Maxima* [?] the open source algebraic computation package does this. See my Maxima file "defns.mac".

We next investigate special cases of x (and their relationship to other choices of μ) that illuminate important features of this class of representations.

(**11.6.11**) Consider now the case $x = (3, 2, 1)$. We have $h = h_1 = 1$ and $h_2 = 2$.
Note that the tableau $T^\mu_p = (1, 2, 3) = 123$ (in the obvious notation) spans a submodule.

(**11.6.12**) If we set $h = x_1 - x_2 = -1$ and $h_2 = x_1 - x_3 = 1$ then $h_1 = x_2 - x_3 = 2$.

11.6.3 Generalised hook lengths and geometry

(**11.6.13**) Hook length: ...

(**11.6.14**) We define $h_{ij}(T^\mu_p)$ as the hook length between the boxes containing i and j in T^μ_p, with $h_{ij}(T^\mu_p) = \infty$ if i, j in different parts. We define $h^0_{ij}(T^\mu_p)$ as the hook length between the boxes containing i and j in T^μ_p, with all Young diagrams overlaying so that the $(1, 1)$-box is in the same position.
For x a d-tuple of complex numbers we define

$$h^x_{ij}(T^\mu_p) \;:=\; h^0_{ij} + x_{\#i} - x_{\#j}$$

where $\#i$ is the position in the tuple corresponding to the Young diagram containing i in T^μ_p.

11.6.4 Connections to Lie theory and Yang–Baxter

To Do!

11.7 Outer products continued — classical cases

(**11.7.1**) Let G, G' be groups and $R_G, R_{G'}$ representations. Then the Kronecker product $R_G \otimes R_{G'}$ is a representation of $G \otimes G'$.

11.7.1 Outer products over Young subgroups

(**11.7.2**) If $\lambda \vdash n$ then

$$S_\lambda := \otimes_i S_{\lambda_i}$$

is a subgroup of S_n (each factor acts nontrivially on a corresponding subset of $\underline{n}$).
Let $\mu \in \Gamma^d_n$ of shape λ. Fix a representation R_ν for each Specht module $\Delta(\nu)$. Then we have a representation $\otimes_i R_{\mu^i}$ of S_λ. Fix a commutative ring K and write M'_μ for the corresponding left KS_λ-module. We define a left KS_n-module by

$$M_\mu \;:=\; KS_n \otimes_{KS_\lambda} M'_\mu$$

This M_μ is called an *outer product* representation.
Our next task is to construct an explicit basis and action for each M_μ.

11.7.2 Outer products over wreath subgroups

Whenever a partition occurs more than once in an outer product representation there is an automorphism acting permuting the identical factors. We can use this to decompose the representation.

11.7.3 The Leduc–Ram–Wenzl representations

As kS_n-modules the Leduc–Ram modules are particular examples of mixtures of the previous two types of modules. (They are of interest via their extensions to Brauer algebra representations, but we shall start by considering them as S_n representations.)

For each n and m such that $n = 2m + k$, and $\lambda \vdash k$ we have the subgroup $S_2 \wr S_m \subset S_{2m} \subset S_n$.

11.8 Finite group generalities

11.8.1 Characters

Let G be a finite group and let $\{\rho_i\}_{i=1,2,...,r}$ be a (complete) set of irreducible representations over $\mathbb{C}$, with dimensions $d_1, d_2, ..., d_r$. Then for each pair i, j and $d_i \times d_j$ matrix C define

$$P_C = \sum_{g \in G} \rho_i(g) C \rho_j(g^{-1})$$

If $P_C \neq 0$ then it intertwines ρ_i and ρ_j,

$$\rho_i(g) P_C = P_C \rho_j(g)$$

Thus by Schur's Lemma $P_C = 0$ unless $i = j$, in which case it is a (possibly zero) scalar multiple of the identity matrix.

In the former case, choosing any pair h, h' and puting $C = e_{hh'}$ (the elementary matrix) we have

$$\sum_g (\rho_i(g))_{kh} (\rho_j(g^{-1}))_{h'k'} = 0$$

for any k, k'. Puting $k = h$ and $k' = h'$ and summing over k and k' we get

$$\sum_g \sum_k \sum_{k'} (\rho_i(g))_{kk} (\rho_j(g^{-1}))_{k'k'} = 0$$

That is, with character $\chi_i(g) = \sum_k (\rho_i(g))_{kk}$,

$$\sum_g \chi_i(g) \chi_j(g^{-1}) = 0$$

Similarly

$$\sum_g \chi_i(g) \chi_i(g^{-1}) = |G|$$

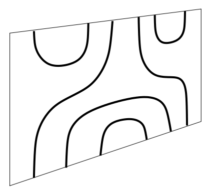
Puting $k = k'$ and summing we get

$$\sum_g \sum_k (\rho_i(g))_{kh} (\rho_j(g^{-1})_{h'k} = 0$$

Chapter 12

The Temperley–Lieb algebra and related problems

Ch:TL



Fix a commutative ring k, and choose an element $\delta \in k$. Then for each $n \in \mathbb{N}_0$ the Temperley–Lieb algebra $T_n(\delta)$ is a certain finite rank k-algebra. There is in particular a notable basis consisting of certain 'diagrams' (the subset of non-crossing diagrams among the 'Brauer diagrams' [**?**] – diagrams representing set partitions of $2n$ vertices into pairs, see later). We already had a first encounter with them in Chapter 2.

Temperley–Lieb algebras appear in several significantly different settings. These settings lead to various distinct applications, and also inform Temperley–Lieb representation theory in different ways. In §12.3 we look at the Hecke algebra context — where Temperley–Lieb algebras are defined by a presentation. In §12.11 we look at the 'diagram algebra' context. In ...

One objective is to describe the structures of these algebras over the complex field (hopefully complementing the method of [104, §7.3]; and see also our §2.1 *et seq*, and in particular §2.6). It is convenient to do this by bringing together features from different realisations. For example, it is relatively easy to determine the dimension of the diagram algebra; and to construct a surjective map from the Hecke quotient to the diagram algebra. ... We continue with representation theory in Ch.13.

Contents

12.1 Introduction/Overview

An informal definition of the Temperley–Lieb algebra is as follows ...

Cf. (1.3.13), §2.1.2, §2.2 ...

Chapters 13,14 contain more on Temperley–Lieb representation theory.

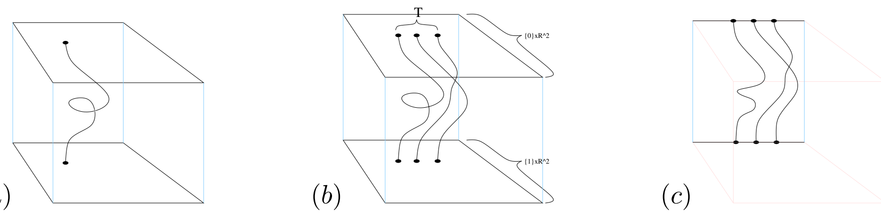
12.2 Braid groups in brief

12.2.1 Geometric Braid groups

ss:braidg

See Crowell–Fox [31], Moise [125], Birman [10], Kassel [85] etc for useful background on braids and isotopy.

(**12.2.1**) A *concrete braid-string* is a continous function $s : [0,1] \to \mathbb{R}^2$ defining a subset $S = S(s)$ of the volume $F = [0,1] \times \mathbb{R}^2 \in \mathbb{R}^3$ by $(t, x, y) \in S$ if $s(t) = (x, y)$.



(a) (b) (c)

Figure 12.1: (a,b) Artists impression of concrete braid-strings; (c) Projection of concrete braid-strings into page.

fig:braidabc

See for example Fig.12.1(a). Note that S is a homeomorphic image of the interval $[0,1]$ in $\mathbb{R}^3$ satisfying a 'no going back' condition. The apparent self-crossing in our picture is an illusion of the projection onto the page.

(**12.2.2**) Given a fixed subset T of $\mathbb{R}^2$ of order n, a set of n concrete braid-strings s_i, and hence subsets S_i, is a *braiding* of concrete braid-strings on T if (i) the subsets S_i do not intersect in F; (ii) $\{s_1(0), s_2(0), ..., s_n(0)\} = \{s_1(1), s_2(1), ..., s_n(1)\} = T$.

The set T is called the base set. For given n we may fix this as $\{(1,0),(2,0),...,(n,0)\}$ here without loss of generality.

(**12.2.3**) We can think of *looking* at a concrete braid so that coordinates (t,x,y) in F appear as height, width and depth respectively. Then for example the braid in which each s_i is a constant function looks like n vertical lines in a box. There is also a braid σ_i in which all except s_i and s_{i+1} are constant, and these two cross over, with s_i passing just in front of s_{i+1} (at distance $1/1000n$ say).

(**12.2.4**) A homeomorphism on volume F is a continuous, invertible and inverse-continuous function (see e.g. §3.3). Examples: (1) $f(t,x,y) = (1-t,x,y)$; (2) $g(t,x,y) = (t,2x,y)$.

An *isotopy* of F is a collection of continuous, invertible and inverse-continuous functions $f_\tau : F \to F$ for $\tau \in [0,1]$, such that $f : F \times [0,1] \to F$ given by $f(\tau, x) = f_\tau(x)$ is continuous.

(**12.2.5**) Two braidings of concrete strings on base set T, braidings S, S' say, are isotopy equivalent if there is an isotopy such that $f_0(S) = S$, $f_\tau(S)$ is a concrete braiding on T for all τ, and $f_1(S) = S'$.

A braiding of strings is a class of braidings of concrete strings under isotopy equivalence. The set of these classes is denoted $\mathfrak{B}(T)$.

Intuitively one thinks here of smooth deformations of one braiding into another while holding the endpoints fixed; as might be realised by moving braided physical strings.

(**12.2.6**) Two concrete braidings with a given base set may be composed to form another by 'vertical' juxtaposition and then rescaling the combined interval $[0,2]$ to $[0,1]$.

Note that this composition on the set of concrete bradings is not associative. In $a(bc)$ the factor a is shrunk once; in $(ab)c$ it is shrunk twice.

This composition passes to a well-defined composition on braidings, satisfying the group axioms.

This group $(\mathfrak{B}(T), *)$ is called the *group of braidings of n strings*, or the group of geometric braids on n strings.

12.2.2 Artin braid groups

ss:abraidg

(**12.2.7**) **Asides on knots.** A *knot* is a subset of $\mathbb{R}^3$ that is the image of the unit circle (in $\mathbb{R}^2$) under a homeomorphism. Two knots are *equivalent* if there is a homeomorphism of $\mathbb{R}^3$ into itself that takes one into the other.

A *polygonal knot* is a knot that is the union of finitely many straight line segments. (In a polygonal knot the endpoints of straight segments are *vertices*.)

A knot is *tame* if it is equivalent to a polygonal knot (else it is *wild*). NB, there exist wild knots.

A *knot projection* of knot K is the image in $\mathbb{R}^2$ given by $p : (x, y, z) \mapsto (x, y)$. A polygonal knot is in *general position* if
(pi) no more than two points of K are projected onto any point of $p(K)$;
(pii) when two points of K are projected onto a point then neither is a vertex.

(**12.2.8**) Define a group $\mathfrak{B}_n$ with generators $g_1, g_2, \ldots g_{n-1}$ and relations $g_i g_j = g_j g_i$ if $|i-j| > 1$,

$$g_i g_j g_i = g_j g_i g_j \qquad \text{if } |i-j| = 1.$$

(**12.2.9**) Theorem. [Artin's Presentation Theorem] *The group $\mathfrak{B}_n$ may be identified with the group of braidings of n strings. The identification takes $g_i \mapsto \sigma_i$.*

12.3 Ordinary Hecke algebras in brief

ss:TLHecke

It is not our intention to study Hecke algebras *per se* here (see e.g. Humphreys [68], Martin [104]). However they do provide one useful *setting* for TL algebras. They also provide access to an intriguing (more general) computational device, Kazhdan–Lusztig polynomials.

Hecke algebras can be defined in various ways, for example: as quotients of braid group algebras; as deformations of the group algebras of Coxeter systems; or as certain limits of the centraliser algebras of quantum group actions on tensor space.

12.3.1 Braid group algebra quotients — the Hecke algebra

ss:braidq

de:HeckeAlg

(**12.3.1**) We work over $\mathbb{C}$ for now. Fix $q \in \mathbb{C}$ and $n \in \mathbb{N}$. Let $H_n = H_n(q)$ denote the *Hecke algebra* over $\mathbb{C}$, the $\mathbb{C}$-algebra that is the quotient of $\mathbb{C}\mathfrak{B}_n$ (as in §12.2.2) by the 'local relation'

$$(g_i - 1)(g_i + q^2) = 0$$

(This parameterisation is used by Martin in [104]. Humphreys [68] uses $(g_i + 1)(g_i - q) = 0$. There are other choices in common use.)

(**12.3.2**) Evidently $H_n(q)$ has one-dimensional representations given by

$$\rho_+(g_i) = 1$$

and

$$\rho_-(g_i) = -q^2.$$

In order to construct corresponding unnormalised idempotent elements of the algebra we need to define a basis. It will be clear that when $q = 1$ the local relation gives the symmetric group, of order $n!$. We shall show in §12.3.2 that $H_n(q)$ is a flat deformation of the group algebra.

(**12.3.3**) The braid relations are homogeneous, so they are not affected by rescaling the generators; while such a rescaling changes the parameters in the local relation. Consider writing the local relation as $\left(q^{-1}g_i - q^{-1}\right)\left(q^{-1}g_i + q\right) = 0$. Setting

$$U_i = q^{-1}g_i - q^{-1} \tag{12.1}$$ eq:urel-1

this relation becomes

$$U_i(U_i + [2]) = 0 \tag{12.2}$$ eq:urel0

and the braid relations

$$U_iU_{i\pm1}U_i - U_i = U_{i\pm1}U_iU_{i\pm1} - U_{i\pm1} \tag{12.3}$$ eq:urel

In other words, these relations (and the corresponding commutation relations) define an isomorphic algebra to $H_n(q)$ (and the algebras may be identified via (12.1), or equivalently $g_i = 1 + qU_i$).

de:TL0001 (**12.3.4**) One way of defining the Temperley–Lieb algebra T_n will be as the quotient of $H_n(q)$ by the relation $U_iU_{i\pm1}U_i - U_i = 0$.

12.3.2 Bourbaki generic algebras

ss:bgeneric

We follow [68] and [**?**, IV §2]. First we introduce a deformation of the group algebra of a Coxeter system.

(**12.3.5**) Let (W, S) be a Coxeter system as in (5.2.10) and k a commutative ring. Choose parameters $q_s, r_s \in k$ $(s \in S)$ such that $q_s = q_{wsw^{-1}}$ for all $w \in W$, and similarly for r_s.

Consider the following relations on the free k-algebra generated by symbols $\{T_w \mid w \in W\}$:

$$T_sT_w = T_{sw} \qquad \text{if } s \in S \text{ and } l(sw) > l(w) \tag{12.4}$$ eq:bnrel1

$$T_s^2 = q_sT_s + r_sT_1 \tag{12.5}$$ eq:bnrel2

where l is the usual Coxeter length function (see (5.2.15)). [1]

(**12.3.6**) Theorem. *Fix a Coxeter system (W, S) and commutative ring k. The quotient H_W of the free algebra by the relations (12.4), (12.5) has basis $\{T_w \mid w \in W\}$.*

(**12.3.7**) For example with $W = S_3$ we must set $q_s = q$ (some q; and we can scale $r_s = r$ by any factor).

(**12.3.8**) Setting $k = \mathbb{Z}[q, q^{-1}]$ and

$$q_s = q - 1$$

and $r_s = q$ this case of H_W is the *Hecke algebra* of W over k.

Evidently H_W is generated by $\{T_s \mid s \in S\}$.

Proposition. The definition of $H_n(q)$ works over $\mathbb{Z}[q, q^{-1}]$. The map given by $\psi : g_i \mapsto T_{\sigma_i}$ extends to an isomorphism $\psi : H_n(q) \to H_{S_n}$.

[1] Equivalently we may substitute the second set of relations by:

$$T_sT_w = q_sT_w + r_sT_{sw} \qquad \text{if } l(sw) < l(w)$$

12.4 On Duality of Hecke algebras with quantum groups

ss:TLdualqg

The category of modules for a k-algebra A is a k-linear category, and hence each end set in this category, $A_M = \hom_A(M, M)$ for each module M, is also a k-algebra. Each of these algebras has its own module category. Under some circumstances it is convenient to treat this collections as an ensemble. Indeed each $\hom_A(M, M')$ is an A_M-module (and a left A_M-right $A_{M'}$ bimodule)[2] by the original category composition.

And sometimes an end set in the end-algebra module category is a quotient of A — a kind of *Schur–Weyl duality*, relating the representation theory of A and A_M.

For Hecke algebras a natural example of a 'dual' comes from quantum groups. In §12.5 we introduce some examples of quantum groups. In §**??** we ...

12.5 Aside on Quantum groups

ss:qgroups

We will need some examples to play with. Here we consider $U_q sl_2$ and $U_q sl_3$.

12.5.1 $U_q sl_2$

Recall (e.g. from Kulish—Reshetikhin [**?**] or Jimbo [80]; see also [104] or [84]) that in case q a complex number that is not a root of unity or zero, then $U_q sl_2$ may be defined as the complex algebra with generators $E, F, K^{\pm 1}$ and relations $KE = q^2 EK$, $KF = q^{-2} FK$ and $[E, F] = (K - K^{-1})/(q - q^{-1})$.

The 'defining' representation, on $V = \mathbb{C}^2$, is

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

From the relations one sees that there are two 1-d representations:

$$\rho_\pm(E) = \rho_\pm(F) = 0, \qquad \rho_\pm(K) = \pm 1$$

This $U_q sl_2$ is roughly speaking a deformation of the universal enveloping algebra of the Lie algebra sl_2 (except of course that the case $q = 1$ is excluded; and there is a second 1-d representation). In particular it can be given the (very powerful) bialgebra property that the UEA has. We may take the coproduct $\Delta : U_q sl_2 \to U_q sl_2 \otimes_{\mathbb{C}} U_q sl_2$ to be given by:

$$\Delta(E) \;=\; E \otimes 1 \;+\; K \otimes E, \qquad \Delta(F) \;=\; F \otimes K^{-1} \;+\; 1 \otimes F, \qquad \Delta(K) \;=\; K \otimes K \tag{12.6}$$

sl2cop

and so on [104]. (One needs to check for example that Δ is an algebra morphism — it is enough to check the Δ-images of the defining relations; and coassociative. For coassociativity we have:

$$(Id \otimes \Delta)(\Delta(E)) = E \otimes (1 \otimes 1) + K \otimes (E \otimes 1 + K \otimes E) = E \otimes (1 \otimes 1) + K \otimes (E \otimes 1) + K \otimes (K \otimes E)$$

$$(\Delta \otimes Id)(\Delta(E)) = (E \otimes 1 + K \otimes E) \otimes 1 + (K \otimes K) \otimes E = (E \otimes 1) \otimes 1 + (K \otimes E) \otimes 1 + (K \otimes K) \otimes E$$

and so on — these are naturally isomorphic (via a 'balanced triple').)

[2] other way round?

(**12.5.1**) Note that the coproduct Δ is not unique. For example, defining $T : a \otimes b \mapsto b \otimes a$ the 'opposite' $T\Delta$ is also a coproduct.

de:type-1

(**12.5.2**) Using the coproduct we can define certain classes of representations (up to isomorphism): under tensoring by $\rho_{\pm}$. There is a transversal of this equivalence called type-1 [3] representations (roughly speaking, K acts on weight vectors as $Kv = +q^{f(v)}v$). Thus, representation theoretically, it is enough to study type-1 (with properties of type-(-1) following immediately).

(**12.5.3**) We also get more general 'tensor products' of representations. For example, from (12.6), E acts on $V \otimes V$ by

$$\rho_{V\otimes V}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_{V\otimes V}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \end{pmatrix}$$

Note that the details depend on our Kronecker product conventions (1.1).

Similarly E acts on $V \otimes_{\mathbb{C}} (V \otimes_{\mathbb{C}} V)$ by

$$\rho_{V\otimes(V\otimes V)}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) +$$

$$\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

and on $(V \otimes_{\mathbb{C}} V) \otimes_{\mathbb{C}} V$ in an 'analogous' way. Note that the modules $V \otimes (V \otimes V)$ and $(V \otimes V) \otimes V$ are naturally isomorphic but not identical, but since Kronecker product is *strictly* associative these matrix representations are not just isomorphic but *identical*.

12.5.2 Intertwiners: arrows in the representation category

(**12.5.4**) The corresponding representation to ρ_{V^2} using the $T\Delta$ product is

$$\rho'_{V^2}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes^{op} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes^{op} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(We mention this because it is part of a construction for the 'quantum Casimir' operators, via the 'R-matrix', which might be useful later.)

[3] See e.g. Lusztig [?], Rosso [?].

Define

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By construction of $T\Delta$ the matrix P is an intertwiner (see e.g. Curtis–Reiner [33, §29]): $P\rho_{V^2} = \rho'_{V^2}P$. But there are also other intertwiners in $\mathrm{Int}(\rho'_{V^2}, \rho_{V^2})$. [4]

(**12.5.5**) For any matrix R such that $R\rho_{V^2} = \rho_{V^2}R$ then $PR\rho_{V^2} = P\rho_{V^2}R = \rho'_{V^2}PR$, so that PR is also an intertwiner . That is, there is an intertwiner for each element of the commutator of ρ_{V^2} (corresponding to elements of the space of $U_q sl_2$-module morphisms from $V \otimes V$ to itself). Conversely, if S is an intertwiner then PS commutes with ρ_{V^2}: $PS\rho_{V^2} = P\rho'_{V^2}S = P\rho'_{V^2}(PP)S = (P\rho'_{V^2}P)PS = \rho_{V^2}PS$.

For example (it is enough to check the intertwining of generators):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The intertwiners P and Q are clearly non-singular. There are also singular choices (in fact P and Q span the space of intertwiners). One can also ask for an intertwiner that gives a solution to the Yang–Baxter equation (the 'quasitriangular' condition). ...

(**12.5.6**) Exercise: Check the following.

A counit algebra map $\epsilon : U_q sl_2 \to \mathbb{C}$ (cf. **??**) is $\epsilon(E) = \epsilon(F) = 0$, $\epsilon(K) = 1$.

A corresponding antipode algebra map $S : U_q sl_2 \to U_q sl_2^{op}$ (cf. **??**) is given by $S(E) = -EK^{-1}$, $S(F) = -KF$, $S(K^{\pm 1}) = K^{\mp 1}$.

Note that $S(S(E)) = S(-EK^{-1}) = -S(K^{-1})S(E) = KEK^{-1} = q^2E$. That is, S is not an involutive antiautomorphism.

We will say more about quantum groups when needed. One task is to extend the definition of $U_q sl_2$ to a more versatile ground ring (allowing specialisation to $q = 1$ for example) by replacing the $[E, F]$ relation by $[E, F] = L$ with $(q - q^{-1})L = (K - K^{-1})$. ... Another task is to generalise to $U_q sl_N$.

12.5.3 $U_q sl_3$

(**12.5.7**) A quick summary of $U_q sl_3$, paralleling our exposition for $U_q sl_2$ is as follows. The relations on generators $E_1,\ E_2,\ F_1,\ F_2,\ K_1^{\pm 1},\ K_2^{\pm 1}$ of $U_q sl_3$ are $K_1E_1 = q^2E_1K_1$, $K_1E_2 = q^{-1}E_2K_1$, ...,

[4] Reminders on 'direction' of interwiners (see e.g. §2.1.2,2.1.3): Let A be a k-algebra. Let M a left A-module, with ordered k-basis $B = \{b_1, b_2, ..., b_n\}$. For $h \in \hom_k(M, M)$, acting on B by $hb_i = \sum_j c_{ij}(h)b_j$, define matrix $C(h) = (c_{ij}(h))$. The map from $\mathrm{Hom}_k(M, M)$ to the ring of $n \times n$ matrices taking h to transpose matrix $C^t(h)$ is a ring and a k-space isomorphism.

If $a \in A$ acts on B as h then $a \mapsto C^t(h)$ is the representation afforded by basis B.

If A, B are representations (corresponding to modules M_A, M_B and bases B_A, B_B) then a matrix X intertwines A to B if $XA(a) = B(a)X$ for all a. (The point is that the roles of A, B are not symmetric.) The set $\mathrm{Int}(A, B)$ of matrices interwining A to B is isomorphic as a k-space to $\mathrm{Hom}_A(M_A, M_B)$.

$K_1F_1 = q^{-2}F_1K_1$, $K_1F_2 = qF_2K_1$, ..., $E_iF_j - F_jE_i = \delta_{ij}(q-q^{-1})^{-1}(K_i - K_i^{-1})$, ..., $F_1^2F_2 + F_2F_1^2 = [2]F_1F_2F_1$.

Note from the relations that

LEMMA. If ρ_q is a representation of U_qsl_3 (or U_qsl_2) over $\mathbb{Q}(q)$ then $\bar{\rho_q}$ given *on generators* by $\bar{\rho_q}(g) = \rho_{1/q}(g)^t$ is also a representation. ■ (Note that this bar operation takes type-1 (in the sense of (12.5.2)) to type-1.)

Note that for U_qsl_2, $\bar{\rho}(E) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\bar{\rho}(F) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\bar{\rho}(K) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$, so $\bar{\rho} \neq \rho$. On the other hand $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ so $\bar{\rho} \cong \rho$.

(**12.5.8**) For U_qsl_3 the 'defining' representation is:[5]

$$\rho_{(1)}(K_1) \;=\; \begin{pmatrix} q & 0 & 0 \\ 0 & 1/q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_{(1)}(K_2) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1/q \end{pmatrix}, \tag{12.7}$$

eq:defsl31

$$\rho_{(1)}(E_1) \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho_{(1)}(E_2) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

We also have, via the bar map:

$$\rho_{(1^2)}(E_1) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho_{(1^2)}(E_2) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho_{(1^2)}(K_1) \;=\; \begin{pmatrix} 1/q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $\rho_{(1)} \ncong \rho_{(1^2)}$ (the reasons for the labels (1) and (1^2) will become apparent shortly). For example:

$$\rho_{(1)}(E_2E_1) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho_{(1^2)}(E_2E_1) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(Note that this also shows that the bar construction can only be applied directly to generators.)

(**12.5.9**) One should compare this with the construction of taking the transpose and antipode of the defining representation.

(**12.5.10**) The one-dimensional representations are $\rho_{++}(E_i) = \rho_{++}(F_i) = 0$, $\rho_{++}(K_1) = \rho_{++}(K_2) = +1$ and similarly for $--$,$+-$ and $-+$.

(**12.5.11**) A coproduct is

$$\Delta(E_i) \;=\; E_i \otimes 1 \;\oplus\; K_i \otimes E_i, \qquad \Delta(F_i) \;=\; F_i \otimes K_i^{-1} \;\oplus\; 1 \otimes F_i, \qquad \Delta(K_i) \;=\; K_i \otimes K_i \tag{12.8}$$

sl3cop

Example: tensoring by ρ_{++} is (naturally isomorphic to) the identity map. Thus looking for a dual to a module M requires to look for a module M' such that $\rho_{M \otimes M'}$ intertwines ρ_{++} — then

[5]check maybe vs mathematica macros: defns.math

checking the (co)evaluation snake-left/right conditions. More precisely, we want $\psi_{le} : M' \otimes M \to \mathbb{C}$ realised by an intertwining row (?) vector μ: $\mu\rho_{MM'} = \rho_{++}\mu$ (since ρ_{++} acts like a scalar, this corresponds to finding a simultaneous left eigenvector for $\rho_{MM'}(X)$ for $X = E_1, E_2, ..., K_2$); and $\psi_{lc} : \mathbb{C} \to M \otimes M'$ realised by an intertwining column vector. (Representation theoretically, for the dual of a module M one usually first takes the k-linear-dual right module then uses the antipode to give this a left-action. Since we need to take care of bases we will try to be more explicit.)

Example:

$$\rho_{V\otimes(1^2)}(E_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} q & 0 & 0 \\ 0 & 1/q & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rho_{V\otimes(1^2)}(E_2) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/q & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right), \quad \rho_{V\otimes(1^2)}(K_1) = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

We can investigate if this intertwines $\rho_{\mathbb{C}} = \rho_{++}$ (in both senses). We are looking for a row vector μ such that $\mu\rho_{V\otimes(1^2)} = \rho_{\mathbb{C}}\mu$ and a column vector ν such that $\rho_{V\otimes(1^2)}\nu = \nu\rho_{\mathbb{C}}$. Since $\rho_{\mathbb{C}}$ acts as a scalar these are both simultaneous eigenvalue problems. Take $\nu^t = (a, b, c, d, e, f, g, h, i)$. From $\rho_{V\otimes(1^2)}(K_1)\nu = \nu$ we have $b = c = d = f = g = h = 0$ (this also satisfies the K_2 constraint). From $\rho_{V\otimes(1^2)}(E_1)\nu = 0$ we have $qa + e = 0$. From $\rho_{V\otimes(1^2)}(E_2)\nu = 0$ we have $qe + i = 0$. Up to a scalar we thus have

$$\nu^t = (q^{-1}, 0, 0, 0, -1, 0, 0, 0, q)$$

Now take $\mu = (\mu_1, \mu_2, ..., \mu_9)$. The K_1 constraint here again gives $\mu_2 = \mu_3 = \mu_4 = \mu_6 = \mu_7 = \mu_8 = 0$. The E_1 constraint gives $\mu_1 + \mu_5/q = 0$ and E_2 gives $\mu_5 + \mu_9/q = 0$. Thus we may take

$$\mu = \nu^t$$

Exercise: Having got this far, we should check the (co)evaluation snake-left/right conditions.

(**12.5.12**) THERE ARE SEVERAL ISSUES TO CHECK HERE:

1. is the dual right? yes, done. (Up to isomorphism. But why this basis?)
2. is the intertwining in (**??**) the right way around (apparently not)?
3. is u in (**??**) the right thing anyway?
4. is the coproduct consistent with the definition of sl3?

...

12.5.4 Tensor space representations of $H_n(q)$

(**12.5.13**) Let n, N be natural numbers, k a field, and $V_N = k^N$. In describing a basis of $V_N^{\otimes n}$ we write 112 for $e_1 \otimes e_1 \otimes e_2$, and so on.

(An ordered basis for $V_3^{\otimes 2}$ is 11,12,13,21,22,21,31,32,33.) Define matrices

$$\mathcal{R}^{\pm} = \mathcal{R}^{\pm}_N = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & q & \pm 1 & 0 & \\ 0 & \pm 1 & q^{-1} & 0 & \\ 0 & 0 & 0 & 0 & \\ & & & & ... \end{pmatrix} \in \mathrm{End}(V_N^{\otimes 2})$$

(here we gave the part acting on basis elements $11, 12, 21, 22$) and $\mathcal{R} = \mathcal{R}_N = \mathcal{R}^+$.

le:rankR (**12.5.14**) Note that the rank of $\mathcal{R}$ is $N!/(N-2)!2!$. That is, rank $\mathcal{R}_2 = 1$, rank $\mathcal{R}_3 = 3$,...

(**12.5.15**) Define

$$R_i^{\pm} = 1_N \otimes ... \otimes \underbrace{\mathcal{R}^{\pm}}_{i-th\ position} \otimes 1_N \otimes ... \otimes 1_N$$

and $R_i = R_i^+$.

de:trep (**12.5.16**) It is easy to check the following.
Lemma. For given n, N, the map given on braid generators by

$$\rho_N : g_i \;\mapsto\; 1 - qR_i \qquad\qquad (i = 1, 2, ..., n-1)$$

extends to a representation of the Hecke algebra $H_n(q)$. (That is, $\rho_N(U_i) = R_i$, for the Hecke generator U_i obeying $U_i^2 = [2]U_i$.) □

...See §12.10.1 and e.g. [104, §5.6] for more.

The representations $\rho_N(H_n)$ have several interesting features not exhibited by H_n itself, and are important objects for study in their own right. For example, at least generically it is easy to check that $\rho_N(H_n)$ gives the whole of the centraliser of a natural action of $U_q sl_N$ on $V_N^{\otimes n}$.

Define

$$H_n^N = H_n/\mathrm{ann}\rho_N \qquad (12.9)$$ eq:defHnN

The k-space V_N can be equipped with an action making it the defining module for sl_N (or $U_q sl_N$). This action extends by the bialgebra property (i.e. by a comultiplication) to an action on $V_N^{\otimes n}$.

(**12.5.17**) Lemma. *The matrices R_i commute with the action of $U_q sl_N$ on $V_N^{\otimes n}$.*

Proof. This can be checked by direct calculation. □

de:Ru2 (**12.5.18**) Consider the example $N = 2$. Noting rank $\mathcal{R}_2 = 1$, we introduce the row-space matrix

$$\mathsf{u} \;=\; (0, -t, t^{-1}, 0)$$

Given a basis of $V \otimes V$ of column vectors, this defines a map $\psi_u : V \otimes V \to \mathbb{C}$ by $\psi_u(v) = u.v$. For $N = 2$ our matrix $\mathcal{R}$ is (up to scalars; and a simple base change) the corresponding (unnormalised) projection in $\mathrm{End}(V \otimes V)$:

$$\mathsf{u}^t\mathsf{u} \;=\; \begin{pmatrix} 0 \\ -t \\ t^{-1} \\ 0 \end{pmatrix} (0, -t, t^{-1}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t^2 & -1 & 0 \\ 0 & -1 & t^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For the normalisation $\mathcal{R}^2 = (u^t u)^2 \propto \mathcal{R}$ note that

$$\mathsf{u}\mathsf{u}^t \;=\; (0, -t, t^{-1}, 0) \begin{pmatrix} 0 \\ -t \\ t^{-1} \\ 0 \end{pmatrix} \;=\; t^2 + t^{-2} \qquad (12.10)$$ eq:uut2

Of course one could have observed this construction of $\mathcal{R}$ and hence the 'tensor space' representation of $H_n(q)$ directly without the remarks about $U_q sl_2$ (and this is the historical order). But it is nice (i.e. useful and pleasing) to have the 'duality' with the quantum group in mind.

(**12.5.19**) Examples. For $N = 2$ we have sl_2 irreducible representations (over $\mathbb{C}$, say) indexed by $\mathbb{N}_0$, or equivalently by Young diagrams of a single row [**?**, **?**]. In these terms we have $V_2 = \square$ and

$$\square \otimes \square \;=\; \square\square + 0 \qquad (12.11)$$ eq:branchingsl2

with dimensions

$$2 \times 2 \;=\; 3 + 1$$

There is in particular an sl_2-module homomorphism in $\mathrm{Hom}_{sl_2}(V \otimes V, \mathbb{C})$ — a map $\psi_\cup : V_2 \otimes V_2 \to \mathbb{C}$ that commutes with the sl_2 action; and similarly a map $\psi_\cap \in \mathrm{Hom}_{U_q sl_2}(\mathbb{C}, V \otimes V)$.

The map $\psi_\cup$ lifts to $U_q sl_2$ (this is not obvious; we have not specified q and the decomposition is not always a direct sum, and the modules are not always irreducible — see later).

In a suitable basis the map $\psi_\cup$ is given for $U_q sl_2$ by $\mathsf{u} \;=\; (0, -t, t^{-1}, 0)$ where $t^2 = q^{-1}$ (and $q = 1$ in the sl_2 case). That is, we have the following.

lem:sl2inter (**12.5.20**) Lemma. *The intertwiner condition*

$$\rho_{\mathbb{C}}(X)\, u \;=\; u\, \rho_{V^2}(X) \qquad \textit{for } X = E, F, K \qquad (12.12)$$ eq:intertwine1

is solved (uniquely up to scalars) by $u = \mathsf{u} = (0, -t, t^{-1}, 0)$. *The intertwiner condition*

$$v\, \rho_{\mathbb{C}}(X) \;=\; \rho_{V^2}(X)\, v \qquad \textit{for } X = E, F, K$$

is solved (uniquely up to scalars) by $v = \mathsf{u}^t$.

Proof. From (12.6), $U_q sl_2$ acts on $V \otimes V$ by

$$\rho_{V^2}(E) = \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{V^2}(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \end{pmatrix}, \quad \rho_{V^2}(K) = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}$$

(note that the details depend on our Kronecker product conventions; if we use the 'other' convention then we need to make $t^2 = q$ instead) and acts on $\mathbb{C}$ by $\rho_{\mathbb{C}}(E) = \rho_{\mathbb{C}}(F) = 0$, $\rho_{\mathbb{C}}(K) = 1$. We have

$$E\,\mathsf{u} \;=\; \rho_{\mathbb{C}}(E)\,\mathsf{u} \;=\; 0.\mathsf{u} \;= (0,0,0,0) \tag{12.13}$$ eq:intertwine1

Indeed since ρ_C is 1-dimensional, for a vector u to intertwine: $\rho_{\mathbb{C}} u = u\rho_{V^2}$ we require u to be a left eigenvector of $\rho_{V^2}(X)$ for $X = E, F, K$, with eigenvector 0,0,1 respectively. Trying $u = (a, b, c, d)$ these constraints give $a = d = 0$ and $b + c/q = 0$. Thus u is as required — indeed it is the unique solution up to scalars. (Note our shorthand in (12.13) — $E\mathsf{u}$ means $\rho_{\mathbb{C}}(E)\mathsf{u} = 0.\mathsf{u}$ and $\mathsf{u}E$ means $\mathsf{u}\,\rho_{V^2}(E)$.)

Similarly, the intertwiner for $\psi_\cap$ is a column vector $v = (e, f, g, h)^t$ obeying $v\rho_{\mathbb{C}} = \rho_{V^2} v$. This says that v is a *right* eigenvector of $\rho_{V^2}(X)$ as above with the same eigenvalues. The constraints turn out to be the same. Thus up to scalars we have $v = \mathsf{u}^t$. □

12.6 On categories and diagram calculi

ss:catdcalc

Here we have in mind a couple of exercises designed to help with subsequent generalisations. One is to try to study 'pictorially' the linear subcategory of the category of matrices over $\mathbb{Z}[t, t^{-1}]$ generated by u and 1_V, allowing the additional operations of transposition and Kronecker product.

Another exercise (closely related!) is to study the subcategory $\mathbf{R}ep(V)$ of the representation category of $U_q sl_2$ with objects of form $V^{\otimes n}$ ($n = 0, 1, 2, ...,$ understood as in §**??**) equipped with the standard ordered basis. In the latter case the arrows are intertwiners, and we have seen in (12.13) that one such intertwiner is u.

It is claimed that, when base-changed to the field $k = \mathbb{C}$ (with t acting by some $t_c \in \mathbb{C}$), $\mathbf{R}ep(V)$ is a rigid monoidal category (a monoidal category with left and right duals, as in **??**); and a multitensor category (a rigid monoidal category that is abelian, as in **??**; k-linear; locally finite and $-\otimes-$-bilinear).

The monoidal property is given by the tensor product. The duals of V are both V itself ...

We will need some preparations.

12.6.1 MacLane's strictness theorem

(**12.6.1**) We claim that the category **Mat** of matrices over $\mathbb{Z}[t, t^{-1}]$ (see (6.1.10)) is a monoidal category (as in §6.4.1) by ordinary multiplication on objects, and the Kronecker product. The object 1 is the monoidal unit.

(**12.6.2**) Exercises/Questions: Is **Mat** strict? Yes — the Kronecker product is strictly associative. In case of $\mathbf{Mat}_k$ (over a field k for now) we note that $\mathbf{V}ec_k$ is not strict, but claim that these two monoidal categories are monoidally equivalent. To investigate this we define functors

$$\begin{array}{rcl} F : \mathbf{Mat}_k & \to & \mathbf{V}ec_k \\ n & \mapsto & k^n \\ A \in \mathbf{Mat}_k(m,n) & \mapsto & A_e : e_i \mapsto \sum_j a_{ij} e_j \end{array}$$

where we have a chosen a basis $\{e_i\}_{i=1,2,\dots,n}$ for each k^n (is this a monoidal functor?) and, picking a basis for every $V \in \mathbf{V}ec_k$ (axiom of choice)

$$\begin{array}{rcl} F' : \mathbf{V}ec_k & \to & \mathbf{Mat}_k \\ V & \mapsto & dim(V) \\ X \in Hom(V,W) & \mapsto & \text{matrix of } X \text{ w.r.t. given bases} \end{array}$$

We see that $F'F\mathbf{Mat}_k = \mathbf{Mat}_k$. We need to check $FF'\mathbf{V}ec_k$. Here $FF'(V) = k^{dim(V)}$, which is isomorphic to V by the natural identification of their given bases. Is this good enough to imply a natural isomorphism of functors $FF' \cong Id_{\mathbf{V}ec_k}$? Yes (proof?!). Indeed $\mathbf{Mat}_k$ is a skeleton of $\mathbf{V}ec_k$, and we have the following.

Theorem. A skeleton (as in §6.3) of any category C is equivalent to C.

(**12.6.3**) The monoidal property, and some facility with strictness, will be very useful in what follows. We have MacLane's strictness Theorem:

Theorem. For each monoidal category there is a monoidally equivalent strict monoidal category.

(**12.6.4**) Example. (1) As above. (2) ...

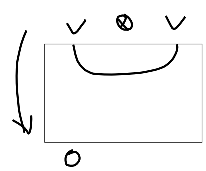
(**12.6.5**) Thus the question is: what is the role of transpose for **Mat**? This is a contravariant functor. Can we use it to define duals (as in **??**) and rigidity (as in **??**)?

The category generated by u is a monoidal subcategory.

12.6.2 Some diagrams

(**12.6.6**) To keep track of the domain and range of the map $\psi_u : V \otimes V \to \mathbb{C}$ as in (12.5.20) (the domain and range are the 'objects' of the map regarded as an arrow in a category) we may draw the map as:

$\mathsf{u} \;=\; (0, -t, t^{-1}, 0) \quad \mapsto$



(12.14) eq:dcr

That is (in this convention), the upper row of the diagram gives the row space; and the lower row the column space, by having a vertex for each factor V.

It turns out that such drawings can encode more, as the instance above suggests. We will further make each vertex a sink/source for an undirected line, so that the vertices will be linked in pairs. A link to the same edge roughly depicts an instance of the map ψ_u; while a map between edges depicts an instance of the identity map $1_V : V \mapsto V$. (The question is: does this naive idea enable is to construct more complex morphisms in some way?)

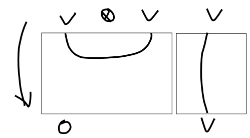
Note that, apart from the basic record of objects, this picture is not encoding the same matrix as a 'chip' picture in tensor calculus. In a chip picture each line between two vertices i, j gives a factor δ_{ij}, so the above picture would correspond to $(1, 0, 0, 1)$. There are some 'diagrammatic rules' that work with the association (12.14) above (we will call them an A_1-*calculus* for now), but they are more subtle than the general isotopies that work for chip pictures, as we shall see.

Considering

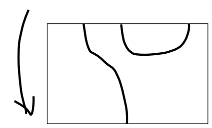
$$\square^{\otimes}\square^{\otimes}\square \;=\; \square\square\square^{+}\square^{+}\square$$

we see (from Schur's Lemma, say) that there should be two independent maps $V^3 \to V$. We have

$$(0, -t, t^{-1}, 0) \otimes 1_2 \quad = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & -t & 0 & t^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & t^{-1} & 0 & 0 \end{array} \right) \quad \mapsto$$



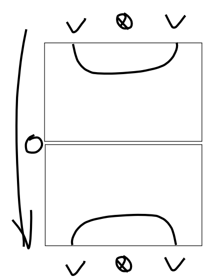
$$1_2 \otimes (0, -t, t^{-1}, 0) \quad = \left(\begin{array}{cccc|cccc} 0 & -t & t^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & t^{-1} & 0 \end{array} \right) \quad \mapsto$$



Exercise: In each case we think of the map as 'fusing' a pair $V \otimes V$ and mapping the remaining V to itself. One might wonder if it is possible to 'fuse the outer pair' for a new map?

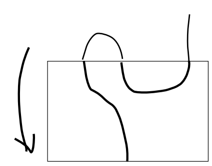
(**12.6.7**) Generalising this calculus slightly, we would like that transpose matrix $\rightsquigarrow$ upsidedown picture! And that composition of maps $(l \xrightarrow{a} m).(n \xrightarrow{b} l) \rightsquigarrow$ stacking of pictures a under b. Firstly we get

$$\mathsf{u}^t\mathsf{u} \;=\; \begin{pmatrix} 0 \\ -t \\ t^{-1} \\ 0 \end{pmatrix} (0, -t, t^{-1}, 0) \quad \mapsto$$



We can then consider

$$\big(1_2 \otimes (0, -t, t^{-1}, 0)\big) \left(\begin{pmatrix} 0 \\ -t \\ t^{-1} \\ 0 \end{pmatrix} \otimes 1_2 \right) \quad \mapsto$$



In fact (using maxima say) we find

$$(1_2 \otimes \mathsf{u})(\mathsf{u}^t \otimes 1_2) \;=\; -1_2$$

(NB Replacing u with $(0, t, 1/t, 0)$ we get $+1_2$ here.)

12.6.3 Towards the case $N = 3$

(**12.6.8**) Next we consider $N = 3$. We introduce [6]

$$u \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -t & 0 & t^{-1} & 0 \\ 0 & 0 & -t & 0 & 0 & 0 & t^{-1} & 0 & 0 \\ 0 & -t & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(**12.6.9**) Here we have

$$u^t u \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & -t & 0 \\ \hline 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ t^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -t & 0 & t^{-1} & 0 \\ 0 & 0 & -t & 0 & 0 & 0 & t^{-1} & 0 & 0 \\ 0 & -t & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & t^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & t^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & t^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

N.B. There are various conventions chosen here. That is (again up to scalars/basis change) $\mathcal{R} = u^t u$. Meanwhile

$$uu^t \;=\; (t^2 + t^{-2})\, 1_3 \qquad \text{(cf. (12.10) and see maxima!)}$$

(**12.6.10**) Note that we have irreducible sl_3-modules indexed by (1- or) 2-row diagrams

$$\square \otimes \square = \square\square + \begin{array}{c}\square \\ \square\end{array}$$

$$3 \times 3 \;=\; 6 + 3'$$

Here the map onto the 3-dimensional subspace is given by (case $t = 1$ of) u^t (the signs depend on conventions for the sl_3 coproduct). We check this next.

`pa:checkeep` (**12.6.11**) We need to check, say, that

$$\Delta(E_1)\, u^t \;=\; \rho_{V \times V}(E_1)\, u^t \;\stackrel{?}{=}\; u^t\, \rho_{(1^2)}(E_1)$$

[6] See the maxima macro file tlm.mac for example, where u =ccc(-t,1/t). See also Appendix ??.

That is, that the intertwiner condition

$$\rho_{V\times V}(X)\, w \;\;=\;\; w\, \rho_{(1^2)}(X)$$

$(X = E_1, E_2, ..., K_2)$ is solved by $w = u^t$.

By the case $X = K_1$ we see that w must agree with u^t in the zero entries.

pa:sl3intproof (**12.6.12**) We now return to checking (12.6.11). From (12.8) and (12.7), again setting $t^2 = q^{-1}$,

$$\Delta(E_1)\, u^t \;=\; \left(\begin{array}{ccc|ccc|ccc} 0 & q & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1/q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & -t & 0 \\ \hline 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ t^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline t^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Delta(E_2)\, u^t \;=\; \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & -t & 0 \\ \hline 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ t^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Meanwhile

$$u^t\, \rho_{(1^2)}(E_1) \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline t^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad u^t\, \rho_{(1^2)}(E_2) \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus the E_1 case is as required. NOW CHECK MORE!...

(**12.6.13**) What about the 'dual' coevaluation property? Do we have $u\rho_{V^2} = \rho_{(1^2)}u$?

(**12.6.14**) Next consider $F_{12} = R_1 R_2 R_1 - R_1$ in this case. Define[7]

$$c \;=\; (0, 0, 0, 0, 0, t^3, 0, t, 0,\ |\ 0, 0, t, 0, 0, 0, 1/t, 0, 0,\ |\ 0, 1/t, 0, 1/t^3, 0, 0, 0, 0, 0)$$

[7] check poss different signs here!

(the basis here is 111,112,113,121,122,123,131,132,133,211,212,213...; the bars are just a guide to the eye). We have

$$F_{12} = c^t c$$

In particular F_{12} has rank 1. (And NB $F_{12} = F_{21}$.)

From a $U_q sl_3$-morphism perspective we interpret c as the map $V^3 \to 0$.

12.6.4 Appendix to (12.6.12)

Also Possibly (!?)

$$\Delta(E_{13})|_{t=1}\; u^t \;=\; \begin{pmatrix} 0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \left(\begin{array}{ccc} 0&0&0\\ 0&0&-t\\ 0&-t&0\\ \hline 0&0&t^{-1}\\ 0&0&0\\ -t&0&0\\ \hline 0&t^{-1}&0\\ t^{-1}&0&0\\ 0&0&0 \end{array} \right)_{t=1}$$

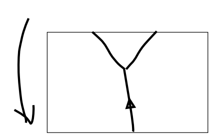
Then we find that (as required with $t = 1$)

$$u^t\, E_{13} \;=\; u^t\, \rho_{(1^2)}(E_{13}) \;=\; \left(\begin{array}{ccc} 0&0&0\\ t&0&0\\ 0&0&0\\ \hline -t&0&0\\ 0&0&0\\ 0&0&0\\ \hline 0&0&0\\ 0&0&0\\ 0&0&0 \end{array} \right)$$

12.6.5 Pictures

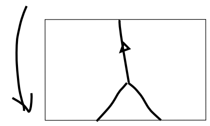
(**12.6.15**) Pictorially here we might consider

$$u \;=\; \begin{pmatrix} 0&0&0&0&0&-t&0&t^{-1}&0\\ 0&0&-t&0&0&0&t^{-1}&0&0\\ 0&-t&0&t^{-1}&0&0&0&0&0 \end{pmatrix} \;\mapsto$$



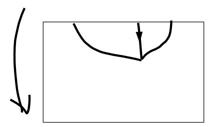
where the arrow means that the target is V' not V. Further consider:

$$u^t \;=\; \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & -t & 0 \\ \hline 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ -t & 0 & 0 \\ \hline 0 & t^{-1} & 0 \\ t^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \mapsto$$



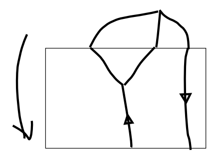
— again note the direction of the arrows; and

$$c \quad \mapsto$$



We then have, for example,

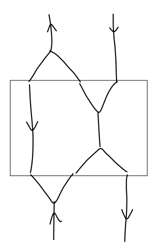
$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -t & 0 & t^{-1} & 0 \\ 0 & 0 & -t & 0 & 0 & 0 & t^{-1} & 0 & 0 \\ 0 & -t & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1_3 \right) c^t \;=\; \delta \begin{pmatrix} 1/t^2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \quad \mapsto$$



where $\delta = t^2 + t^{-2}$ (CHECK this — I'm recalling it from maxima but the fine details need a check.) We can check this 'particle-antiparticle creation' process a different way – see maxima computations...

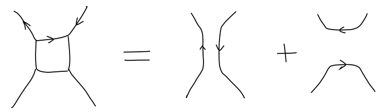
Note that we can remove a bubble on a line at cost of a factor $[2] = \delta$. We can (CHECK) remove a vacuum bubble at cost a factor [3]. What about other non-tree structures in pictures?...

(**12.6.16**) Next we want to consider $\mathcal{H} \;=\; (u^t \otimes 1_3)(1_3 \otimes \mathcal{R}_3)(u \otimes 1_3)$:

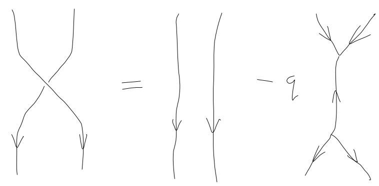


We can do this in maxima. Or... since $R_1R_2R_1 - R_1 \;=\; c^t c$ we have $(u^t \otimes 1)(R_1R_2R_1 - R_1)(u^t \otimes$

$1) \;=\; (u^t \otimes 1)c^t c(u^t \otimes 1)$ which gives $\mathcal{H} - 1 \otimes 1 \;=\; cup - cap$:



(**12.6.17**) We can also incorporate braid pictures, using (12.1), which looks like:



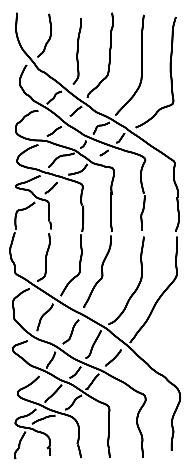
(NB wrong coefficient -CHECK!).

12.7 Aside: A little 'diagrammatic' representation theory

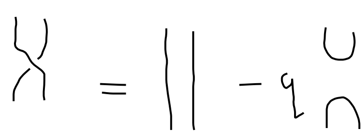
12.7.1 Action of a central element in T_n (and H_n^N)

ss:TLcentral1

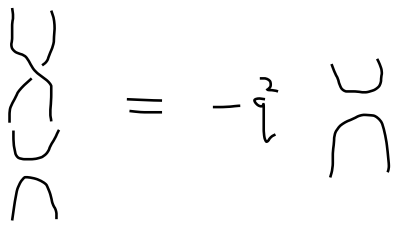
Noting that T_n (resp. H_n^N) is a quotient of the braid group B_n we may consider the action of the central double-twist braid element M^2

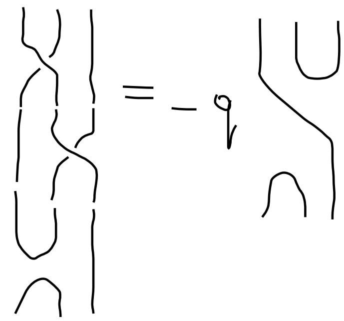


on indecomposable modules. This action can be computed using some hybrid diagrammatic rules, where crossings are understood as linear combinations of TL diagrams. First recall that the quotient map takes the braid generator g_i to $g_i \mapsto 1 - qU_i$. Informally we can generalise our diagrams for TL elements to incorporate this as:



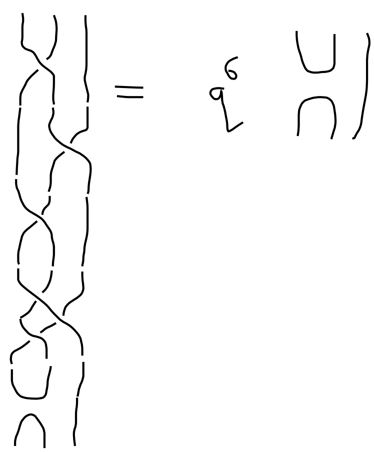
This gives us actions of braids on TL diagrams (and half-diagrams). In particular we have 'move 1' and 'move 2':

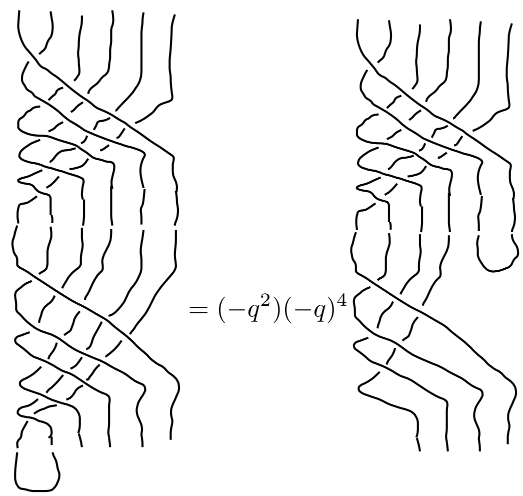




Note that the braids *look* like partition diagrams, but we *cannot* consider these as partition diagrams any more!

Applying the moves we get, for example,





We can think of the computation for the action of M^2 as passing the 'U' from the bottom-left through the various braids, first using move-1 ($-q^2$); then move-2 $(n-2)$ times $((-q)^{(n-2)})$; then a 'right-to-left over' version of move-2 $(n-2)$ times $((-q)^{(n-2)})$; then move-1 again. This gives a factor q^{2n} altogether; and what is left to act is M^2_{n-2} — the double-twist from B_{n-2} — on the remaining part of the basis element. (Thus if there is another 'U' then we will get a factor $q^{2(n-2)}$, and so on.)

In this way we can easily compute the action of M^2 on a basis element for any one of our modules from Fig.2.1. Besides the moves, the other feature is that because of the quotient by which the modules are defined, a braid acts like 1 on parallel lines in a module basis element.

The results are given in Fig.12.2. For $b \in D^{\mathrm{TL}}_n(l)$ we have:

$$M^2 b = q^{(n-l)(n+l+2)/2} b$$

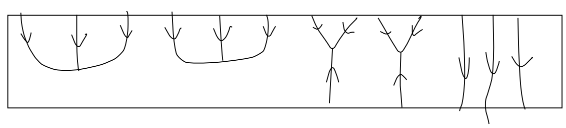
$n \setminus l$	0	1	2	3	4	5	6
0	1						
1		1					
2	q^4		1				
3		q^6		1			
4	q^{12}		q^8		1		
5		q^{16}		q^{10}		1	
6	q^{24}		q^{20}		q^{12}		1

Figure 12.2: Scalars by which M^2 acts on indecomposable modules $\Delta_n^{TL}(l)$. fig:Mabt001

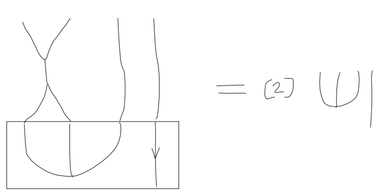
Note in particular that the actions are all by powers of q, and that for given n they are all by different powers of q. By (**??**) this tells us that no two of these modules are in the same block unless q is a root of unity.

12.7.2 Aside: action of a central element in case $N = 3$

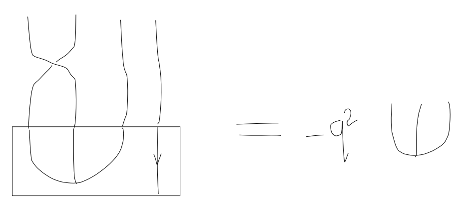
The analogues of half-diagram modules in $N = 3$ are modules generated by a suitable 'fixed-charge' sector of the tensor space action on elements like



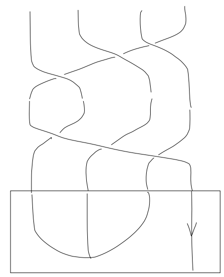
The full basis is difficult to draw in general (see Fig.**??**). But for the action of central elements M^2 we need only look at the eigenvalue. For example, noting



so that

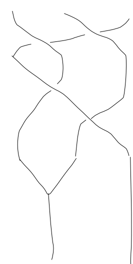


we can absorb all the (six) other $g_i \rightsquigarrow -q^2$ factors from the twist $M^2 \in H_4$ and consider



Here we are looking for the analogue of the factor in move-2 (in §12.7.1).

In order to compute this it is convenient (??) to compute the following



12.8 Tableau Representations of Hecke algebras

s:orthogonalform

From definition (12.3.1) we see that the Hecke algebra is a deformation of the symmetric group. Many representations can be obtained by deforming symmetric group representations. A good (i.e. interesting) starting point is outer-product representations (representations induced from representations of Young subgroups — which one might assume to be known by induction on n and elementary facts on direct products).

(**12.8.1**) For $d \in \mathbb{N}$, denote by Γ_n^d the set of all d-tuples $\lambda = (\lambda^1, \lambda^2, ..., \lambda^d)$ of Young diagrams ('multipartitions') with $|\lambda| = \sum_{i=1}^d |\lambda^i| = n$. (See also §11.6.1.)

Fix $\lambda \in \Gamma_n^d$. A *tableau of shape* λ is any arrangement of symbols $1, 2, .., n$ in the n boxes of λ.

A tableau is *standard* if each component tableau λ^i is standard. Denote the set of all standard tableaux of shape λ by T^λ.

(**12.8.2**) Number the rows of multipartition λ by placing the whole of the component diagram λ^{i+1} under λ^i for all i, and numbering the rows from top to bottom. Then define a total order $<$ on standard tableaux of shape λ by setting $T < U$ if the highest number which appears in different rows of T and U is in an earlier row in U.

(**12.8.3**) Let T be a tableau. For $i \in \{1, 2, \ldots, n-1\}$, let $\sigma_i = (i\ i+1) \in \Sigma_n$, the symmetric group of degree n. We define $\sigma_i(T)$ to be the tableau obtained by interchanging i and $i+1$. In this way we get an action of Σ_n on the set of all tableaux of shape λ. Note that this action does not necessarily take a standard tableau to a standard tableau.

(**12.8.4**) Let $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. For $i, j \in \{1, 2, \ldots, n\}$ and $T \in \Gamma_n^d$ let $h_{ij}^x = h_{ij}^x(T)$ denote the *generalised hook length* between the symbols i and j in T. Thus h^x_{ij} is given by:

$$h^x_{ij} = h^0_{ij} + x_{\#i} - x_{\#j},$$

where h^0_{ij} is the usual hook length obtained by superimposing the component tableaux of T containing i and j, and $\#i$ is the number of the component containing i in T. See [123] (note that there is a typographical error in this paper at the relevant point) and also [104, p.244].

(**12.8.5**) Recall the S_n action on certain tableau in (11.8). A trivial re-write of this is:

$$\sigma_i \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{h} \begin{pmatrix} -1-h & -(h-1) \\ -(h+1) & 1-h \end{pmatrix} \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix} \qquad (12.15)$$

eq:YBx2

where $h = x_{p(i)} - x_{p(i+1)}$. The following generalisation can be seen as 'quantising' this in the 'right' way to get a Hecke representation.

prop:outerproduct pa:10.6

(**12.8.6**) PROPOSITION. [123] *Let $\lambda \in \Gamma^d_n$. Then the set T^λ is a basis for a left H_n–module R^λ. For $i \in \{1, 2, \ldots, n-1\}$ and $T \in T^\lambda$, the action is as follows:*
(a) If $i, i+1$ lie in the same row of T then $g_i T = T$.
(b) If $i, i+1$ lie in the same column of T then $g_i T = -q^2 T$.
(c) If neither (a) nor (b) hold, then $\sigma_i(T)$ is also standard. Let $h = h^x_{i,i+1}$. Then the action is given by

$$g_i \begin{pmatrix} T^\lambda_p \\ \sigma_i(T^\lambda_p) \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{q}{[h]} \begin{pmatrix} [h+1] & [h-1] \\ [h+1] & [h-1] \end{pmatrix} \right) \begin{pmatrix} T^\lambda_p \\ \sigma_i(T^\lambda_p) \end{pmatrix},$$

provided $T < \sigma_i(T)$.

REMARK. This is a (multiple) deformation of the classical symmetric group case motivated partly by the role of the Hecke algebra in computational statistical mechanics; partly by the challenge of interpolating between 'tensor space' representations, Young representations and Specht modules; and also having connections with the representation theory of affine Hecke algebras.
(Young's orthogonal form (see e.g. [13, IV.6]) involves an action via symmetric matrices related to those above via conjugation).

We remark that the action is only well-defined provided $[h]$ never vanishes.

We shall illustrate this construction using a simple case, which will turn out to factor through the Temperley–Lieb algebras.

12.9 Temperley–Lieb algebras from Hecke algebras

pa:10.7

(**12.9.1**) We now restrict attention, in Prop. 12.8.6, to the case in which λ has exactly two components, each consisting of exactly one row. We can represent $T \in T^\lambda$ by an n-tuple $a = (a_1, a_2, \ldots, a_n)$ with entries in $\{1, 2\}$, defined by the condition that $i \in \lambda^{a_i}$ for all $i \in \{1, 2, \ldots, n\}$. Such a tuple can be regarded as a walk of length n in $\mathbb{Z}^2$ starting at the origin. The ith step of the walk consists of adding the vector $(1, 1)$ if $a_i = 1$ or adding the vector $(1, -1)$ if $a_i = 2$.

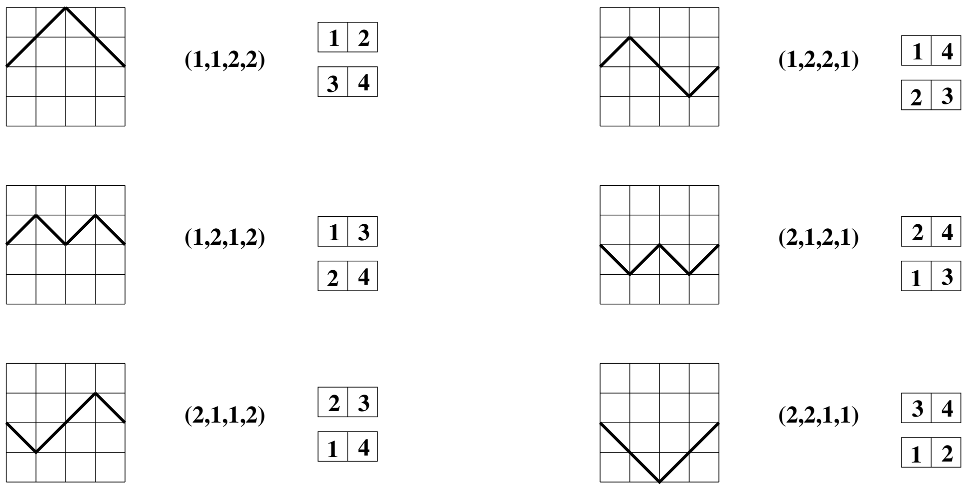


Figure 12.3: Standard tableaux of shape $((2),(2))$ and the corresponding walks. fig:examplewalks

(**12.9.2**) Example. For example, if $n = 4$ and each component of λ is a row of length 2, the elements of T^λ and the corresponding tuples and walks are as shown in Figure 12.3.

(**12.9.3**) We note that in the walk realisation of a standard tableau T, σ_i swaps a pair of steps $(1,2)$ with the pair $(2,1)$, i.e. a local maximum is swapped with a local minimum or vice versa. Thus, in order for there to be mixing between two basis elements as in Proposition 12.8.6(c), the corresponding walks must agree in all but their ith and $(i+1)$st steps, and in each diagram separately the second coordinate (or *height*) after $i-1$ steps and after $i+1$ steps must coincide. In fact, it is not hard to show that the height coincides with the usual hook length $h^0_{i,i+1}$. If $T < \sigma_i(T)$ then $h^x_{i,i+1}$ is equal to the sum of $x_1 - x_2$ and the height of the walk after $i-1$ steps.

(**12.9.4**) If this value is 1, it follows from the description of the action in case (c) that the elements are not actually mixed. It follows that, if we choose x so that $x_1 - x_2 = 1$, there is an action of H_n on the set of (standard tableaux corresponding to) walks which do not go below the horizontal axis given by the formulas in Proposition 12.8.6. In fact, in this case, the action cannot be extended to the whole of T^λ since the action is not defined for hook length zero.

(**12.9.5**) Similarly, if we set $x_1 - x_2 = 2$, only the walk $(2,2,1,1)$ is decoupled from the rest. In other words, changing the value of x allows us to define a module for H_n with basis elements corresponding to walks which do not go below a certain "exclusion" line.

pa:10.12 (**12.9.6**) The walks we have been considering can be regarded as walks on $\mathbb{Z}$ (with edges joining integers with difference 1), by projecting onto the first coordinate. Thus, in summary, we have extracted an H_n-module with a basis of walks on $\mathbb{Z}$ which only visit vertices on a certain subgraph, from the formal closure of a Zariski-open set of modules (that is, actions depending on a parameter) whose bases consist of walks on a larger subgraph. The decoupling of the subgraph, in this sense, is determined by the structure of the graph.

pa:10.13 (**12.9.7**) The case $x_1 - x_2 = 1$ is special in that the decoupled module is irreducible for generic values of q. It is an analogue of the boundary of the dominant region in the Weyl group construction in Lie theory. The most interesting step, however, is the next one. We now fix $x_1 - x_2 = 1$, and also specialise q to be an lth root of unity, so that $[l] = 0$. In this situation, there is a further decoupling: we obtain a module whose basis corresponds to walks whose height is bounded above by $l-1$. In other words, we now only include walks that lie between two 'walls': the lines given by setting the second coordinate to 0 and $l-1$. It can be shown that this module is simple in this specialisation. Such simple modules are otherwise very hard to extract, but here their combinatorics is manifested relatively simply.

12.9.1 Presentation of Temperley–Lieb algebras as Hecke quotients

ss:TLpresent

Fix a commutative ring k and $q \in k$. We defined $T_n = T_n(q)$ as the quotient of $H_n(q)$ by

$$U_i U_{i\pm 1} U_i = U_i.$$

12.9.2 Tensor space representations

ss:TLtensors

Cf. §12.4.

12.10 Diagram categories

ss:TLcat1

Confer also monoidal categories for example in §6.4.1; and partition categories in §1.3 et seq.

Humanistically, a 'diagram' is a collection of open and (possibly filled) closed lines on the page (i.e. a subset of $\mathbb{R}^2$ with some topological structure), representing some objects and relationships between them. So far this setup is very vague, but in as much as diagrams do convey information, it can necessarily be made precise (see §**??** and §12.11 for examples). We then proceed with the simple observation that it may be possible to separate such a diagram into two subdiagrams without losing data. The reverse of this separation might then amount to a way of composing suitable diagrams. This hopefully sounds categorical as well as diagrammatic.

The TL algebra shows up in various such constructions, and the diagram setting is in turn a useful tool in studying the TL algebra, as we shall see.

For one example, this collection of lines (in a diagram) may well represent a 3D geometrical shape, so it may be, in part, an embedding of lines in 3D onto 2D. A simple version of this is the embedding of knots as braids. For this reason, one source of categories with a form of diagram calculus is braided tensor categories (with duals). A major source of such categories is the representation theory of quasi-triangular bialgebras, such as quantum groups. We say a bit about these in §12.10.1.

12.10.1 Relation to quantum groups

ss:qgrpx

See for example Joseph [82], Kassel [85], [23], [135].

Fix a field R, and recall the definition of R-algebra from Section 9.1. Recall that an R-bialgebra $(A, \nabla, \nu, \Delta, \epsilon)$ is an R-algebra (A, ∇, ν) with coassociative comultiplication Δ and counit $\epsilon : A \to R$

(that are morphisms of algebras). A bialgebra with

$$\Delta(a) = \sum_i l(a)_i \otimes r(a)_i$$

is cocommutative if the opposite coproduct

$$\Delta^o(a) = \sum_i r(a)_i \otimes l(a)_i$$

coincides with the coproduct. For example a group algebra is cocommutative, since $\Delta(a) = a \otimes a$.

(**12.10.1**) A bialgebra is *quasicocommutative* if there is an element $\mathcal{R}$ of $A \otimes A$ such that

$$\mathcal{R}\Delta^o(a) = \Delta(a)\mathcal{R}$$

This is like the intertwiner of Kronecker product matrices, except that it works at the algebra level. On this basis such an $\mathcal{R}$ is sometimes called a *universal $\mathcal{R}$-matrix*.

(**12.10.2**) A *quasitriangular bialgebra* (or braided bialgebra) is a quasicocommutative bialgebra such that, writing $\mathcal{R} = \sum_i L_i \otimes R_i$, we have

$$\sum_i \sum_j l(L_i)_j \otimes r(L_i)_j \otimes R_i = (\sum_m L_m \otimes 1 \otimes R_m)(\sum_n 1 \otimes L_n \otimes R_n) = \mathcal{R}_{13}\mathcal{R}_{23}$$

and similarly 'reversed': $(Id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$.

This formulation looks clunky, but we have

(**12.10.3**) THEOREM. *If A is a quasitriangular bialgebra then*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

Given $\mathcal{R}$ and any two A-modules M, N we have

$$\begin{aligned} \mathcal{R}_{MN} : M \otimes N &\to N \otimes M & (12.16) \\ m \otimes n &\mapsto \sum_i r_i n \otimes l_i m & (12.17) \end{aligned}$$

eq:intertwine12

This map can be shown to be an isomorphism of A-modules. It makes the category $A - \text{mod}$ into a braided tensor category (as defined in Section 6.4.1).

12.11 Temperley–Lieb diagram algebras

ss:TLdiagram

Recall that a *concrete* (n, m)-*Brauer diagram* is a pictorial representation of a pair partition of a set of $n + m$ objects. The objects are drawn as marked points on a rectangular frame (n of them usually drawn on one edge; and m on the opposing edge); and the representation is drawn in the interior — usually a collection of connecting lines and possibly other objects.

A few moments thought will yield some ground rules as to what kind of picture leads to an unambiguous representation. For example distinct connecting lines may have to cross, so it must be possible to resolve such crossings.

(**12.11.1**) Two concrete diagrams are in the same class if they represent the same partition. An (n,m)-*Brauer diagram* is a class of concrete diagrams, or any representative thereof.

(**12.11.2**) An (n,m)-*Temperley–Lieb diagram* is an (n,m)-Brauer diagram without crossing lines, or a class of such diagrams with respect to some specified equivalence relation. (Note that this is not necessarily the same relation as before, since there are representatives of all classes of concrete diagrams except the empty diagrams that have crossings. However this need not concern us here, until we come to consider certain *generalisations* of the TL algebra.)

12.11.1 TL diagram notations and definitions

ss:TLdnotat

Let $T^o(n,m)$ be the set of (n,m)-Temperley–Lieb diagrams, up to boundary order preserving isotopy. Let $T^o(n,l,m)$ be the subset of Temperley–Lieb diagrams with l propagating lines.

For example there is a single element $1_1 \in T^o(1,1)$, $u = \boxed{\cup} \in T^o(2,0)$, $u^* = \boxed{\cap} \in T^o(0,2)$, and $U \in T^o(2,0,2)$. There is also a single element $1_0 \in T^o(0,0)$.

For d a *concrete* diagram we may write $[d]$ for the diagram, i.e. the diagram up to isotopy, i.e. the isotopy class of concrete diagrams.

(**12.11.3**) By convention we draw TL diagrams with their vertices on the N and S edges of the frame. We then define

$$\boxtimes : T^o(n,m) \times T^o(n',m') \to T^o(n+n', m+m')$$

by side-by-side concatenation. Note that this is well-defined and associative.

We define $u_1 \in T^o(n,n)$ by
$$u_1 = U \boxtimes 1_1 \boxtimes 1_1 \boxtimes ... \boxtimes 1_1.$$

(**12.11.4**) For $[d] \in T^o(l,m)$ and $[d'] \in T^o(m,n)$ we may pick representatives (e.g. d and d' respectively) and then form a new concrete 'diagram' by *vertical* concatenation. We denote this stack of concrete diagrams by $d|d'$. Note that there are representatives such that the m vertices match up in *vertical* concatenation. The combined 'diagram' then *determines* a (representative of a) TL diagram
$$c(d,d') = [d|d']_o$$
in $T^o(l,n)$. Here $d|d'$ may contain closed loops and $[]_o$ means to omit these (and then pass to the class).

pr:TL defined

(**12.11.5**) PROPOSITION. *The TL diagram $[d|d']_0$ is independent (up to isotopy) of any further details of the choice of representatives of d and d' (so the notation $c([d],[d'])$ is well-defined). The combined diagram* may not be a TL diagram per se*, since closed loops may appear, but the* number *$n(d,d')$ of closed loops appearing is also independent of the choice of representatives.*

(**12.11.6**) Now fix a commutative ring R and $\delta \in R$. For $n \in \mathbb{N}_0$ let $\mathtt{T}_n = \mathtt{T}_n(\delta)$ be the TL diagram algebra over R with basis $T^o(n,n)$ and multiplication given by

$$dd' \;=\; \delta^{n(d,d')}\; c(d,d')$$

(**12.11.7**) PROPOSITION. *If R is a field then $\mathtt{T}_n$ is a finite-dimensional unital associative algebra.*
$\square$

le:propag (**12.11.8**) Lemma. *If $\#(d)$ denotes propagating number then $\#(d) = \#([d])$ and*

$$\#(dd') \leq \#(d)$$

12.11.2 Isomorphism with Temperley–Lieb algebras

Comparing with the presentation in §12.9.1 we see that

pr:TLsurj (**12.11.9**) Proposition. *The map*

$$\Psi : U_i \mapsto u_i$$

defines an algebra homomorphism $\Psi : T_n \to \mathtt{T}_n$ (where T_n is as defined in §12.9.1) ... provided that

$$\delta = [2]_q = q + \frac{1}{q}$$

Furthermore this homomorphism is surjective.

Proof. To check that Ψ is an algebra homomorphism one checks by drawing pictures that

$$\Psi(U_1)\Psi(U_2)\Psi(U_1) = \Psi(U_1U_2U_1) = \Psi(U_1)$$

and so on. The map is surjective if a set of generators of $\mathtt{T}_n$ lie in the image. One can show that the diagrams u_i generate $\mathtt{T}_n$ by showing that any diagram contains in its class a diagram that looks like a stack of these diagrams. □

One can check that the two algebras have the same dimension (see §12.11.3), and hence

th:TLT (**12.11.10**) Theorem. *The map $\Psi : T_n \to \mathtt{T}_n$ is an isomorphism (see e.g. [104, Cor.10.1]).*

Henceforth we will usually write T_n for an algebra in this class, and use diagrams and words, and δ and q parameterisations, interchangeably.

12.11.3 TL diagram counting

ss:TLdc

Here we write down and enumerate the TL diagrams in a way which will be helpful in representation theory. In particular we construct an augmented truncated Pascal triangle which encodes every diagram. The construction is conveniently summarized by its low-rank part — Fig.12.4.

It will be clear that every element in every $T^o(n,l,l)$ arises in this construction, and that every element in every $T^o(n,m)$ is constructed by combining elements of this form. In particular

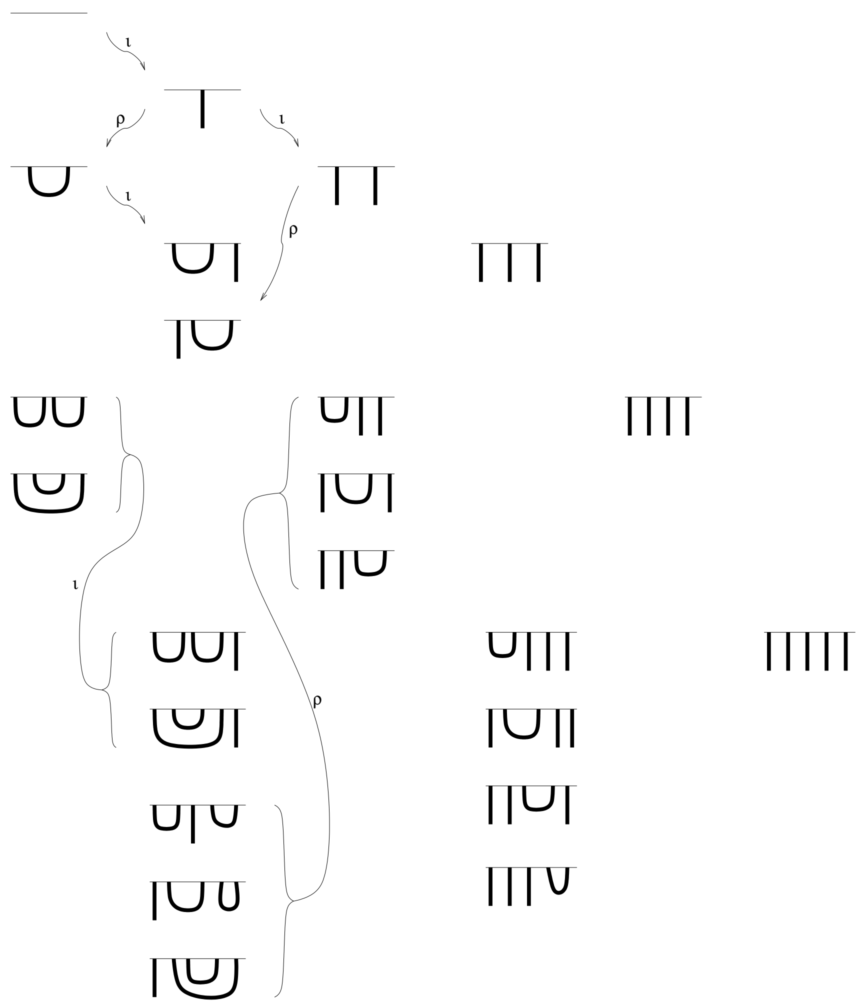
th:TLcata (**12.11.11**) Theorem. *The orders of the sets $T^o(n,n)$ are the Catalan numbers.* ...

12.11.4 Back to the TL isomorphism theorem

Here we follow [104]. By Prop.12.11.9 and Th.12.11.11 we know a lower bound on the dimension of T_n. We will saturate this bound by studying the word problem. In particular we will construct a spanning set of words of Catalan order, as in [104, §6.4 Def.25].

For given l consider the ideal generated by $U_1U_3...U_{2l-1}$.

...



p:dpasc1

Figure 12.4: Elaborated truncated Pascal's triangle for diagrams. Here we have only drawn the northern edge of the frame rectangle for each diagram. Later we will associate a specific representation theoretic meaning to this modification.

Chapter 13

On representation of Temperley–Lieb algebras

Ch:TLrep

We continue with the notations of Chapter 12. In Chapter 12 we constructed many representations of TL algebras over various ground rings. Here our first aim is 'reductive' representation theory — i.e. the simple and projective modules, and the decomposition matrices, for these algebras over algebraically closed fields. (NB We also addressed the cases in characteristic 0 in Chapter 2.) The approach here is closely related to that used for the partition and Brauer algebras in Chapters 16,18.

Later we consider closely related topics such as Ringel duality (§13.6) and fusion (§13.6.1).

Broadly speaking the strategy for computing decomposition matrices over a given field is to consider the TL algebras as sitting in a modular tower (a tower of modular systems) in the sense of **??** and [57, §6]. In this case the integral ground ring is $\mathbb{Z}[\delta]$. The case over the field of fractions of $\mathbb{Q}[\delta]$ is semisimple, so we have lifts of the 'generic' simple modules to the integral case, and reductions of these to the case of interest. A module of this type that is in addition a quotient of an indecomposable projective module is called a Δ-module (and there are dual/analogous ∇-modules). Here we have various constructions of such modules, each with desirable properties — and furthermore we can show that they are isomorphic to each other — see §13.2.

One type of construction for Δ-modules is by 'globalisation' of modules from lower n in the tower — see §13.4. There is more than one globalisation route, and it is an interesting question when these give isomorphic modules.

Induction/restriction. Alcove geometry. ...

13.1 Representations of Temperley–Lieb diagram algebras

ss:TLrep

For R, δ, n given, we may here write A for the TL algebra $\mathtt{T}_n \cong T_n$ as in (12.11.10).

13.1.1 Tower approach: Preparation of small examples

(This subsec is a placeholder for now.)

13.2 Temperley–Lieb Δ-modules

ss:TLD

Remark: We work over various ground rings here. We generally end up working over a field k with $\delta \in k$ — the ‘artinian’ setting. In this setting the case $\delta = 0$ tends to be slightly different from the rest. This difference is not of great intrinsic interest here. However we will often comment on it, as a comparative device illuminating the general methodology.

13.2.1 ‘Diagram’ representations

(**13.2.1**) Note that both algebras T_0 and T_1 are isomorphic to the ground ring R.

(**13.2.2**) Suppose $n > 1$ and consider the left ideal T_nU_1. This ideal has a basis of diagrams (in bijection, for example, with the diagram basis of T_{n-1}) over any ring.

$$T_nU_1 \;=\; R\boxed{\cap}\boxtimes T(n,n-2) \;=\; R\boxed{\cap}\boxtimes\left(\bigcup_{l=n-2}^{0} T(n,l,n-2)\right) \tag{13.1}$$

eq:idealTL

Diagrams in the ideal T_nU_1 have at most $n-2$ propagating lines. There are also diagrams with fewer propagating lines (provided that $n > 3$), and these span a submodule. Thus there is a module in A-mod which is the quotient of AU_1 by this submodule. This quotient, denoted $\Delta_{n-2}(n)$, has basis the set of diagrams in AU_1 with precisely $n-2$ propagating lines. The action of the algebra on this basis is rather like the action on these diagrams if they were regarded as elements of A, except that if the nominal outcome is a diagram with fewer propagating lines then the action gives zero. We may similarly construct a module $\Delta_{n-4}(n)$ from AU_1U_3, and so on.

For $0 \leq 2m \leq n$ we have defined

$$\Delta_{n-2m}(n) \;=\; T_nU_1U_3...U_{2m-1}/\; T_nT^o(n,n-2m-2,n)$$

(by convention M/N means $M/M\cap N$) cf. §12.11.1.

pr:Deltabasis

(**13.2.3**) Proposition. *Module $\Delta_{n-2m}(n)$ has a basis of the set of diagrams in $T_nU_1U_3...U_{2m-1} = R\boxed{\cap\cap...\cap}\boxtimes T(n,n-2m)$ with $n-2m$ propagating lines (with a diagram understood as a class in the quotient). That is, the basis is $\boxed{\cap\cap...\cap}\boxtimes T(n,n-2m,n-2m)$. The action of a diagram on this basis is as* in *the algebra except if the nominal outcome has fewer propagating lines then it is zero.*

(**13.2.4**) Note that the $\boxed{\cap\cap...\cap}\boxtimes -$ does not necessarily play a great role. We may sometimes omit it in the following. In which case the basis is $T(n,n-2m,n-2m)$.

(**13.2.5**) Example. $\Delta_1(5) = R\{\boxed{\cup\cup|}, \boxed{\cup|\cup}, \boxed{|\cup\cup}, \boxed{\bigcup|}, \boxed{|\bigcup}\}$ ...

13.2.2 Bimodules and functors

a:TLinflatebasis

(**13.2.6**) Note that T_nU_1 becomes a left T_n right T_{n-2} bimodule, by allowing T_{n-2} to act on the last $n-2$ strings on the right. (In our convention the schematic for this is as in figure 13.1, where the legs on which T_{n-2} is to act by diagram composition are indicated.) This means that we

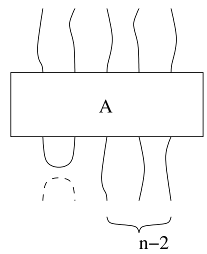


Figure 13.1: Schematic for $T_n U_1$. fig:TL bimod act

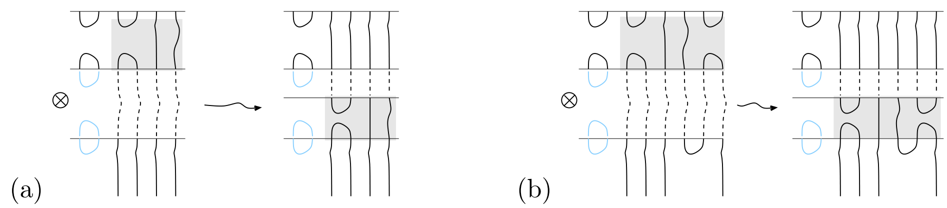


fig:kill123 Figure 13.2: (a) Illustration of element of $T_6 U_1 \otimes_{T_4} \Delta_4(4)$. Dashed lines indicate parts than can pass across by the tensor product. Hence this element is zero (since $U_1\Delta_4(4) = 0$). (b) Similar for $T_8 U_1 \otimes_{T_6} \Delta_4(6)$ (here $U_1\Delta_4(6) \neq 0$ but $x\Delta_4(6) = 0$ if $x \in T_6$ has less than 4 propagating lines, for example $U_1 U_5 \Delta_4(6) = 0$).

can construct a module $AU_1 \otimes_{T_{n-2}} M$ for any M in T_{n-2}-mod. In particular we can construct $AU_1 \otimes_{T_{n-2}} \Delta_{n-2m}(n-2)$ $(m = 1, 2, ...)$ and study it.

This module has elements of form $\sum_i a_i \otimes b_i$ (in the sense of **??**), and is spanned by the subset of these strictly of form $a \otimes b$ where a is taken from the diagram basis for AU_1 and b is taken from the nominal diagram basis for (say) $\Delta_{n-4}(n-2)$. See Fig.13.2 (in case $\Delta_4(4)$); which also shows that this spanning set is not independent.

In case $\delta \neq 0$ a similar approach is to note that $U_1 T_n U_1 \cong T_{n-2}$ is an isomorphism of algebras, and that $T_n U_1$ is thus also a right T_{n-2}-module via this isomorphism. We shall return to this later.

preparH (**13.2.7**) Proposition. *Let the ground ring R be a field k, with $\delta \in k$. For $n > 4$, and for $n = 4$*

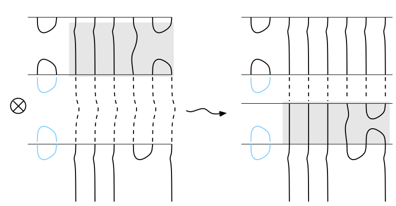


fig:kill1234 Figure 13.3: Illustration of *non-vanishing* element of $T_8 U_1 \otimes_{T_6} \Delta_4(6)$. Consider the action of T_8 on this, cf. $\Delta_4(8)$.

and $\delta \neq 0$,

$$T_n U_1 \otimes_{T_{n-2}} \Delta_{n-4}(n-2) \;\cong\; \Delta_{n-4}(n)$$

Proof. We compare the given spanning set on the left (from 13.2.6 above) with the basis on the right (from 13.2.3 above). First we need a Lemma.

(**13.2.8**) LEMMA. *Suppose a to be in the subset of AU_1 diagrams with less than $n-4$ propagating lines. Then a can be constructed as cd where c is another element of the basis of AU_1 and d is an element of the T_{n-2} which acts on AU_1, and which has less than $n-4$ propagating lines.* ■

We can thus see that in this case $a \otimes b$ can be written in the form $cd \otimes b \sim c \otimes db$. Thus $a \otimes b \sim 0$ and these cases can be excluded from the spanning set.

If we consider the subset of AU_1 diagrams with $n-2$ propagating lines we can see that $a \otimes b$ can be written in the form $a \otimes fg \sim af \otimes g$, where g is another element of the basis of Δ and f is an element of T_{n-2} which has $n-4$ propagating lines. (Caveat: this is perhaps not obvious; and may assume that $\delta \neq 0$.) Thus af has $n-4$ propagating lines. Thus the module is spanned by objects $a \otimes b$ in which a has $n-4$ propagating lines and b is a given element. Without loss of generality this element can be chosen to have the last $n-4$ lines propagating (in the nominal sense). Then again $a \otimes b \sim 0$ unless the last $n-4$ lines (on the side of the right action) in a are propagating. We can now construct a surjective homomorphism from this module onto $\Delta_{n-4}(n)$ by simply concatenating $a \otimes b \mapsto ab$, where ab is to be understood in $\Delta_{n-4}(n)$. But the degree of what remains of the basis of AU_1 from our reduction to a smaller spanning set coincides with the rank of $\Delta_{n-4}(n)$, so this is an isomorphism. □

(**13.2.9**) Exercise: What goes wrong when $\delta = 0$ and $n = 4$? The rank of $\Delta_2(4)$ is 2. Meanwhile $\Delta_0(2)$ has a basis consisting of a single diagram; and $T_4 U_1$ has a basis of 5 diagrams. In the construction we thus have a spanning set of 5 elements — but 2 of these are manifestly 0 when $\delta = 0$. What happens to the others? (Cf. The Brauer algebra case in (18.4.4) *et seq.*)

(**13.2.10**) An interesting exercise is to consider the contravariant dual modules (with respect to the upside-down diagram antiautomorphism). And ask what happens to them under $T_n U_1 \otimes -$.

(**13.2.11**) Another interesting exercise is as follows. Note that the sequence of left T_n-modules

$$T_n U_1 \;=\; R\boxed{\cap}\boxtimes T(n,n{-}2) \;=\; R\boxed{\cap}\boxtimes\left(\bigcup_{l=n-2}^{0} T(n,l,n-2)\right) \supset\; R\boxed{\cap}\boxtimes\left(\bigcup_{l=n-4}^{0} T(n,l,n-2)\right) \supset \ldots$$

are also right T_{n-2}-modules (and bimodules). Thus the sections, denoted $\mathfrak{T}^m(n,n-2)$ (and having bases $T(n,m,n-2)$), are also bimodules. For example $\mathfrak{T}^{n-2}(n,n-2)$ is a direct sum of copies of the trivial module as a right module. Thus in particular we may define a functor $\mathfrak{T}^{n-2}(n,n-2)\otimes -$. What happends to Δ-modules under this functor?

13.3 Δ-filtration of projective modules

ss:TLDfilt

(This sec is a placeholder for now.)

13.4 Idempotent subalgebras, F and G functors

Proposition 13.2.7 illustrates that the functor $T_n U_1 \otimes_{T_{n-2}} -$ gives a natural connection of certain modules of T_n with the modules of T_{n-2}. We can use this to determine the representation theory of T_n largely from lower rank cases. We examine aspects of this strategy next. In case U_1 can be replaced by an idempotent (as for example if $\delta \neq 0$), a large part of the strategy is well described in the literature, such as in Green[57], in a more general setting. In this Section we review the general machinery. We add a bit of novelty by using our example to also look at its limitations!

(**13.4.1**) Let A be an algebra over a field k, and $e \in A$ idempotent. Then Ae is a left A- right eAe-bimodule. Define functors

$$\begin{aligned} F : A - \mathrm{mod} &\rightarrow eAe - \mathrm{mod} &(13.2)\\ M &\mapsto eM &(13.3)\end{aligned}$$

F functor

$$\begin{aligned} G : eAe - \mathrm{mod} &\rightarrow A - \mathrm{mod} &(13.4)\\ N &\mapsto Ae \otimes_{eAe} N &(13.5)\end{aligned}$$

G functor

F exact

Proposition 13.1. *The functor F is an exact functor.*
(I.e. it takes a short exact sequence to a short exact sequence.)

G right inverse

Proposition 13.2. *The functor G is a right inverse to F.*
(I.e. $F(G(N)) \cong N$.)

13.4.1 Aside on non-exactness of functor G

ss:TLG

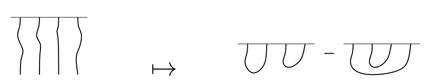
Note that G is right exact by Proposition 8.5.20. An example illustrating its failure to be left exact in general is as follows.

(**13.4.2**) Example. We work with $A = T_6(1)$, the Temperley–Lieb algebra (over $\mathbb{C}$) with $\delta = 1$, and $e = U_1$. We use the notation $T_{n_1} \boxtimes T_{n_2} \hookrightarrow T_{n_1+n_2}$ for the usual 'parabolic' subalgebra. We note that $e \in T_2$ and that $T_4 \to e \boxtimes T_4 \subset T_6$ defines an isomorphism of T_4 and eAe.

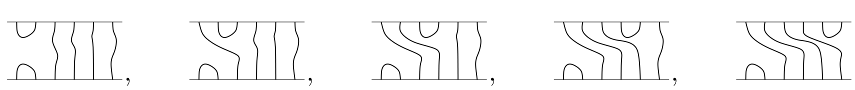
First note that in the $n = 4$ Temperley–Lieb algebra with $\delta = 1$ there is a sequence of standard modules

$$0 \rightarrow \Delta_4(4) \xrightarrow{\psi} \Delta_0(4)$$

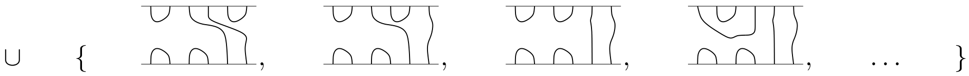
exact at $\Delta_4(4) = T_4/T_4U_1T_4$, given in diagram shorthand by

$\psi :$ 

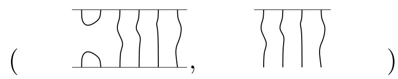
(note that the action of the algebra on the diagram on the left involves a quotient of the ordinary algebra action on diagrams). However we can show that the image under G is not an injection. The image of $\Delta_4(4)$ in $eAe-\mathrm{mod}$ can be represented as $\Delta_4(4)$ itself, noting that e acts like 1. The bimodule Ae includes elements such as

$\{$  $\}$ (13.6)

D4-6

 (13.7) D4-6x

(and many with fewer than four propagating lines). The image $G(\Delta_4(4)) = Ae \otimes_{eAe} \Delta_4(4)$ thus contains elements like

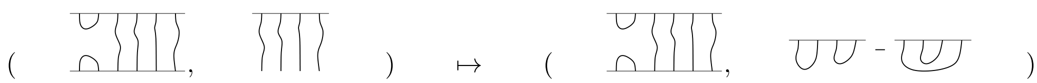


In particular the set of such objects built using the elements from the list in (13.6) on the left-hand side is spanning, since elements of Ae with fewer propagating lines kill $\Delta_4(4)$ (see Fig.13.2). Indeed this set in (13.6) is a basis, since $G(\Delta_4(4)) = \Delta_4(6)$.

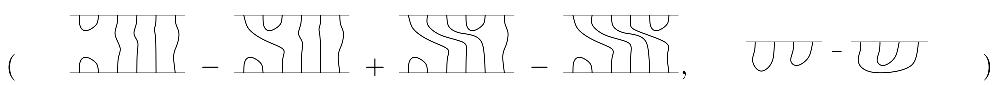
The image $G(\Delta_0(4)) = Ae \otimes_{eAe} \Delta_0(4)$ contains elements like

and is mapped isomorphically to $\Delta_0(6)$ by the multiplication map.

Finally we have

$G(\psi)$: 

and so on. The image on the right is to be understood to lie in $G(\Delta_0(4)) = \Delta_0(6)$ (NB, the expression on the right would mean something else as the image of the submodule) so again there is a multiplication map (as in $a \otimes d \mapsto ad$), which is an isomorphism. It is easy to check that

 $\mapsto 0$

under this isomorphism, thus the preimage indicated by the sum on the left-hand side of this construct lies in the kernel of $G(\psi)$.

13.4.2 More general properties of F and G

ss:green here

We continue here with the assumptions of §13.4. See also Green[57, §6.2].

(**13.4.3**) Lemma. *(Suppose G left-adjoint to F.) Let $N \in eAe-mod$. Then $End_{eAe}(N) \cong End_A(G(N))$.*

Proof. By left-adjointness

$$\mathrm{Hom}_A(G(N), M) \cong \mathrm{Hom}_{eAe}(N, F(M))$$

Take $M = G(N)$:

$$\mathrm{Hom}_A(G(N), G(N)) \cong \mathrm{Hom}_{eAe}(N, F(G(N)))$$

but $F(G(N)) \cong N$. □

On the other hand, a finite dimensional module is indecomposable if and only if it has a local endomorphism ring. Thus N is indecomposable if and only if $G(N)$ is indecomposable.

proper submodule (**13.4.4**) LEMMA. *If $N \in eAe-mod$ simple and M a proper submodule of $G(N)$, then $F(M) = 0$.*

Proof. Else $F(M) \cong N$ by simplicity, and $M \supseteq AeM = A(e \otimes N) = G(N)$, a contradiction. □

(**13.4.5**) For $M \in A-\mathrm{mod}$ let M_e be the largest submodule of M contained in $(1-e)M$. Thus $F(M_e) = 0$.

GL indecomp1 **Proposition 13.3.** *If $N \in eAe-mod$ simple then $G(N)$ has simple head. The unique maximal proper submodule is $G(N)_e$.*
(I.e. if L is a simple factor below the head it is also a simple A/AeA-module.)

Proof. We have $F(G(N)_e) = 0$, so by proposition 13.1 the exactness of

$$0 \to G(N)_e \to G(N) \to G(N)/G(N)_e \to 0$$

implies exactness of

$$0 \to 0 \to F(G(N)) \to F(G(N)/G(N)_e) \to 0, \tag{13.8}$$ FG exact seq

i.e.

$$F(G(N)/G(N)_e) \cong F(G(N)) \cong N$$

by proposition 13.2. Thus $G(N)/G(N)_e$ is not zero, and so $G(N)_e$ is a *proper* submodule. But by lemma 13.4.4, every proper submodule is contained in $G(N)_e$. □

FM simple **Proposition 13.4.** *If $M \in A-mod$ simple, and $eM \neq 0$, then $F(M) = eM$ is simple.*

Proof. If $M' = eM'$ is a nonzero eAe-submodule of eM then it is also a k-submodule of M. Since M is simple as an A-module we have $AM' = AeM' = M$. Thus $eM = eAeM' = M'$. That is, eM does not have a proper eAe-submodule. □

(**13.4.6**) Let $M \in A-\mathrm{mod}$ and $eM \neq 0$. There is an A-module map from $G(F(M))$ to M given by

$$a \otimes em \mapsto aem.$$

If M is simple, then $AeM = M$ and this map is surjective. Since this is a surjective map onto a simple module, the kernel is the maximal proper submodule of $G(F(M))$. But since $F(M)$ is simple this is $G(F(M))_e$, by proposition 13.3. Thus

invers* **Proposition 13.5.** *If $M \in A-mod$ simple, and $eM \neq 0$, then*

$$head\ (G(F(M))) \cong M.$$

Thus $F(M) \cong F(M')$ with M, M' simple implies $M = M'$.

We have

th:simples **Theorem 13.6.** *(See also 1.7.29.) Let A be an algebra over a field k, and $e \in A$ idempotent. Let $\{L(\lambda) \mid \lambda \in \Lambda^e\}$ be a full set (up to isomorphism) of simple modules in $eAe-mod$, and $\{L_A(\lambda) \mid \lambda \in \Lambda^0\}$ a full set of simple modules in $A/AeA-mod$. Then the disjoint union $\{L_A(\lambda) = head\ (G(L(\lambda))) \mid \lambda \in \Lambda^e\} \cup \{L_A(\lambda) \mid \lambda \in \Lambda^0\}$ is a full set of simple modules in $A-mod$.*

13.4.3 Decomposition numbers

Let $\{L_x \mid x \in \Lambda\}$ be a complete set of simple A-modules. Let $e \in A$ be an idempotent and Λ' the set of $x \in \Lambda$ such that $eL_x \neq 0$. Note that eL_x is simple.

(**13.4.7**) Lemma. *For any A-module M a composition series*

$$M = M_0 \supset M_1 \supset ... \supset M_l = 0$$

passes to a series

$$eM = eM_0 \supseteq eM_1 \supseteq ... \supseteq eM_l = 0$$

where, as eAe-modules:

$$e(M_{j-1}/M_j) \cong \; eM_{j-1}/eM_j \tag{13.9}$$ eq:issue8

Removing a term whenever $eM_{j-1} = eM_j$ (which is whenever the section in the M-series is killed by e) we have a composition series for eM; and

$$[eM : eL_x] = [M : L_x]$$

Proof. Note that each section in the initial eM series is simple or zero (Prop.13.4). The only other non-trivial step is (13.9), but this follows by the exactness of F. □

—

(**13.4.8**) Remark. We shall continue to investigate the properties of the setup described in this section in §22.1.1.

13.5 Decomposition numbers for the Temperley–Lieb algebra

ss:TLDN

By 13.2.7, $G\Delta_\lambda(n-2) \cong \Delta_\lambda(n)$ for any n. Similarly $F\Delta_\lambda(n) \cong \Delta_\lambda(n-2)$, except for the case $F\Delta_{(n)}(n) = 0$. Thus any map $\psi : \Delta_\lambda(n) \to \Delta_\mu(n)$. necessarily has $\lambda > \mu$; and F can be applied (perhaps repeatedly) to ψ to give a 'seed' map $F\psi : \Delta_\lambda(m) \to \Delta_\mu(m)$ where $m = |\lambda|$, so that Δ_λ is the trivial module.

It follows that any ψ can be detected by looking for occurences of the trivial module as a submodule of $\Delta_\mu(m)$ — the signal of a corresponding seed map. We call $m - |\mu|$ the *length* of ψ.

13.5.1 Ordinary induction and restriction

(**13.5.1**) Besides F, G we may introduce induction and restriction functors with respect to the 'natural' algebra injection $T_{n-1} \hookrightarrow T_n$ (add a line to each diagram on the right).

13.5.2 Germ maps

Using induction and restriction functors and Frobenius reciprocity one can show that each seed $F\psi : \Delta_{(m)}(m) \to \Delta_\mu(m)$ of length greater than 2 implies the existence of a 'germ' map $\psi' : \Delta_{(m')}(m') \to \Delta_{(m'-2)}(m')$ for some $m' < m$. Thus all Δ-module maps ψ may be determined by locating the germ maps.

(**13.5.2**) It is a routine matter to determine the germ maps in the complex ground field cases. There is such a map whenever $[m-1]=0$. There are a number of ways to show this.

...

13.5.3 Asides on $\delta = 0$

...

13.6 Ringel dualities with $U_q sl_2$

ss:TLRingel

Proceeding from §12.4 we know the following. Here $V_N = k^N$ denotes an N-dimensional vector space.

(**13.6.1**) THEOREM. *The map ρ_2 as in (12.5.16), defines a faithful representation of $T_n(q)$ on vector space $V_2^{\otimes n}$.*

Proof. See e.g. [105] — this deals with the case over the complex field, but the same argument works more generally. □

(**13.6.2**) Fix $q \in k$ and $k = \mathbb{C}$ and define $\mathfrak{U} = U_q sl_2$. Recall that this is a quantum group, that is, a quasi-triangular Hopf algebra. We include divide powers [**?**].

As a $U_q sl_2$-module let V_N denote the generically irreducible N-dimensional module. That is, the Weyl module evaluated at q.

(**13.6.3**) LEMMA. *There is an action of $U_q sl_2$ on $V_2^{\otimes n}$ with which the ρ_2 action of $T_n(q)$ commutes. Here V_2 is the defining $U_q sl_2$-module; and the action on $V_2^{\otimes n}$ is constructed using the bialgebra property.*

(**13.6.4**) THEOREM. *Let ground field $k = \mathbb{C}$. For any $q \in \mathbb{C}$,*

$$T_n(q) \cong End_{U_q sl_2}(V_2^{\otimes n}).$$

Indeed, except in case $[2]=0$, $V_2^{\otimes n}$ is a full-tilting $U_q sl_2$-module over the complex field, and the action ρ_2 makes $T_n(q)$ the Ringel dual with respect to this module.

(**13.6.5**) Proofs of these claims.

Since the actions commute and $T_n(q)$ acts faithfully we know that the centraliser algebra $A = \mathrm{End}_{U_q sl_2}(V_2^{\otimes n})$ contains an algebra isomorphic to $T_n(q)$. Generically both actions are semisimple and as a $\mathfrak{U}$,A-bimodule we have

$$V_2^{\otimes n} \cong \bigoplus_\lambda L_\lambda^A \otimes L_\lambda^{\mathfrak{U}}$$

where $\dim(L_\lambda^A)$ is the multiplicity of $L_\lambda^{\mathfrak{U}}$ in $V_2^{\otimes n}$ as a $\mathfrak{U}$-module. These multiplicities are determined by the Littlewood–Richardson rule:

(**13.6.6**) 'Classically', i.e. at $q=1$, (and hence generically, or at the level of characters) one may compute the tensor product $V_N^{\otimes n}$ of $U_q sl_2$-modules using Young diagrams and the Littlewood-Richardson rule. Firstly one represents V_N as the Young diagram $\lambda = (N-1)$. Given any V_N then for $M \leq N$ one obtains:

$$V_N \otimes V_M = V_{N+M-1} \oplus V_{N+M-3} \oplus ... \oplus V_{N-M+1}$$

The multiplicities agree with the dimensions of the simple modules for $T_n(q)$ (by the corresponding restriction rules). This proves the result in the generic case.

In the exceptional cases we need to recall the structures of the algebras.

...

13.6.1 Fusion of Temperley–Lieb algebras

ss:TLfus

.

Since the action of every element t of $T_n(q)$ commutes with the action of $U_q sl_2$ on $V_2^{\otimes n}$, then $tV_2^{\otimes n}$ is a $U_q sl_2$-submodule. In particular idempotent elements of $T_n(q)$ are projections. Generically one can check that

$$e_{(1^2)} V_2^{\otimes 2} \cong V_3$$

where, with $\delta = [2]_q$,

$$e_{(1^2)} = 1 - \frac{1}{\delta} U_i$$

Indeed we claim this holds over $\mathbb{Q}(q)$. This means that (in suitable notation)

$$(e_{(1^2)} \otimes e_{(1^2)} \otimes ... \otimes e_{(1^2)}) V_2^{\otimes 2n} \;=\; e_{(1^2)}^{\otimes n} V_2^{\otimes 2n} \;=\; (e_{(1^2)} V_2^{\otimes 2})^{\otimes n} \;\cong\; V_3^{\otimes n}$$

We are interested in $\mathrm{End}_{U_q sl_2}(V_N^{\otimes n})$ (here in case $N = 3$).

...

Evidently $V_3^{\otimes n}$ is an ETE-module...

...

Recall for algebra A with idempotent $e \in A$ that $M \mapsto eM$ defines an exact functor from $A - \mathrm{mod}$ to $eAe - \mathrm{mod}$.

$$eAe$$

In this section we review the Martin–Saleur analysis of $\mathrm{End}_{U_q sl_2}(V_N^{\otimes n})$, where V_N denotes the generically irreducible N-dimensional module [117]. This uses the theory of Drinfeld–Jimbo quantum groups (see e.g. [**?**, 80, 84, **?**]), quasiheredity [**?**, 39], Donkin's version of quasiheredity [**?**], Ringel duality [138], results of Martin–McAnally [113], and various other tools.

(**13.6.7**) Remark. The original paper on $\mathrm{End}_{U_q sl_2}(V_N^{\otimes n})$ [117] is rather relaxed about treating cases requiring degenerations of Morita equivalence. For completeness we will aim to explain the degenerate cases more explicitly. Recall Ringel's notion, for a quasihereditary algebra B, of a *characteristic tilting module*

$$T = \sum_{\lambda} T_\lambda$$

(a minimal full tilting module). The game is, in the case where there is a simple tilting module, to push this to the degeneration T' where this simple module has multiplicity zero. Note that T' is still faithful as well as tilting (but not full tilting). Let T^x be a full tilting module, and $A^x = \mathrm{End}_B(T^x)$. Thus the algebras A^x are Morita equivalent quasihereditary algebras. Whereas $A = \mathrm{End}_B(T)$ is a basic algebra among the collection of algebras A^x, the algebra $A' = \mathrm{End}_B(T')$ is not in this Morita equivalence class (and is not in general quasihereditary). The classes are, however, closely connected. They are largely treated together in the original paper.

(**13.6.8**) Recall that $\text{ann}_{\mathfrak{U}} V_N^{\otimes n}$ denotes the annihilator in $\mathfrak{U}$ of $V_N^{\otimes n}$. Define

$$\mathfrak{U}_n^N = \mathfrak{U}/\text{ann}_{\mathfrak{U}} V_N^{\otimes n}.$$

the quotient of $\mathfrak{U}$ acting faithfully on $V_N^{\otimes n}$.

cl:gen NM (**13.6.9**) CLAIM: For generic q, $\mathfrak{U}_n^N \cong \mathfrak{U}_m^M$ if $n, m > 1$ and $n(N-1) = m(M-1)$.

Proof. For generic q the issue is simply to determine which irreducible representations of $U_q sl_2$ appear in $V_N^{\otimes n}$ (using the Littlewood–Richardson rule, say [**?**]). Let us start with some examples. The first table here shows the multiplicites of each V_N in V_2^n:

2^n	n	$\emptyset$	□	□□	□□□	□□□□
1	0	1				
2	1		1			
4	2	1		1		
8	3		2		1	
16	4	2		3		1

Note here that only the representations of the right 'parity' appear, but for fixed parity every representation up to V_{n+1} appears. The next table shows V_3^n (cf. e.g. [104, p.329]):

3^n	n	$\emptyset$	□	□□	□□□	□□□□		
1	0	1						
3	1			1				
9	2	1		1		1		
27	3	1		3		2	1	
81	4	3		6		6	3	1

Here only even parity ever appears and (for $n > 1$) all such representations up to V_{2n+1} ($2n$ boxes) appear.

Comparing $(N-1)n = (2-1)(n_2)$ with $(M-1)m = (3-1)(n_3)$ we see that so long as n_2 is even then $V_{n_2+1} = V_{2n_3+1}$ when $n_2 = 2n_3$ as required. But of course n_2, n_3 are integers so solutions always have n_2 even. Other cases are similar. □

(**13.6.10**) Next we consider the non-generic cases over the complex field. If V_N is tilting then $V_N^{\otimes n}$ is a direct sum of indecomposable tilting modules (see e.g. [**?**]). The structure of these is known in our case, so it becomes a combinatorial exercise to work out their multiplicities. The key issue here is if any multiplicities of *these* are zero.

Here are some cases in point.

(**13.6.11**) With $[2] = 0$ the structure of standard and tilting modules is as follows.

$$\begin{array}{rccccccc}
\text{lambda=} & 0 & (1) & (2) & (3) & (4) & (5) & \\
\text{dim=} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{simple content=} & 1 & 2 & \begin{matrix}2\\1\end{matrix} & 4 & \begin{matrix}3\\2\end{matrix} & 6 & \begin{matrix}4\\3\end{matrix} \\
\text{tilting=} & 1 & 2 & \begin{matrix}1\\2\\1\end{matrix} & 4 & \begin{matrix}2\\1\;\;3\\2\end{matrix} & 6 & \begin{matrix}3\\2\;\;4\\3\end{matrix}
\end{array}$$

The module V_2 is tilting here, so V_2^2 is also tilting. It contains $\Delta_{(2)} = V_3$ but not V_5 and hence must contain the indecomposable tilting module $T_{(2)}$. But then $V_2^2 = T_{(2)}$ by a dimension count, so the multiplicity of T_0 is zero. This should be compared with the generic case above.

With $[2] = 0$ the module V_3 is not tilting. This means that V_3^n is not tilting, which complicates matters somewhat. We shall omit this case for now. (We can construct a 'nearby' tilting module by suitable gluing on a copy of the trivial module...)

Until further notice we shall restrict to cases in which V_N is simple tilting. This condition is satisfied by avoiding q-value,N-value pairs such that $[m] = 0$ for $m < N$. We call this the *simple tilting condition*.

(**13.6.12**) With $[3] = 0$ the structure of standard and tilting modules is as follows.

$$\begin{array}{rccccccc}
\text{lambda=} & 0 & (1) & (2) & (3) & (4) & (5) & \\
\text{dim=} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{simple content=} & 1 & 2 & 3 & \begin{matrix}2\\2\end{matrix} & \begin{matrix}4\\1\end{matrix} & 6 & \begin{matrix}3\\4\end{matrix} \\
\text{tilting=} & 1 & 2 & 3 & \begin{matrix}2\\2\\2\end{matrix} & \begin{matrix}1\\4\\1\end{matrix} & 6 & \begin{matrix}4\\1\;\;3\\4\end{matrix}
\end{array}$$

In this case $V_2^2 = T_0 \oplus T_{(2)} = \Delta_0 \oplus \Delta_{(2)}$;

$$V_2^3 = T_1 \oplus T_{(3)} == 1.2 + 1.\begin{matrix}2\\2\\2\end{matrix}$$

(in the obvious shorthand); and

$$V_2^4 = 1.T_0 \oplus 3.T_{(2)} \oplus 1.T_{(4)} = 1.1 + 3.3 + 1.\begin{array}{c}1\\4\\1\end{array} \tag{13.10}$$ eq:v24

However

$$V_3^2 = 0.T_0 \oplus 1.T_{(2)} \oplus 1.T_{(4)} = 0.1 + 1.3 + 1.\begin{array}{c}1\\4\\1\end{array}$$

by a similar dimension-count argument to before. Here we have $(2-1)n_2 = (3-1)n_3$ with $n_3 = 2$ so $n_2 = 4$. Comparing with (13.10) we see that the tilting content is fundamentally different.

However, the modules V_2^4 and V_3^2 have the same annihilator here. This is because the missing module in V_3^2 is simple, and this simple appears as a composition factor elsewhere in V_3^2.

In summary Claim 13.6.9 holds in this case for $[3] = 0$.

(**13.6.13**) Next we aim to show that Claim 13.6.9 holds in general, so long as the simple tilting condition holds. We will do this by considering elementary combinatorial properties of the endomorphism algebra

$$\mathcal{T}_n^N = \mathrm{End}_{\mathfrak{U}}(V_N^n)$$

(the TL algebra or one of its generalisations). (However see also [45, §4.1] for example.)

Our argument locates tilting modules with multiplicity zero in V_N^n by locating 'Specht' modules of the endomorphism algebra that are isomorphic with the trivial module in the given specialisation. This works because (a) every fibre (under globalisation) of Specht module maps starts with a 'seed' map from the trivial module; and (b) in order for a Specht module *not* to have a head which gives a simple module with the same label, its simple content must be asignable to another label, thus the seed map from the trivial module must be an isomorphism. (Strictly speaking one also needs to use some facts about restriction rules. We will demote these for now.)

Note that the multiplicities of tilting modules in V_N^n (fixed N) are not fixed as n varies. However if some T_λ has multiplicity zero for some n then it has multiplicity zero for all n. (Explain why?)

We have argued that a necessary condition for a zero multiplicity is that the corresponding Specht module has rank 1 in the first layer in which it appears. The occurence of rank 1 Specht modules is quite rare. We have precisely one at level $n = 1$ — the trivial module itself (in general the trivial module is $\lambda = n(N-1)$). At level $n = 2$ all the Specht modules occuring have rank 1. We have $\lambda = 0, (2), ...(2(N-1))$ occuring. At level $n = 3$ only $\lambda = 0, (2(N-1))$ have rank 1 (consider, for example, the generic restriction rules). Thereafter only the trivial module has rank 1 (again by the restriction rules).

We need to show that among this small set of rank 1 modules those which can be isomorphic to the trivial module in some specialisation are the 'leftmost' element in their block. (This shows that the corresponding tilting module is simple as we require to show.) The only inobvious case is level $n = 2$.

Imposing the simple tilting condition we find that all but possibly the trivial module appearing in level $n = 2$ lie in the fundamental alcove, and hence correspond to simple tilting modules as required.

(**13.6.14**) We claim that we have now established the claim 13.6.9 above for all q for which the simple tilting condition holds.

(**13.6.15**) The next challenge is to investigate the conditions under which the Martin–Saleur 'dual and Morita equivalent' property holds (the vertical maps in the main commutative diagram [117, (1)]).

The property holds whenever V_N^n is full tilting, as shown in [105] or [113]. It evidently does not hold otherwise, since the dual does not have enough simples. However a useful degeneration (sufficient to determine representation theory) holds, as we now explain.

The degeneration works as follows. The TL-like algebra is 'almost' Morita equivalent, in the sense that one must allow one formal simple module of dimension-0. This is simply a trick which allows us to keep all the bookkeeping of the non-degenerate cases.

...

Chapter 14

Yet more representation of Temperley–Lieb algebras

Ch:TLplus1

In Chapter 12 we determined the characters of irreducible representations of the Temperley–Lieb algebras in all cases over $\mathbb{C}$. We also determined the ordinary quivers in all cases, which determines the complete structure up to isomorphism. This yields as complete a description of the fundamental invariants in TL representation theory as one could wish for. This is not *necessarily* the end of the story however, as far as certain applications are concerned. One might ask to study, perhaps up to a suitable notion of isomorphism, constructions that give representations for all n.

Such constructions might have interesting special properties (made interesting by Physics considerations perhaps); and only work for specific values of δ.

Examples can be found in various references, such as: [104], [103], [**?**, **?**] ...

Other relevant refs: Soergel, ..., Beilinson, Ginzburg, Soergel 1996 Koszul duality patterns in rep theory. ...

14.1 Potts representations

Potts representations of TL are obtained by restricting Potts representations of the partition algebra. These are most succinctly described categorically. See **??**. The partition category P is dual-monoidally generated by the singleton s (the unique partition in $hom_P(1,0)$), the identity in $hom_P(1,1)$ and the single part partition Γ in $hom_P(1,2)$. The Q-state Potts representation is the monoidal functor to the subcategory of **Vect** generated by $V = \mathbb{C}^Q$ given on the natural basis by

$$s \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$$

and (for example in case $Q = 2$)

$$\Gamma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Write $\otimes$ for the monoidal product in P, and $\star : hom_P(n,m) \to hom_P(m,n)$ for the flip antiautomorphism 'dual'.

The TL category T is generated by s and $a = \Gamma\Gamma^\star \in \hom_P(2,2)$:

$$\Gamma\Gamma^\star \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(**14.1.1**) The irreducible content of the Potts representations is determined in [102].

14.1.1 Generalised Potts functor representations

The Potts functor restricts to the Brauer subcategory. At the Brauer level it is amenable to a generalisation, which then survives to the TL level. This was studied in [**?**]. It has its origins in the Brauer *algebra* case studied by Benkhart [**?**, **?**]. The idea is to replace the orthogonal or symplectic form under which the Brauer part of the Potts functor is invariant with a mixed orthosymplectic form.

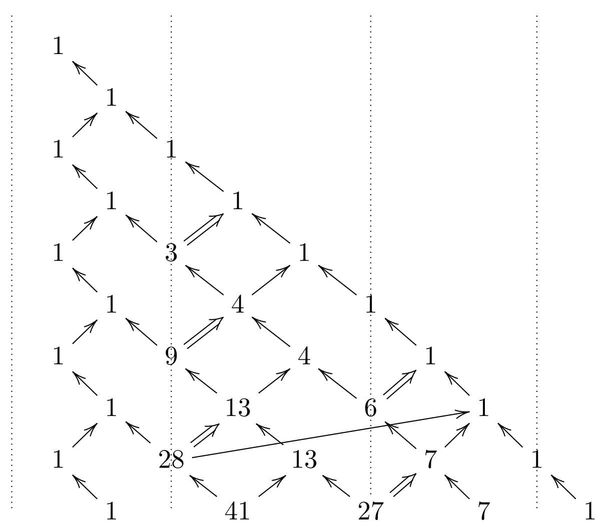
...

14.2 Simple modules

14.2.1 Restriction of simple modules

Recall that the restriction rule in the generic case is elementary.

The Beraha cases are more interesting. In case $\delta = 1$ for example we have



The matrix form is

$$\left(\begin{array}{cc|ccc|ccc|c}
0 & 1 & & & & & & & \\
1 & 0 & & & & & & & \\
\hline
0 & 1 & 0 & 2 & & & & 1 & \\
0 & 0 & 1 & 0 & 1 & & & & \\
0 & 0 & 0 & 1 & 0 & & & & \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 2 & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ddots
\end{array}\right)^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \\ 4 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

The spectral radius of the first block is 1. It follows that the asymptotic growth rate of dimensions (going vertically down the table) is 1 in the first two columns; and 4 in all other cases.

In the generic case the downward walks on the diagram from 0 to λ lift readily to a basis of the λ simple module. In general it is interesting to consider how such bases can arise.

...

14.3 Recall notation for graphs

ss:graph20

Some elementary properties of graphs will be useful in what follows.

Recall (e.g. from §3.5) that a graph G is a set G_0 of 'vertices' and a set G_1 of 'edges' and a pair of maps:

$$\begin{tikzcd} G_1 \arrow[r, shift left, "in"] \arrow[r, shift right, "out"'] & G_0 \end{tikzcd}$$

or equivalently an adjacency matrix $\mathsf{A}^G = (\mathsf{A}^G_{ij})_{i,j \in G_0}$ (for a concrete matrix a total order on G_0 is required) and an edge label set $G(i,j)$ for each ordered vertex pair, with $|G(i,j)| = \mathsf{A}^G_{ij}$. Note

$$G_1 \;=\; \sqcup_{i,j} G(i,j)$$

We assume G comes with a prefered 'base' vertex $0 \in G_0$. A graph is *path-connected* if there is a directed walk between any ordered pair of vertices.

(**14.3.1**) Associated to path-connected A^G is a Perron-Frobenius eigenvalue λ^G, and a choice of positive Perron-Frobenius eigenvector v^G (see e.g. [104]), such that $v^G_0 = 1$.

(A matrix M is *Perron* if some power M^L is positive. Then M^L has a unique largest magnitude eigenvalue which is positive, and a positive eigenvector unique up to scalars. This PF eigenvector is also an eigenvector of M and orthogonal to all eigenvectors for other eigenvalues; and hence also for $M-1$. Thus a matrix A has a PF eigenvector if $A+1$ is Perron. In particular the adjacency matric of a path-connected graph has a PF eigenvector.)

pa:A3

(**14.3.2**) Example (see also §**??**): For the A_3 graph

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \end{tikzcd}$$

(with base vertex 1, say) we have $\mathsf{A}^G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix} = \pm\sqrt{2} \begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}$$

The other ($\lambda = 0$) eigenspace is spanned by $(1, 0, -1)^t$.

For the A_∞ graph

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \rightleftarrows \ldots$$

we have, for any q,

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ [2] \\ [3] \\ \vdots \end{pmatrix} = [2] \begin{pmatrix} 1 \\ [2] \\ [3] \\ \vdots \end{pmatrix}$$

This works because of the recursion

$$[2][n] = [n+1] \; + \; [n-1]$$

or equivalently because of the $n = 0$ node in the fourier sine-series solution to the translation invariant double-infinite graph case:

$$\ldots \rightleftarrows 1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \rightleftarrows \ldots$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Indeed the 3-vertex example above can be considered as the fourier series in a box (2-node) version of this (the second boundary condition constrains q).

In particular $\Omega \; := \; \mathsf{A}^{A_\infty^\infty}$ is an operator on Hilbert space with basis $\mathbb{Z}$, acting as

$$\Omega \, \underline{n} \; = \underline{n-1} + \underline{n+1}.$$

Thus Ω is translation invariant — it commutes with the shift operator Σ^+ defined by $\Sigma^+ \underline{n} = \underline{n+1}$. For any complex α set $v^\alpha = \sum_n e^{\alpha n} \underline{n}$. Then evidently

$$\Sigma^+ v^\alpha = e^{-\alpha} v^\alpha$$

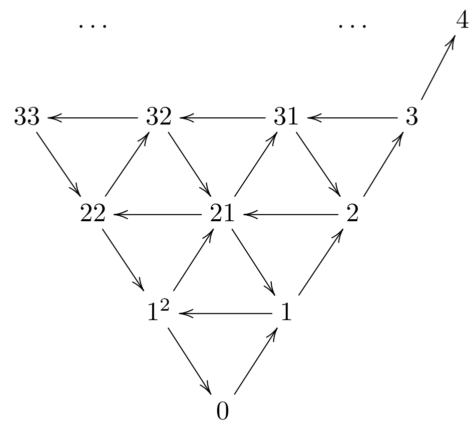
i.e. the 'fourier transform' solves the eigenvalue problem for Σ^+. For Ω we have

$$\Omega \overbrace{\sum_n e^{\alpha n} \underline{n}}^{v^\alpha :=} \; = \; \sum_n e^{\alpha n} (\underline{n-1} + \underline{n+1}) \; = \; \sum_n (e^{\alpha} e^{\alpha(n-1)} \underline{n-1} + e^{-\alpha} e^{\alpha(n+1)} \underline{n+1}) \; = \; (e^{\alpha} + e^{-\alpha}) \sum_n e^{\alpha n} \underline{n}$$

The eigenvalues for v^α and $v^{-\alpha}$ are the same, so we can use $w^\alpha_\pm = v^\alpha \pm v^{-\alpha}$ as eigenvectors. Here w^α_- has coefficient of $\underline{0}$ equal zero. Meanwhile $\Omega^+ \; := \; \mathsf{A}^{A^\infty}$ is an operator on Hilbert space with basis $\mathbb{N}$, acting for $n \neq 1$ by $\Omega^+ \underline{n} = \underline{n-1} + \underline{n+1}$.

Now (with sl_2 representation theory in mind) note that Ω^+ has an eigenvalue 2 with eigenvector $v = (1, 2, 3, 4, ...)$. The check is that if the m-th term is m then we need $2m = (m+1) + (m-1)$ (and that the 0-th term vanishes). One should compare this with the Weyl character formula **??**, which gives the dimensions of irreps of sl_2! ...Noting that our adjacency matrix gives the tensor product rule for tensoring with the defining 2-d representation in this case.

(**14.3.3**) What about the graph



The positions on this graph may be indexed by m_1, m_2, corresponding to moving up-right and left respectively, starting at 0 (we omit any second 0 here, in the figure itself). Consider the ansatz

$$v_{m_1,m_2} = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$$

(the Weyl character formula for sl_3). One readily confirms that

$$\mathsf{A}^G v \;=\; 3v$$

Is there a q-version?

14.3.1 Pairings and forms on graphs

(**14.3.4**) Let G be a graph, and a an edge. We write ${}_v a_w$ to emphasise that edge a has $in(a) = v$ and $out(a) = w$, i.e. that $a \in G(v, w)$.

(**14.3.5**) A pairing σ on G is an (involutive?) bijection from $G(i, j)$ to $G(j, i)$ for each pair i, j. Note that a pairing does not exist unless $|G(i, j)| = |G(j, i)|$ for every pair i, j.
Example: For the A_n example above, and in general when $|G(i, j)| = |G(j, i)| = 1$ for every pair i, j, there is only one possible pairing.

For given pairing σ on G we may write $y' = \sigma(y)$ for edge y. That is

$${}_w y'_v = \sigma({}_v y_w).$$

(**14.3.6**) For given pairing σ on G define a map

$$w_+^G : G_1 \times G_1 \to \mathbb{C}$$

as follows. Define $w_+^G(a, b) = 1$ if $a = \sigma(b)$; and zero otherwise. That is $w_+^G(a, b) = \delta_{a,\sigma(b)}$.

14.4 Generalised walk-basis representations

ss:GWBR

One way to think of the Andrews–Baxter–Forrester (ABF) representations (interactions-round-a-face (IRF) representations) of Hecke and TL ([103], and cf. [?]) is as representations on bases of walks on certain graphs.

14.4.1 Towards 'categorified' TL walk reps

Almost a 'defining' representation of the Temperley–Lieb algebra is the action on tensor space $V^{\otimes n}$, with $V = k^2$, in **??**. This is the action that commutes with the $U_q sl_2$ action. We showed in **??** that we can think of the cup and cap parts of the TL diagram as birth and death processes in $U_q sl_2$ representation theory — the map (uniquely defined up to a scalar) from $V \otimes_{\mathbb{C}} V$ to the trivial module $V^0 = \mathbb{C}$ is 'death', and v.v. ... Thus the TL category (for $\delta = q + q^{-1}$) is realised inside the $U_q sl_2$ rep tensor category.

Tensor space decomposes, as a linear representation, into Young modules Y_λ — it is a manifest direct sum simply by arranging (not changing) the basis. And there is a natural class of isomorphic modules to each Young module, with (if $k = \mathbb{C}$ say) a natural continuous parameter h sweeping among them. This construction is singular for certain special values h, but well defined on an open subset. For suitable values of h then Specht modules manifest inside each Y_λ in one of Young's canonical forms, with matrix coefficients depending on hook lengths in corresponding tableau.

A tableau representation (§**??**) can be considered as actions on a space with a basis of walks on the A_∞ graph. Such a space is not a tensor space in the usual sense — tensoring spaces over the ground field — but is a tensor space if we allow tensoring over a suitably enhanced commutative ring, as follows.

de:path98

(**14.4.1**) Let G be a graph as in §14.3 above. Define unital $\mathbb{C}$-algebra $G_0^{\mathbb{C}}$ by a primitive idempotent decomposition $1 = \sum_{G_0} e_v$ and $G_0^{\mathbb{C}} = \mathbb{C}\{e_v\}_{v \in G_0}$. (The basic algebra that is the path algebra, in the sense of §1.5.2, of the trivial graph with vertex set G_0.)

Note that $G_1^{\mathbb{C}} := \mathbb{C}G_1$ is a subset of the path algebra of G over $\mathbb{C}$ that is a $G_0^{\mathbb{C}}$-bimodule by restriction of the path algebra action.

(**14.4.2**) Consider $\mathbb{C}G_1$ as a space with a basis of '1-step walks'. Then $\mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1$ (understood as our usual tensor product of a left and right ring-module, as in 1.7.26 or §8.4) is a space with a basis of 2-step walks; and so on:

$$\mathbb{C}G_1^n \;:=\; \mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1 \otimes ...$$

(of course the tensor product is not strictly associative, but we side-step this issue for now). Note that for $n = 0, 1, 2, 3, ...$:

$$\dim \mathbb{C}G_1^n \;=\; \sum_{ij} ((\mathsf{A}^G)^n)_{ij} \tag{14.1}$$

For example $G = \; 1 \rightleftarrows 2 \rightleftarrows 3$ (as in (14.3.2)) obeys $(\mathsf{A}^G)^0 = 1_3$ and $(\mathsf{A}^G)^{m+2} = 2(\mathsf{A}^G)^m$ for $m \geq 1$; so that it has dimension sequence 3,4,6,8,12,... (we write out the first few of these explicitly below).

14.4.2 Some homs in a bimodule category (create/birth;death)

(**14.4.3**) For a graph G, we define certain 'creation' and 'annihilation' operators in the subcategory $\mathbf{Vect}_G$ of $\mathbf{Vect}$ with objects $\mathbb{C}G_1^n$. (Strictly speaking we aim for the subcategory in which homs are $G_0^{\mathbb{C}}$-module morphisms; with the category monoidal by the $\otimes_{G_0^{\mathbb{C}}}$ operation. So not the usual monoidal structure from $\mathbf{Vect}$.)

Fix a graph G with a pairing σ. Note that edge-pairs of form ${}_ia_j \otimes {}_jb_k$ give a basis W_2^G of $\mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1$. We may write ab for $a \otimes b$. The general form of a $G_0^{\mathbb{C}}$-module morphism from $G_0^{\mathbb{C}}$ to $\mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1$ is then:

$$e_v \mapsto \sum_{ab \in W_2^G} \alpha_{ab}^v ({}_ia_j \otimes {}_jb_k)$$

but evidently coefficient $\alpha_{ab}^v = 0$ unless the walk ab begins and ends in v. Note that any left $G_0^{\mathbb{C}}$-module is also a right module since $G_0^{\mathbb{C}}$ is commutative (we might call this the 'left-to-right' construction); but here we understand the right action of $G_0^{\mathbb{C}}$ on $G_1^{\mathbb{C}}$, and hence on $\mathbb{C}G_1^n$, as the natural right action, which is *distinct* from the left-to-right action.

We define one specific vector space map of this form by taking the coefficient of every possible path constructed from a pairing $\sigma(y)y$ to be 1, up to an overall factor:

$$\begin{aligned} b : \mathbb{C}G_0 &\rightarrow \mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1 && (14.2)\\ e_v &\mapsto \frac{1}{v_v^G} \sum_{y \in G(-,v)} y'y && (14.3)\end{aligned}$$

(recall $y' = \sigma(y)$; and note that we need $v_v^G \neq 0$ here).

By the same token, the general form of a $G_0^{\mathbb{C}}$-module morphism from $\mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1$ to $G_0^{\mathbb{C}}$ is

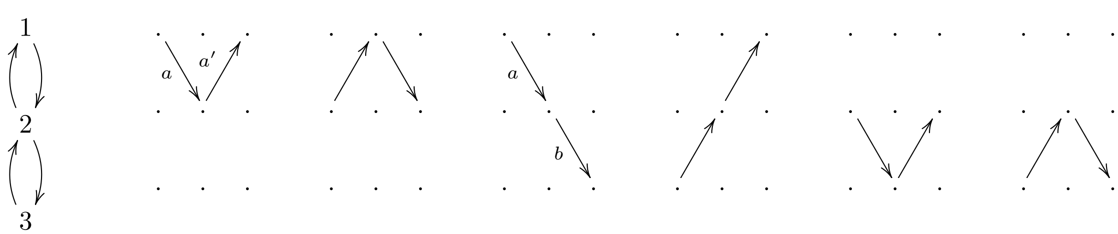
$${}_ia_j \otimes {}_jb_k \;\mapsto\; \sum_v \beta_v^{ab} e_v$$

but $\beta_v^{ab} = 0$ unless $v = i = k$. Next define

$$\begin{aligned} d : \mathbb{C}G_1 \otimes_{\mathbb{C}G_0} \mathbb{C}G_1 &\rightarrow \mathbb{C}G_0 && (14.4)\\ {}_ia_j \otimes {}_jb_k &\mapsto v_j^G w_+^G(a,b)\, \delta_{ik}\, e_i && (14.5)\end{aligned}$$

We should check that these are well-defined $G_0^{\mathbb{C}}$-module maps.
We would also like that the composition $d \circ b = 1_{G_0^{\mathbb{C}}}$. See (14.4.7) below.

(**14.4.4**) Example: Consider $1 \rightleftarrows 2 \rightleftarrows 3$ as in (**??**), with the enforced pairing. We have $\mathbb{C}G_1^2 := \mathbb{C}G_1 \otimes_{G_0^{\mathbb{C}}} \mathbb{C}G_1; := \mathbb{C}\{{}_1a_2 \otimes {}_2a_1',\ {}_2a_1' \otimes {}_1a_2,\ {}_1a_2 \otimes {}_2b_3,\ {}_3b_2' \otimes {}_2a_1',\ {}_2b_3 \otimes {}_3b_2',\ {}_3b_2' \otimes {}_2b_3\}$ which we can depict as



For example

$$\begin{aligned} d : {}_1a_2 \otimes {}_2a'_1 &\mapsto \sqrt{2}\delta_{11} w^G_+(a,a')e_1 \; = \; +\sqrt{2}e_1 \\ d : {}_2a'_1 \otimes {}_1a_2 &\mapsto 1e_2 w^G_+(a',a) \; = \; e_2 \\ d : {}_1a_2 \otimes {}_2b_3 &\mapsto \sqrt{2}\delta_{12} e_1 w^G_+(a,b) \; = \; 0 \end{aligned}$$

$$\begin{aligned} b : e_1 &\mapsto +{}_1a_2 \otimes {}_2a'_1 \\ b : e_2 &\mapsto \frac{1}{\sqrt{2}}({}_2a'_1 \otimes {}_1a_2 + {}_2b_3 \otimes {}_3b'_2) \end{aligned}$$

Altogether, in this case the maps b, d act on the bases by:

$$b \mapsto \begin{pmatrix} +1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{+1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d \mapsto \begin{pmatrix} +\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix} = \sqrt{2}b^t$$

We use $b|d$ to denote composition in the opposite to the usual function order — this opposite order being more natural and convenient in the diagram visualisation. (This means that we should consider operators as acting from the right.) Thus

$$b|d \; \mapsto \; \begin{pmatrix} \sqrt{2} & & \\ & \sqrt{2} & \\ & & \sqrt{2} \end{pmatrix}, \qquad d|b \; \mapsto \; \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & +\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

(**14.4.5**) Exercise: compute the action of $b \otimes_{G_0^{\mathbb{C}}} 1_{G_1^{\mathbb{C}}}$ on the appropriate basis.

This operator takes a vector in $G_0^{\mathbb{C}} \otimes_{G_0^{\mathbb{C}}} G_1^{\mathbb{C}}$ as input (with basis $e_1 \otimes a, e_2 \otimes a', e_2 \otimes b, e_3 \otimes b'$ identified with a, a', b, b') and gives a vector in $\mathbb{C}G_1^3$ as output. Let us order the basis of the latter as 1212, 1232, 2121, 2323, 2123, 2321, 3232, 3212 (in the obvious shorthand). We have

$$b \otimes_{G_0^{\mathbb{C}}} 1_{G_1^{\mathbb{C}}} \; \mapsto \; \begin{pmatrix} \sqrt{2} & & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ & & & & & & & \sqrt{2} \end{pmatrix} \qquad 1_{G_1^{\mathbb{C}}} \otimes_{G_0^{\mathbb{C}}} d \; \mapsto \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & \\ 1 & & & \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & & 1 \\ & & & 1 \end{pmatrix}$$

...!!!???

(**14.4.6**) Meanwhile then:

$$(b \otimes 1_{G_1^{\mathbb{C}}})(1_{G_1^{\mathbb{C}}} \otimes d) \mapsto \left(\begin{pmatrix} +1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{+1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes 1_4 \right) \left(1_4 \otimes \begin{pmatrix} +\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \right) = 1_4$$

Note that the $\otimes$ 'shorthand' on the image side is *not* the Kronecker product (we are not using the usual monoidal product in **Vect**).

14.4.3 Properties of homs in the b, d construction

papadgobd1 (**14.4.7**) In general we have

$$b|d = \lambda^G 1_{G_0^{\mathbb{C}}}$$

since

$$e_i \xrightarrow{b} \frac{1}{v_i} \sum_{y \in G(-,i)} {}_i y'_q \otimes {}_q y_i \xrightarrow{d} \frac{1}{v_i} \sum_{y \in G(-,i)} d({}_i y'_q \otimes {}_q y_i) = \frac{1}{v_i} \sum_{y \in G(-,i)} v_q w_+^G(y', y) e_i$$

$$= \frac{1}{v_i} \left(\sum_{y \in G(-,i)} v_q \right) e_i = \frac{1}{v_i} \left(\lambda^G v_i \right) e_i \ = \ \lambda^G e_i$$

— here we use that $\sum_q (A^G)_{iq} v_q = \lambda^G v_i$ (and write v_i for v_i^G).

(**14.4.8**) The crucial aspect for functoriality is the 'snake' property:

$$(b \otimes 1_{G_1^{\mathbb{C}}})(1_{G_1^{\mathbb{C}}} \otimes d) = +1_{G_1^{\mathbb{C}}}$$

To check this we look at the action on ${}_v y_w \equiv e_v \otimes_0 {}_v y_w$:

$$e_v \otimes_0 {}_v y_w \xrightarrow{b \otimes_0 1_{G_1^{\mathbb{C}}}} \left(\frac{1}{v_v} \sum_{z \in G(-,v)} z' z \right) {}_v y_w = \left(\frac{1}{v_v} \sum_{z \in G(-,v)} z' z_v y_w \right) \xrightarrow{1_{G_1^{\mathbb{C}}} \otimes d} \left(\frac{1}{v_v} \sum_{z \in G(-,v)} {}_v z'_q d({}_q z_{vv} y_w) \right)$$

$$= \left(\frac{1}{v_v} \sum_{z \in G(-,v)} {}_v z'_q \, x_v w_+^G(z, y) \delta_{qw} e_w \right) = \left(\frac{1}{v_v} \sum_{z \in G(-,v)} {}_v z'_q \, x_v \delta_{z,y'} \delta_{qw} e_w \right) = \ {}_v y_w \otimes_0 \ e_w$$

Note how we identify congruent tensor products in regarding this as the identity map!

14.4.4 The generic case

In the generic case we consider the infinite commutative algebra

$$A = \mathbb{C}\{\underline{n} \;:\; n \in \mathbb{Z}\}$$

and

$$V = \mathbb{C}\{{}^{n}\overset{n+1}{\rightarrow},\ {}^{n}\overset{n+1}{\leftarrow} \;:\; n \in \mathbb{Z}\}$$

and note that $V = {}_AV_A$ by the obvious actions. We get a birth operator

$$b : A \to V^{\otimes_A 2}$$

$$\underline{n} \mapsto \frac{1}{x_n}({}^{n}\overset{n+1}{\rightarrow} \otimes {}^{n}\overset{n+1}{\leftarrow} \ +)$$

and a death operator

$$d : V^{\otimes_A 2} \to A$$

14.5 On Temperley–Lieb algebra, group theory and the Potts model

Here we review some results from Martin [101]. This paper developed representation theory of the TL algebras in a formalism that is now somewhat anachronistic. However it also explored aspects of the problem that benefit from this treatment.

In the paper, the TL relations are given initially (following Temperley and Lieb 1971 [145]), for each n and for operators $\{U_i,\ i = 1, 2, ..., 2n-1\}$, as

$$U_i^2 = \sqrt{\mathsf{q}}U_i \tag{14.6}$$ $\boxed{\texttt{eq:TL00}}$

$$U_iU_{i\pm1}U_i = U_i \tag{14.7}$$ $\boxed{\texttt{eq:TL000}}$

$$U_iU_j = U_jU_i \qquad (|i-j| > 1) \tag{14.8}$$ $\boxed{\texttt{eq:TL0000}}$

The parameter q here agrees with the form appearing in the q-state Potts representation (on an n-site wide lattice).
(In the paper the symbol q is used, but this is not consistent with conventions elsewhere in these notes. Elsewhere the symbol Q is used, but here we will try to stay closer to the conventions in the paper, changing only the font.)
Another parameterisation that will be useful is in terms of θ, where

$$e^{\theta} + e^{-\theta} = \sqrt{\mathsf{q}} \tag{14.9}$$

A key feature of this setting is the extent of dependence on n. The physical context suggests to emphasise n-stable aspects. The operators U_i are, in a suitable sense, localised near i, and act trivially far from i. (Thus we have the relation (14.8).) This leads to representation by monoidal functors (as we have already seen several times), but also means that such constructions can be 'gamed', for example as we will see here.

The representation-theoretic ‘groupification’ strategy is to turn focus from the operators U_i to operators of form

$$t_i = 1 - K(\mathsf{q})U_i$$

which are more ‘group-like’, with

$$t_i^{-1} = 1 - \tilde{K}(\mathsf{q})U_i$$

where

$$\tilde{K} + K = \sqrt{\mathsf{q}}\tilde{K}K$$

The best-known such t_i is given by

$$K(\mathsf{q}) = e^{\theta}$$

from which the t_i operators obey the braid relations. From an algebraic perspective these operators also have only two eigenvalues — we say that they give reps of the Hecke quotients of the braid groups. But staying strictly with the group perspective for now, this aspect is not generally manifest... ...One exception is in the case $\mathsf{q} = 4$, where $t_i^2 = 1$ — yielding representations of the symmetric groups (here denoted Σ_{2n}, or Σ_{l+1} using the rank variable $l = 2n - 1$; indeed we can generalise to include the $l = 2n$ operator case, and hence to any number of operators l).

The Potts representation is given by

$$U_{2i-1} = 1_{\mathsf{q}} \otimes ... \otimes A \otimes 1_{\mathsf{q}} \otimes ... \otimes 1_{\mathsf{q}}$$

— n factors, where the i-th tensor factor is

$$A = \frac{1}{\sqrt{\mathsf{q}}} \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ \vdots & \vdots & ... & \vdots \\ 1 & 1 & ... & 1 \end{pmatrix}$$

and

$$U_{2i} \;=\; \frac{1}{\sqrt{\mathsf{q}}} \sum_{r=1}^{\mathsf{q}} B_i(r)\; B_{i+1}(r)^{\dagger}$$

where

$$B_i(r) = 1_{\mathsf{q}} \otimes ... \otimes C(r) \otimes 1_{\mathsf{q}} \otimes ... \otimes 1_{\mathsf{q}}$$

— n factors, where the i-th tensor factor is given by

$$(C(r))_{jk} = \delta_{jk} \exp(\frac{2\pi i r(j-1)}{\mathsf{q}})$$

That is

$$C(r) = \begin{pmatrix} 1 & & & \\ & (\exp(\frac{2\pi i}{\mathsf{q}}))^r & & \\ & & (\exp(\frac{2\pi i}{\mathsf{q}}))^{2r} & \\ & & & ... \end{pmatrix} = C(1)^r = \begin{pmatrix} 1 & & & \\ & (\exp(\frac{2\pi i}{\mathsf{q}})) & & \\ & & (\exp(\frac{2\pi i}{\mathsf{q}}))^2 & \\ & & & ... \end{pmatrix}^r$$

For example for $\mathsf{q} = 2$

$$C(1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

so

$$B_i(1)\ B_{i+1}(1)^\dagger = 1_2 \otimes ... \otimes \left(\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes 1_2 \right) \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \right) \otimes ... \otimes 1_2$$

$$= 1_2 \otimes ... \otimes \left(\begin{pmatrix} 1 &&& \\ & -1 && \\ && 1 & \\ &&& -1 \end{pmatrix} \begin{pmatrix} 1 &&& \\ & 1 && \\ && -1 & \\ &&& -1 \end{pmatrix} \right) ... = ... \begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& 1 \end{pmatrix} ...$$

(check Kronecker conventions, we normally try to use the basis order as indicated by 11, 12, 21, 22 - later places changing quicker) and

$$U_{2i} = \ 1_{\mathsf{q}} \otimes ... \otimes \sqrt{\mathsf{q}} \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 1 \end{pmatrix} \otimes ... \otimes 1_{\mathsf{q}}$$

and similarly for general q.

Part IV

Partition

Chapter 15

Background from computational statistical mechanics

ss:pvector

Partition categories come from computational statistical mechanics (see e.g. [104]), so it is helpful to briefly establish this context before proceeding with Ch.16. The context is intrinsically interesting; and it also helps to understand choices made in the construction. (See also for example Martin–Woodock [119]; [**?**], and [110].) But it is safe for the reader to *skip this section* and pass directly to §16.1.

Let G be a graph of the type where the vertex set is V_G and the edge set E_G comes with a map $\mathbf{E}_G : E_G \to \mathtt{P}_2(V_G)$ as in §3.1.2 (see also §3.5). Fix $Q \in \mathbb{N}$. The *Q-state Potts Hamiltonian* is $H_G : \underline{Q}^{V_G} \to \mathbb{R}$ given by

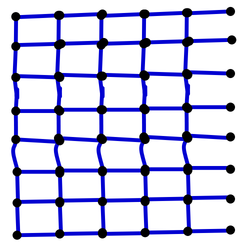
$$H_G(\sigma) = \sum_{\langle ij \rangle \in E_G} \delta_{\sigma_i, \sigma_j}$$

(here we write $\langle ij \rangle$ for $e \in E_G$ such that $\mathbf{E}_G(e) = \{i, j\}$, and σ_v for $\sigma(v)$).
The first object of interest is the partition function

$$Z_G = \sum_{\sigma \in \hom(V_G, \underline{Q})} \exp(\beta H_G(\sigma))$$

regarded as a function of the variable β. This β plays the role of inverse temperature. For analytical purposes we set $x = \exp(\beta)$, so that our Z_Gs here are polynomials in x.

Typically we are interested in cases where G represents atoms and interactions in something like a cubical lattice of side length 10^6, thus with $|V_G| \sim 10^{18}$. A toy model for this (much smaller, and square) can be indicated pictorially:



(15.1) eq:toyx

Note in particular here that we have not given explicit vertex labels, and that Z_G does not depend on the details of the vertex label set V_G.

15.0.1 Computation

The rest of our exposition is concerned with the computation of Z_G for certain interesting types of graph; and limits for certain sequences of graphs. The graph is not of intrinsic interest here, it is only a convenient way to codify the interactions between physical degrees of freedom (represented here by the vertices) and hence the Hamiltonian.

Now, keeping Q fixed, recall $x = \exp(\beta)$. Note that the partition function Z_G is an element of $\mathbb{Z}[x]$ (since H_G only takes values in $\mathbb{N}_0$). Let V_{ext} be a subset of V_G. Define a 'vector' $Z_G^{V_{ext}} \in \hom(\hom(V_{ext}, \underline{Q})), \mathbb{Z}[x])$ whose f-th component (i.e. with f an element of $\hom(V_{ext}, \underline{Q})$) is

$$\left(Z_G^{V_{ext}} \right)_f \;=\; \sum_{g \in Q^{V_G} s.t. g|_{V_{ext}} = f} \exp(\beta H_G) \tag{15.2}$$

de:Zf

where $Q^V := \hom(V, \underline{Q})$, and $g|_{V_{ext}} = f$ means g agrees with f on the subset.

Note here that verification of $g|_{V_{ext}} = f$ does depend on which vertex is which in V_{ext}. It will be convenient to have a total order on V_{ext} and then the state f can be given simply by writing out the sequence of images in this order. Thus $f = 12121112$ for example, if V_{ext} is the final column of vertices in (15.1), ordered from top to bottom. Indeed 'locally' V_{ext} of order n might as well be identified with $\underline{n}$ (although care will be needed with this approach later). Thus we have the vertex labels of V_{ext} specified element-wise; with the remaining vertices unlabelled.

(**15.0.1**) Suppose that graph G may be decomposed into G' and G'' with vertices V_{ext} in common, but no edges in common. Then

$$Z_G = \sum_{f \in \hom(V_{ext}, \underline{Q})} \left(Z_{G'}^{V_{ext}} \right)_f \left(Z_{G''}^{V_{ext}} \right)_f$$

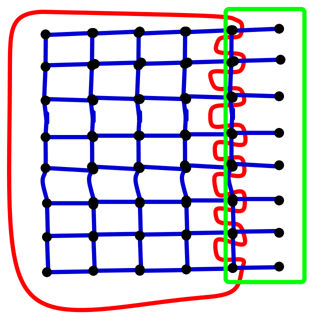
Proof/context. This is elementary here, but it is worth considering the details for contextualisation. For any graph and any vertex subset chain $V_G \supset V' \supset V^x$ we have a decomposition $V_G = V' \cup V''$ where

$$V'' = (V_G \setminus V') \cup V^x$$

and a decomposition of G into G' the full subgraph on V' and G'' the full subgraph on $V_G \setminus V'$ made into a graph on V'' by adding the vertices of V^x as disconnected vertices. Thus every graph has the decomposition property for every chain. ... [finish me]

(**15.0.2**) Now suppose we have V_{in}, V_{out} such that $V_{ext} = V_{in} \cup V_{out}$. We can organise $f \in \hom(V_{ext}, \underline{Q})$ as $f_{in} \in \hom(V_{in}, \underline{Q})$ $f_{out} \in \hom(V_{out}, \underline{Q})$. Note that these may overlap or even be the same. We thus have $\left(Z_G^{V_{in},V_{out}}\right)_{f_{in},f_{out}}$ (we may write $\left(Z_G^{V_{in}\cup V_{out}}\right)_{f_{in},f_{out}}$) defined analogously to $\left(Z_G^{V_{ext}}\right)_f$. Thus $Z_G^{V_{in},V_{out}}$ is a 'matrix' in the same sense as the partition vector above. (Note that if V_{in}, V_{out} do overlap then some partial sums will be empty.)

Suppose that graph G decomposes into G' and G'' with vertices V_{mid} in common, but no edges in common.
Example with G' indicated by a red cordon and G'' a green one:



(the red cordon is meandering merely to avoid any of the edges inside the green one).

Let $V'_{ext} \subset V_{G'}$ be $V'_{ext} = V'_{in} \cup V'_{out}$ such that $V'_{out} = V_{mid}$. Similarly let $V''_{ext} \subset V_{G''}$ be $V''_{ext} = V''_{in} \cup V''_{out}$ such that $V''_{in} = V_{mid}$. Set

$$V_{ext} = V'_{in} \cup V''_{out}.$$

We have

$$\left(Z_G^{V_{ext}}\right)_{f,g} = \sum_{h \in Q^{V_{mid}}} \left(Z_{G'}^{V'_{in}\cup V_{mid}}\right)_{f,h} \left(Z_{G''}^{V_{mid}\cup V''_{out}}\right)_{h,g} \tag{15.3}$$ eq:pmatx

Typically $V'_{in}, V_{mid}, V''_{out}$ may be disjoint in V_G, but not necessarily.

15.0.2 Dichromatic polynomial

The dichromatic polynomial for a graph G is a recasting of the Q-state Potts partition function which has Q as an indeterminate.

This is useful from a number of perspectives. One is that it presents a variation of the computational problem, useful for context and comparison.

Observe that we can write

$$Z_G = \sum_{states\ \sigma} x^{H_G(\sigma)}$$

[finish me]

15.0.3 Exercise: the calculus expressed categorically

As an exercise, we can see this setup as a kind of subcategory Z^Q of the category $\mathbf{Mat}^Q$ over the ring $\mathbb{Z}[x]$. (From this perspective we are joining underlying graphs rather than cutting one up. There is more information in a graph and a 'cut' than in the nominal components after cut, so this information has to be retained somehow.) The object class is $\mathbb{N}_0$ and the n,m-morphisms are partition matrices associated to some graph G as above with the in and out subsets of V_G 'identified' with (distinct copies of) $\underline{n}$ and $\underline{m}$ respectively. Note here that a morphism would arise by a specific identification, but the identification is not recorded as such in the morphism.

Note that we need to check that the **Mat** category composition closes on partition matrices; that the identity morphisms are present; how the monoidal composition works, and so on. But let us start with an example to illustrate the 'identification' process.

As an extreme example, we can consider $G = (\{1,2\}, \emptyset)$, that is just two isolated vertices with no edges, together with some identifications as follows. With the identity map identifications on both sides we have the morphism, i.e. the matrix, Id_4 in $\mathsf{Z}^2(2,2)$. With the identity map identification of in states and the obvious flip map for out states we have $\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$ in $\mathsf{Z}^2(2,2)$.

Here we have in mind the 'basis' 11,12,21,22 for rows; and for columns. Let us consider an example requiring more thought about conventions. ... Probably we need to be at least in $\mathsf{Z}^3(3,3)$ ...

15.0.4 And more categories therein

Guided by the computational statistical mechanics context we can abstract to a *relative graph category* $\mathfrak{G}_0$ (a category of graphs as morphisms, with subsets of vertices as objects, composed by graph gluings at these subsets; different, note, from the category of graphs and graph morphisms). The idea is that (fixing Q) each graph yields a partition function; and each relative graph yields a partition matrix; and so the composition of partition matrices as in (15.3) draws back to an underlying relative graph composition. There are a few possible ways of setting up the relative graph composition. Informally one version of this is as follows.

We take $\mathbb{N}_0$ as object class and (n,m)-graphs as morphisms. An (n,m)-graph is a graph G together with injective 'structure' maps $\lambda_i : \underline{n} \hookrightarrow V_G$ and $\lambda_o : \underline{m}' \hookrightarrow V_G$. We write $\mathfrak{G}_0(n,m)$ for the set of (n,m)-graphs. The composition

$$* : \mathfrak{G}_0(l,m) \times \mathfrak{G}_0(m,n) \to \mathfrak{G}_0(l,n)$$

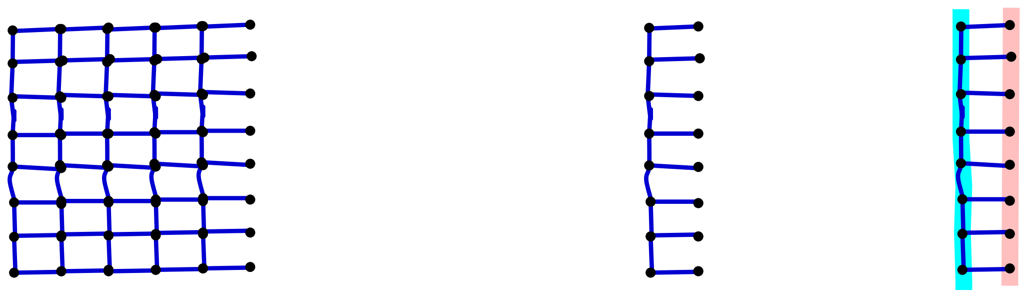
is given on $((G,\lambda_i,\lambda_o),(G',\lambda'_i,\lambda'_o))$ by first forming the disjoint union graph; then forming the partition of vertices by putting $\lambda_o(i)$ with $\lambda'_i(i)$ for all $i \in \{1,2,...,m\}$; then forming the quotient graph with respect to this partition. Finally we take

$$(G,\lambda_i,\lambda_o) * (G',\lambda'_i,\lambda'_o) = (G \sqcup G'/\sim, \lambda_i, \lambda'_o).$$

Let us explain the terms and check that this is well-defined. (Then we can give examples and check associativity.)

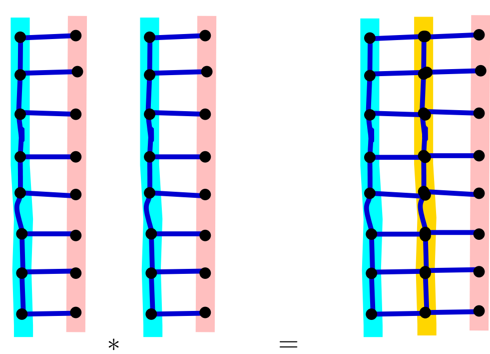
Pictures serve as a natural shorthand for specifying graphs of this type, much as one could visualise a crystal lattice. The lattice refers to the regular pattern of positions of atoms - represented by vertices. The edges represent the largest of the pairwise interactions, which are assumed to depend on separation, so the edges merely indicate the pairs that are close neighbours - a component that is strictly unnecessary if the positions can be represented accurately.

We typically want to consider graphs with some kind of translational symmetry, for example of the form indicated on the left here:



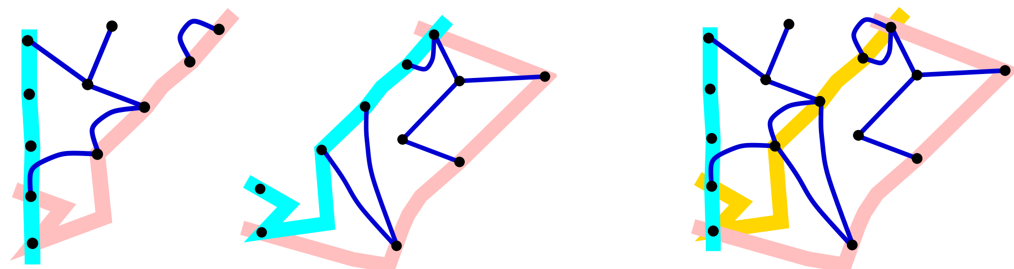
— the vertices are unlabelled here, but they could be given labels for example by using their coordinates (x, y) in the plane. We decompose such graphs into several 'layer' graphs as shown on the right, made relative with in and out vertices as indicated by cyan and pink highlight bands respectively. We have in mind specifically that the highlight band indicates in-vertex (resp. out-vertex) label 1 at the top, and so on proceeding downwards.

Composition is as here:



with in and out vertices again as indicated by the cyan and pink highlights respectively; and composition outcome as shown on the right, identifying 'middle' layer vertices (gold highlight) in the appropriate order.

Note however that the formalism allows more complicated cases:



The first picture here represents a relative graph in which λ_i maps to V_G hitting the vertices in the blue line in order from top to bottom. And λ_o maps to V_G hitting the vertices in the pink line in order from the top to the other end. Thus in particular the vertex labelled by λ_i as 4 is labelled by λ_o as 6.

Remark: It is probably not necessary to allow changes of order between intersecting in and out like this. Or indeed to allow them to intersect at all unless $n = m$. ...

Comments: ...

... [section to do! poss. following from p.101 of [?].]

Chapter 16

On representations of the partition algebra

ch:pa

Contents

For each commutative ring k and $\delta \in k$ the partition algebras over k may be constructed as the end-algebras of a certain k-linear category P with object class $\mathbb{N}_0$ [106, 108]. The category P, and its end-algebras, arise naturally in computational statistical mechanics [104].

The category P is useful in representation theory (both for the representation theory of its end-algebras and more generally), so we construct it first here, in §16.1.

In §16.2 we collect some useful arithmetic properties of P. The main focus of the rest of the Chapter is on illustrating aspects of representation theory of the partition algebras P_n ($n \in \mathbb{N}_0$) — in particular illustrating techniques that are useful in the wider context of 'diagram algebras' (the initial layout of the Chapter matches that of the corresponding Chapter 18 on Brauer algebras, with the idea that they may instructively be compared and contrasted).

We largely aim at the determination of Cartan decomposition matrices over various fields k — the fundamental invariants of representation theory in this setting (thus reviewing, for example, the decomposition matrix results of [108]). In §16.3 we construct two kinds of analogues of Specht modules [144, 77, 76] that are useful in treating 'modular towers' for P. (These are modules that are defined over an 'integral' ground ring that become simple modules upon base change to a suitable field.)

In §16.4 we discuss various globalisation functors. These facilitate the treatment of decomposition matrices for all ranks simultaneously, and of 'translation' functors — the modular tower formalism. But we will also describe properties of the 'algebra' of such functors. Another component of translation functors is Frobenius reciprocity (of induction and restriction between layers of a tower). And this leads us to consider, essentially in an aside, in §16.6, aspects of the combinatorics of set partitions.

In §16.8 we look at the basics of representation theory for specific k, combinatorial and geometric 'linkage' and so on (cf. e.g. [79, 68]) ... In §16.9 we review the basics of an approach via generalised Schur algebras [120]. Further bibliographic notes are in §16.11.

Recall the Glossary in 3.1.2.

16.1 Partition categories

ss:pcat

We can use a 'diagram calculus' to describe composition of 'morphisms' in the partition category. We do this in §16.1.2 *et seq*, but it is convenient to start more formally.

We recall the following notations from §3.2.3 (see also §1.3, §2.7).

For S a set, P_S is the set of partitions of S. For $n, m \in \mathbb{N}_0$ then $\underline{n} = \{1, 2, ..., n\}$, $\underline{n}' = \{1', 2', ..., n'\}$ and so on. Then $\mathsf{P}_{n,m} := \mathsf{P}_{\underline{n} \cup \underline{m}'}$, and $\mathsf{P}_n := \mathsf{P}_{n,n}$. Also E_S is the set of equivalence relations on S (and we may apply the 'standard bijection' $\mathsf{E}_S \leftrightarrow \mathsf{P}_S$ without further comment).

16.1.1 Bare partition categories

For a, b equivalence relations, we define ab as the transitive closure of the relation $a \cup b$:

$$ab \;:=\; \overline{a \cup b}$$

If $b \in \mathsf{P}_{l,m}$ then $b' \in \mathsf{P}_{\underline{l}' \cup \underline{m}''}$ is obtained by adding a (further) prime to every element. For example $\{\{1\}, \{1', 2', 3'\}\}' = \{\{1'\}, \{1'', 2'', 3''\}\}$.

Notation. When no ambiguity arises, we sometimes tidy up partitions by using the convention from [104], which writes $\{\{1, 3\}, \{1', 2, 2', 3', 4'\}\}$ as $(13)(1'22'3'4')$ and so on. Then our overall-prime transformation above becomes: $(1)(1'2'3')' = (1')(1''2''3'')$.

If $c \in \mathsf{P}_{\underline{n} \cup \underline{l}' \cup \underline{m}''}$ then $r(c) \in \mathsf{P}_{\underline{n} \cup \underline{m}''}$ is obtained by restriction. Example: $r((121'')(1')(2'3)) = (121'')(3)$.

If $c \in \mathsf{P}_{\underline{l} \cup \underline{m}''}$ then $u(c) \in \mathsf{P}_{\underline{l} \cup \underline{m}'}$ is obtained by removing a prime from double-primed elements. Example: $u((121'')(3)) = (121')(3)$.

We then define a product $\mathsf{P}_{n,l} \times \mathsf{P}_{l,m} \to \mathsf{P}_{n,m}$ by

$$a \circ b = u(r(ab'))$$

Note that ab' uses the implicit identification of partitions and equivalence relations (in both directions).

Example:

$$(121') \circ (12')(1') = u(r(\overline{(121') \cup (1'2'')(1'')})) = u(r((121'2'')(1''))) = u((122'')(1'')) = (122')(1').$$

Remark. Some different conventions for composition are possible ('conventions' in the sense that an isomorphic construction arises). Here, in summary, if we compose as $a \circ b$ then the unprimed elements from a end up as the unprimed elements of $a \circ b$; and the primed elements of b end up as the primed elements. The primed elements of a and unprimed elements of b are discarded.

For example, both $(1) \circ (1')$ and $(1') \circ (1)$ make sense. Here we have $(1) \circ (1') = (1)(1')$ and $(1') \circ (1) = ()$ (the unique partition of the empty set). [-use this to check conventions later!]

(16.1.1) THEOREM. *The triple* $\mathsf{P}^\bullet = (\mathbb{N}_0, \mathsf{P}_{n,m}, \circ)$ *gives a category.*

Proof. We will treat this as a special case of the following Theorem 16.1.5. □

(16.1.2) If $p \in \mathsf{P}_S$ and T a set, then $\#^T(p)$ denotes the (cardinal) number of parts of p which contain only elements of T.

(16.1.3) Now consider a similar formal categorical triple

$$\mathsf{P}^o = (\mathbb{N}_0, \mathsf{P}_{n,m} \times \mathbb{N}_0, *)$$

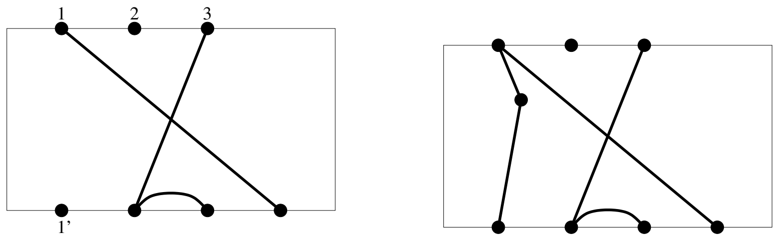


Figure 16.1: Graph/Partition diagram examples for $\Gamma_{3,4}$. fig:part diag gr

where $*$ is defined as follows. Consider $a*b$ for $a=(a_1,a_2)$, $b=(b_1,b_2)$, with $a_1 \in \mathsf{P}_{n,l}$ and $b_1 \in \mathsf{P}_{l,m}$. Then

$$(a_1,a_2)*(b_1,b_2) \;=\; (a_1 \circ b_1,\; a_2+b_2+\#^{\underline{l}'}(a_1 b_1'))$$

We call the $\#^T(-)$ in this setting the *vacuum number*.

(**16.1.4**) Example. Let $a = (\{\{1\},\{1'\}\},0)$ and $b = (\{\{1\},\{1'\}\},0)$. Then $b_1 = \{\{1\},\{1'\}\} \in \mathsf{P}_{1,1}$ and $b_1' = \{\{1'\},\{1''\}\}$. Thus $a_1 b_1' = \{\{1\},\{1'\},\{1''\}\}$ and $a*b = (\{\{1\},\{1'\}\},1)$.

th:p cat (**16.1.5**) THEOREM. *The triple* P^o *is a category.*

Proof. Observe that P^o has the formal structure of a category, so it remains to check the axioms, as in §6.1. The main issue is associativity of $*$ (and the other issue is identity elements). Partition diagrams provide a convenient way to give (or follow) the argument, and are indeed generally useful, so we discuss these next. (For a direct approach see [106]. See also §3.2.3.) We complete the proof in (16.1.24).

16.1.2 Graphs, partition diagrams: towards proof of theorem 16.1.5

ss:pdiagramat

A visualization for elements of $\mathsf{P}_{n,m}$ proceeds via certain graphs as follows. (See also §1.3.2.)

16.1.3 Graphs revisited

(**16.1.6**) Aside. (See also §3.5.) Here a *proper digraph* $G=(V,E,s,t)$ is two sets V,E, and two maps $s,t : E \to V$; and a *digraph* is a class of proper digraphs under bijective mappings of the 'edge' set E - that is, the edge labels are forgotten.

In other words a digraph G is equivalent to a map μ_G from $V \times V$ to $\mathbb{N}_0$ recording the multiplicities of edges between pairs of vertices.

(**16.1.7**) A *proper graph* $G=(V,E,I)$ is two sets V,E, and a relation $I \subset V \times E$ such that for each $e \in E$ the set there are either one or two v's such that $(v,e) \in I$. And a *graph* is a class of proper graphs under bijections of E.

In other words a graph G on V is equivalent to a map $\mu_G : V \times V \to \mathbb{N}_0$ that is symmetric: $\mu_G(v,v') = \mu_G(v'v)$.

(**16.1.8**) Relative graphs. The idea here is to capture formally the idea of transfer matrices.

For $n, m \in \mathbb{N}_0$, an *nm-pregraph* is a graph on some V together with injective 'structure' maps $\lambda : \underline{n} \to V$ and $\lambda' : \underline{m'} \to V$.

Note that $\lambda(\underline{n})$ and $\lambda'(\underline{m'})$ may not be disjoint. For $s \in V \setminus (\lambda(\underline{n}) \cup \lambda'(\underline{m'}))$ we may write i_s for this vertex (i for 'internal').

An *nm-relative graph*, or just nm-graph, is a class of nm-pregraphs under bijections of V that commute with the structure maps.

By convention, the structure maps will be assumed to be inclusions unless otherwise stated.

(**16.1.9**) We write $\Gamma_{n,m}$ for the set of nm-graphs.

Observe that this $\Gamma_{n,m}$ is a set, not merely a class in the set-theoretic sense.

16.1.4 Drawings of graphs

[the following is now unfinished cf the above...]

(**16.1.10**) Draw a rectangular frame, and draw n vertices on the northern edge (labelled $1, 2, ..., n$ in the natural order); m vertices on the southern edge; and finitely many vertices in the interior. Consider a graph on these drawn vertices. Specifically an n, m-graph. The vertices labelled by λ are those represented on the northern edge; and λ' on the southern edge. If a vertex is labelled by both we draw a 'thick' line between the two formal vertices. (In practice we will be able to avoid such thick lines almost entirely; and indeed make both structure maps inclusions.) We draw the graph with edges drawn as intervals regularly embedded [31] in the rectangle (i.e. such that each edge touches the vertices only at its end-points; and touches the rectangle-boundary at most at its end-points; and at most two edges touch at any other point, and if so then transversally — as in figure 16.1).

Such a drawing (of an embedding) determines the graph, up to the specifics of the interior vertices. We will consider these vertices as unlabelled (or equivalently that such a 'graph' is really a class of graphs corresponding to all possible interior labellings).

(**16.1.11**) We write $\Gamma_{n,m}$ for the set of such 'graphs'.

(**16.1.12**) Example. The second drawing in Fig.16.1 gives a 3,4-graph with vertex set $V = \{1, 2, 3\} \cup \{i_1\} \cup \{1', 2', 3', 4'\}$, where i_1 stands for any suitable label.

(**16.1.13**) Example. The graph in the middle of (1.28) may be given as follows.
Firstly the vertices

$$V = \{1, 2, 3, 4\} \cup \{i_1, i_2, i_3, i_4, i_5\} \cup \{1', 2', 3', 4'\}$$

where the first and last subsets are given by the $\Gamma_{4,4}$ context, and the middle one is a choice (i_2 is chosen here with i for 'interior' and 2 for the second interior vertex in English reading order in the picture! - recall these names are significant only intrinsically to the graph - for example in giving edge data as we will do next; from an external perspective they are arbitrary).
Then the edges

$$E = \{1{-}4,\ 1{-}i_1,\ 3{-}i_2,\ i_1{-}i_5,\ i_1{-}4', i_2{-}2', i_5{-}1', 2'{-}3'\}$$

- here we write an edge connecting $\{i, j\}$ simply as $i{-}j$. Here there are no multiple edges. We can write $i{-}j^a$ or $i \overset{a}{-} j$ to indicate when there are $a > 1$ edges.

16.1.5 Composition of graphs

de:Gaacomp

(**16.1.14**) [before the drawn version maybe do the set-theory version??] We aim to construct a composition

$$\star : \Gamma_{n,l} \times \Gamma_{l,m} \to \Gamma_{n,m}$$

In particular for (d, d') on the LHS, we claim that there exists $d'' \in \Gamma_{n,m}$ such that it can be decomposed into d''_1 and d''_2 with l vertices - vertex subset $\underline{l}''$ say - in common and no edges in common; such that replacing $\underline{l}''$ with $\underline{l}''$ in the obvious way takes d''_1 to d; and replacing $\underline{l}''$ with $\underline{l}$ in the obvious way takes d''_2 to d'. [...finish me.]

Consider a graph/drawing d for a graph in $\Gamma_{n,l}$, and d' for a graph in $\Gamma_{l,m}$. We call these *juxtaposable* if they can be juxtaposed in such a way that the l vertices on the l-vertex edges meet up (and become 'interior' vertices of the resultant graph/drawing). It will be evident that juxtaposable representatives exist, so take in particular any such pair d, d'. Now form their juxtaposition. Call this new graph dd'.

Note that dd' has three primary rows of vertices (together with any interior vertices of d and d'): a top row of n; a 'middle' interior row of l; and a bottom row of m. By forgetting/internalising the l labels, we thus have $dd' \in \Gamma_{n,m}$. In other words we have a composition

$$\star : \Gamma_{n,l} \times \Gamma_{l,m} \to \Gamma_{n,m} \tag{16.1}$$ eq:Gaa1

(we write dd' for $d \star d'$).

(**16.1.15**) Exercise. Make some examples of compositions. For example transliterate the pictures in (1.28) to the in-line realisation as above. (Observe that the pictures encode the calculus faithfully, and are relatively easy to follow! That is perhaps why it is sometimes called diagram calculus.)

le:GaaaId

(**16.1.16**) In the case of $\Gamma_{n,n}$ note that we can in principle allow the unspecified height of the frame used in the construction to pass to zero, allowing the northern and southern runs of n vertices to coincide. In this case there is no 'room' for any edges, and in this sense the graph is forced - call it I_n.
The partition of vertices into connected components here (as used in (16.1.19)) is necessarily simply into the degenerated pairs $\{i, i'\}$ (in the sense that there is a walk of length zero from the vertex with label i to the vertex with label i').
Allowing this degenerated kind of graph gives us an identity element for the composition (16.1) in rank n.

le:Gaaassoc

(**16.1.17**) Now consider $d \in \Gamma_{n,l}$, $d' \in \Gamma_{l,l'}$ and $d'' \in \Gamma_{l',m}$ (say), and hence $(dd')d''$. Observe that this lies in $\Gamma_{n,m}$ by forgetting the labels in the l, l' layers. Next consider $d(d'd'')$. This is the same as $(dd')d''$ up to the names of the interior vertices. But we forget these names, so the two expressions give the same element of $\Gamma_{n,m}$.

(**16.1.18**) In light of 16.1.16 and 16.1.17 we have a category

$$\Gamma \ = \ (\mathbb{N}_0, \Gamma_{n,m}, \star).$$

ss:pcatcong

16.1.6 Congruences on category Γ and proof of theorem 16.1.5

pa:picon1

(**16.1.19**) A graph such as in Γ_{nm} describes a partition: two vertices are in the same part if there is a chain of edges between them.

Note that the map from the set Γ_{nm} of such graphs (or indeed of such drawings) to the set of partitions of the vertices is surjective. Of course it is not injective in general.

(**16.1.20**) Such a graph, G say, determines a set $b(G)$ of symmetric binary relations on its vertices: there is a pair $(x, y) \in b(G)$ iff there is a direct edge (x, y) in G. One may then form the reflexive, transitive closure of $b(G)$. This determines the partition of the vertices — which may be restricted to a partition of the edge vertices.

If G is such an embedded graph (or graph, or picture) we shall write $[G]$ for the corresponding partition in $\mathsf{P}_{n,m}$. We can go further and write $[[G]] \in \mathsf{P}_{n,m} \times \mathbb{N}_0$ for the element (a_1, a_2) where $a_1 = [G]$ and a_2 is the vacuum number (in the obvious sense in this case - the number of connected components of G involving only unlabelled vertices). Observe again that $[[-]] : \Gamma_{n,m} \to \mathsf{P}_{n,m} \times \mathbb{N}_0$ is surjective, since we can add isolated connected components to G to achieve any value for the second component of $[[G]]$ - which we can write as $[[G]]_2$.

Indeed a graph G on any collection of vertices together with a suitably injective 'structure' map λ_G from $\underline{n} \cup \underline{m}'$ to the vertex set of G defines an element of $\mathsf{P}_{n,m}$ similarly (see e.g. [119]). We have in mind something slightly less general for now however.

(**16.1.21**) Consider the frame construction as above. An 'exterior component' is a graph component that meets the rectangle-boundary vertices.

[Delete:] Although $[d]$ is still well-defined for such a graph d, it will be convenient to *exclude* as representatives of partitions graphs with interior components disconnected from the exterior components.

(**16.1.22**) In P^o, the composition $a * b$, for the object triple (n, l, m) say, can then be envisaged as follows. Consider a graph/drawing d such that $[d] = a_1$ and $[[d]]_2 = a_2$; and a d' such that $[d'] = b_1$ and $[[d']]_2 = b_2$. Now form dd' as in (16.1.14).

(**16.1.23**) Suspending the label-forgetting momentarily in $\star$ in (16.1) (just so we have names to call the l-vertices by), the graph dd' thus defines, in particular, a partition $p_-(dd')$ of $\underline{n} \sqcup \underline{l} \sqcup \underline{m}$, and hence in a natural way an element $p(dd')$ of $\mathsf{P}_{\underline{n} \cup \underline{l}' \cup \underline{m}''}$. Specifically

$$p_{n,l,m}(dd') = ab'$$

To see this, pick an x in $\underline{n} \cup \underline{l}' \cup \underline{m}''$ and note that $x \sim^{ab'} y$ if (and only if) there is a path from x to y in dd'.

Comparing the constructions for $[-]$ and dd' with the definition of $\circ$ we see (1) that

$$[dd'] = [d] \circ [d']$$

and (2) that there may be some components of the graph disconnected from the top and bottom edges; and (if d, d' did not have any such) the number of these is the vacuum number. More generally the vacuum number of the composite will be $a_2 + b_2$ plus any new contributions created in composition - given by the bare vacuum number. That is

$$[[dd']] = [[d]] * [[d']]$$

In other words, $[[-]]$ defines a congruence on Γ.

pf:p cat (**16.1.24**) *proof of Theorem 16.1.5:* Recall from (16.1.17) that

$$(dd')d'' = d(d'd'')$$

for any suitably juxtaposable triple of graphs. Now note that the computation of $a * (b * c)$ and of $(a * b) * c$ can be done by drawing the same diagram $dd'd''$ (where $[[d'']] = c$). This verifies associativity of $*$.

□

(**16.1.25**) We can go further. Define

$$[[[-]]] : \Gamma_{n,m} \to \mathsf{P}_{n,m} \times \mathbb{N}_0 \times \mathbb{N}_0$$

by $[[[-]]]_1 = [-]$; $[[[-]]]_2 = [[-]]_2$; $[[[-]]]_3 = ...$ FINISH ME!

16.1.7 More partition categories: k-linearity; generalisations

ss:Pcat

(**16.1.26**) Let k be a commutative ring. We may formally extend P^o to a k-linear category. For each $\delta \in k$ the relation

$$(a_1, b_1 + 1) \;\sim\; \delta(a_1, b_1)$$

defines a congruence on the k-linear category, and hence a quotient category P_k^δ.

$$\mathsf{P}_k^\delta = (\mathbb{N}_0, k\mathsf{P}_{n,m}, \cdot)$$

Note that the end-sets, $\hom_{\mathsf{P}_k^\delta}(n, n)$, in the category P_k^δ are k-algebras. For any given k, the n-th case has basis $\mathsf{P}_{n,n}$ and is the partition algebra $P_n = P_n(\delta)$ over k.

(**16.1.27**) In particular the 'integral' version of the partition algebra is the case over the ring $k = \mathbb{Z}[\delta]$. From here there are two aspects to the base change to an algebra over a field: the choice of field k and the choice of δ in k. We have possible intermediate steps: base change to ring $k[\delta]$ (k now some field); base change to $\mathbb{Z}$ (a $\mathbb{Z}[\delta]$-algebra by fixing $\delta \in \mathbb{Z}$).

Each of these ground rings is a PID and hence a Dedkind domain, and hence amenable to a P-modular treatment (in the sense of Brauer's general approach to 'modular' representation theory [14]). Furthermore the $k[\delta]$ case turns out to be semisimple over the field of fractions, see **??**, so we have constructions for sets of integral Δ-modules which are pivotal in Brauer-Humphreys reciprocity — that is, the Cartan matrix can be computed via Δ-decomposition matrices.

(**16.1.28**) For more details on this see §**??** (the Brauer algebra case) for now (or cf. say Benson [**?**]).

(**16.1.29**) In our construction we forced the top and bottom vertex sets of $P_n(\delta)$ to be disjoint. However this is not necessary. We define algebra $P_{n+}(\delta)$ similarly, except that the basis is the subset of elements of $\mathsf{P}_{n+1,n+1}$ is which $n + 1$ and $n + 1'$ are identified (or equivalently are always in the same part). Thus P_{n+} has a basis of partitions of a set of $2n + 1$ objects.

(**16.1.30**) There are a large number of other interesting generalisations and subcategories of the partition category. For example, the product closes on the span of partitions of at most two parts. Mazorchuk calls the corresponding algebra the *rook* or *partial Brauer algebra*, so we have the 'rook Brauer category'. We shall write $RB_{n,m}$ for the corresponding subset of $\mathsf{P}_{n,m}$, and so on.

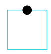
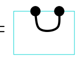
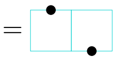
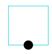
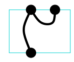
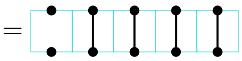
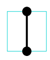
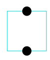
$\mathsf{v} = \{\{1\}\} =$ 	$\mathsf{U} = \{\{1,2\}\} =$ 	$\mathsf{u} = \mathsf{v} \otimes \mathsf{v}^\star =$ 
$\mathsf{v}^\star = \{\{1'\}\} =$ 	$\Gamma = \{\{1,2,1'\}\} =$ 	$\mathsf{u}_1 := \mathsf{u} \otimes 1 \otimes 1 \otimes ... \otimes 1 =$ 
$1 = \{\{1,1'\}\} =$ 	$\sigma = \{\{1,2'\},\{2,1'\}\}$	$\mathsf{u}_2 := 1 \otimes \mathsf{u} \otimes 1 \otimes ... \otimes 1$
$\mathsf{u} = \{\{1\},\{1'\}\} =$ 	$\square = \{\{1,2,1',2'\}\}$	$\mathsf{e} := \mathsf{U} \otimes \mathsf{U}^\star$

Table 16.1: Set partitions: examples and notations tab:part1-15

16.1.8 Examples and useful notation for set partitions

This section is reproduced from §1.3.

dedsifdbp-15 (**16.1.31**) See Table 16.1 for examples and notations.

Given a partition p of some subset of $N(n,m) = \underline{n} \cup \underline{m'}$, take $p^\star$ to be the image under toggling the prime.

de:pa tensor (**16.1.32**) Define partition $p_1 \otimes p_2$ by side-by-side concatenation of diagrams (and hence renumbering the p_2 factor as appropriate). See Table 16.1 for examples.

de:pnotations-15 (**16.1.33**) We say a part in $p \in \mathsf{P}_{n,m}$ is *propagating* if it contains both primed and unprimed elements. Write $\mathsf{P}_{n,l,m}$ for the subset of $\mathsf{P}_{n,m}$ with l propagating parts; and $\mathsf{P}^l_{n,m}$ for the subset of $\mathsf{P}_{n,m}$ with at most l propagating parts. Thus

$$\mathsf{P}^l_{n,m} = \bigsqcup_{l=0}^{l} \mathsf{P}_{n,l,m} \qquad \text{and} \qquad \mathsf{P}_{n,m} = \bigsqcup_{l=0}^{n} \mathsf{P}_{n,l,m}.$$

E.g. $\mathsf{P}_{2,2,2} = \{1 \otimes 1, \sigma\ \}$, $\mathsf{P}_{2,1,1} = \{\mathsf{v} \otimes 1, 1 \otimes \mathsf{v}, \Gamma\ \}$, $\mathsf{P}_{2,0,0} = \{\mathsf{v} \otimes \mathsf{v}, \mathsf{U}\ \}$ and

$$\mathsf{P}_{2,1,2} = \mathsf{P}_{2,1,1}\mathsf{P}_{1,1,2} = \{\mathsf{u} \otimes 1, 1 \otimes \mathsf{u}, \mathsf{v} \otimes 1 \otimes \mathsf{v}^\star, \mathsf{v}^\star \otimes 1 \otimes \mathsf{v}, \Gamma\Gamma^\star, ...\ \}.$$

Note that $\mathsf{P}_{n,n,n}$ spans a multiplicative subgroup:

$$\mathsf{P}_{n,n,n} \cong S_n \qquad (16.2)$$ eq:PnSnsub-15

where S_n is the symmetric group as in (1.30), Ch.11, (1.4.66), et seq.

(**16.1.34**) Define $L : \mathsf{P}_{n,l,m} \to S_l$ by deleting all but the (top and bottom row) leftmost elements in each propagating part, and renumbering consecutively. Define

$$\mathsf{P}^L_{n,l,m} \ = \ \{p \in \mathsf{P}_{n,l,m} \mid L(p) = 1 \in S_l\}$$

(**16.1.35**) Fix k and $\delta \in k$. We have $P_0 \cong k$, $P_1 = k\{1, \mathsf{u}\}$ and

$$P_2 = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \mathsf{P}_{2,0,2}) = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \{\mathsf{U} \otimes \mathsf{U}^\star, (\mathsf{v} \otimes \mathsf{v}) \otimes \mathsf{U}^\star, (\mathsf{v} \otimes \mathsf{v})^\star \otimes \mathsf{U}, \mathsf{u} \otimes \mathsf{u}\}).$$

We have $\mathsf{u}^2 = \delta\mathsf{u}$ and $\mathsf{v}^\star\mathsf{v} = \delta\emptyset$ and $\mathsf{v}\mathsf{v}^\star = \mathsf{u}$.

16.2 Basic properties of partition categories

ss:partition9

(**16.2.1**) We make
$$\mathsf{P} \;:=\; \cup_{n,m} \mathsf{P}_{n,m}$$
a monoid $(\mathsf{P}, \otimes)$ by lateral composition (as in (16.1.32)).

(**16.2.2**) Note that there is a unique element in $\mathsf{P}_{1,0}$. Let us define v, v^* as the unique elements of $\mathsf{P}_{1,0}$ and $\mathsf{P}_{0,1}$ respectively. Write
$$\mathsf{u} \;:=\; \mathsf{v} \otimes \mathsf{v}^* \quad \in \mathsf{P}_{1,1}.$$

(**16.2.3**) Write 1_n for the identity element in P_n. If we write $\mathsf{u} \in \mathsf{P}_{n,n}$ we shall mean $\mathsf{u} \otimes 1_{n-1}$. We shall extend this notation in the obvious way to other elements.

(**16.2.4**) There is a sequence of unital algebra injections
$$P_n \subset P_{n+} \subset P_{n+1}$$
— the first is by $p \mapsto p \otimes 1_1$; the second is inclusion.

de:hashp

(**16.2.5**) Define
$$\# : \mathsf{P}_{n,m} \to \mathbb{N}_0$$
by $p \mapsto \#$ propagating parts (as in (16.1.33)).

16.2.1 Initial filtration: Propagating ideals

(**16.2.6**) Define $\mathsf{P}^l_{n,m} \;:=\; \mathsf{P}_{n,l} \circ \mathsf{P}_{l,m} \quad \subset \mathsf{P}_{n,m}$ as in (16.1.33). Note that this is the subset of partitions with at most l propagating parts. Define
$$\mathsf{P}^{=l}_{n,m} \;:=\; \mathsf{P}_{n,l,m} \;=\; \mathsf{P}^l_{n,m} \setminus \mathsf{P}^{l-1}_{n,m}$$

Automatically then, we have the following.

de:Pfilt1

(**16.2.7**) Proposition. For any k, $\delta \in k$,
$$k\mathsf{P}_{n,n} \supset k\mathsf{P}^{n-1}_{n,n} \supset k\mathsf{P}^{n-2}_{n,n} \supset \ldots \supset k\mathsf{P}^0_{n,n} \qquad (16.3)$$
eq:Ppure1

is a chain of two-sided ideals in P_n. The l-th ideal, $k\mathsf{P}^{n-l}_{n,n}$, is generated by $\mathsf{u}^{\otimes l} \otimes 1_{n-l}$:
$$k\mathsf{P}^{n-l}_{n,n} \;=\; P_n(\mathsf{u}^{\otimes l} \otimes 1_{n-l})\, P_n$$
or indeed by any partition with $n - l$ propagating parts.
The l-th section, counting from the *right*, has basis $\mathsf{P}^{=l}_{n,n}$. Let us write $\mathsf{P}^{l/}_{n,n}$ for this section. □

(**16.2.8**) Write
$$P_n^{l+1/} \;=\; P_n^{/l} \;:=\; P_n / k\mathsf{P}^l_{n,n}$$
for the quotient algebra. (Note the significance of position of the slash.)

pr:Pbimnm

(**16.2.9**) Proposition. A bimodule $\mathsf{P}^{l/}_{n,m}$ may be defined similarly to $\mathsf{P}^{l/}_{n,n}$.

Proof. This is a section in an analogous sequence to (16.3). □

(**16.2.10**) In particular $\mathsf{P}^{l/}_{n,l}$ has the nice property that its basis $\mathsf{P}^{=l}_{n,l}$ consists of all diagrams in $\mathsf{P}^{l}_{n,l}$ such that each vertex on the bottom edge is in a distinct propagating part.

Note that $(\mathsf{P}^{=n}_{n,n}, \circ)$ gives a copy of the symmetric group S_n.

lem:flipinvheck (**16.2.11**) Consider $\mathsf{P}^{=l}_{l,n} \circ \mathsf{P}^{=l}_{n,l}$ for a moment. Evidently some of these diagrams lie in $\mathsf{P}^{=l}_{l,l}$ and some in $\mathsf{P}^{<l}_{l,l}$. Note however the following.

LEMMA. The particular products of form $p \circ p^\star$ in $\mathsf{P}^{=l}_{l,n} \circ \mathsf{P}^{=l}_{n,l}$ all equal 1_l in $\mathsf{P}^{=l}_{l,l}$.

Proof. The propagating parts in p take the form $\{i, j'_1, j'_2, ...\}$ for $i = 1, 2, ..., l$; and for each such part there is a corresponding part $\{j_1, j_2, ..., i'\}$ in $p^\star$. Thus the composition gives a part $\{i, i'\}$ for each i. □

(**16.2.12**) Note that there is a unique element in $\mathsf{P}^{=1}_{1,1}$, denoted 1_1. There are two elements in $\mathsf{P}^{=2}_{2,2}$, one of which is $1_2 := 1_1 \otimes 1_1$. Write (12) for the other element.

(**16.2.13**) We may consider the parts of $p \in \mathsf{P}_{n,m}$ that meet the top set of vertices to be totally ordered by the natural order of their lowest numbered elements (from the top set). We may define a corresponding order for parts that meet the bottom set of vertices. We say that p is *non-permuting* if the subset of propagating parts has the same order from the top and from the bottom.

For $l \leq n$ let $\mathsf{P}^{1_l}_{l,n}$ denote the subset of $\mathsf{P}^{=l}_{l,n}$ of non-permuting partitions; and analogously $\mathsf{P}^{1_l}_{n,l}$ for the corresponding subset of $\mathsf{P}^{=l}_{n,l}$. We may write $\mathsf{P}^{||}_{l,n}$ for either when unambiguous. That is, for $l \leq n$,

$$\mathsf{P}^{1_l}_{l,n} = \mathsf{P}^{L}_{l,l,n}.$$

de:Psect1 (**16.2.14**) Now consider the section $\mathsf{P}^{l/}_{n,n}$ as a left-module. We have the left-module isomorphism

$$\mathsf{P}^{l/}_{n,n} \;\cong\; \bigoplus_{w \in \mathsf{P}^{||}_{l,n}} \mathsf{P}^{l/}_{n,l} w$$

Every summand is isomorphic to $\mathsf{P}^{l/}_{n,l}$.

(**16.2.15**) PROPOSITION. *Regarded as a set of classes in the obvious way, $\mathsf{P}^{=l}_{n,l}$ gives a basis for $\mathsf{P}^{l/}_{n,l}$. (We may write $[\mathsf{P}^{=l}_{n,l}]$ to emphasise the set of classes $[b]$.)* □

16.2.2 Polar decomposition

pa:usefac3 (**16.2.16**) The set $\mathsf{P}^{=l}_{n,l}$ has a useful factorisation:

$$\mathsf{P}^{=l}_{n,l} = \mathsf{P}^{||}_{n,l} \circ \mathsf{P}^{=l}_{l,l} \tag{16.4}$$ eq:psbas

where $\mathsf{P}^{=l}_{l,l} \;\cong\; S_l$.

(**16.2.17**) Indeed for $l < n, m$

$$\mathsf{P}^{=l}_{n,m} = \; \mathsf{P}^{||}_{n,l} \circ \mathsf{P}^{=l}_{l,l} \circ \mathsf{P}^{||}_{l,m} \tag{16.5}$$ eq:psbas2

(**16.2.18**) In fact (given our particular definition of non-crossing) the factorisation $p = l_p \circ c_p \circ r_p$ of $p \in \mathsf{P}_{n,m}$ as above is unique.

We call this the *polar decomposition* of p.

...

16.2.3 I-Functors

lem:Pfree (**16.2.19**) Lemma. Fix k and $\delta \in k$. Consider the left P_n- right P_l bimodule $\mathsf{P}^{l/}_{n,l} = k[\mathsf{P}^{=l}_{n,l}]$ (Prop.16.2.9). The restriction of the P_l action to S_l makes it a P_n-kS_l-bimodule. It is free as a right kS_l-module.

Proof. Freeness follows from (16.2.16). □

(**16.2.20**) Proposition. Fix k and $\delta \in k$. For each $n \geq l \in \mathbb{N}_0$ the functor

$$I^n_l : kS_l - \text{mod} \;\to\; P_n - \text{mod}$$

$$M \;\mapsto\; \mathsf{P}^{l/}_{n,l} \otimes_{kS_l} M$$

is exact.

16.3 Partition algebra $\Delta^{\cdot\cdot}$-modules and Δ-modules

ss:pecht Here we suppose k and $\delta \in k$ fixed. (For the following §16.3.1 cf. also §18.2.1.)

16.3.1 Symmetric group Specht modules (a quick reminder)

ss:PSp (**16.3.1**) For $\lambda \vdash l$ let $\mathcal{S}_\lambda$ denote the corresponding kS_l Specht module. See e.g. §11.2.4.

de:PDdef-1 (**16.3.2**) For each $\lambda \vdash l$ let us choose an element $f_\lambda \in kS_l$ such that

$$\mathcal{S}_\lambda = kS_l f_\lambda$$

is the corresponding Specht module (cf. e.g. James–Kerber [78, Lem.7.1.4] or James [75]). Suitable elements are given for example in §11.2.4.

16.3.2 Two Δ-module constructions

(**16.3.3**) For $|\lambda| \leq n$ define a P_n-module:

$$\Delta^{\cdot\cdot}_n(\lambda) \;=\; I^n_l \mathcal{S}_\lambda$$

pr:PDbasis (**16.3.4**) Proposition. *For each basis b_λ of $\mathcal{S}_\lambda$ then*

$$b_{\Delta^{\cdot\cdot}_n(\lambda)} \;:=\; \{a \otimes_{kS_l} b \mid (a,b) \in [\mathsf{P}^{||}_{n,l}] \times b_\lambda\}$$

is a basis for $\Delta^{\cdot\cdot}_n(\lambda)$.

Proof. Note from (16.4) that $b_{\dot\Delta_n(\lambda)}$ is spanning. Independence follows from the freeness property as in (16.2.19), since $\mathsf{P}^{||}_{n,l}$ is a basis for the free module. (Cf. also Prop.18.2.11.) □

de:PDdef1 (**16.3.5**) Given f_λ as in (16.3.2), define a subset of $\mathsf{P}^{l/}_{n,l}$

$$\dot\Delta_n(\lambda) \;:=\; \mathsf{P}^{l/}_{n,l} f_\lambda$$

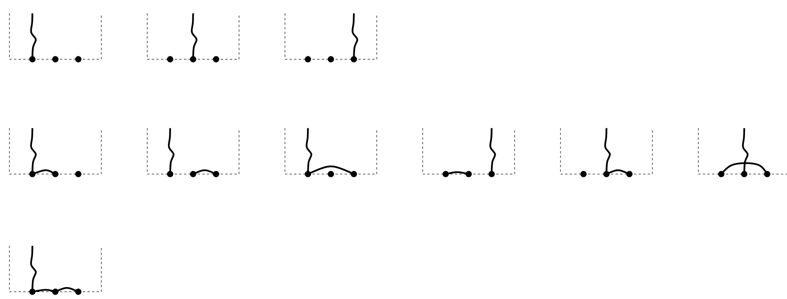
This is a sub-P_n-module of $\mathsf{P}^{l/}_{n,l}$.

(**16.3.6**) Including $f_\lambda \in kS_l$ in P_l in the obvious way allows us to draw a picture for $\Delta_n(\lambda)$.
Exercise: CLARIFY THIS! (Hint: keep in mind the defining quotient.)

pr:PDbas1 (**16.3.7**) Proposition. *For each basis b_λ of $\mathcal{S}_\lambda$ there is a basis $\mathsf{P}^{||}_{n,l} \times b_\lambda$ of $\Delta_n(\lambda)$.*

Proof. Note that the module is spanned by elements of form abf_λ where $a \in \mathsf{P}^{||}_{n,l}$ and $b \in S_l$ (consider (16.4)). For independence note the freeness property from Lem.16.2.19. □

(**16.3.8**) An example of a basis for a Δ-module is as follows. For $n = 3$, $\lambda = (1)$:



By 16.3.7 the basis elements for $\Delta_n(\lambda)$ with $|\lambda| = l$ say, are elements of $\mathsf{P}^{||}_{n,l}$ with S_l Specht module basis elements 'attached'. A convenient way to attach is to label the legs on the l end with the appropriate Young sequences (the sequences associated to the basis of Young tableau in **??**). In our example there is only one trivial Young sequence and we omit it.

Suppose we fix a basis b_λ of the Specht module and associate its elements to Young sequences. Then write $\mathsf{P}^{b_\lambda}_n$ for the corresponding basis of $\dot\Delta_n(\lambda)$.

(**16.3.9**) Proposition.

$$\Delta(\lambda) \cong \dot\Delta_n(\lambda)$$

Proof. Consider the bases in (16.3.4) and (16.3.7). □

16.3.3 Δ-filtration of the regular module (and beyond)

(**16.3.10**) Note from (16.2.7), (16.2.14) and (16.3.5) that if k is such that kS_l has a Specht filtration (e.g. if $k \supseteq \mathbb{Q}$) then the regular P_n module has a filtration by Δ-modules, and indeed a filtration in which all the modules labelled with partitions of a given degree are consecutive. Indeed, if $k = \mathbb{C}$ (or at least contains $\mathbb{Q}$ so that kS_l is semisimple) these modules of fixed degree do not extend each other (so can be arranged, among themselves, in any order in the filtration).

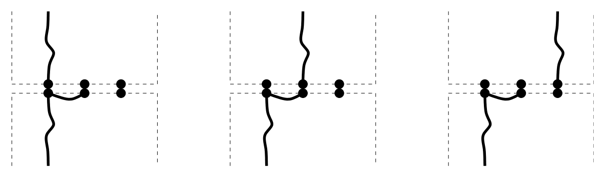
16.3.4 On simple modules, labelling and Brauer reciprocity

For now this section is here to match the B_n one: §18.3.4. It is a straightforward exercise to determine a labelling scheme for simple modules here for example.

ss:Pn-cvform

16.3.5 Contravariant form on $\Delta_n(\lambda)$

(**16.3.11**) If $k \supseteq \mathbb{Q}$ then each of these Δ-modules $\Delta_n(\lambda)$ can be considered as a primitive-idempotently generated left-ideal in a quotient algebra. Thus by e.g. (**??**) there is a contravariant form defined on these modules. The form (a, b) is computed by juxtaposing b with $a^\star$. For example



These three yield δ, δ, and 0 respectively.

In general there is some choice in the form coming from the choice of basis/form on the $S_{|\lambda|}$ Specht module associated to λ. By default we use the form described in §10.1.7 and in (11.4.5) *et seq*, and the diagonal version as in (10.8).

(**16.3.12**) Examples: Fig.16.2 and Fig.16.3. Note that a concrete form comes in Fig.16.3 on fixing a form for the S_3 Specht module part, such as that in (11.4.5) *et seq*.

(**16.3.13**) In studying the form it will be convenient to introduce some notation:

Let b^o denote the set of isolated (non-propagating) connected parts in $b \in \mathsf{P}_n^{b_\lambda}$. Write $a^o \leq b^o$ if a^o, b^o have the same underlying set and a^o is a refinement of b^o, that is if every edge in a^o (in the obvious sense) is also in b^o.
We also define a partial order on the basis by $a < b$ if $|a^o| < |b^o|$.

We arrange the basis $\mathsf{P}_n^{b_\lambda}$ into blocks in which the diagram part is fixed and only the b_λ part is varying.

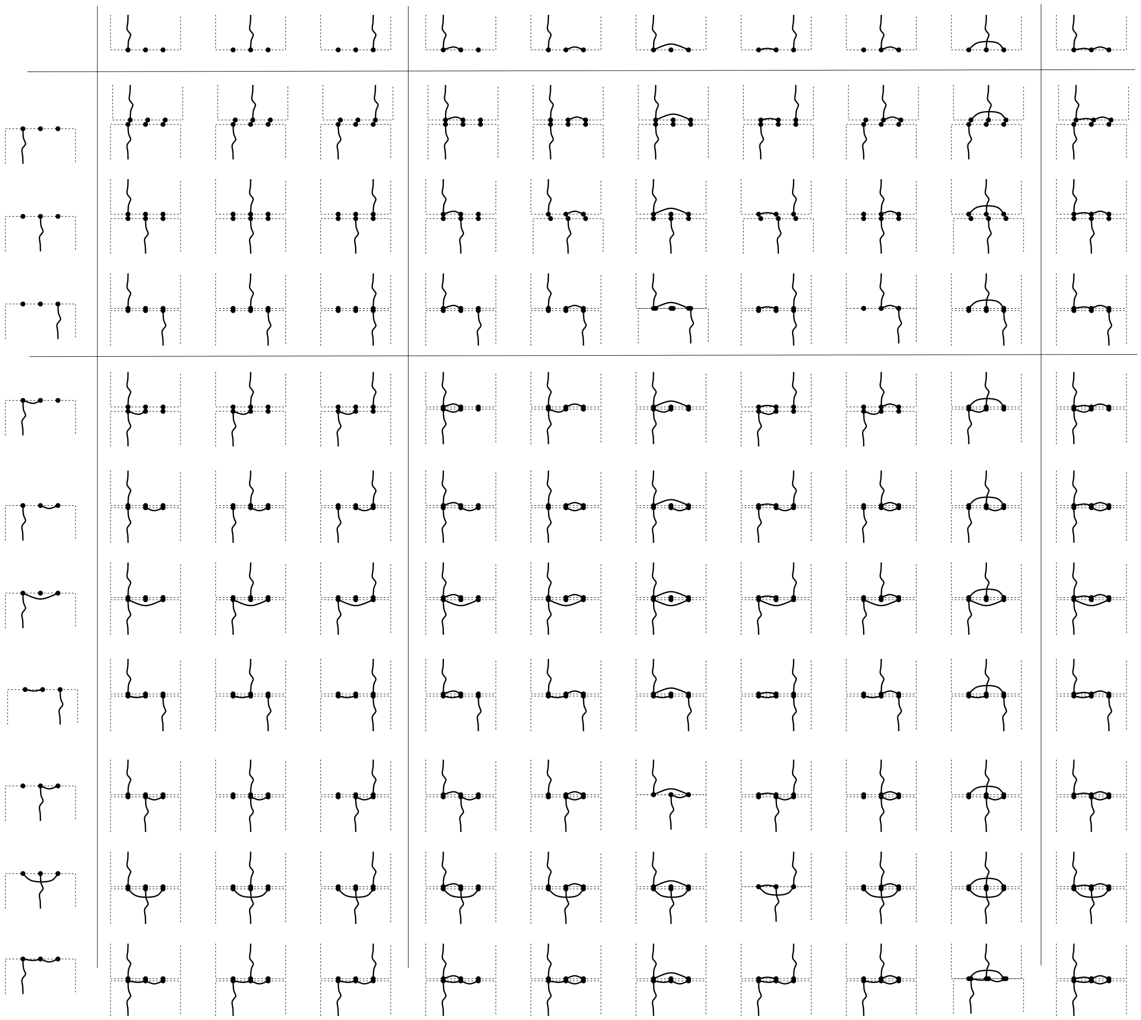
(**16.3.14**) Lemma. Note that the blocks on the block diagonal of the gram matrix are then diagonal if the seed basis b_λ can be chosen orthogonal (since the flip acts as inversion on the 'symmetric subgroup' — cf. Lem.16.2.11). □

de:Pn-Gn (**16.3.15**) For given b_λ define $G_n(\lambda)$ as the gram matrix of the form with respect to the basis $\mathsf{P}_n^{b_\lambda}$, totally ordered in a way consistent with the above. That is:

$$G_n(\lambda) \;=\; \left((a,b)\right)_{a,b \in \mathsf{P}_n^{b_\lambda}}$$

(see e.g. Fig.16.2 and 16.3; and see §16.7.6 for more examples).

(**16.3.16**) Proposition. This form is non-degenerate for generic δ (i.e. over a ring $k[\delta]$ where k is a field of char.0).



$$G_3((1)) \;=\; \left(\begin{array}{ccc|cccccc|c}
\delta^2 & 0 & 0 & \delta & \delta & \delta & 0 & 0 & 0 & 1 \\
0 & \delta^2 & 0 & \delta & 0 & 0 & 0 & \delta & \delta & 1 \\
0 & 0 & \delta^2 & 0 & 0 & \delta & \delta & \delta & 0 & 1 \\
\hline
\delta & \delta & 0 & \delta & 1 & 1 & 0 & 1 & 1 & 1 \\
\delta & 0 & 0 & 1 & \delta & 1 & 1 & 0 & 1 & 1 \\
\delta & 0 & \delta & 1 & 1 & \delta & 1 & 1 & 0 & 1 \\
0 & 0 & \delta & 0 & 1 & 1 & \delta & 1 & 1 & 1 \\
0 & \delta & \delta & 1 & 0 & 1 & 1 & \delta & 1 & 1 \\
0 & \delta & 0 & 1 & 1 & 0 & 1 & 1 & \delta & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

Figure 16.2: Example gram matrix calculation for $\Delta_3((1))$. fig:pb31xl

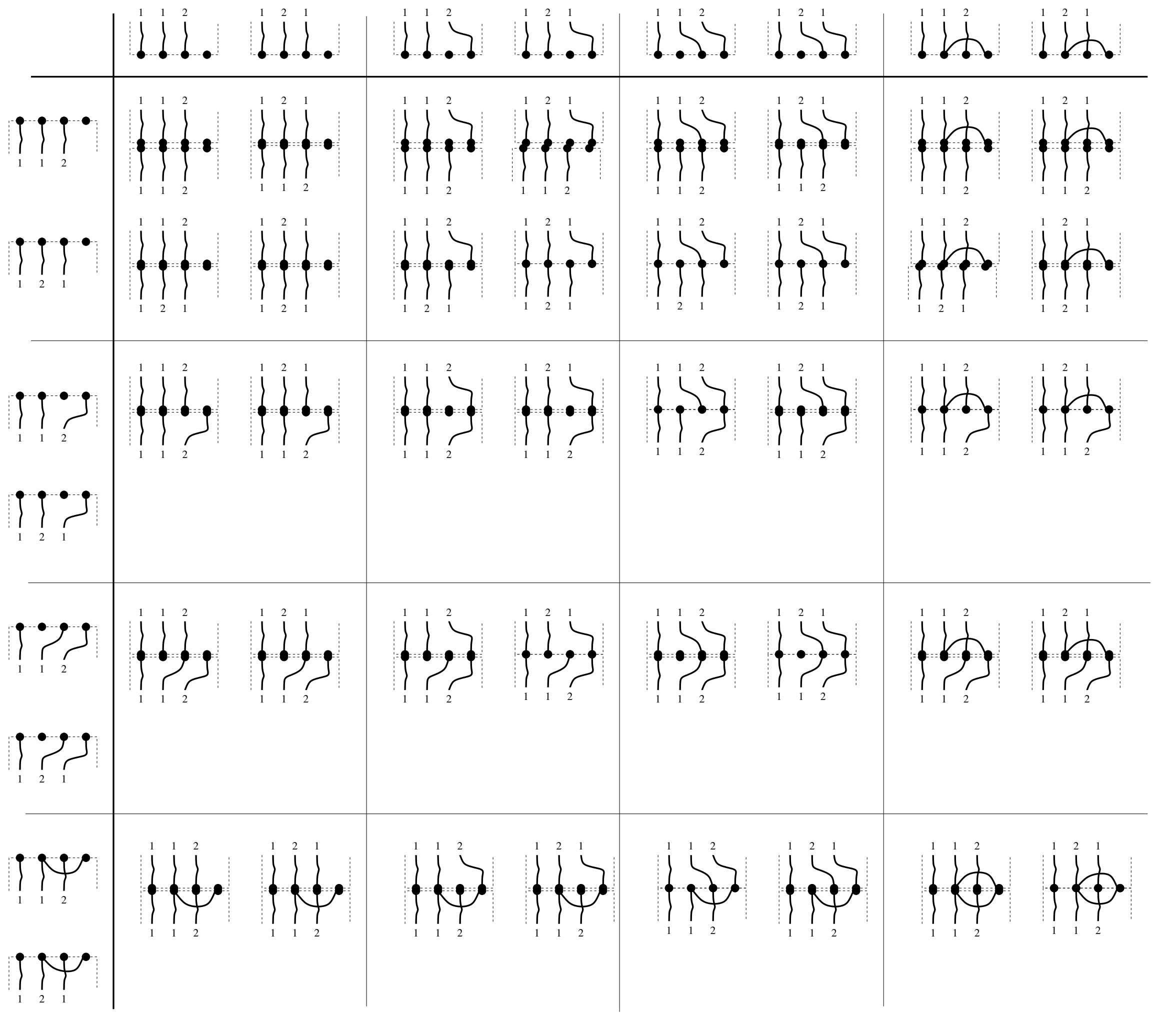


Figure 16.3: Partial gram matrix for $\Delta_4((2,1))$ with entries blocked according to the symmetric group $(2,1)$ part. Note that each block-diagonal block is a power of δ times the corresponding symmetric group Specht gram matrix. However off-diagonal blocks do not necessarily take this form (see the figure for both zero and nonzero examples).

fig:pgram43

Proof. Remark: This is Prop.12 in [106] (where the outline proof is, however, somewhat terse). See also [117] and [**?**] for example.

To verify the proposition now consider the following points.

(1) Each product of basis elements (a, b) gives either (i) some power of δ together with a rank-l permutation diagram (such as the identity diagram — see Fig.16.3 for examples) with Young sequences at both ends or (ii) 0 by the quotient.

(2) The maximum power of δ in the row $(-, b)$ is realised by (b, b). Then the power here is $\delta^{|b^o|}$. Fixing b, this power is also realised in general by other (a, b), but a necessary condition is that (i) the underlying sets of a^o and b^o agree; and (ii) $a^o \leq b^o$.

To see (i) note that otherwise a propagating line meets b^o in composition and the number of isolated connected components goes down.

To then see (ii) note that if $a^o \not\leq b^o$ then by definition there is an edge in a^o not in b^o — but then the number of connected components in the composite goes down.

(3) Next we claim that if the underlying sets of a^o and b^o agree then composition is 0 by the quotient unless the propagating parts all agree (i.e. unless we are on the block diagonal).

To see this note that if the propagating parts do not all agree then an edge in a propagating part in either a or b is not present in the other, in which case it combines two propagating parts in composition.

(4) Since k is char.0 we may take the basis b_λ to be orthogonal and normalizable.

Note that the product of diagonal terms at the top of the partial order strictly dominates any other Leibniz product from these rows. On discarding all these rows and corresponding columns, the product of diagonal terms strictly dominates any other Leibniz product from the rows in the next layer in the partial order. And so on. The proposition now follows from the Laplace expansion. □

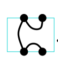
16.4 Globalisation functors

ss:globp

Here we apply to P_n the idempotent-localisation machinery developed in §1.7.5-1.8.1. For this we need suitable idempotents. (Aside: From certain perspectives, such as quasiheredity, there are minor differences in P_n representation theory depending on whether $\delta = 0$. In §16.4.2 we touch on this point, as a framing device for our main discussion.)

16.4.1 Choice of localisation idempotents

de:Boopp

(**16.4.1**) Recall $\square =$ . Define $\square_1 = \square \otimes 1 \otimes 1 \otimes ... \otimes 1 \in P_n$. Define $\square_i$ analogously. Note the following identities: $\mathsf{u}_1 \square_1 \mathsf{u}_1 = \mathsf{u}_1$, $\square_1 \mathsf{u}_1 \square_1 = \square_1$.

Define $y = \mathsf{v}^\star \Gamma$, so $y_1 = \square_1 \mathsf{u}_1$. We have

$$y_1 y_1 = y_1$$

Note that $\square_1, \mathsf{u}_1$ and y_1 all have $n-1$ propagating lines. Note that the ideal generated by any $z \in \mathsf{P}_{n,l,n}$ contains $\mathsf{P}^l_{n,m}$.

de:Prest

(**16.4.2**) There are a number of useful (conjugate) inclusions of P_{n-1} in P_n. The 'natural' inclusion $P_{n-1} \hookrightarrow P_n$ is given by

$$p \mapsto p \otimes 1.$$

The 'unnatural' inclusion is $p \mapsto 1 \otimes p$.

Note that the unnatural P_{n-1} commutes with u_1. Thus for example $P_n\mathsf{u}_1$ is a right P_{n-1}-module via the unnatural action.

Note that $P_n\mathsf{u}_1$ and $P(n, n-1)$ are isomorphic vector spaces. They are also isomorphic as left P_n-modules; as right P_{n-1}-modules (via the unnatural action); and as bimodules.

If $n > 0$ (or $\delta \neq 0$) then there is a vector space isomorphism $P_n \;\cong\; \mathsf{u}_1 P_{n+1} \mathsf{u}_1$ given by $p \mapsto \mathsf{u} \otimes p$. Then $P_n\mathsf{u}_1$ is also a right P_{n-1}-module via the following Lemma (where it applies).

We claim that (where the Lemma applies) these two actions are isomorphic.

de:Poopp (**16.4.3**) Lemma. Provided δ is invertible then the vector space isomorphism

$$P_n \;\cong\; \mathsf{u}_1 P_{n+1} \mathsf{u}_1$$

given by $p \mapsto \mathsf{u} \otimes p$, is an algebra isomorphism. □

In case δ invertible then, $P_{n+1}\mathsf{u}_1$ is a right-P_n-module and the functor

$$G_{\mathsf{u}} \;:=\; P_{n+1}\mathsf{u}_1 \otimes_{P_n} - \qquad (16.6)$$ eq:Gu1

fully embeds P_n-mod in P_{n+1}-mod — cf. (1.7.28). (We will give examples of applications in (16.4.17).)

By the same token, for $n > 0$, $\mathsf{P}_n \cong \mathsf{u}_1 \square_1 \mathsf{P}_{n+1} \mathsf{u}_1 \square_1$ (given by $p \mapsto (\mathsf{u} \otimes p)\square_1$) and

$$P_n \cong y_1 P_{n+1} y_1$$

(given by $p \mapsto \square_1(\mathsf{u} \otimes p)$) are algebra isomorphisms for any δ.

16.4.2 Aside: Long G functors — $\otimes_{C_n}$ versus category composition

ss:PlonG

For 'propagating' object orders on k-linear categories look for example at [**?**, 111] (or maybe Green—Martin's construction [59]).

We are interested here in partition categories as giving examples of modular towers of algebras (1.8.1), (**??**) [57]. In particular we are interested in the effect on representations (particularly Δ-modules as in §16.3) of various kinds of tower 'elevator' functor. Let us think more generally for a moment. Let $\mathcal{R}$ be the category of rings and bimodules. Thus each bimodule ${}_SB_T \in \mathcal{R}(S,T)$ defines a functor

$${}_SB_T \otimes_T - \;:\; T - \mathrm{mod} \;\;\to\;\; S - \mathrm{mod}$$

Such a functor offers the possible prospect of a useful relationship between the two module categories, but this obviously depends very much on ${}_SB_T$. Consider a k-linear category C, and the various algebras $C_y = C(y, y)$. The basic elevator functors in C are the functors of the form

$$C(x, y) \otimes_{C_y} - \;:\; C_y - \mathrm{mod} \;\;\to\;\; C_x - \mathrm{mod}$$

Just as before these functors offers the prospect of a useful relationship between the two module categories — depending on the properties of the functor. This raises the question of what we can say about $C(x, y)$ as a C_y-module. For instance, When is it projective (cf. (1.7.28) for example)? (Aside: Note that this construction is defined for arbitrary k, so in a modular context elevated modules may have 'positive Δ-characters'.)

Recall the notation $C^y(x,z) := C(x,y)C(y,z) \subseteq C(x,z)$. (These $C^y(x,z)$ are also bimodules, so we can investigate them as C_z-modules too.)

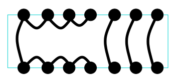
de:kgood (**16.4.4**) Let us say that a k-linear category C is *k-immaculate* at object x if $C(x,y)$ is projective as a left $C(x,x)$-module for all y.

(**16.4.5**) We say a k-linear category C is *fair* if $C(x,y) \otimes_{C(y,y)} C(y,z) \cong C(x,z)$ whenever $C(x,y)C(y,z) = C(x,z)$ (in the category composition).
More generally we could ask, when is $C(x,y) \otimes_{C(y,y)} C(y,z) \cong C^y(x,z)$?
We could also ask about chaining such bimodules together...

To see what are some interesting questions to ask here, we pass back to our example.

Partition category case

(**16.4.6**) As before we suppose k and $\delta \in k$ fixed (but arbitrary). Define partitions

$H_m =$  $= \{\{1,2,...,m,1',2',...,m'\}\} \otimes 1_{n-m} \quad \in P_n$

$$J_m = \{\{1\},\{1'\},\{2\},\{2'\},...,\{m\},\{m'\}\} \otimes 1_{n-m}$$

Note that H_m is an idempotent in P_n.

le:projhom (**16.4.7**) Lemma. *For $l > 0$ the map $P(m+l,l) \cong P(m+l,m+l)H_{m+1}$ given by*

$$p \mapsto ((\mathsf{v}^\star)^{\otimes m} \otimes p)H_{m+1}$$

is an isomorphism of left P_{m+l}-modules.

Proof. Firstly this is a set isomorphism. The inverse is to 'forget' the first m primed vertices, which are all connected and connected to $(m+1)'$ in every basis element on the RHS. From this it will also be clear that these vertices play no active role in the algebra action from the left, leaving the action identical on each side. □

(**16.4.8**) ("We see from the above that category P is not k-immaculate , for any 'interesting' (k,δ), and this is not the definition that we are looking for!!!" — Why?) What about fairness?

le:factorset (**16.4.9**) Lemma. *For category P with any k we have $P(n,m) = P(n,l)P(l,m)$ so long as $l \geq m$ or $l \geq n$.* □

le:tensorvcat (**16.4.10**) Lemma. *Suppose $n \geq m \geq l$ and consider $P(n,m) \otimes_{P_m} P(m,l)$. The multiplication in the category P (as in (16.4.9)) provides a multiplication map $\mu : a \otimes b \mapsto ab$, that defines a bimodule isomorphism*

$$P(n,m) \otimes_{P_m} P(m,l) \xrightarrow{\sim} P(n,l)$$

unless δ is a non-unit, $l = 0$ and $m = 1$.

Proof. Recall that multiplication is balanced (8.4.6), so μ is well-defined by the universal property of tensor products (8.9.2). It is surjective by (16.4.9). We need to show that it does not have a kernel.

By (16.4.7) we have that ${}_{P_m}P(m,l)$ is projective for $l > 0$ (if $\delta \neq 0$ then $l = 0$ may be included similarly, using J_m as (pre)idempotent in Lemma (16.4.7)). Now use the property of the multiplication map (8.9.2).

For $\delta = 0$ and $l = 0$ neither H_{m+1} nor the alternate idempotent are defined. The natural spanning set is $\{a \otimes b \mid a \in \mathsf{P}(n,m),\ b \in \mathsf{P}(m,0)\}$. We need to show that we end up with a basis in bijection with $\mathsf{P}_{n,0}$.

Write ω_m for the all-singleton partition in $\mathsf{P}(m,0)$. In $\mathsf{P}(m,m)$ write h^i_m for the partition with part $\{i, 1', 2', ..., m'\}$ and other parts singletons. Note

$$\omega_m = h^i_m \omega_m$$

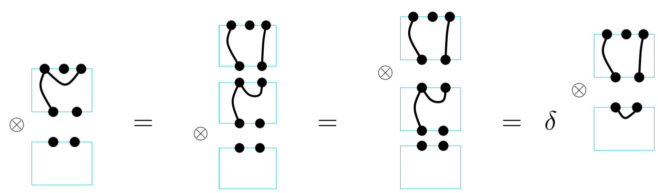
One can see that a smaller spanning set is elements of form $a \otimes \omega_m$, where a has at least one propagating part. Next note that any such $a \otimes \omega_m$ equals $a' \otimes \omega_m$ where a' has all but one propagating part 'cut': consider $a \otimes \omega_m = a \otimes h^i_m \omega_m = a h^i_m \otimes \omega_m$. Indeed the propagating part can be chosen to be the part containing 1. If it does not then we can first add this as a second propagating part:

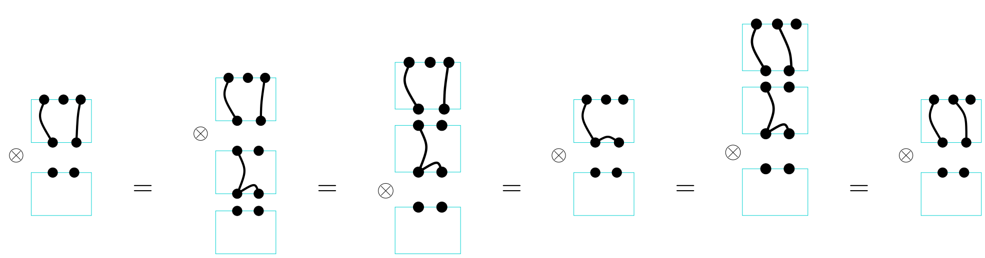
$$a \otimes \omega_m = a^x h^m_m \otimes \omega_m = a^x \otimes \omega_m$$

where a^x is obtained from a by removing all primed elements but m' from the original propagating part and adding $\{1', 2', ..., m-1'\}$ to the part containing 1. Then cut the original by applying h^1_m. NB this requires two propagating parts in the intermediate step, so $m \geq 2$.

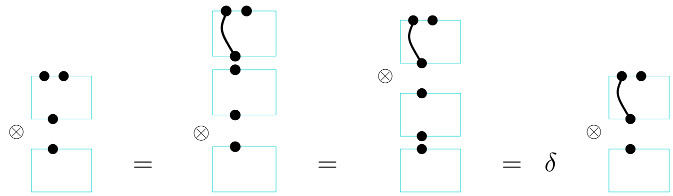
Note that we now have a spanning set in bijection with $\mathsf{P}(n,0)$ (and which passes to $\mathsf{P}(n,0)$ under μ). Specifically the elements are $a \otimes \omega_m$ where a is obtained from $a_- \in \mathsf{P}(n,0)$ by appending $\{1', 2', ..., m'\}$ to the part containing 1. $\square$

Example: Consider the case $m = 2$. Linear dependences between elements from the natural spanning set can arise, for example, like this:





To see that the case $m = 1$ must be excluded when $\delta = 0$ consider



There are 5 elements in the spanning set, and 2 in a basis for the nominal target (isomorphic to $P(2, 0)$). The above shows that we can remove the one on the left. By symmetry there is a chain from this ending in the LR image of the one on the right, so if $\delta \neq 0$ then all three of these are linearly dependent. However if $\delta = 0$ then the one on the right and its LR image cannot be made dependent in this way. The remaining two span a 1d subspace similarly to the above. Thus if $\delta = 0$ we have 3 independent elements. □

(**16.4.11**) Exercise: Generalise this to other diagram categories. Cf. e.g. (18.4.2) and (18.4.5) for the Brauer algebra case.

(**16.4.12**) Aside: Let us come back to the question of projectivity of $P(n, m)$ as a left P_n-module. Obviously every module is projective when P_n is semisimple, so we should look at non-semisimple examples (although there are plenty of these that are projective, as Lemma **??** shows).

Example: When $\delta = 0$ the left P_1-module $P(1, 2)$ is not projective over $\mathbb{C}$. To see this note that $P(1, 2)$ has dimension 5, but there is only one indecomposable projective and this has dimension 2.

Exercise: say more about ${}_{P_1}P(1, 2)$.

(**16.4.13**) CLAIM: Fix field k and $\delta \in k$. The partition category $P = \mathsf{P}_k^\delta$ is fair if $\delta \neq 0$ and almost fair otherwise.

Proof. First note Lem.16.4.9. Given the flip symmetry (and some elementary cases) it is enough to consider $n > l > m$.

...

16.4.3 G-functors and Δ-modules

We have several kinds of 'standard' modules with useful properties. We construct them and show how they are related.

(**16.4.14**) We have several functors between P_n-module categories. For example:

$$G_n^m = P(m, n) \otimes_{P_n} - : \; P_n - \text{mod} \to P_m - \text{mod}$$

and in particular $G_n = G_n^{n+1}$ and $F_n = F_n^{n-1}$;
and also G_u from (16.4.3).

By (**??**) we have that $G_n \cong G_\mathsf{u}$ provided that ...

(**16.4.15**) Consider the 'long inflation' from the $\lambda \vdash l$ Specht module $\mathcal{S}_\lambda = \Delta_l(\lambda)$, that is:

$$\Delta_n(\lambda) \; := \; G_l^n \Delta_l(\lambda).$$

Is it clear that this is defined integrally? What is a basis? (What are the problems with determining if a construction 'works' integrally? A construction might make sense over any base, but not be free over every base... Examples?)

(**16.4.16**) Proposition. *For $\lambda \vdash l$ and $n > l$*

$$G_l^n \Delta_l(\lambda) \cong \Delta_n(\lambda)$$

Proof. By (**??**) we can work with $\Delta_n(\lambda) = I_l^n \mathcal{S}_\lambda$. So we are comparing $G_l^n I_l^l -$ with $I_l^n -$ here.

We first argue that we can essentially identify $I_l^l \mathcal{S}_\lambda$ with $\mathcal{S}_\lambda = kb_\lambda$ as k-modules.

Unpacking we have $G_l^n I_l^l - = k\mathsf{P}(n,l) \otimes_{P_l} I_l^l -$. Compare the bases: For $I_l^n \mathcal{S}_\lambda$ we have the basis

$$b_\Delta = \{a \otimes_{kS_l} b \mid a \in \mathsf{P}^l(n,l);\ b \in b_\lambda\}$$

by (**??**). For $G_l^n I_l^l \mathcal{S}_\lambda$ we have the subset given by the image of the set $\mathsf{P}(n,l) \times b_\lambda$ in $k\mathsf{P}(n,l) \otimes_{P_l} I_l^l \mathcal{S}_\lambda$. It will be clear that this subset is spanning.

If $l = 0$ this set is a basis and is in natural bijection with b_Δ.

More generally, the set $\mathsf{P}(n,l) \times b_\lambda = \mathsf{P}^{=l}(n,l) \times b_\lambda \cup \mathsf{P}^{<l}(n,l) \times b_\lambda$. Let us consider the image of $\mathsf{P}^{<l}(n,l) \times b_\lambda$. By (**??**) a partition in $\mathsf{P}^{<l}(n,l)$ with $l > 0$ can be expressed in the form $du_1 d'$ where $d \in \mathsf{P}(n,l)$ and $u_1, d' \in \mathsf{P}(l,l)$.

TRUE?!

But then $du_1 d' \otimes_{P_l} b_\lambda = d \otimes_{P_l} u_1 d' b_\lambda = 0$

...

pr:GD1 (**16.4.17**) Proposition. *Provided δ invertible (or $n > 2$) we have*

$$G\Delta_n(\lambda) = \Delta_{n+1}(\lambda)$$

Proof. First recall from (**??**) that $G_l^{l+2} \cong G.G$ ...

16.5 Induction and restriction along $P_n \hookrightarrow P_{n+1}$

ss:PartIndRes

Cf. §18.5. References here are [106, 107, 108, 109].

We note that the restriction of a Δ-module $\Delta_n(\lambda)$ has a Δ-filtration (in the sense of **??**), and give the filtration factors. We deduce corresponding results for induction.

[finish me!]

16.6 Enumerating set partitions and diagrams

ss:Pcomb

In this section we discuss enumerations of the sets $\mathsf{P}_{\underline{n}}$ of set partitions (noting that these sets are, essentially, the bases of our various partition algebras).

One idea, for example, is to parallel the Robinson–Schensted (RS) correspondence (see e.g. Knuth [?]), regarded as an enumeration of the symmetric group S_n (noting that the Young graph facilitates an enumeration of Young tableau). There are various ways to do this. We describe one that is natural from a representation theory perspective (see e.g. [?]).

In more directly combinatorial terms the sizes of the sets $\mathsf{P}_{\underline{n}}$ ($n = 0, 1, 2, 3, 4, 5, ...$) are given by:

$$1, 1, 2, 5, 15, 52, ...$$

These numbers $B(n) = |\mathsf{P}_{\underline{n}}|$, are known as the *Bell numbers*. Also useful are the subsets of partitions into a fixed number of parts, $\mathsf{P}_{\underline{n}}(l)$ and the Stirling numbers $S(n, l) = |\mathsf{P}_{\underline{n}}(l)|$; partitions into at most l parts; and various related combinatorics (Stirling numbers). For these we essentially follow [104, Ch.8] — see §16.7.7 here (but these structures are all classical - see e.g. the references in [104]).

exa:partition3

(**16.6.1**) Example. For $n = 3$ we have

$$\mathsf{P}_{\underline{3}}(1) = \{\{\{1,2,3\}\}\}, \qquad \mathsf{P}_{\underline{3}}(2) = \{\{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\},\ \{\{1\},\{2,3\}\}\}$$

$$\mathsf{P}_{\underline{3}}(3) = \{\{\{1\},\{2\},\{3\}\}\}$$

(**16.6.2**) First note that a partition p of a set S can be described by giving the restriction to some subset S', denoted $p|_{S'}$; the restriction to the complement of S' in S; and the details of the connections between the parts thus described. These connection details must give the list of parts in $p|_{S'}$ that are connected; the equinumerate list of parts for the complement; and a bijection between these lists. If a canonical list order is fixed (for any such list), then the bijection may be represented by an element of the symmetric group. Via the RS correspondence this element may be represented as an ordered pair of Young tableaux. We may think of giving the first tableau to S' and the second to the complement. In this way, p is split into two 'halves'.

(**16.6.3**) A *half-partition* is an ordered partition of a set partition into two parts, called *non-propagating* and propagating respectively.

16.6.1 Generalising Robinson-Schensted correspondence

(**16.6.4**) Consider the graph G shown in figure 16.4 (figure taken from Marsh–Martin[99]).

(**16.6.5**) A vertex in G is labelled by a pair consisting of a natural number n (or $n+$) and a Young diagram. The vertex labelled (n, λ) consists in the set of ordered pairs where the first element is a half-partition of $\underline{n}$ with $|\lambda|$ propagating parts; and the second part is a Young tableau of shape λ. If the label is $(n+, \lambda)$ then one takes instead a half-partition of $\underline{n+1}$, but requires that $n + 1$ itself lies in a propagating part.

(**16.6.6**) We shall use shortly the following construction (again see [99]). Let S_ν be an S_{l-1}-Specht module, with a basis of standard tableau of shape ν, and $\mathbb{C}S_l \otimes_{\mathbb{C}S_{l-1}} S_\nu$ the induced module. This has a basis of elements of form $\pi_k \otimes T$ ($k \in \{0, 1, 2, ..., l-1\}$) where, in cycle notation,

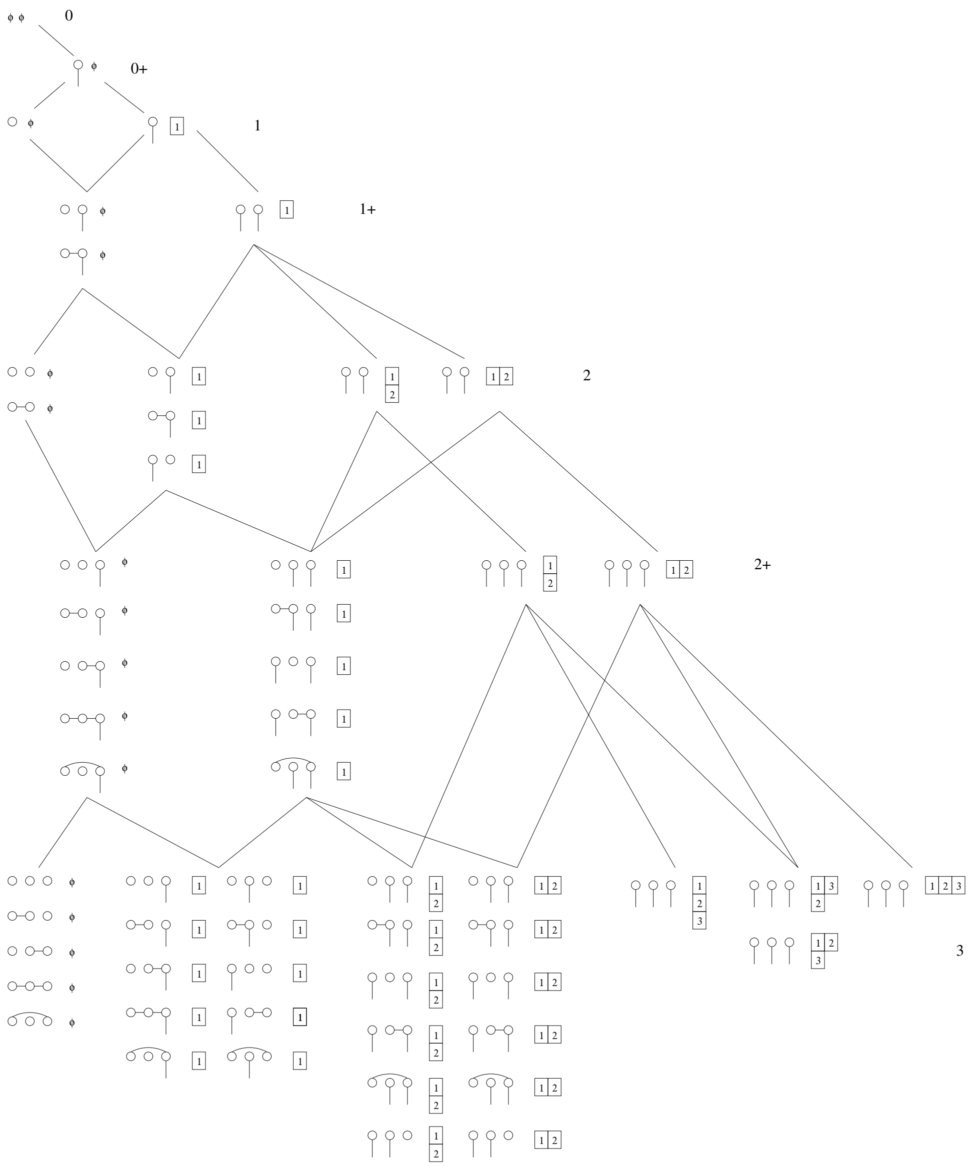


Figure 16.4: The array of 'half-partitions'. fig:cat set part

$\pi_k := (l\ l-1\ ...\ l-k)$ is a coset representative for $S_{l-1} \subset S_l$ and T is a tableau from the standard tableau basis above. The induced module also has a basis of standard tableau of various shapes corresponding to its Specht content (i.e. shapes of form $\nu + e_i$). Thus one may choose a bijection between these bases. Such a bijection gives a correspondence between the subset of standard tableau of shape $\lambda = \nu + e_i$ and some subset of the elements of form $\pi_k \otimes T$.

(**16.6.7**) We now introduce maps describing how to construct each vertex set of G from those in the layer above in G. (Firstly we shall define some maps, and then later we shall show that they do the job.)

There is a map from (n, λ) to $(n+, \lambda)$ adding a propagating part $\{n+1\}$ and keeping the same tableau. (Note that no permutation of the part involving $n+1$ is possible if $n+1$ is common to both halves.)

There is a map from (n, λ) to $(n+, \lambda - e_i)$ [1] as follows. Let (h, T') be the pair to be mapped. Then T' has shape λ and our bijection in (**??**) above takes this to some element $\pi_k \otimes T$. In this case add $n+1$ to the k-th propagating part of h, and replace T' with T.

There is a map from $(n+, \lambda)$ to $(n+1, \lambda)$ that makes the part containing $n+1$ non-propagating, and leaves the tableau unchanged.

There is a map from $(n+, \lambda)$ to $(n+1, \lambda + e_i)$ that leaves the half-partition unchanged and inserts $|\lambda| + 1$ into the new box in the Young diagram, leaving the rest unchanged.

(**16.6.8**) Theorem. *This is everything.*

Proof. ...

16.7 Potts functors and some applications

Computation for Potts models is the starting point for partition categories. As such the Potts representations are fundamental. But they are also directly useful in representation theory.

16.7.1 Preparations

s:partitionPotts

Recall that the partition algebras P_n (for $n \in \mathbb{N}$) may be defined over $\mathbb{Z}[\delta]$, where they are of finite rank; and hence over any ring that is a $\mathbb{Z}[\delta]$-algebra, by base change. For example we can work over $\mathbb{C}[\delta]$ - a $\mathbb{Z}[\delta]$-algebra by inclusion; or over $\mathbb{C}$ by evaluating δ as some element $\delta_c \in \mathbb{C}$ (we may write $\mathbb{C}[\delta_c \in \mathbb{C}]$ for $\mathbb{C}$ made a $\mathbb{Z}[\delta]$-algebra in this way, for example $\mathbb{C}[2]$, or simply write $\mathbb{C}$ if no ambiguity arises, leaving the action of δ implicit).

Indeed the pair of rings $(\mathbb{C}[\delta], \mathbb{C}[\delta_c \in \mathbb{C}])$ represent a modular system for the representation theory of $P_n(\delta_c \in \mathbb{C})$ - the algebra over the latter. Over the field of fractions of the former these algebras are semisimple. The simple modules can be labeled with $\lambda \in \Lambda^n$, the set of integer partitions of ranks up to n. There are integral lattice forms of the simple modules which thus define standard modules over $\mathbb{Z}[\delta]$, and hence by base change over any ground ring.

Once the ground ring is fixed, the standard modules may be denoted $\mathcal{S}_n(\lambda)$, where label λ is an integer partition of rank up to n. Over $\mathbb{C}[\delta]$ these modules are quotients of indecomposable projectives. Upon specialisation to $\delta_c \in \mathbb{C}^\times$ they have pairwise distinct simple heads. We write $\mathcal{D}_n^{\delta_c}(\lambda)$ for the simple head of $\mathcal{S}_n(\lambda)$ in the specialisation evaluating δ at δ_c.

[1] WHICH IS FIDDLY — CHECK IT!

(**16.7.1**) *Remark*: Here we use the language of classical Brauer modular representation theory (see e.g. Curtis–Reiner [33]). The modules $\mathcal{S}_n(\lambda)$ also coincide with cellular cell modules and quasi-hereditary standard modules but we will not need this technology. Their most natural construction is via the partition category over the corresponding ground ring, which we briefly recall below.

(**16.7.2**) In this section we will write $\mathtt{P}_{n,m}$ for the set of set partitions of two rows of n and m vertices respectively. (In §16.1 we write this as $\mathsf{P}_{n,m}$. But here it will sometimes be convenient to emphasise the distinction between morphisms and basis elements in $\mathsf{P}(n,m)$. If we do write a partition $p \in \mathtt{P}_{n,m}$ as an element of $\mathsf{P}(n,m)$ we will mean $1.p$, that is $\sum_{r \in \mathtt{P}_{n,m}} \alpha_r r$ with $\alpha_r = \delta_{r,p}$.)

(**16.7.3**) It will be convenient to review a simple example. Fix n and a ground field k (with the property of $\mathbb{Z}[\delta]$-algebra). Consider the partition basis of P_n — the set $\mathtt{P}_{n,n}$ of set partitions of two rows of n vertices. Let E_1 denote the partition in the partition basis of P_n in which there is only one part (denoted b_n when n is not fixed, as in (**??**) above). And let E_0 denote the partition, hence partition basis element, in which the top row of n vertices are one part, and the bottom row another part:

$$E_0 = \{\{1,2,...,n\},\{1',2',...,n'\}\} \tag{16.7}$$ eq:E0

Recall the chain of ideals $P_n \supset P_n E_1 P_n \supset P_n E_0 P_n$. We immediately have $E_1 E_1 = E_1$ and $E_1 P_n E_1 = k E_1$ mod. $P_n E_0 P_n$. Thus E_1 is a primitive idempotent in the corresponding quotient. We also have $E_0 E_0 = \delta E_0$, and $E_0 P_n E_0 = \delta k E_0$, so E_0 passes to an unnormalised idempotent in any specialisation for which δ has an inverse. In these cases then, the left ideal $P_n E_0$ is indecomposable projective. In fact $\mathcal{S}_n(\emptyset) = P_n E_0$; and $\mathcal{S}_n((1)) = P_n E_1 / P_n E_0 P_n$.

de:PnE0basis (**16.7.4**) Observe that $P_n E_0$ has basis the subset of the partition basis of P_n consisting of all partitions in which the bottom row of vertices exactly make one of the parts. The basis is thus constructed by taking all partitions of the top row of n vertices. It will be convenient to organise these partitions according to the number of parts. There is of course one with one part, for any n. The partitions into two parts or less are double-counted by the power set of the set of vertices, so there are 2^{n-1} of these. For the three parts case, and higher cases, things are a bit more complicated, and we will return to these shortly.

16.7.2 Potts functors for partition categories

This is an opportune moment to consider the Potts functors (see e.g. [104]). First we recall the source and target categories.

(**16.7.5**) Fix a commutative ground ring k. Then **Mat** denotes the monoidal category of matrices over k with Kronecker product as monoidal product.

(**16.7.6**) With the same ground ring k, and for fixed $Q \in \mathbb{N}$, let $[Q]_{\mathsf{k}}$ denote the natural image of Q in k. For example if $Q = 1$ and characteristic $char(\mathsf{k}) = 2$ then $[Q]_{\mathsf{k}}$ is the class $\{..., 7, 5, 3, 1, -1, -3, -5, ...\}$. Then $\mathsf{P} = \mathsf{P}(Q)$ denotes the partition category with $\delta_c = Q$ (strictly speaking this is the natural image of Q in k, and only Q on the nose if $\mathsf{k} \supset \mathbb{Z}$). Thus $\mathsf{P}(Q) = \mathsf{P}(Q + char(\mathsf{k}))$ and so on. This $\mathsf{P}(Q)$ is essentially thus $\mathsf{P}_{\mathsf{k}}^{\delta_c}$ as in §16.1.7.

de:PottsFunctor (**16.7.7**) For fixed $Q \in \mathbb{N}$ the *Potts functor* is a strict monoidal functor

$$\mathfrak{P} : \mathsf{P}(Q) \to \mathbf{Mat}.$$

- writing $\mathfrak{P} = \mathfrak{P}^{(Q)}$ if we need to emphasise Q (since $\mathfrak{P}$ will depend on Q on the nose, not only up to congruence in k). The functor may be given by giving the images of generators as follows.

(**16.7.8**) Observe that the category P is linearly-monoidally generated by the object 1 and
the unique partition elements in $\mathsf{P}(1,0)$ and $\mathsf{P}(0,1)$;
the single-part elements in $\mathsf{P}(1,2)$ and $\mathsf{P}(2,1)$;
and the elementary permutation element in $\mathsf{P}(2,2)$.
Remark. The proof is routine.
Remark. The single-part element in $\mathsf{P}(2,2)$ can be used instead of $\{\{1,2,1'\}\}$ and $\{\{1,1',2'\}\}$.
Remark. This does not give a presentation.

(**16.7.9**) The image under $\mathfrak{P}$ of the generating object 1 is Q. So the image of an element in $\mathsf{P}(i,j)$ lies in $\mathbf{Mat}(Q^i, Q^j)$. The rows of a matrix in $\mathbf{Mat}(Q^i, Q^j)$ are indexed by words in $\underline{Q}^i \cong \hom(\underline{i}, \underline{Q})$; and the columns by words in $\underline{Q}^j$.

The images for $Q = 2$ of the generators above are
[write out the mechanics of this in general. perhaps bra/ket?]

$$\mathfrak{P}(\{\{1\}\}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathfrak{P}(\{\{1'\}\}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{16.8}$$

eq:Potts1

$$\mathfrak{P}(\{\{1,1',2'\}\}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{P}(\{\{1,2,1'\}\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{16.9}$$

$$\mathfrak{P}(\{\{1,2'\},\{2,1'\}\}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16.10}$$

eq:Potts12

In other words, noting that each matrix entry position corresponds to a Q-state Potts configuration (or Q-colouring) on the corresponding vertex set, all vertices in the same part must have the same ‘colour’ for an entry 1, otherwise the entry is 0. (The “Potts rule”.)

(**16.7.10**) Examples. Derived from (16.8)-(16.10) above, keeping $Q = 2$, we have

$$\mathfrak{P}(\{1,2,...,n\}) \;=\; \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 & 1 \end{pmatrix}^T \tag{16.11}$$

$$\mathfrak{P}(\{1,2,...,n,1'\}) = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{pmatrix}^T \tag{16.12}$$

(transposes).

pr:Pottsss (**16.7.11**) **Proposition**. For $Q \in \mathbb{N}$ and $k = \mathbb{C}$ the Potts representation of $P_n(Q)$ is semisimple.
Proof. Exercise. $\square$

16.7.3 Potts $\mathfrak{S}_Q$ symmetry and Schur–Weyl duality contexts

pr:SWdual01 (**16.7.12**) Observe that the functor $\mathfrak{P}^{(Q)}$ is invariant under the diagonal action of the symmetric group $\mathfrak{S}_Q$ (the element $\sigma_1 \in \mathfrak{S}_Q$ takes word $w = 11..1$ to $\sigma_1 w = 22..2$ and so on). This is simply the Potts symmetry [104], but in algebraic terms, for (σ) the matrix form of an element in $\mathfrak{S}_Q$, and $p \in \mathsf{P}(n,m)$

$$\left(\ \sigma\ \right)^{\otimes n} \mathfrak{P}(p) \;=\; \mathfrak{P}(p) \left(\ \sigma\ \right)^{\otimes m} \tag{16.13}$$ eq:SW00

In other words, writing Y for the natural representation of $\mathfrak{S}_Q$ as $Q \times Q$ permutation matrices; and writing $\Delta(\lambda)$ for the λ Specht module of $\mathfrak{S}_Q$, elements of $\mathsf{P}(n,m)$ intertwine the $\mathfrak{S}_Q$ representations Y^n and Y^m. The details of this depend on Q, but many aspects work 'for all Q'. For example, in terms of Specht modules

$$Y \cong \Delta(Q-1,1) + \Delta(Q)$$

so by Schur's Lemma the space of intertwiners with the trivial module $Y^0 \cong \Delta(Q)$ is usually 1-dimensional. Note that $\mathsf{P}(1,0)$ is also 1-dimensional, with spanning element $\{\{1\}\}$. As noted, $\mathfrak{P}((1)) = (1,1,...,1)$ (Q entries), and this indeed intertwines Y and the trivial module.

(**16.7.13**) The examples of (16.13) that we will use concern $p \in \mathsf{P}_{\underline{n}}$. For example, with $Q = 2$ and for any partition p of $\underline{n}$ we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes n} \mathfrak{P}^{(2)}(p) \;=\; \mathfrak{P}^{(2)}(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes 0} \;=\; \mathfrak{P}^{(2)}(p)$$

and so in particular the vector entry $\mathfrak{P}^{(2)}(p)_{11..1}$ obeys $\mathfrak{P}^{(2)}(p)_{11..1} = \mathfrak{P}^{(2)}(p)_{22..2}$.

Before proceeding let us express this a different way.

(**16.7.14**) Fix a field k, and recall that **Vect** is the category of k-spaces. For G a group and V a kG-module then $\mathbf{Vect}_{G,V}$ is the subcategory with objects $k, V, V^2, ...$ and homs commuting with the diagonal action of G, i.e. $f : V^m \to V^n$ such that

$$f\sigma v = \sigma f v \qquad \forall \sigma \in G$$

Note that this inherits the tensor structure from **Vect**. [-say more here.]

We can fix an ordered basis for V, whereupon our category can be expressed as a subcategory of **Mat** rather than **Vect**. And then our Potts functor can be expressed as

$$\mathfrak{P}^{(Q)} : \mathsf{P} \to \mathbf{Vect}_{\mathfrak{S}_Q, Y}.$$

And in particular that this is surjective.

(**16.7.15**) A neat second example of this is to consider $G = \mathbb{Z}_d \wr \mathfrak{S}_Q$ and V the representation replacing perm matrix entries with d-th roots of unity. Here $\mathbf{Vect}_{G,V}$ will be a subcategory of $\mathbf{Vect}_{\mathfrak{S}_Q, Y}$. So if we want $\mathfrak{P}$ to hit this target we will have to replace the source P with a subcategory.

A partition-basis morphism in P is called d-tonal if for every part the difference between the number of north and south elements is zero mod d. These partitions span a subcategory, P^d.

...

[-time to say a bit more about [119]; and about Specht decomp of V^n; and about the signed-perm case]

16.7.4 Potts $\mathfrak{S}_Q$ symmetry and applications

lem:norbits (**16.7.16**) In light of the $\mathfrak{S}_Q$ invariance, the dimension of the image of $\mathsf{P}(n,0)$ under $\mathfrak{P}$ is thus bounded above by the number of orbits of the $\mathfrak{S}_Q$ action on $\underline{Q}^n$.

We will see that this number is the number of partitions of $\underline{n}$ into at most Q parts, and that the bound is saturated.

pa:dimD0 (**16.7.17**) In $P_n(Q)$ the idempotent form of the partition E_0 is $\frac{1}{Q}E_0$ and $Trace(\mathfrak{P}(E_0)) = Q$. We deduce that the simple head of the left module $P_nE_0 \cong \mathsf{P}(n,0)$ occurs exactly once as a summand of the Potts representation $\mathfrak{P}(P_n = \mathsf{P}(n,n))$.

pa:2bas (**16.7.18**) Indeed for $Q = 2$ it is straightforward to see that the 2^{n-1} basis elements of P_nE_0 corresponding to (up to) two-part partitions have independent images in $\mathfrak{P}$; and that elements of higher order then do not. Thus the dimension of the simple head is 2^{n-1}.

To illustrate this, and the generalisation to all Q that we will use, we first introduce a bit more notation.

pa:(123) (**16.7.19**) We may follow [104, Ch.8] in writing (12)(3) for the set $\{\{1,2\},\{3\}\}$, and so on, just to avoid curly-bracket overload.

(**16.7.20**) Given a set S of finite sets, and $Q \in \mathbb{N}$, we write $S\lfloor_Q$ for the subset of sets of order Q. Thus

$$S = \bigsqcup_Q S\lfloor_Q \tag{16.14}$$

(**16.7.21**) Recall we write $\mathtt{P}_{ij}$ for the partition basis of the morphism set $\mathsf{P}(i,j)$ (later we may abbreviate $\mathtt{P}_{nn}$ as $\mathtt{P}_n$). Thus there is a unique element in each $\mathtt{P}_{ij}\lfloor_1$. And $\mathtt{P}_{n0}\lfloor_2 \cup \mathtt{P}_{n0}\lfloor_1$ yields the basis of the image under $\mathfrak{P}$ of $P_nE_0 \cong \mathsf{P}(n,0)$ in (16.7.18) above (when $Q = 2$), noting (16.7.4).

Example:

$$\mathtt{P}_{30}\lfloor_1 = \{(123)\}, \qquad \mathtt{P}_{30}\lfloor_2 = \{(12)(3),(13)(2),(1)(23)\}, \qquad \mathtt{P}_{30}\lfloor_3 = \{(1)(2)(3)\} \tag{16.15}$$ eq:P123

(**16.7.22**) *Aside on notation.* Cf. (16.6.1) where the same sets have the notation $\mathsf{P}_{\underline{n}}(l)$. This notation becomes overloaded in the present setting, so we use the alternative.

(**16.7.23**) For any set S a function $f : S \to T$ to any target defines an equivalence relation, and hence a partition $\varpi(f)$ of S, by $a \sim b$ if $f(a) = f(b)$. Thus in particular there is a partition $\varpi(w)$ of $\underline{n}$ for every word w in $\underline{Q}^n$. For example $\varpi(112) = (12)(3)$.

Observe that the image of the map $\varpi : \underline{Q}^n \to \mathtt{P}_{\underline{n}}$ is $\sqcup_{i=1}^Q \mathtt{P}_{\underline{n}}\lfloor_i$. That is,

$$\varpi : \underline{Q}^n \to \sqcup_{i=1}^Q \mathtt{P}_{\underline{n}}\lfloor_i$$

is well-defined and surjective.

lem:orb1 (**16.7.24**) Observe here that the set $\varpi^{-1}(\varpi(w))$ is the orbit of w under the diagonal $\mathfrak{S}_Q$ action as in (16.7.12). For example, when $Q = 2$, $\varpi^{-1}((12)(3)) = \{112, 221\}$; and when $Q = 3$, $\varpi^{-1}((1234)) = \{1111, 2222, 3333\}$ and $\varpi^{-1}((12)(3)(4)) = \{1123, 1132, 2213, 2231, 3312, 3321\}$.

We can choose representatives for the orbits by using dictionary order. Fixing Q and $p \in \sqcup_{i=1}^{Q} \mathsf{P}_{\underline{n}}\lfloor_i$, let us write $\varpi^-(p)$ for the representative in $\varpi^{-1}(p)$. For example $\varpi^-((12)(3)) = 112$.

eg:123 (**16.7.25**) Considering the partitions of $\underline{3}$ organised as in (16.15) we have, when $Q = 2$,

$$\mathfrak{P}((123)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16.16}$$ eq:123

$$\mathfrak{P}((12)(3)) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{16.17}$$

In fact this is the first instance so far in which the Kronecker convention in **Mat** is relevant to how we write the image. Here we say that the possible states of vertices 123 in the $Q = 2$ state case are ordered as 111, 112, 121, 122, 211, 212, 221, 222. (All other cases will be clear from this.) This is sometimes called aB convention since in writing $A \otimes B$ the first block we write is given by B multiplied by the first entry of A.

We also have

$$\mathfrak{P}((13)(2)) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\mathfrak{P}((1)(23) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \tag{16.18}$$ eq:1-23

Clearly these (16.16-16.18) are all independent of each other. Meanwhile

$$\mathfrak{P}((1)(2)(3)) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

is not then also independent.

(**16.7.26**) Example with $Q = 3$. The states of vertices 123 in case $Q = 3$ are given by $\underline{3}^3 = \{111, 112, 113, 121, 122, 123, 131, 132, 133,\ 211, 212, 213, 221, 222, 223, 231, 232, 233,\ 311, 312, 313, 321, 322, 323, 331, 332, 333\}$
The orbits of the diagonal action of $\mathfrak{S}_3$ are 111, 222, 333, 112, 113, 221, 223, 331, 332, 121, 131, 212, 232, 313, 323, 122, 133, 211, 233, 311, 322, 123, 132, 213, 231, 312, 321. Thus we only need to give the five vector entries in a transversal of this partition to give any $\mathfrak{P}(p)$.

Representatives of orbits in $\underline{3}^5$ (243 words altogether) can be organised according to integer partitions of 5 into at most 3 parts ((5), (41), (32), (311), (221)). A complete list of representatives are: 11111, 11112, 11121, 11211, 12111, 21111, 11122, 11212, 11221, 12112, 12121, 12211, 21112, 21121, 21211, 22111, 11123, 11213, 11231, 12113, 12131, 12311, 21113, 21131, 21311, 23111, 11223, 11232, 11322, 12123, 12132, 13122, 12213, 12312, 13212, 12231, 12321, 13221, 31122, 31212, 31221. Note, one in an orbit of size 3 and 40 in orbits of size 6 — a 41d space. Of course these correspond to the 41 set partitions of $\underline{5}$ of size at most 3.

Thus to give $\mathfrak{P}((1)(2)(3)(45))$ we only need to give $\mathfrak{P}(p)_w$ for w in the lists above. For example $\mathfrak{P}(p)_{11111} = 1$; $\mathfrak{P}(p)_{11112} = 0$, $\mathfrak{P}(p)_{11121} = 0$, $\mathfrak{P}(p)_{11211} = 1$ and so on.

(**16.7.27**) Let S be any given set, and $p, q \in \mathsf{P}_S$. Write $p \prec q$ (or $q \succ p$) to denote that partition p is less refined than q. For example $(123) \prec (12)(3) \prec (1)(2)(3)$. Note that this gives a partial order on P_S.

lem:basis1 (**16.7.28**) Observe that $\mathfrak{P}(\varpi(w))_w = 1$ (recall words w index vector elements here, cf. (16.7.12)). And observe that for every partition $q \not\succeq \varpi(w)$ we have $\mathfrak{P}(q)_w = 0$. This verifies the independence of the set of images of partitions into at most Q parts. But with (16.7.24) it also yields that the images of these partitions form a spanning set, taking account of the diagonal action of $\mathfrak{S}_Q$.
Specifically in our example (16.7.25) above we have $\varpi^{-1}((123)) = \{111, 222\}$, $\varpi^{-1}((12)(3)) =$

$\{112, 221\}$, $\varpi^{-1}((1)(23)) = \{122, 211\}$, $\varpi^{-1}((13)(2)) = \{121, 212\}$, whose union is $\underline{2}^3$, so $\mathfrak{P}((1)(2)(3))$ must be *dependent*.

pr:dimDDD

(**16.7.29**) In other words we have proved the following.

Proposition. For $Q \in \mathbb{N}$, the simple module dimension $\dim(\mathcal{D}_n^Q(0)) = \sum_{l=1}^{Q} S(n,l)$. □

(**16.7.30**) Observe in the above 'Potts rule' (see (16.7.7)) that different parts may have different colours, but of course if there are only two colours then if there are more than two parts some of them will have the same colour, and potentially be indistinguishable from cruder partitions. We deduce that $\mathfrak{P}$ has a significant kernel.

Consider in particular the representation of $P_n(Q)$ given by $\mathfrak{P}$. This is a representation on $(\mathbb{C}^Q)^{\otimes n}$, and hence of dimension Q^n. Furthermore the image is a *-algebra and hence a semisimple quotient of $P_n(Q)$. Indeed the corresponding quotient tower gives a Jones basic construction in the sense, for example, of [54] (in fact two such).

An example of an element in the kernel is the unnormalised Young antisymmetrizer of rank $Q+1$ (in P_n for any $n > Q$ by the natural inclusion). We write this for now as $\mathsf{f}_{Q+1} = \sum_{g \in S_n} (-1)^{len(g)} g$. For example $\mathsf{f}_2 = (11')(22') - (12')(21')$. Evidently this vanishes in $\mathfrak{P}$ when $Q = 1$. More generally the image of f_{Q+1} vanishes whenever we have symmetry under the exchange of two vertices among $Q + 1$ - which holds in $\mathfrak{P}$ for any element of P_n since we only have Q colours with which to distinguish vertices.

(**16.7.31**) Observe also that the partition category action commutes with the action of the symmetric group $\mathfrak{S}_Q$ permuting the 'colours' (as opposed to the vertices) - the diagonal action on the corresponding tensor spaces. This is the Potts symmetry. Indeed these actions are in (an appropriate generalisation of) Schur–Weyl duality.

(**16.7.32**) Recall the Bell numbers $B(n) = |\mathtt{P}_{n0}|$ and the Stirling numbers (of the second kind) $S(n,l) = |\mathtt{P}_{n0}\lfloor_l|$ (see e.g. [104, Ch.8] or [106] for realisations in our setting). We have

$$B(n) = \sum_l S(n,l)$$

16.7.5 Asides on the Potts image of Δ-modules and SW duality

(**16.7.33**) Let us consider the simple head $\mathcal{D}_n^Q(1)$ of the $\lambda = (1)$ Specht module for a moment. Here we can consider the Specht module as the left module (or categorical left ideal) generated by the partition $p = (12....n1')$, modulo the submodule generated by $(12...n)(1')$ (see e.g. (16.3.5)). Starting with rank $n = 2$ the basis (noting the quotient) can be taken to be (121'), (11')(2), (1)(21').

The image under $\mathfrak{P}^2$ is

$$\mathfrak{P}((121')) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathfrak{P}((11')(2)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathfrak{P}((1)(21')) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16.19}$$

eq:Pottsimagex

$$\mathfrak{P}((12)(1')) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathfrak{P}((1)(2)(1')) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathfrak{P}((11')(2)+(1)(21')-2(121')) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

In the abstract one sees that $(11')(2) + (1)(21') - 2(121')$ generates a 1d submodule when $Q = 2$. And in the image this is clearly in the image of the no-propagating-line submodule. In other words the (necessarily isomorphic) image of the simple module $\mathcal{D}_2^2(1)$ under $\mathfrak{P}^2$ has a basis of any two of the elements in (16.19). Note that the basis here is *not* given simply by the image of the partitions with at most two parts.

Exercise: generalise to all n, and Q, and work out the combinatorics cf. $\lambda = 0$.

Solution first steps: [say something about general n for the symmetries noted below...; including $n = 2$!] Staying with $Q = 2$ for now, we have at $n = 3$ (recall our basis convention gives the order 111, 112, 121, 122, 211, 212, 221, 222)

$$\mathfrak{P}((1231')) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{P}((121')(3)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{P}((131')(2)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{P}((1)(231')) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathfrak{P}((11')(2)(3)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{P}((1)(21')(3)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{P}((1)(2)(31')) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Meanwhile the images of the partitions with $(1')$ a singleton can easily be seen to span matrices that are both top-to-bottom and left-to-right flip symmetric (a 4d subspace). All image matrices are π-rotation symmetric, so the whole image is at most 8d.

(**16.7.34**) Another result of passing interest is the restriction of such simple modules to modules for the natural symmetric group S_n subalgebra. Since both partitions (1)(2) and (12) are invariant under the σ_1 action then $\mathcal{D}_2^2(0)$ decomposes as two copies of the trivial module: $(2) \oplus (2)$. Since (1)(21') and (11')(2) are not invariant, then $\mathcal{D}_2^2(1)$ decomposes as $(2) \oplus (1^2)$.

Exercise: Generalise to all n and compare with Schur–Weyl duality!
Do we want to use the tensor-Hom adjunction? Somehow this is connected to $\mathbf{Set}(\underline{n}, \underline{2})$ when $Q = 2$, and $\mathbf{Set}(\underline{n}, \underline{Q})$ more generally. ...

...

16.7.6 Representation theory and an application of Pr.16.7.29

ss:Pn-repapp

(**16.7.35**) Over $\mathbb{C}$ the algebras P_n are generically semisimple, so much of their representation theory is encoded in the Bratelli diagram (or a suitably enhanced version thereof) for the tower of algebras $P_n \subset P_{n+1}$ in the generic case: see Fig.16.5.

The Figure encodes a lot of information. However here we will only need some specific observations. Meanwhile, the gram matrices $\mathrm{Gram}_n(\lambda)$ for the standard modules $\mathcal{S}_n(\lambda)$ also contain a lot of information, but again we will only need parts of it that are readily accessible. To illustrate, we give some examples explicitly. Note (e.g. from §16.3.5) that the modules $\mathcal{S}_n(\lambda)$, and hence their gram matrices, can be defined over $\mathbb{C}[\delta]$ (even $\mathbb{Z}[\delta]$).

See §10 for a review of relevant gram matrix methods.

(**16.7.36**) Example. The (standard-basis) Gram matrix $\mathrm{Gram}_n(0)$ of the $P_n(\delta)$ module $\mathcal{S}_n(\emptyset)$ for $n = 2, 3, 4$ is given in Fig.16.6. (Cf. §16.3.5. NB, in (16.3.15) our $\mathrm{Gram}_n(\lambda)$ is written $G_n(\lambda)$.)

The corresponding determinants are: $\det(\mathrm{Gram}_2(0)) \;=\; \delta^2(\delta-1)$;

$$\det(\mathrm{Gram}_3(0)) \;=\; \delta^5(\delta-1)^4(\delta-2), \qquad\qquad \det(\mathrm{Gram}_4(0)) \;=\; \delta^{15}(\delta-1)^{14}(\delta-2)^7(\delta-3).$$

(**16.7.37**) The matrix rep $\mathcal{S}_3(0)$ with the given ordered basis is given by

$$(A_1)\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix} = \begin{pmatrix}A_1(1)(2)(3)\\A_1(12)(3)\\A_1(13)(2)\\A_1(1)(23)\\A_1(123)\end{pmatrix} = \begin{pmatrix}\delta(1)(2)(3)\\(1)(2)(3)\\(1)(2)(3)\\\delta(1)(23)\\(1)(23)\end{pmatrix} = \begin{pmatrix}\delta&&&&\\1&&&&\\1&&&&\\&&&\delta&\\&&&1&\end{pmatrix}\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix}$$

$$(A_{12})\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix} = \begin{pmatrix}A_{12}(1)(2)(3)\\A_{12}(12)(3)\\A_{12}(13)(2)\\A_{12}(1)(23)\\A_{12}(123)\end{pmatrix} = \begin{pmatrix}(12)(3)\\(12)(3)\\(123)\\(123)\\(123)\end{pmatrix} = \begin{pmatrix}&1&&&\\&1&&&\\&&&&1\\&&&&1\\&&&&1\end{pmatrix}\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix}$$

$$(\sigma_{23})\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix} = \begin{pmatrix}\sigma_{23}(1)(2)(3)\\\sigma_{23}(12)(3)\\\sigma_{23}(13)(2)\\\sigma_{23}(1)(23)\\\sigma_{23}(123)\end{pmatrix} = \begin{pmatrix}(1)(2)(3)\\(13)(2)\\(12)(3)\\(1)(23)\\(123)\end{pmatrix} = \begin{pmatrix}1&&&&\\&&1&&\\&1&&&\\&&&1&\\&&&&1\end{pmatrix}\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix}$$

$$(\sigma_{12})\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix} = \begin{pmatrix}\sigma_{12}(1)(2)(3)\\\sigma_{12}(12)(3)\\\sigma_{12}(13)(2)\\\sigma_{12}(1)(23)\\\sigma_{12}(123)\end{pmatrix} = \begin{pmatrix}(1)(2)(3)\\(1)(23)\\(12)(3)\\(13)(2)\\(123)\end{pmatrix} = \begin{pmatrix}1&&&&\\&1&&&\\&&&1&\\&&1&&\\&&&&1\end{pmatrix}\begin{pmatrix}(1)(2)(3)\\(12)(3)\\(13)(2)\\(1)(23)\\(123)\end{pmatrix}$$

where $A_1 = (1)(1')(22')(33')$, and so on. The corresponding intertwining is:

$$\begin{pmatrix}\delta&&&&\\1&&&&\\1&&&&\\&&&\delta&\\&&&1&\end{pmatrix}\begin{pmatrix}\delta^3&\delta^2&\delta^2&\delta^2&\delta\\\delta^2&\delta^2&\delta&\delta&\delta\\\delta^2&\delta&\delta^2&\delta&\delta\\\delta^2&\delta&\delta&\delta^2&\delta\\\delta&\delta&\delta&\delta&\delta\end{pmatrix} = \begin{pmatrix}\delta^3&\delta^2&\delta^2&\delta^2&\delta\\\delta^2&\delta^2&\delta&\delta&\delta\\\delta^2&\delta&\delta^2&\delta&\delta\\\delta^2&\delta&\delta&\delta^2&\delta\\\delta&\delta&\delta&\delta&\delta\end{pmatrix}\begin{pmatrix}\delta&1&1&&\\&&&&\\&&&&\\&&&\delta&1\\&&&&\end{pmatrix}$$

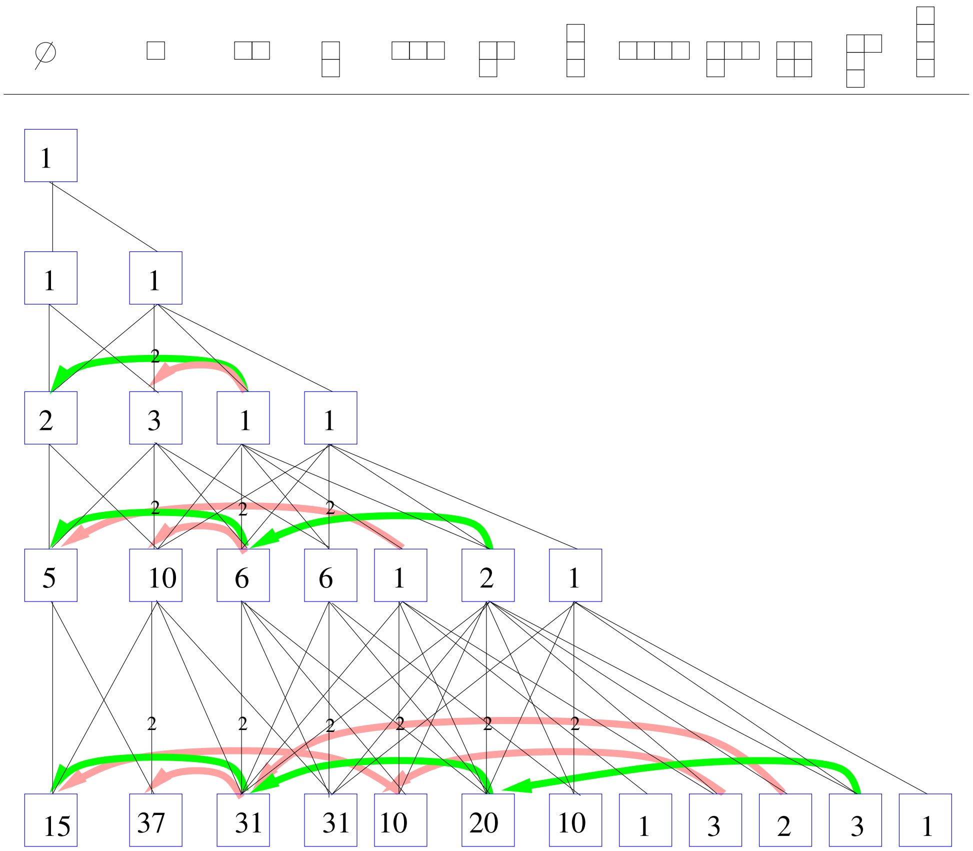


Figure 16.5: Augmented Bratelli diagram for $P_0 \subset P_1 \subset ... \subset P_4$. Vertices are standard modules with index as shown at the top of their column; and dimension shown in the box. Black edges indicate restriction rules (with multiplicities) so dimensions can be checked. Green arrows indicate morphisms for $\delta = 1 \in \mathbb{C}$ (see main text for commentary). Pink arrows indicate morphisms for $\delta = 2$. (Morphisms for other δs demoted to next Figure to avoid clutter.)

fig:PnBratelli

$$
\begin{pmatrix} \delta^2 & \delta \\ \delta & \delta \end{pmatrix}
\begin{pmatrix}
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta \\
\delta^2 & \delta^2 & \delta & \delta & \delta \\
\delta^2 & \delta & \delta^2 & \delta & \delta \\
\delta^2 & \delta & \delta & \delta^2 & \delta \\
\delta & \delta & \delta & \delta & \delta
\end{pmatrix}
\begin{pmatrix}
\delta^4 & \delta^3 & \delta^3 & \delta^3 & \delta^3 & \delta^3 & \delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta \\
\delta^3 & \delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta & \delta & \delta^2 & \delta^2 & \delta & \delta & \delta \\
\delta^3 & \delta^2 & \delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta & \delta^2 & \delta & \delta^2 & \delta & \delta^2 & \delta & \delta \\
\delta^3 & \delta^2 & \delta^2 & \delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta & \delta & \delta^2 & \delta & \delta^2 & \delta^2 & \delta & \delta \\
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^3 & \delta^2 & \delta^2 & \delta & \delta & \delta^2 & \delta^2 & \delta & \delta & \delta^2 & \delta \\
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^3 & \delta^2 & \delta & \delta^2 & \delta & \delta & \delta^2 & \delta & \delta^2 & \delta \\
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^2 & \delta^3 & \delta^2 & \delta & \delta & \delta & \delta & \delta^2 & \delta^2 & \delta \\
\delta^2 & \delta^2 & \delta & \delta & \delta & \delta & \delta^2 & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta & \delta \\
\delta^2 & \delta & \delta^2 & \delta & \delta & \delta^2 & \delta & \delta & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta \\
\delta^2 & \delta & \delta & \delta^2 & \delta^2 & \delta & \delta & \delta & \delta & \delta^2 & \delta & \delta & \delta & \delta & \delta \\
\delta^2 & \delta^2 & \delta^2 & \delta & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta^2 & \delta & \delta & \delta & \delta \\
\delta^2 & \delta^2 & \delta & \delta^2 & \delta & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta^2 & \delta & \delta & \delta \\
\delta^2 & \delta & \delta^2 & \delta^2 & \delta & \delta & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta^2 & \delta & \delta \\
\delta^2 & \delta & \delta & \delta & \delta^2 & \delta^2 & \delta^2 & \delta & \delta & \delta & \delta & \delta & \delta & \delta^2 & \delta \\
\delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta & \delta
\end{pmatrix}
$$

Figure 16.6: Gram matrices for $\mathcal{S}_n(0)$ for $n = 2, 3, 4$. fig:gram234

$$
\begin{pmatrix}
 & 1 & & & \\
 & 1 & & & \\
 & & & & 1 \\
 & & & & 1 \\
 & & & & 1
\end{pmatrix}
\begin{pmatrix}
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta \\
\delta^2 & \delta^2 & \delta & \delta & \delta \\
\delta^2 & \delta & \delta^2 & \delta & \delta \\
\delta^2 & \delta & \delta & \delta^2 & \delta \\
\delta & \delta & \delta & \delta & \delta
\end{pmatrix}
=
\begin{pmatrix}
\delta^3 & \delta^2 & \delta^2 & \delta^2 & \delta \\
\delta^2 & \delta^2 & \delta & \delta & \delta \\
\delta^2 & \delta & \delta^2 & \delta & \delta \\
\delta^2 & \delta & \delta & \delta^2 & \delta \\
\delta & \delta & \delta & \delta & \delta
\end{pmatrix}
\begin{pmatrix}
 & & & & \\
1 & 1 & & & \\
 & & & & \\
 & & & & \\
 & & 1 & 1 & 1
\end{pmatrix}
$$

- the σ_{ij} identities will be clear. The Smith form of a matrix has the same rank, so

$$
\mathrm{Gram}_3(0) \rightsquigarrow \begin{pmatrix}
\delta(\delta-1)(\delta-2) & & & & \\
 & \delta(\delta-1) & & & \\
 & & \delta(\delta-1) & & \\
 & & & \delta(\delta-1) & \\
 & & & & \delta
\end{pmatrix}
$$

is forced here.

pa:gram-det-order (**16.7.38**) The general form of the gram determinant is somewhat non-trivial. But it is easy to see that the exponent of factor δ is the dimension of the module (which is the Bell number $B(n)$); and the total polynomial degree is $\sum_{l=1}^{n} lS(n,l)$ (coincidentally also $\dim(\mathcal{S}_n((1)))$) — this is the sum of the exponents down the main diagonal, by the composition rule; and it is straightforward to check that this gives the highest order term in the Laplace expansion.

pa:simpleS0 (**16.7.39**) On the other hand we observe that the polynomial must, on representation theory grounds, contain factors as follows. (NB, we sometimes write $\mathcal{S}_n(0)$ for $\mathcal{S}_n(\emptyset)$ - purely for aesthetic reasons. The head of $\mathcal{S}_n(\lambda)$ when working over a field depends on the specialisation $\delta \rightsquigarrow \delta_c$. For given δ_c we write $\mathcal{D}_n^{\delta_c}(\lambda)$ for the head.) For the factors $(\delta - 1)$ we observe that dimension of head of $\mathcal{S}_n(0)$ when $\delta_c = 1$ is $\dim(\mathcal{D}_n^1(0)) = 1$. We can observe this directly in our low rank diagram in Fig.16.5; or in general by inspecting the Potts representation. Thus the rank of the morphism (unique up to scalars, since necessarily head to socle) from standard module $\mathcal{S}_n(0)$ to costandard module $\nabla_n(0)$ is 1. It follows that all but one entry in the Smith form has a factor $(\delta - 1)$. So

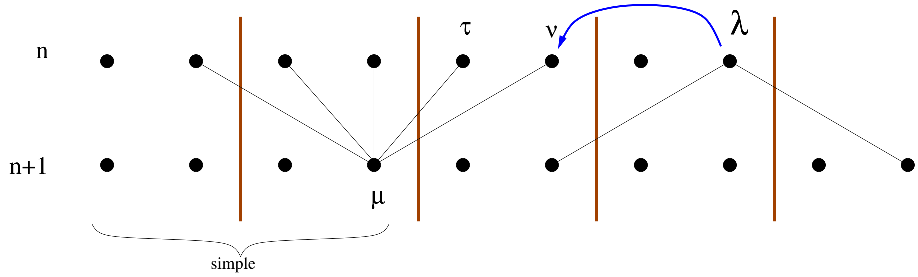


fig:FrobR111 Figure 16.7: Schematic for Induction (downward cone on the right), Restriction (cone on the left) and hence reciprocity. Nodes represent standard modules at levels n and $n{+}1$ as indicated. Vertical lines separate different ranks.

there are at least $\dim(\mathcal{S}_n(0))$-1 such factors in the determinant. That is, writing

$$\det(\mathrm{Gram}_n(0)) = \prod_{\delta_c} (\delta - \delta_c)^{\alpha_{\delta_c}}$$

we have a bound on the exponent: $\alpha_1 \geq \dim(\mathcal{S}_n(0)) - 1$. A similar argument gives a bound $\dim(\mathcal{S}_n(0)) - \dim(\mathcal{D}_n^Q(0))$ on the number of factors of each form $(\delta - Q)$ for $Q \in \mathbb{N}$. The formula for $\dim(\mathcal{D}_n^Q(0))$ becomes more complicated. For example $\dim(\mathcal{D}_n^2(0)) = 2^{n-1}$. In general we have

$$\dim(\mathcal{D}_n^Q(0)) \;=\; \sum_{l=1}^{Q} S(n,l)$$

(arguing as in (16.7.17 - 16.7.28)) and hence

$$\dim(\mathcal{S}_n(0)) - \dim(\mathcal{D}_n^Q(0)) \;=\; \sum_{l=1}^{n} S(n,l) - \sum_{l=1}^{Q} S(n,l) \;=\; \sum_{l=Q+1}^{n} S(n,l)$$

Summing we obtain an overall bound for the polynomial degree

$$degree(|\mathrm{Gram}_n(0)|) \;\geq\; \sum_{Q} \sum_{l=Q+1}^{n} S(n,l) \;=\; \sum_{l=1}^{n} l S(n,l)$$

By our earlier observation in (16.7.38) this is saturated. In other words all factors come from these Q values. Thus this gram determinant does not vanish for any other δ value. Thus:

Proposition. The P_n module $\mathcal{S}_n(0)$ is simple for all n for all other δ values. □

pa:indicator (**16.7.40**) **Proposition**. If $\delta_c \neq 0$ and $\mathcal{S}_n(0)$ is simple for all n then P_n is semisimple for all n.

Proof. We argue by Frobenius reciprocity using properties of the standard module restriction rules (as indicated in Fig.16.5 for example), following [115]. For brevity we assume the construction and basic properties of the standard modules $\mathcal{S}_n(\lambda)$ with λ a suitable integer partition. In particular if a module M has a filtration by these modules we say that it has an $\mathcal{S}$-filtration.

Firstly note that if P_n is not semisimple then there must be a homomorphism $\psi : \mathcal{S}_n(\lambda) \to \mathcal{S}_n(\nu)$ for some distinct λ, ν; in particular with $|\lambda| > |\nu|$ (the latter condition comes from localisation functors and Maschke's Theorem; or equivalently from quasiheredity). And hence in particular some $\mathcal{S}_n(\nu)$ is not simple. So it is enough to prove that all $\mathcal{S}_n(\nu)$ are simple. We do this by induction on $|\nu|$.

We will use the reciprocity

$$\mathrm{Hom}(Ind\, \mathcal{S}_n(\lambda), \mathcal{S}_{n+1}(\nu)) \cong \mathrm{Hom}(\mathcal{S}_n(\lambda), \mathrm{Res}\mathcal{S}_{n+1}(\nu)) \tag{16.20}$$ eq:FR

By the usual $P_{n-1} \hookrightarrow P_n$ Induction/Restriction rules we have

$$Ind\, \mathcal{S}_n(\lambda) \leftsquigarrow +_{\rho\in\lambda\pm1} \mathcal{S}_{n+1}(\rho), \qquad \mathrm{Res}\mathcal{S}_{n+1}(\lambda) \leftsquigarrow +_{\rho\in\lambda\pm1} \mathcal{S}_n(\rho), \tag{16.21}$$ eq:Indres

where $\leftsquigarrow$ indicates $\mathcal{S}$-filtration and a covering set of filtration factors; and $\lambda \pm 1$ indicates a set of factors with weights of rank between $|\lambda| + 1$ and $|\lambda| - 1$.

So now suppose for an induction that modules $\mathcal{S}_n(\mu)$ of ranks $|\mu|$ up to l are simple for all n. The case we have concluded in (16.7.39) above - rank $l = 0$ - will be the base of induction. Excluding $\delta = 0$ this implies no maps ψ in to $\mathcal{S}_n(\mu)$ (apart from self-isomorphism) for any n, so then in particular for every μ of rank $|\mu| = l$ we have $\mathrm{Hom}(Ind\ \mathcal{S}_n(\lambda), \mathcal{S}_{n+1}(\mu)) = 0$ for all λ of rank at least $|\mu|+2$, by (16.21) and Schur's Lemma. But suppose (for a contradiction) some $\mathcal{S}_n(\nu)$ fig:FrobR111 with ν of rank $l+1$ is not simple, and hence has a map in, from $\mathcal{S}_n(\lambda)$ say. See Fig. . The factors in any $\mathrm{Res}\mathcal{S}_n(\mu)$ as above are either in their own block by the inductive assumption, or of equal rank to ν and hence not involved in extensions with $\mathcal{S}_n(\nu)$, and for a suitable μ they include $\mathcal{S}_n(\nu)$. Thus there is a map in $\mathrm{Hom}(\mathcal{S}_n(\lambda), \mathrm{Res}\mathcal{S}_{n+1}(\nu))$. This then contradicts (16.20). We deduce that every $\mathcal{S}_n(\nu)$ with ν of rank $l+1$ is simple, which completes the inductive step. □

(**16.7.41**) Given (16.7.40) and (16.7.39) we have the following.

Theorem. The algebra P_n is semisimple over $\mathbb{C}$ unless $\delta \in \mathbb{N}_0$. □

16.7.7 Potts functors, Potts configurations and set partitions II

:partitioncombi2

This section is taken from [104, Ch.8]. In particular we use the notation (12) for $\{\{1,2\}\} \in \mathsf{P}_{\underline{2}}$; and (1)(2) for $\{\{1\},\{2\}\}$, and so on, to avoid curly-bracket overload on the eye.

(**16.7.42**) Next we define, for each $Q \in \mathbb{N}$, a function

$$\pi_Q : \mathbf{Set}(\underline{n}, \underline{Q}) \to \mathsf{P}_{\underline{n}}$$

Consider a function $f \in \mathbf{Set}(\underline{n}, \underline{Q})$ (a Q-state Potts configuration of $\underline{n}$). Each such f yields a partition $\pi_Q(f) \in \mathsf{P}_{\underline{n}}$ by $i \sim j$ if $\overline{f}(i) = f(j)$.

Evidently the image of π_Q contains only partitions into at most Q parts, but for sufficiently large Q ($Q \geq n$) all partitions are realised.

Example. We may understand a sequence such as 1123 as giving a function f (for $Q \geq 3$) by $1123 = f(1)f(2)f(3)f(4)$. Then $\pi_3(f = 1123) = (12)(3)(4)$. And indeed $\pi_4(f = 1123) = \pi_5(f) = ... = (12)(3)(4)$.

(**16.7.43**) The table in Fig.16.8 gives exactly one representative f for each partition. For partitions of n one uses the table up to row n. The ith row encodes the colour ascribed to element $i \in \underline{n}$.

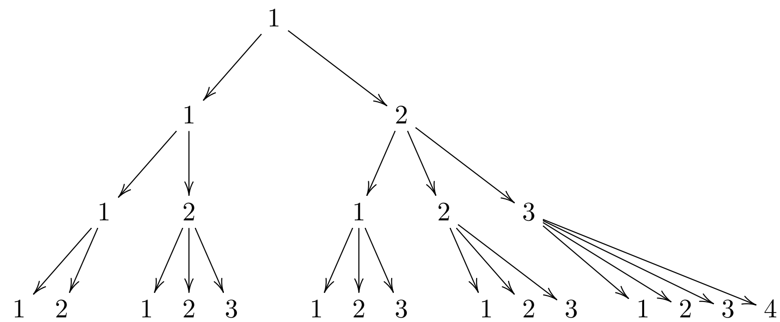


fig:branchio Figure 16.8: A colouring for each partition; and a total order on partitions. The diagram up to row n gives the set of partitions of $\underline{n}$. For example the path down giving sequence 1123 gives the images of vertices 1,2,3,4 in a function f, and then $\pi_n(f) = (12)(3)(4)$.

We thus address the elements of $\underline{n}$ using their 'natural' order. But we use the S_Q symmetry of the colour set $\underline{Q}$ to ascribe colours. Thus:
We colour 'vertex'/node/element 1 with colour 1.
We can then colour vertex 2 with colour 1 or 2 (to give functions corresponding to the partitions (12) and (1)(2) respectively).
We can colour vertex 3 with colour 1 or 2, or colour 3 can be used if 2 already used.
And for each subsequent vertex we can colour with one of the colours already used, or the next new colour.

(**16.7.44**) Consider the diagram in Fig.16.9. This iterating construction gives all the Sterling numbers. This follows from the above Fig.16.8 by considering the restrictions to the parts containing only up to 'colour' Q, for the various values of Q, and then the number of configurations in each line coming from the line above.

For example, if we used Q colours up to line $n-1$ then our configuration can be continued to line n in Q ways still using Q colours. While if we used $Q-1$ colours so far then there is exactly one way to proceed making a Q colour configuation - by giving vertex n colour Q. ...

Or alternatively it follows by noting that ...

(**16.7.45**) Note from the algebra composition rule that the numbers in column i of row n in Fig.16.9 (i.e. $S(n,i)$) give the number of entries δ^i in the main diagonal of the corresponding gram matrix.

(**16.7.46**) We deduce in particular that the gram-matrix is non-singular (and hence the standard module simple) for every δ_c value except those used in the bounding sum. This is because those used in the sum already saturate the bound on degree of the determinant polynomial - any further singularity would require a further vanishing polynomial factor, thus breaking the bound.

Later we will show using Frobenius reciprocity that if $\mathcal{S}_n(0)$ is simple for all n at δ_c then every P_n is semisimple at this δ_c value.

(**16.7.47**) Aside. Note that the total degree is indeed the dim of $\mathcal{S}_n(1)$ - although we have not needed to use this per se. It follows since for a Q-part partition in the basis of $\mathcal{S}_n(0)$ there are Q ways to make it into a $\mathcal{S}_n(1)$ element, by making one of the parts propagating.

$$
\begin{tikzcd}
1 \arrow[d] \arrow[dr] & & & & & \\
1 \arrow[d] \arrow[dr] & 1 \arrow[d, "2"] \arrow[dr] & & & & \\
1 \arrow[d] \arrow[dr] & 3 \arrow[d, "2"] \arrow[dr] & 1 \arrow[d, "3"] \arrow[dr] & & & \\
1 \arrow[d] \arrow[dr] & 7 \arrow[d, "2"] \arrow[dr] & 6 \arrow[d, "3"] \arrow[dr] & 1 \arrow[d, "4"] \arrow[dr] & & \\
1 \arrow[d] \arrow[dr] & 15 \arrow[d, "2"] \arrow[dr] & 25 \arrow[d, "3"] \arrow[dr] & 10 \arrow[d, "4"] \arrow[dr] & 1 \arrow[d, "5"] \arrow[dr] & \\
\vdots & \cdots & {} & {} & {} & \cdots
\end{tikzcd}
$$

Figure 16.9: Tower computing Stirling numbers $S(n,l)$ by $S(n,l) = S(n-1,l-1) + lS(n-1,l)$. Rows are $n = 1, 2, 3, ...$ Columns are $l = 1, 2, 3, ...$

fig:stirling1

16.8 Representation theory for specific $\delta \in k$

ss:reppk

This section is based on [108]

16.9 Representation theory via Schur algebras

ss:schup

This section is designed as a companion to Martin–Woodcock [119]. This is an approach to partition algebra representation theory using generalised Schur algebras, motivated by certain n-stability properties of tensor product rules for symmetric group representations. We start by recalling some notations used in [119].

16.9.1 Local notations

Here Λ is the set of all compositions:

$$\Lambda \;=\; \{\lambda : \mathbb{N} \to \mathbb{N}_0 \mid supp(\lambda) \; finite\}$$

$$\Lambda_0 \;=\; \{\lambda : \mathbb{N}_0 \to \mathbb{N}_0 \mid supp(\lambda) \; finite\}$$

(caveat: this is not the notation we use elsewhere). The elements of Λ and Λ_0 are called *weights*. If Γ is any set of weights and $Q \in \mathbb{R}$ then $\Gamma(Q)$ is the subset of weights of degree Q.

The *dominant weights* Λ^+ (and Λ_0^+) are the set of not-strictly descending weights. There is a natural action of the symmetric group $S_{\mathbb{N}}$ (resp. $S_{\mathbb{N}_0}$) on Λ, for which Λ^+ is a fundamental region.

For $\lambda \in \Lambda$ and $Q \in \mathbb{N}_0$ (with $Q \geq |\lambda|$) define $\lambda^{(Q)} \in \Lambda_0(Q)$ by

$$\lambda^{(Q)} := (Q - |\lambda|, \lambda_1, \lambda_2, ...)$$

(**16.9.1**) Define

$$\mathcal{A} \;=\; \{g \in \mathbb{Q}[v] \mid g(\mathbb{Z}) \subset \mathbb{Z}\}$$

Note that this is a subring — the ring of 'numerical polynomials'. For example,

$$\binom{v}{i} = \frac{v(v-1)...(v-i)}{i!} \;\in \mathcal{A}$$

Indeed

$$\mathcal{A} \;=\; \bigoplus_{i \geq 0} \mathbb{Z}\binom{v}{i} \;=\; \mathbb{Z} \,\oplus\, \mathbb{Z}v \,\oplus \mathbb{Z}\frac{v(v-1)}{2} \oplus ...$$

16.9.2 The Schur algebras

Here $n \in \mathbb{N}$ and K is an infinite field. Following Green [57] we define $I(n,r) = \hom(\underline{r}, \underline{n})$; an action of S_r on the right on $I(n,r)$ by place permutation; and an equivalence relation on $I(n,r)^2$ by $ij \sim kl$ if there is a $w \in S_r$ such that $k = iw$ and $l = jw$. We write $I(n,r)^{2'}$ for a fixed but arbitrary transversal of $I(n,r)^2/\sim$.

Function $c_{ij} : GL_n(K) \to K$ takes $g \in GL_n(K)$ to its i,j-entry. The set of all functions $f : GL_n(K) \to K$ is an algebra via $(f+f')(g) = f(g) + f'(g)$ and $(ff')(g) = f(g)f'(g)$. Following Green [57] we define $A_K(n)$ as the subalgebra generated by the c_{ij}s. The subspace of elements expressible as homogeneous polynomials of degree r in the c_{ij}s is denoted $A_K(n,r)$. In fact $A_K(n,r)$ has a coalgebra structure.

(**16.9.2**) We then define the *Schur algebra*

$$S_K(n,r) = \mathrm{Hom}_K(A_K(n,r), K)$$

This has K-basis $\{\zeta_{ij} \mid i,j \in I(n,r)^{2'}\}$. The multiplication is

$$\zeta_{ij}\zeta_{kl} \;=\; \sum_{p,q} Z(i,j,k,l,p,q)\zeta_{pq} \tag{16.22}$$ bidet mult

where ...

Put

$$E_k \;=\; \oplus_{i \geq 0} k e_i$$

(**16.9.3**) The *global Schur algebra of degree Q* is

$$S_k(Q) \;=\; \mathrm{End}^{fin}_{kS_Q}(E_k^{\otimes Q})$$

There are sub-k-modules of $E_k^{\otimes Q}$ of form

$$M_k(\lambda) \;=\; k\{e_i \mid wt_0(i) = \lambda\,\}$$

so that

$$E_k^{\otimes Q} \;=\; \bigoplus_{\lambda\in\Lambda_0(Q)} M_k(\lambda)$$

is a decomposition of kS_Q-modules.

Let $\xi_\lambda \in S_k(Q)$ be the idempotent projecting onto $M_k(\lambda)$. For $n\in\mathbb{N}$ let

$$\xi = \sum_{\lambda\,:\,|\lambda|=Q;\;\lambda_i=0\;for\;i\geq n} \xi_\lambda .$$

Then

$$S_k(n,Q) \;=\; \xi S_k(Q)\xi \tag{16.23}$$ glob schur

(**16.9.4**) If $n\geq Q$ then (16.23) defines a Morita equivalence of $S_k(Q)$ with $S_k(n,Q)$.

(**16.9.5**) For $i,j\in I(\mathbb{N}_0,Q)$ write $(i,j)\sim(k,l)$ if the pairs are conjugate under the right S_Q-action on $I(\mathbb{N}_0,Q)^2$. Let $\xi_{ij}\in S_k(Q)$ be

$$\xi_{ij} : e_m \mapsto \sum_{(i,j)\sim(l,m)} e_l$$

Multiplication of these elements is essentially the same as for the ζ_{ij} in (16.22).

(**16.9.6**) We now take a kind of inverse limit of large Q.
Let $\mathcal{T}_\mathcal{A}$ be the free $\mathcal{A}$-module with basis $\{\xi_{ij} \mid (i,j)\in I(\mathbb{N}_0,\mathbb{N})^2/\sim\}$.

(**16.9.7**) Proposition. *There are unique elements $\hat{Z}(i,j,l,m,p,q)$ such that*

$$\xi_{ij}\xi_{lm} \;=\; \sum_{(p,q)} \hat{Z}(i,j,l,m,p,q)\xi_{pq}$$

(where the sum is over a transversal of $I(\mathbb{N}_0,\mathbb{N})^2/\sim$) makes $\mathcal{T}_\mathcal{A}$ an associative $\mathcal{A}$-algebra without identity.

(**16.9.8**) Example. An $i\in I(\mathbb{N}_0,\mathbb{N})$ is an infinite list of integers, almost all zero. In writing them we may omit trailing zeros. Thus 11=11000, 11111, 101, 0011 are all examples. We write elements ξ_{ij} as bracketed pairs in this notation, with i over j, such as

$$\xi_{ij} \;=\; \begin{bmatrix} 11000 \\ 11111 \end{bmatrix} \;=\; \begin{bmatrix} 10100 \\ 11111 \end{bmatrix}$$

Then for example

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix}\begin{bmatrix} 11111 \\ 11000 \end{bmatrix} \;=\; \ldots$$

may be computed by considering a 'general Q' case of the finite problem. This has a given pq on the right only if there is an s such that $(11000,11111)\sim(p,s)$ and $(s,q)\sim(11111,11000)$ (we continue to omit trailing zeros even in the general-finite case). Note that any such s must have five 1s, but there are potentially many possible distributions, depending on p,q. We may fix $p=11000$ in the transversal. There are then various possibilities for q.

Clearly there are solutions when $p = q = 11000$. This requires that s has five 1s, with the first two in the first two positions, so there are (as it were) $(v-2)(v-3)(v-4)/6!$ possibilities.

Another possibility for q is then $q = 101$. Here s must start 111, but the remaining two 1s can go anywhere: $(v-3)(v-4)/2$ possibilities.

The last possibility in the transversal is $q = 0011$. Here s must start 1111, but the remaining 1 can go anywhere: $(v-4)$ possibilities.

Altogether we have

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix} \begin{bmatrix} 11111 \\ 11000 \end{bmatrix} = \binom{v-2}{3} \begin{bmatrix} 11 \\ 11 \end{bmatrix} + \binom{v-3}{2} \begin{bmatrix} 110 \\ 101 \end{bmatrix} + \binom{v-4}{1} \begin{bmatrix} 1100 \\ 0011 \end{bmatrix}$$

(**16.9.9**) For k a commutative ring and $Q \in \mathbb{Z}$ we write $k^{(Q)}$ for k made into an $\mathcal{A}$-algebra via evaluation of polynomials at Q.

When $Q \in \mathbb{N}$ there is an isomorphism between suitable finite pieces of $\mathcal{T}_{k^{(Q)}}$ and $S_k(Q)$:

$$\left(\sum_{\lambda \in \Lambda[Q/2]} \xi_\lambda \right) \mathcal{T}_{k^{(Q)}} \left(\sum_{\lambda \in \Lambda[Q/2]} \xi_\lambda \right) \cong S_k(\Gamma, Q)$$

where

$$\Gamma =$$

16.9.3 The global partition algebra as a localisation

The idea is to identify the Potts module U_k (for fixed Q), viewed as a right S_Q-module, with a summand of the defining module $E_k^{\otimes Q}$ of $S_k(Q)$ and then to "take limits".

16.9.4 Representation theory

(**16.9.10**) Here F is a commutative ring that we shall specify shortly. For any such F, and $Q \in \mathbb{Z}$, we write $F^{(Q)}$ for F made into a $\mathcal{A}$-algebra or $\mathbb{Z}[v]$-algebra (say) by evaluating polynomials at $v = Q$ (see [119, (3.2),(3.8)]).

Now let $R \in \mathbb{N}_0$ and $F = \mathbb{F}_p^{(R)}$ for some characteristic $p > 0$ (precise choice of which will eventually not matter). Our first objective is to say something about the modules of the global Schur algebra $\mathcal{T}_k$, where k is an $\mathcal{A}$-algebra which is a field of char.0 in which element v maps to $R \in \mathbb{N}_0$. Under suitable circumstances, simple modules for $\mathcal{T}_F$ are obtained by reduction mod.p of those for $\mathcal{T}_k$. We can thus study $\mathcal{T}_k$ (at the level of characters, say) by studying $\mathcal{T}_F$. But $\mathcal{T}_F$ in turn can be studied by studying a suitable collection of ordinary Schur algebras, and hence via the representation theory of the general linear groups.

For $\nu \in \Lambda_0^+$ with support at most in positions 0 through n (note that for each ν this just sets a lower bound for n), let $\Delta_F(\nu)$ denote the Weyl module for the F-group scheme GL_{n+1} (rows and columns of matrices indexed from 0). Let $\Delta_F^l(\nu)$ denote the l-th term in the Jantzen filtration [79, II.8] of $\Delta_F(\nu)$. Write $e_0, e_1, ..., e_n$ for the standard ordered basis in the weight lattice $\mathbb{Z}^{n+1}$.

(**16.9.11**) Set

$$Q = R + p.$$

Now fix n (some $n >> 0$, say) and set $\rho = \rho_n = (n, n-1, ..., 0)$. If $\nu = \lambda^{(Q)}$ then the Jantzen sum formula [79, II.8.19] gives:

$$\sum_{l>0} ch\ \Delta^l_F(\nu) \;=\; \sum_{\substack{1 \leq j \leq n \\ \langle e_0 - e_j, \nu + \rho_n \rangle > p}} \chi(\nu(j)) \qquad (16.24)$$

eq:Jantzen sum

where $\chi(\mu) = ch\ \Delta_F(\mu)$ if μ dominant and $\chi((ij).\mu) = -\chi(\mu)$; and

$$\begin{aligned} \nu(j) \;&=\; (0j).\nu + p(e_0 - e_j) \\ &=\; (\lambda_j - j, \lambda_1, ..., \lambda_{j-1}, Q - |\lambda| + j, \lambda_{j+1}, ...) + (p, 0, ..., 0, -p, 0, ...) \end{aligned}$$

Here

$$w.\lambda \;:=\; w(\lambda + \rho_n) - \rho_n$$

(**16.9.12**) Remarks. (1) The dot action is used here so that the nominal index scheme for modules is the natural scheme for GL. One could work with ρ-shifted weights from the start, whereupon the dot action would be replaced by ordinary reflections.
(2) The $+(p, 0, ..., -p, 0, ...)$ in $\nu(j)$ makes it the image of ν in an affine wall. However, $\lambda^{(Q)} = \lambda^{(R+p)}$ has a p in the 0-th term, so $(0j).\nu$ has a p in the j-th term, which is then just moved back to the first term by $+(p, 0, ..., -p, 0, ...)$.

(**16.9.13**) Let us examine the sum on the RHS in (16.24). We have

$$\langle e_0 - e_j, \nu + \rho \rangle \;=\; (Q - |\lambda| + n) - (\lambda_j + n - j)$$

so there is a j-term in the sum on the RHS in (16.24) iff

$$R + j > |\lambda| + \lambda_j$$

If there is no such j then $\Delta_F(\nu)$ is simple. If there is such a j, let i be the least such. Then $\nu(i)$ fails to be dominant iff

$$R + i - 1 = |\lambda| + \lambda_{i-1}$$

If this holds, then not only is $\nu(i)$ non-dominant, but it lies on the $(i-1, i)$-reflection wall:

$$\nu(i) \;= (..., \lambda_{i-1}, \lambda_{i-1} + 1, ...)$$

so $\chi(\nu(i)) = 0$. It follows that $\chi(\nu(j)) = 0$ for all $j > i$ too, so again $\Delta_F(\nu)$ is simple.

On the other hand if $\nu(i)$ is dominant then, noting that $R+i > |\lambda|+\lambda_i$ implies $R+j > |\lambda|+\lambda_j$ for all $j > i$, we have

$$\sum_{l>0} ch\ \Delta^l_F(\nu) \;=\; ch\ \Delta_F(\nu(i)) \;+\; \sum_{j>i} \chi(\nu(j))$$

Is $\nu(i+1)$, say, dominant? We can bypass this question.

The sum formula for $\Delta_F(\nu(i))$ in this case involves:

$$\langle e_0 - e_i, \nu(i) + \rho \rangle \;=\; (\lambda_i - i + p) \;-\; (R + i - |\lambda|) + i \;<\; p$$

$$\langle e_0 - e_{i+1}, \nu(i) + \rho \rangle \; = \; (\lambda_i - i + p) \; - \; (\lambda_{i+1} - (i+1)) \; > \; p$$

so j gives a contribution iff $j > i$:

$$\sum_{l>0} ch \; \Delta_F^l(\nu(i)) \; = \; \sum_{j>i} \chi((\nu(i))(j))$$

In this case

$$(\nu(i))(j) \; = \; (\lambda_j - j + p, ..., Q + i - |\lambda|, ..., \lambda_i - i + j, ...)$$

(displaying positions $1, i, j$). In fact

$$(\nu(i))(j) \; = \; (ij).\nu(j)$$

Thus

$$\sum_{l>0} ch \; \Delta_F^l(\nu) \; = \; ch \; \Delta_F(\nu(i)) \; - \; \sum_{l>0} ch \; \Delta_F^l(\nu(i)) \; = \; ch \; L_F(\nu(i)) \; - \; \sum_{l>1} ch \; \Delta_F^l(\nu(i))$$

Since the LHS is a non-negative sum of simple characters the nominally negative part must vanish, and we have

$$\Delta_F^1(\nu) = L_F(\nu(i))$$

(**16.9.14**) JOB. Recast all this in the P-natural ρ-shift setting.

(**16.9.15**) EXAMPLES. The simplest examples is $\lambda = \emptyset$, $R = 0$. In principle we need to choose n and p. We note (a) that this can always be done; and (b) that the choice plays no subsequent role. Indeed the primeness of p is a vestige of the Schur algebra 'finesse' that allows us to use the Jantzen sum formula. With this in mind, we shall take n large and shift so that the first p-affine reflection wall parallel to $(0i)$ (each i) is drawn as if the 'non-affine' wall. This corresponds, combinatorially, to setting $p = 0$ — we must then remember that the $(0i)$ wall drawn is not at the boundary of the 'dominant region'. We will also apply (an n-independent version of) the ρ-shift derived above to weights at the outset, so we can replace the dot action of the 'Weyl group' by the ordinary action. After the ρ-shift we have the embedding:

$$\Lambda \to \mathbb{Z}^{\mathbb{N}_0}$$

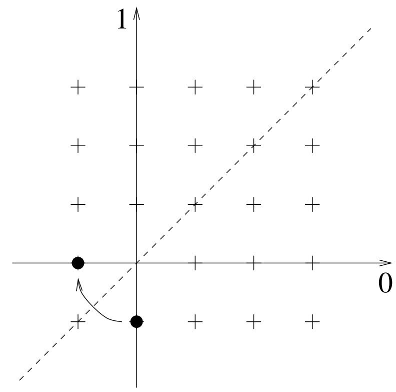
$$\mathfrak{e}_R : \lambda \mapsto (R - |\lambda|, \lambda_1, \lambda_2, ...) + (0, -1, -2, -3, ...)$$

so

$$\emptyset^{(0)} \; \mapsto \; (0, -1, -2, -3, ...)$$

We can represent this diagramatically by projecting onto some i, j-subspace, such as the $0, 1$-

subspace:



The figure also shows

$$(01)(0,-1,-2,-3,...) \;=\; (-1,0,-2,-3,...)$$

Note that since we are working with the ρ-shifted weight we use the simple reflection, not the dot action. Note that $\mathfrak{e}_R$ is invertible:

$$\mathfrak{e}_o^{-1}(-1,0,-2,-3,...) = \;(1)$$

Let us consider the images of some other weights:

λ	$\mathfrak{e}_0(\lambda)$	$\mathfrak{e}_1(\lambda)$	$\mathfrak{e}_2(\lambda)$
0	$(0,-1,-2,-3,...)$	$(1,-1,-2,-3,...)$	$(2,-1,-2,-3,...)$
(1)	$(-1,0,-2,-3,...)$	$(0,0,-2,-3,...)$	$(1,0,-2,-3,...)$
(2)	$(-2,1,-2,-3,...)$	$(-1,1,-2,-3,...)$	$(0,1,-2,-3,...)$
(1^2)	$(-2,0,-1,-3,...)$	$(-1,0,-1,-3,...)$	$(0,0,-1,-3,...)$

Note that some of these weights do not look 'dominant', but this is because we have omitted $+p$ from the 0-th term. Note that $\mathfrak{e}_0(2)$ lies on the (02)-wall. Recall that this is an affine wall in the GL setting (with large p):

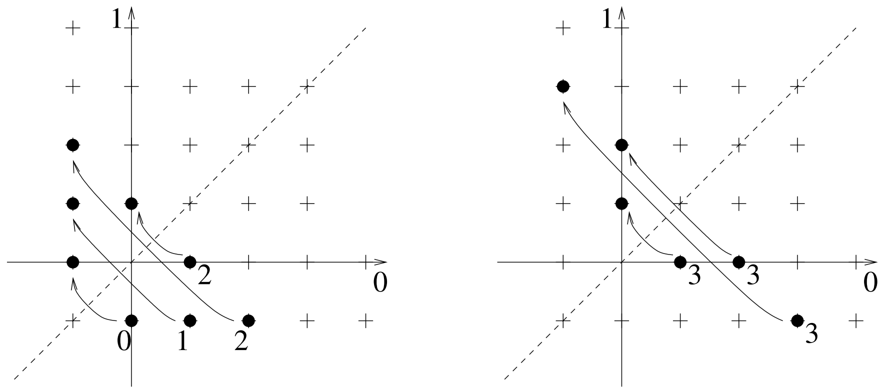
$$(2) \mapsto (p-2,1,-2,-3,...)$$

so does not imply that $\Delta_F(2)$ has vanishing character. However, it follows that all images under reflections of form $(0i)$ lie on an (ij)-wall. This implies that $\Delta_F(2)$ has vanishing radical. Note indeed that every λ lies on a wall — the (first affine) $(0\,|\lambda|)$-wall, unless it takes the form $\lambda = (1^m)$ for some m.

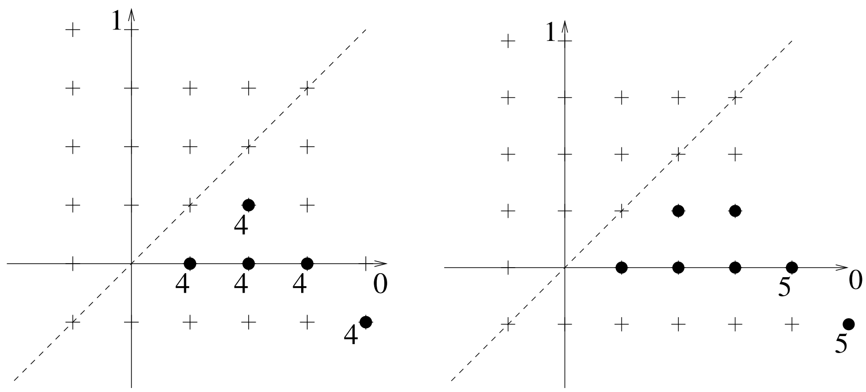
Similarly both $\mathfrak{e}_1(1)$ and $\mathfrak{e}_1(1^2)$ give simple modules.

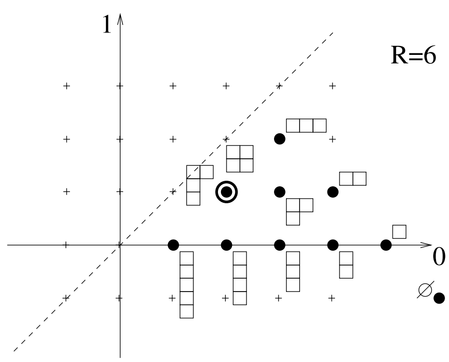
(**16.9.16**) LEMMA. *(I) each block consists either of a singleton, or of a chain of weights.*
(II) for given R each chain block begins with a weight which is a partition of R with the first row removed.

The first two elements in each chain for $R = 0, 1, 2, 3$ are here:



The first elements in each chain for $R = 4, 5, 6$ are here:





In the latter figure ($R = 6$) we recall the original labels for these elements. Note that position in the 0,1-projection is no longer sufficient to distinguish all the weights depicted here.

16.9.5 Alcove geometric charaterisation

If we consider the $S_{\mathbb{N}}$ parabolic in $S_{\mathbb{N}_0}$ then the weights $\cup_R \mathfrak{e}_R(\Lambda)$ (a disjoint union???) all lie in the 'dominant' fundamental chamber. If we maintain our convention of considering $p = 0$ then dominance with respect to the 0-th position is not imposed.

(1) The weights that label the various chain blocks are the fully dominant weights — the weights in the fundamental alcove (in the sense of the Coxeter group/parabolic above).

(2) The other weights in the chain blocks are images of these weights in the various $(0, j)$-walls (which I guess are not walls intersecting facets of the fundamental alcove), which can also be realised via simple reflection chains of the form $(i\; i+1)...(23)(12)(01)$.

(3) The weights in the fundamental chamber that lie on a wall (i.e. on a $0, j$-wall) are singletons. Because of the $\mathfrak{e}_R$ embedding we use for $\mathcal{T}_k$ and the partition algebra (cf. that used for the Brauer algebra, say), there are many of these. If we make n large compared to R then almost every block is a singleton.

16.9.6 More

What about walks and so on? Can we embed the walks on the multiplicity-free Bratelli diagram in the same setup?

(1) can we embed the index set for the odd-partition algebras in the same setup?

16.10 Notes and references

The partition algebra first appears in the context of the partition vector formalism for Potts models in computational Statistical Mechanics [103, 104] ('partition vector' refers to a vector of partition functions, not to set partitions). In this setting it is a quotient of a case of the so-called graph Temperley–Lieb algebras. It appears as a focus for study in its own right in the 1992 Yale preprints YCTP-P33-92 and YCTP-P34-92 [106, 114].

16.10.1 Notes on the Yale papers on the partition algebra

A set of subsets of a set M *covers* M if its union is M.

First we focus on the partition category in the form introduced in [106, §7]. There the set P_M of partitions of a set M is denoted S_M. Further, for q a covering set of subsets, $\mathcal{Q}(q)$ is defined as the transitive closure.

(**16.10.1**) Fix a field k. For sets $N \subseteq M$ define

$$In_N : S_M \to k(Q)S_N$$

by $In_N(p) = Q^{f_N(p)} p|_N$, where $p|_N$ is the restriction of p to N and $f_N(p)$ is the number of parts of p not intersecting N.

(**16.10.2**) With $N \subset M \cup M'$ define $Ag : S_M \times S_{M'} \to S_{M\cup M'}$ by $Ag(A, B) = \mathcal{Q}(A \cup B)$; and composition $\mathcal{P}_N$ by commutativity of

$$\begin{tikzcd} S_M \times S_{M'} \arrow[rr, "\mathcal{P}_N"] \arrow[dr, "Ag"'] & & k(Q)S_N \\ & S_{M\cup M'} \arrow[ur, "In"'] & \end{tikzcd}$$

(**16.10.3**) It will be clear how to use this composition to define a composition on, say, $S_{\underline{m}\cup\underline{n'}} \times S_{\underline{n}\cup\underline{l'}} \to k(Q)S_{\underline{m}\cup\underline{l'}}$ (by mapping $\underline{n} \cup \underline{l'} \to \underline{n'} \cup \underline{l''}$ and then using $\mathcal{P}_{\underline{n'}}$ directly). This then extends $k(Q)$-linearly to define composition in the partition category. .

16.10.2 Aside on notation

Suppose $M = \{1, 2, ..., n, 1', 2', ..., n'\}$ and let $m = 2n = |M|$. Where convenient and unambiguous write $S_m = S_M$.

Define an equivalence relation on S_m by $A \sim B$ if they are the same up to a perm of the propagating 'lines'.

16.11 Bibliographic notes

ss:pabib

To do. Cf. e.g. [81, ?, ?].

Part V

Brauer

Chapter 17

On representations of the Brauer algebra

ch:BrauerI

The Brauer algebra [15] provides a good testing ground for some of the technniques we met in Chapters 1-9.

In this Chapter we first note some motivations for studying the Brauer algebra. In §17.2 we define the algebra and associated 'diagram categories' in terms of bases of Brauer diagrams; and determine some useful properties of these diagrams. In §17.5 we summarize a general approach to Brauer algebra representation theory. In the appendix (§17.6 *et seq*), as a kind of extended overview of the next few chapters, we summarize several key results from the literature.

In Chapter 18 we study general aspects of the representation theory of the Brauer algebra, such as the construction and properties of a special class of 'integral' modules. In Chapter 19 we study the representation theory over $\mathbb{C}$. In Chapters 20,21 we study the *explicit* construction of simple modules over $\mathbb{C}$.

We include bibliographic notes later. Further extensive references and an exposition of Brauer algebra representation theory over the complex field are also given in the sequence of papers of Cox, De Visscher, Martin [28, 29, 30].

Contents

17.1 Context of the Brauer algebra

ss:brauer context

The Brauer algebra is a device proposed by Brauer [15] to study invariant theory of the orthogonal groups, generalising Schur–Weyl duality.

17.2 Brauer diagrams and diagram categories

ss:Brauerdefn

We recall some notation from §3.2. For $n \in \mathbb{N}$ define $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n'} = \{1', 2', ..., n'\}$. For S a set, $\mathsf{P}(S)$ is the power set; P_S the set of partitions of S; J_S the set of pair partitions of S; and $J_{n,m} = J_{\underline{n} \cup \underline{m'}}$.

We shall use $J_{n,n}$ as a basis for $B_n(\delta)$, but we also introduce a useful diagram calculus.

(**17.2.1**) For given $n > 1$, we define some particular pair partitions in $J_{n,n}$:

$$U_{ij} = \{\{1, 1'\}, \{2, 2'\}, ..., \{i, j\}, \{i', j'\}, ..., \{n, n'\}\} \tag{17.1}$$

Uij

$$(ij) = \{\{1, 1'\}, \{2, 2'\}, ..., \{i, j'\}, \{i', j\}, ..., \{n, n'\}\}$$

We shall call pair partitions of this form *generators*.

(**17.2.2**) An (n, m)-*Brauer diagram* is a representation of a pair partition of a row of n and a row of m vertices, arranged on the top and bottom edges (respectively) of a rectangular frame. Each part is drawn as a line (a smooth or piecewise-linear embedding of $[0, 1]$ in $\mathbb{R}^2$), joining the corresponding pair of vertices, in the rectangular interval. Each line touches the frame only at its endpoints, and there transversally (i.e. with tangent disinct from the frame). The set of embedded lines is *regular* (if two lines meet at x they have distinct tangents at x [31]).

See Figure 17.1 for an example. Note that Brauer diagrams are particular kinds of examples of partition diagrams as in §16.1.2.

Note that there are a continuum of different ways of representing a given non-empty partition in this way. However, it will be clear that an (n, m)-Brauer diagram can be used to represent an element of $J_{n,m}$. Again see Figure 17.1 for an example.

For d such a diagram, write $[d]$ for the corresponding pair partition. (We will shortly justify suspending this distinction.)

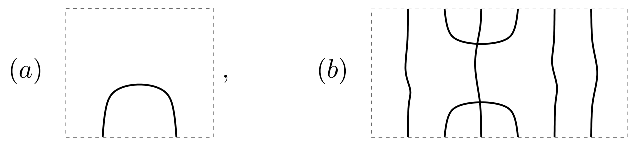


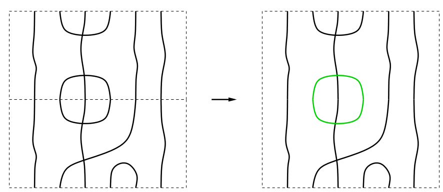
Figure 17.1: (a) A $(0,2)$-Brauer diagram; (b) a $(6,6)$-Brauer diagram, representing $U_{24} \in J_{6,6}$. fig: Br1

(**17.2.3**) Recall (e.g. from (6.1.22)) the *bare* Brauer category

$$\mathbf{Br} = (\mathbb{N}_0, J_{n,m}, \circ)$$

As noted, Brauer diagrams can be used to describe the 'homs' in this category. They can also be used to compute compositions, as we show next.

(**17.2.4**) Suppose d, d' are Brauer diagrams such that $[d] \in J_{n,m}$ and $[d'] \in J_{m,l}$. If the vertices are suitably spaced, then d and d' may be juxtaposed so as to make each of the m lower vertices in d coincide with the corresponding upper vertex in d', as illustrated on the left here:



Write dd' for this juxtaposition. Note that a pair partition of the vertices on the exterior of the combined frame may be read off from dd', in direct analogy to (3.4). Write $[dd']$ for this partition, regarded as an element of $J_{n,l}$.

lem:welldef (**17.2.5**) Lemma. *Suppose d, d' are Brauer diagrams as above.*
(1) The partition $[dd']$ depends only on $[d]$ and $[d']$.
(2) Composition of partitions in the category **Br** *may be computed by juxtaposition of diagrams. That is, $[d] \circ [d'] = [dd']$.*
(3) Another feature of dd' that depends only on $[d]$ and $[d']$ is the number $\#^\circ(dd')$ of internal closed loops formed (such as the one in the example).

Proof. This is effectively a special case of the corresponding partition algebra result in §16.1.2. □

(**17.2.6**) We define a category $\mathbf{Br}^\bullet = (\mathbb{N}, J_{n,m} \times \mathbb{N}_0, \bullet)$ by the composition

$$([d], i) \bullet ([d'], i') = ([dd'], i + i' + \#^\circ(dd'))$$

(a diagram calculus for this uses Brauer diagrams with closed loops, sometimes called Brauer pseudodiagrams).

(**17.2.7**) Fix a ring k, and $\delta \in k$. Noting Lemma (17.2.5)(3), we define a deformation of the k-linear extension of **Br** by modifying the category composition to:

$$[d] * [d'] = \delta^{\#^\circ(dd')}[dd']$$

(and extending k-linearly). We denote this category

$$\mathbf{Br}_\delta^k = (\mathbb{N}_0, kJ_{n,m}, *),$$

or just $\mathbf{Br}_\delta$ if k is fixed.

(**17.2.8**) Lemma. *For any commutative ring k and $\delta \in k$ the category $\mathbf{Br}_\delta^k$ is a quotient of the k-linear extension of $\mathbf{Br}^\bullet$, given by*

$$([d], i) \mapsto \delta^i [d]$$

diagram hom sets (**17.2.9**) We write $\mathbf{Br}(m,n)$ for $J_{m,n}$ realised as the set of (m,n)-Brauer diagrams up to equivalence. Hereafter a Brauer 'diagram' generally means a diagram up to equivalence. Thus the hom-set in the k-linear category $\mathbf{Br}_\delta^k$ can equally well be considered to be $k\mathbf{Br}(m,n)$.

(**17.2.10**) Example. The sets $\mathbf{Br}(0,0)$, $\mathbf{Br}(1,1)$, $\mathbf{Br}(2,0)$ and $\mathbf{Br}(0,2)$ each have a single element, here denoted 1_0, 1_1, u and u' respectively.

de:brauer alg (**17.2.11**) The Brauer algebra $B_n(\delta)$ over k is the free k-module with basis $\mathbf{Br}(n,n)$ and the $\mathbf{Br}_\delta^k$ category composition (i.e. replacing each closed loop formed in composition by a factor δ).

17.2.1 Remarks on the ground ring and Cartan matrices

(**17.2.12**) Remark. The basic, 'integral' version of the Brauer algebra is the case over the ring $k = \mathbb{Z}[\delta]$. From here there are thus two aspects to the base change to an algebra over a field: the choice of field k and the choice of δ. More precisely this is the choice of field k equipped with the structure of $\mathbb{Z}[\delta]$-algebra. Thus we have possible intermediate steps: base change to ring $k[\delta]$ (k some field); base change to $\mathbb{Z}$ (a $\mathbb{Z}[\delta]$-algebra by fixing $\delta = d \in \mathbb{Z}$).

Each of these ground rings is a PID and hence a Dedekind domain, and hence amenable to a P-modular treatment (in the sense of Brauer's general approach to 'modular' representation theory of finite dimensional algebras, described variously in Benson [7], Curtis–Reiner [33], Brauer [16] and so on). This means that the Cartan matrix $C = ([P_i : L_j])_{i,j}$ can be computed via a Δ-decomposition matrix D. See §9.3.2 for a description of the general setting.

(**17.2.13**) Roughly speaking, cf. [16], the setup is as follows. Let B be an algebra over $\mathbb{Z}[\delta]$ (we are thinking of one of the Brauer algebras).
(I) There is a valuation on $k[\delta]$ (any field k — we are thinking of $k = \mathbb{C}$, and will use this example hereafter); and hence an absolute value. It then follows that there is a completion of $\mathbb{C}[\delta]$ to a complete DVR, call it R. (Since R simply extends $\mathbb{Z}[\delta]$ we have a version of B over R, which we could call B_R.)
(NB According to Curtis–Reiner [33] the various composition multiplicities we shall compute do not depend on the completion — it is needed only to satisfy certain existence requirements in the proof.)
(II) Now pick $\delta_c \in \mathbb{C}$ and consider an element $(\delta - \delta_c)$ in $\mathbb{C}[\delta]$ (and hence in R) and consider the quotient ring by the ideal I so generated. Since the ideal is maximal in R (see e.g. **??** in Ch.7) the quotient is a field, call it $\bar{R}$.
(III) Next suppose that there is an extension $\hat{R}$ of R (typically just the field of fractions), such that $B_{\hat{R}}$ is semisimple (with simple modules $\hat{\Delta}_i = \hat{R} \otimes_R \Delta_i$ for some set of R-free B-modules Δ_i).

So far we have (a) here:

$$(a)\quad \begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ \hat{R} & & \bar{R} \end{tikzcd} \qquad (b)\quad \begin{tikzcd} & P, e' \arrow[dl] & \\ M & & P, e \arrow[ul] \end{tikzcd} \qquad (c)\quad \begin{tikzcd} & \Delta \arrow[dr] & \\ \hat{\Delta} \arrow[ur] & & \bar{\Delta} \end{tikzcd}$$

Let us write L_j for a complete set of simple $B_{\bar{R}}$-modules (so far unknown). And $\bar{\Delta}_i = \bar{R} \otimes_R \Delta_i = R/I \otimes_R \Delta_i$. (NB we have $\bar{\Delta}_i = \Delta_i / I\Delta_i$, right!?) We have some well-defined (but unknown) multiplicities

$$d_{ij} = [\bar{\Delta}_i : L_j]$$

(NB This notation d_{ij} is exactly as used by Benson [7]. We shall mainly adhere to a slightly different notation — $D_{ji} = d_{ij}$.) Now consider an indecomposable projective $B_{\bar{R}}$-module P.

(**17.2.14**) CLAIM: Let $ee = e \in B_{\bar{R}}$. Then there is an idempotent e' in B whose image in $B_{\bar{R}}$ is e.

ASIDE: The point of this observation is that there is a projective $B_{\bar{R}}$-module $B_{\bar{R}}e$, and a projective B-module Be' that, by the observation, reduces to it. On the other hand, Be' is also a lattice inside a $B_{\hat{R}}$-module M, say. Since $B_{\hat{R}}$ is semisimple, *this* module decomposes as a sum of simple $B_{\hat{R}}$-modules. The multiplicity of $\hat{\Delta}_i$ in M can be expressed as

$$[M : \hat{\Delta}_i] \;=\; \dim_{\hat{R}} \hom_{B_{\hat{R}}}(M, \hat{\Delta}_i)$$

On the other hand:
CLAIM: $\hom_B(Be', \Delta_i)$ is an R-form for $\hom_{B_{\hat{R}}}(M, \hat{\Delta}_i)$, so its rank agrees with the dimension.
(Certainly $\hom_B(Be', \Delta_i)$ sits inside. But is $\hom_B(Be', \Delta_i)$ even a free R-module? We have (from earlier) $\hom_B(Be', \Delta_i) \cong e'\Delta_i$. Does this help?)

(**17.2.15**) It follows that there is a notion of a multiplicity of Δ_i in P_j (even if there is not a filtration by such modules on the $B_{\bar{R}}$ side). One then shows that $[P_j : \Delta_i] = [\Delta_i : L_j]$, and by bringing in partial data from each side, eventually computes C.

17.3 Properties of the Brauer diagram basis

ss:diagram99

(**17.3.1**) Extending the equivalence of diagrams from (17.2.9), there is a set bijection between $\mathbf{Br}(m, n)$ and a set of pair partitions similarly drawn on a disk with marked boundary point. The $m + n$ vertices are drawn around the disk clockwise from the marked point. For any $i \in \mathbb{Z}$ such that $m - i, n + i \in \mathbb{N}_0$ this induces a bijection $\mathcal{R}_i : \mathbf{Br}(m, n) \to \mathbf{Br}(m - i, n + i)$. For example, if $i = 1$ this moves a single vertex ('ambient isotopically') from the top edge clockwise to the bottom edge.

17.3.1 Manipulation of Brauer diagrams: lateral composition

(**17.3.2**) For $m, n, r, s \in \mathbb{N}_0$, define a product

$$\boxtimes : \mathbf{Br}(m, n) \times \mathbf{Br}(r, s) \;\to \mathbf{Br}(m + r, n + s)$$

by placing diagrams side by side. Hence define an injection adding propagating lines $\{\{m+1, n+1'\}, ..., \{m+r, n+r'\}\}$:

$$\begin{aligned} i_{m+1,m+r} : \mathbf{Br}(m,n) &\hookrightarrow \mathbf{Br}(m+r, n+r) \\ D &\mapsto D \boxtimes 1_r \end{aligned}$$

(**17.3.3**) The map $d \mapsto d \boxtimes 1_1$ defines an inclusion of algebras $B_{n-1}(\delta) \hookrightarrow B_n(\delta)$ $(n \geq 1)$. By this restriction, any B_n-module (such as $k\mathbf{Br}(n,m)$) is also a B_{n-1}-module.

(**17.3.4**) We may define an injection of $\mathbf{Br}(n,n) \hookrightarrow \mathbf{Br}(n, n+2)$ by $d \mapsto \mathcal{R}_1(d \boxtimes 1_1)$. This is the same as $d \mapsto d \boxtimes u'$.

(**17.3.5**) A non-crossing Brauer diagram is called a Temperley–Lieb diagram. Using both the category and the lateral composition the elements 1_1, u and u' generate all such diagrams. There are three elements in $\mathbf{Br}(2,2)$, including a crossing diagram. If we write x for this, then:

(**17.3.6**) Proposition. *Any Brauer diagram may be realised using the two compositions $\circ$ and $\boxtimes$ (or indeed $*$ and $\boxtimes$) on the diagrams $\{1_1, u, u', x\}$.*

Proof. Note first that the realisation of any TL diagram with $\{1_1, u, u'\}$ is easy. Note next that $\{1_1, x\}$ generate all the S_n subgroups. Using these it is easy to see that the number of crossings in a diagram can be reduced to zero. This reduces to the TL case. □

17.3.2 Ket-bra diagram decomposition

ss:ket bra

de:prop line (**17.3.7**) A *propagating line* in a (pseudo)diagram is a line from top to bottom (a special case of that in (16.2)). Define $\#^p : \mathbf{Br}(n,m) \to \mathbb{N}_0$ so that $\#^p(d)$ is the number of propagating lines.

(**17.3.8**) We write $\mathbf{Br}^{\leq l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ with $\leq l$ propagating lines.

(**17.3.9**) Lemma. *Using the bare composition*

$$\mathbf{Br}^{\leq l}(m,n) = \mathbf{Br}(m,l) \circ \mathbf{Br}(l,n)$$

Thus

$$k\mathbf{Br}^{\leq l}(m,n) = k\mathbf{Br}(m,l) * k\mathbf{Br}(l,n)$$

unless $m, n = 0$, $l \geq 2$, in which case

$$k\mathbf{Br}(m,l) * k\mathbf{Br}(l,n) = \delta k\mathbf{Br}^{\leq l}(0,0)$$

(**17.3.10**) We write $\mathbf{Br}^{l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ with l propagating lines; and $\mathbf{Br}^{1_l}(m,n)$ for the subset of these in which none of the l propagating lines cross.

(**17.3.11**) Lemma. *The identity diagram in $(\mathbf{Br}(l,l), \circ)$ (and hence in $(k\mathbf{Br}(l,l), *)$ for any k, δ) is denoted 1_l. The pair $(\mathbf{Br}^{l}(l,l), *)$ is isomorphic to the symmetric group S_l.*

(**17.3.12**) For this reason we sometimes call elements of $\mathbf{Br}^{l}(l,l)$ (any l) *permutations.*

pa:dec1 (**17.3.13**) Since an element of $\mathbf{Br}(m,n)$ is specified by the list of pair partitions it depicts, it is specified in particular by the decomposition of this list into (i) pairs on the top row; (ii) pair on the bottom row; (iii) pairs from top to bottom. We have immediately the following Lemma.

lem:braket (**17.3.14**) Lemma. *(*Bra-ket Lemma*) Let $m, n \geq l$. The category composition defines bijections:*

$$\mathbf{Br}^{1_l}(m,l) \;\times\; \mathbf{Br}^{l}(l,l) \;\stackrel{\sim}{\to}\; \mathbf{Br}^{l}(m,l) \tag{17.2}$$ 1_lx(l,l)

$$\mathbf{Br}^{1_l}(m,l) \;\times\; \mathbf{Br}^{l}(l,l) \;\times\; \mathbf{Br}^{1_l}(l,n) \stackrel{\sim}{\to} \mathbf{Br}^{l}(m,n) \tag{17.3}$$ 1_lx(l,l)2

We call this ket-bra decomposition *or* polar decomposition. □

(**17.3.15**) The *ket* $|d\rangle$ of $d \in \mathbf{Br}(m,n)$ is the projection of its preimage (in (17.3)) into $\mathbf{Br}^{1_l}(m,l)$.

(**17.3.16**) Remark. Note the opposite equivalence of our various categories. By this equivalence, every 'left' result has a right version. In general we shall only state one version explicitly.

iagram-principal lem:gpa1 (**17.3.17**) Lemma. *Fix $m, n \geq l$.*
(I) For any $d, d' \in \mathbf{Br}^{l}(m,l)$, there exists a permutation $w \in \mathbf{Br}^{m}(m,m)$ such that $wd = d'$ (hereafter we may omit the composition symbol, where no ambiguity arises).
(II) For any $d \in \mathbf{Br}^{l}(m,n)$ and $d' \in \mathbf{Br}^{\leq l}(m,n)$ there exist diagrams $w \in B_m$ and $w' \in B_n$ such that $d' = wdw'$.

Proof. (I): This follows from (17.3.13). (II): This follows similarly. Note first that the case of $d' \in \mathbf{Br}^{l}(m,n)$ is a direct extension of (I). If $d' \in \mathbf{Br}^{\leq l-2}(m,n)$ then d has at least two propagating lines, and any two such can be replaced by a 'cup and cap', with all other pairs unchanged, by multiplying by a suitable diagram d''. That is, $dd'' \in \mathbf{Br}^{l-2}(m,n)$. Now use the first part again, at level $l-2$. □

17.4 Idempotent diagrams and subalgebras in $B_n(\delta)$

ss:brauer ids

We shall have a convention of using capital letters for elements of Brauer algebras defined by generators (and hence making sense in any sufficiently large Brauer algebra); and using lower-case for elements defined as specific diagrams in specific algebras.

In this section the category $\mathbf{Br}_\delta$, and each B_n, is defined over an arbitrary commutative ring k with $\delta \in k$.

(**17.4.1**) The product $E_1 = U_{12}U_{23}$ in B_n defines an element in $\mathbf{Br}^{n-2}(n,n)$ for any $n \geq 3$. For $m \in \mathbb{N}$, the product

$$\overline{E}_m := \; U_{12}\, U_{34} \;...\; U_{2m-1\;2m}$$

defines an element in $\mathbf{Br}^{n-2m}(n,n)$ for any $n \geq 2m$;

$$E_m := (U_{12}U_{34}...U_{2m-1\;2m})(U_{23}U_{45}...U_{2m\;2m+1}) = U_{2m-1\;2m}E_{m-1}U_{2m\;2m+1}$$

defines an element in $\mathbf{Br}^{n-2m}(n,n)$ for any $n \geq 2m+1$. (See Figure 17.2(a) for an example.)

lem:id9 (**17.4.2**) Lemma. *All the elements E_m are idempotent.*

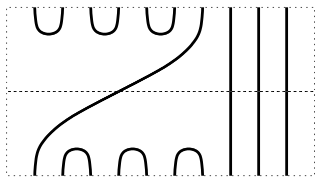
Proof. Draw a diagram. □

(**17.4.3**) One finds, for example, that $E_m = (u^{\boxtimes m} \boxtimes 1_1 \boxtimes (u')^{\boxtimes m}) \boxtimes 1_r \in B_n$ for a suitable choice of r (such that $n = 2m+1+r$). In particular

$$e_3^1 := \; u \boxtimes 1_1 \boxtimes u' = U_{12}U_{23}.$$

in $\mathbf{Br}^1(3,3)$. Note that

$$(1_1 \boxtimes (u')^{\boxtimes m})(u^{\boxtimes m} \boxtimes 1_1) \;=\; 1_1 \tag{17.4}$$ eq:1uu1

$E_3 \quad =$ , $\quad (1_1 \boxtimes (u')^{\boxtimes 3}) \boxtimes 1_3 \quad =$ 

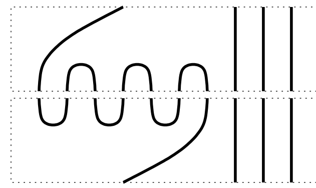
$((1_1 \boxtimes (u')^{\boxtimes 3})(u^{\boxtimes 3} \boxtimes 1_1)) \boxtimes 1_3 \quad =$ 

Figure 17.2: (a), (b), (c). E_3

(**17.4.4**) Lemma. *Any 'bra' in $\mathbf{Br}^l(l,n)$ with $l > 0$, may be completed to an idempotent.*

Proof. Apply Lemma (17.3.17).

lem:EB (**17.4.5**) Lemma. *Fix k and δ. For $n, m > 0$, consider $E_m \in B_{n+2m}$ (note that the condition $n > 0$ is needed for this to make sense). Then*
(I) $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m) \subset \mathbf{Br}(n+2m, n+2m)$ is a k-basis of $E_m B_{n+2m}$. Thus
(II) there is an isomorphism of left B_n right B_{n+2m}-bimodules

$$\begin{aligned} k\mathbf{Br}(n, n+2m) &\xrightarrow{\sim} E_m B_{n+2m} && (17.5)\\ d &\mapsto u^{\boxtimes m} \boxtimes d && (17.6) \end{aligned}$$

(Note that the left action of B_n on $E_m B_{n+2m}$ is defined by this map. It corresponds to a certain specific inclusion of B_n into B_{n+2m}.)

Proof. (I): By construction $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$ is the set of all elements of $\mathbf{Br}(n+2m, n+2m)$ that have $u^{\boxtimes m}$ in the top-left-hand corner. Since E_m has this property, the right ideal $E_m B_{n+2m}$ is spanned by diagrams with this property, and so any such spanning set is contained in $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$. On the other hand, by (17.4), $E_m d = d$ for every $d \in u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m) \subset \mathbf{Br}(n+2m, n+2m)$. Thus $E_m(u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)) = u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$.
(II): The bimodule structure on $k\mathbf{Br}(n, n+2m)$ is clear. On the other side, the right action is clear. The easiest way to see the left action is to note that the algebra $E_m B_{n+2m} E_m$ acts, and that we shall establish an isomorphism with B_n shortly. (A direct proof is also possible.) $\square$

(**17.4.6**) Note that various other idempotents will also work in place of E_m, in general (both here and subsequently).

lem:EBE (**17.4.7**) Lemma. *Fix commutative ring k and $\delta \in k$. For $n, m > 0$ the map defined by*

$$\begin{aligned} \Psi_E : B_n &\xrightarrow{\sim} E_m B_{n+2m} E_m && (17.7)\\ d &\mapsto E_m(1_{2m} \boxtimes d) E_m && (17.8) \end{aligned}$$ eq:B=EBE

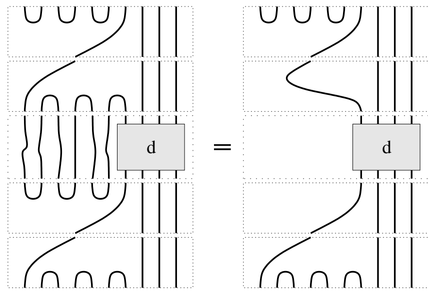


Figure 17.3: Image of $d \in B_4$ in $E_3 B_{10} E_3$ under the map in Lemma 17.4.7. EBE

is an isomorphism of k-algebras. Every diagram on the right has $u^{\boxtimes m}$ in the top left-hand corner, and $(u')^{\boxtimes m}$ starting from the second position in the bottom left-hand corner. Removing these cups and caps from a diagram leaves a diagram in B_n. This process defines the inverse map.

Proof. The proof of isomorphism as k-modules is analogous to (17.4.5)(I). See Figures 17.3 and 17.4 for illustrations of the key calculation in verifying the algebra structure.

(**17.4.8**) Lemma. *(I) For $m, n > 0$,*

$$k\mathbf{Br}^{\leq n}(n+2m, n+2m) = B_{n+2m} E_m B_{n+2m}$$

(II) For $m > 0$, there is an idempotent in $k\mathbf{Br}^0(2m, 2m)$ if and only if δ is a unit in k.

Proof. (I): We have $E_m \in \mathbf{Br}^n(n+2m, n+2m)$. Now use Lemma (17.3.17). (II): Every diagram composition from $\mathbf{Br}^0(2m, 2m)$ creates a closed loop. □

Examples

(**17.4.9**) We define the 'cup' map $c : \mathbf{Br}(m, n) \hookrightarrow \mathbf{Br}(m+2, n)$ by $d \mapsto u \boxtimes d$, and extend this k-linearly to a morphism of free k-modules. Observe that $\mathbf{Br}(m, n)$ is a right-B_n-module; and that $c(k\mathbf{Br}(m, n))$ is a right-B_n-submodule of $k\mathbf{Br}(m+2, n)$ that is isomorphic to $\mathbf{Br}(m, n)$.

We define the 'herniation' map $h : \mathbf{Br}(1, 1) \hookrightarrow \mathbf{Br}(1, 3) \times \mathbf{Br}(3, 1)$ by $1_1 \mapsto (1_1 \boxtimes u', u \boxtimes 1_1)$ (see figure 17.5). Note that this map is inverted by composition: $\circ(h(1_1)) = 1_1$. This may be applied more widely: for example, note that there is a trivial bijection $\mathbf{Br}(1, 3) \to \mathbf{Br}(1, 1) \times \mathbf{Br}(1, 3)$; applying $(h, 1)$ to this, and then applying c (i.e applying $(c, 1, 1)$) we have figure 17.5 (where the shaded region represents any element of $\mathbf{Br}(1, 3)$). Multiplying out, this shows an injection of $\mathbf{Br}(1, 3) \hookrightarrow U_{12} U_{23} B_3$. But the steps are reversible, so we see that, as a right B_3-module we have

$$U_{12} U_{23} B_3 \;\cong\; k\mathbf{Br}(1, 3) \tag{17.9}$$ eq:B3

(pictorially, the right action corresponds to acting with diagrams from $\mathbf{Br}(3, 3)$ from *below*).

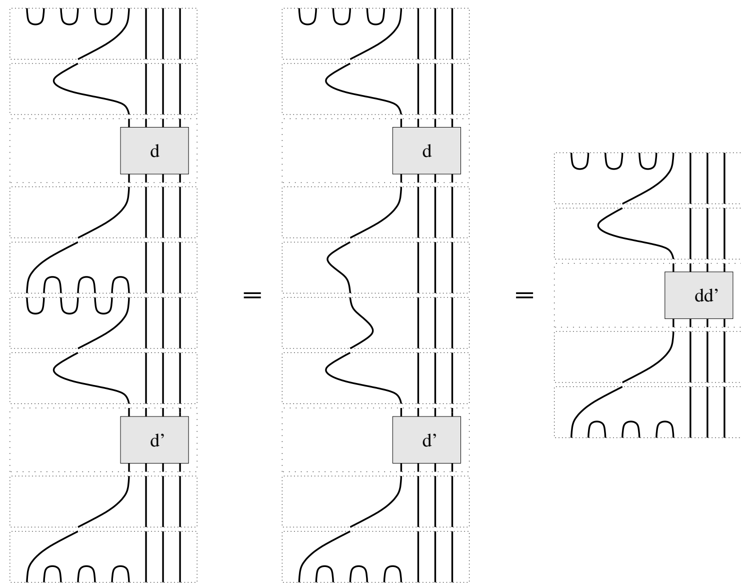


Figure 17.4: Schematic for proof of algebra isomorphism $B_4 \cong E_3 B_{10} E_3$. EBE3

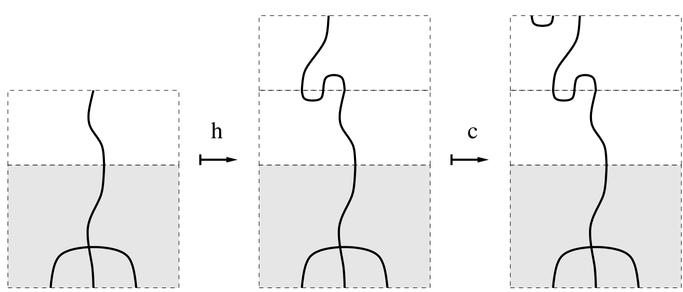


Figure 17.5: Illustrating the isomorphism $U_{12}U_{23}B_3 \cong k\mathbf{Br}(1,3)$. fig:hernia1

17.5 Introduction to Brauer algebra representations

ss:IntBrRep

Fix a field k. For each natural number n and parameter $\delta \in k$, the Brauer algebra $B_n(\delta)$ is a finite dimensional algebra, with a basis of pair partitions of the set $\{1, 2, ..., 2n\}$. (It is a certain deformation (from $\delta = 1$) of the k-linearisation of our example in (6.1.22). We describe it in (17.2.11).)

17.5.1 Reductive and Brauer-modular representation theory

ss:reductive01

A finite dimensional algebra over a field presents us with the following tasks in representation theory.

Firstly we have the 'reductive' representation theory — the aspects of representation theory concerned with the extraction of simple characters from the study of the regular module:
(1) There are finitely many isomorphism classes of simple modules — index these.
(2) Describe the blocks (the RST closure of the relation on the index set for simple modules given by $\lambda \sim \mu$ if L_λ and L_μ are composition factors of the same indecomposable projective module).
(3) Describe the composition multiplicities of indecomposable projective modules.

Then there are various more 'constructive' tasks such as:
(4) Describe the composition series of indecomposable projective modules.
(5) give explicit constructions for simple modules.
And then there are more esoteric tasks which we shall pass over here.

Accordingly $B_n(\delta)$ presents us with these tasks.

(**17.5.1**) A finite dimensional algebra over a field has a finite collection of (isomorphism classes of) simple modules $L = L(\lambda)$ and corresponding indecomposable projective modules $P = P(\lambda)$.
(If the radical $J = 0$ then $L = P$. Otherwise, the correspondence is either characterised by $L = P/JP$; or equivalently by taking a minimal projective cover of L.
Exercise: verify this equivalence.)
For the Brauer algebra B_n (for given k, δ) in particular we write $L_n(\lambda)$ and $P_n(\lambda)$ if we need to emphasise n.

The Brauer algebras also have *Brauer-Specht modules* $\Delta_n^k(\lambda)$ (sometimes we just call these Specht modules). This means the following.
For each n there is
(A) a $\mathbb{Z}[v]$-algebra $B_n^{\mathbb{Z}}$, free of finite rank as a $\mathbb{Z}[v]$-module, that passes to each Brauer algebra by base change (making k a $\mathbb{Z}[v]$-algebra by $v \mapsto \delta$); and
(B) a collection of modules $\Delta_n = \{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ for this algebra (here Λ^n is the set of integer partitions of $n, n-2, ..., 0/1$, as we shall see) that are $\mathbb{Z}[\delta]$-free modules of known rank, thus defining

$$\Delta_n^k(\lambda) \; = \; k \otimes_{\mathbb{Z}[v]} \Delta_n(\lambda)$$

for each k and $\delta \in k$ (note that the notation assumes δ is given); and that there is a choice of field k ($k = k^o$, say) extending $\mathbb{Z}[v]$ for which $\{\Delta_n^k(\lambda)\}_{\lambda \in \Lambda^n}$ is a complete set of simple modules.

(**17.5.2**) The above conditions do not determine the Δ-modules completely (but see e.g. Benson [7]). We give a construction shortly.

The idea is firstly the generalisation of Brauer's approach to the modular representation theory of a finite group G (N.B., the ubiquity of Brauer's name here is mathematically coincidental). This essentially starts with the irreducible representations of G over $\mathbb{C}$ (or some smaller extension of $\mathbb{Q}$), the *ordinary* irreducibles, and uses integral lattices in these in the role for which we shall use Δ-modules. That is, it uses them to construct p-modular systems.

Secondly the idea is to generalise in particular the Specht modules of symmetric group representation theory (ordinary irreducibles which are defined over $\mathbb{Q}$, and can even be constructed by base change from modules over $\mathbb{Z}$).

(**17.5.3**) Hereafter we may abuse notation slightly by writing $\mathbb{Z}[\delta]$ for $\mathbb{Z}[v]$ when k is to be a $\mathbb{Z}[v]$-algebra by $v \mapsto \delta \in k$.

(**17.5.4**) CLAIM: The index set Λ^n for the Δ-modules of B_n contains an index set for simple modules over any k and δ (indeed for each k, δ, a certain subset of heads of suitably constructed Brauer-Specht modules is a complete set of simple modules, and the labels for simple modules may be obtained in this way).
This general k assertion is not verified here. The idea is that it should work similarly to the kS_n case.

(**17.5.5**) As we shall see, when $k = \mathbb{C}$ as a field (note that this does not fix the action of δ *per se*) the claim holds for each δ, and indeed each of these simple index sets, here denoted $\Lambda^{n,\delta}$, coincides with Λ^n, unless $\delta = 0$, in which case the index set for simples has one less element when n is even: we can take $\Lambda^{n,0} := \Lambda^n \setminus \Lambda^0$. Either way, note the inclusions $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$.

(**17.5.6**) As in (9.3.11), or [57, §6.6], we define the Brauer-Specht module decomposition matrix for $B_n(\delta)$ over k (note that k is a field with a structure of $\mathbb{Z}[v]$-algebra) by

$$d_{ij} \;=\; [k \otimes_{\mathbb{Z}[v]} \Delta_n(i) : L_n(j)]$$

(we shall also use $D_{ji} = d_{ij}$), the composition multiplicity, where $\{L_n(j)\}_{j \in \Lambda^{n,\delta}}$ is the set of simple $B_n(\delta)$ modules over k.

As we shall see, while the matrix D is not necessarily square, it is lower unitriangular in an order that respects the inclusions $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$.

As usual we define the Cartan decomposition matrix C (for given k, δ) by

$$C_{ij} \;=\; [P_n(i) : L_n(j)]$$

Recall from the general machinery of §**??** that there is a $B_n^{\mathbb{Z}}$-module Π_i — strictly speaking a module over some k, δ-dependent $\mathfrak{p}$-adic extension of $\mathbb{Z}[v]$ — that passes to each $P_n(i)$ by base change (note that Π_i depends on k, δ, but we are here holding these fixed), and hence a corresponding module over the field k^o of fractions, Π_i^o. We have from this, in principle, another natural collection of invariants D^o associated to k, δ, given by:

$$D^o_{ij} \;:=\; [\Pi_i^o : \Delta_n^o(j)]$$

However, as we know on general grounds $D = D^o$.

17.5.2 Globalisation and towers of recollement

(**17.5.7**) Another important organisational property is that there is an idempotent $e \in B_n$ (any ground ring, at least for sufficiently large n) such that $eB_ne \cong B_{n-2}$, so that (cf. [57, §6.6]) we can construct an inverse limit for the sets $\{\Lambda^{n,\delta}\}_n$, and a corresponding limit for C and D, so that any fixed n case can be extracted by projection. The set limit is simply Λ (i.e. corresponding to the obvious inclusions), and the projections are correspondingly transparent. By the unitriangular property it is just a truncation (noting, as we shall see, that $C = DD^T$).

We have from Green §6.6 that e defines an inclusion of $\Lambda^m \subset \Lambda^{m+2}$ and $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$ (any m), such that d_{ij} does not depend on n whenever i, j belong in the appropriate set. Since $D_{ji} = d_{ij}$ we have a corresponding stability property of D, and hence of C.

In fact we claim that we can prove this stability of D directly, and strengthen this to a statement about filtration multiplicities in a filtration of each $P_n(\lambda)$ by Δ-modules.

(**17.5.8**) Fix k, δ. By general properties of P-modular systems we can 'lift' each $P_n(i)$ to a projective module over a suitable ground ring and hence define a corresponding module over the field that makes B_n semisimple. This module obviously has a well-defined decomposition into Δ-modules. (At least if all fields have char.0) This defines a set of invariants of $P_n(i)$ that will provide a character formula in terms of Δ-characters. Let us write

$$(P_n(i) : \Delta_n(j))_{\mathbb{Z}}$$

for these invariants. In the theory of quasihereditary algebras one has a similar notation: $(P_n(i) : \Delta_n(j))$. It will be helpful to compare these.

Separately from the P-modular and q.h. settings, $P_n(i)$, or indeed any module, might have a 'weak' Δ-filtration (i.e. a filtration without necessarily well-defined multiplicities). If this is in fact a ('strong') Δ-filtration (i.e. a filtration with well-defined multiplicities), then this would define another set of invariants. (Note that if the Δ_n are a basis for the Grothendieck group then well-definedness is forced, since any filtration determines a character. This is usually, but not always, the case for us.)

Separately from this again, Δ_n, *or some subset*, will be a basis for the Grothendieck group (for now we are still working in char.0). The choice of a given subset gives another character formula. In Lie theory/q.h. there is a rather analogous setup in which the Δ_n are always a basis for the Grothendieck group. In this case there can only be one Δ-character formula for any module, so the various formal invariants considered above would all coincide. In this case, if M has a Δ-filtration it is automatically strong and $(M : \Delta_n(j))$ is then *defined* by

$$[M] = \sum_j (M : \Delta_n(j)) \, [\Delta_n(j)]$$

in the Grothendieck group (see e.g. Donkin (Appendix) [45]).

(**17.5.9**) An interesting question arises when the basis for the Grothendieck group is smaller (in our case with $k = \mathbb{C}$ this is just the case $\delta = 0$), so that the two sets of invariants differ. The question is: are the filtration multiplicities well-defined? Are they forced to agree with the well-defined numbers $(- : -)_{\mathbb{Z}}$?

An extra ingredient that we have available is knowledge about the *order* in which Δ-modules must appear in a filtration. Can we use this?

We shall return to this later.

(**17.5.10**) Anyway, our approach to task (3) is to compute the $(P_n(i) : \Delta_n(j))$, by computing each number in the first n-level in which it appears. We make heavy use of the solution to task (2) (via the study of a fortuitous choice of central elements of B_n); and the properties of Δ-modules under ordinary induction (noting that induction takes projectives to projectives).

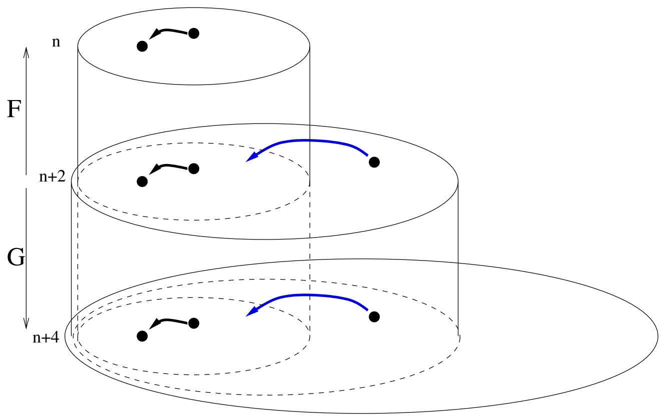


Figure 17.6: fig:functors Schematic for embeddings of B_n-module categories.

17.5.3 Overview of the Chapter

In (17.2.11) we define the Brauer algebra $B_n = B_n(\delta)$ (initially over $\mathbb{Z}[\delta]$ with δ indeterminate, and hence for each choice of commutative ground ring k and $\delta \in k$ by base change). In §17.3 we prove some Lemmas on Brauer diagram manipulation that we use later.

In §17.4 we construct some useful idempotents and idempotent subalgebras. In particular we note an idempotent $e \in B_n$ $(n > 2)$ such that

$$B_{n-2} \cong eB_ne \qquad (n > 2)$$

while

$$B_n/B_neB_n \;\; \cong kS_n$$

It follows that an index set for simple B_n-modules may be obtained as the disjoint union of an index set for B_{n-2} and one for S_n (e.g Λ_n if $k = \mathbb{C}$). We label simple modules accordingly. It follows from the first isomorphism that $M \mapsto eM$ defines an exact functor from B_n-mod to B_{n-2}-mod. Then for all simple modules $L_\lambda = L^n_\lambda$ such that $eL_\lambda \neq 0$ we have that eL_λ is simple and

$$(M : L_\lambda) \; = \; (eM : eL_\lambda)$$

In (18.2.5) we construct a certain special set of B_n-modules called Δ-modules. In fact we give three constructions, showing isomorphism via the contruction of a basis in each case.

In (18.2.21) we show that there is an extension of the ring $k = \mathbb{Z}[\delta]$ over which these Δ-modules are a complete set of simple modules, and B_n is semisimple, for every n. (From this we note that the simple decomposition matrices (D_{ij}) of these modules over a suitable choice of k, δ determine the Cartan decomposition matrices over that k, δ.)

We show more generally that these modules provide a basis for the Grothendieck group — the character ring — of each algebra. We also show that they filter projective modules. (Note that the Δ-content of projectives is reciprocal data to the simple content of Δ-modules.)

We construct functors $F : B_{n+2} - \text{mod} \to B_n - \text{mod}$ and $G : B_n - \text{mod} \to B_{n+2} - \text{mod}$ that allow us to embed the category B_n-mod in B_{n+2}-mod. These functors thus define another class of modules — the G-images of B_n-modules in B_{n+2}-mod. We show that (with one manageable exception over $\mathbb{C}$) this embedding takes the Δ-module $\Delta_n(\lambda)$ to the Δ-module $\Delta_{n+2}(\lambda)$; and takes indecomposable projectives to indecomposable projectives similarly. Thus the Δ-content of projective modules is, once defined, stable with n. We determine each new such datum by an induction on n, using also the induction functor associated to the natural inclusion $B_n \hookrightarrow B_{n+1}$ (see later); known restriction rules for Δ-modules; and projection onto blocks (which we also determine).

17.6 Appendix: Bibliographic notes

ss:ABibnotes

The next two Chapters will be concerned with the general and complex representation theory of the Brauer algebra. Here we assemble a brief summary of results from Auslander [4] and particularly from CDM [28]–[30] and references therein.

17.6.1 Summary

Sets of B_n-modules indexed by Λ^n and satisfying 'generic irreducible' criteria can be found in [17]. Important properties of these modules are determined in [147, 46, **?**].

The blocks in all cases over $\mathbb{C}$ are determined in [28]. Here they are described, for each δ, as an equivalence relation on Λ^n (or indeed Λ). Indeed there is a poset structure, refining $(\Lambda, \subseteq)$, for which this relation is the RST closure. This poset is in a suitable sense locally finite (as an inverse limit of finite posets), and so has a transitive reduction. An algorithmic procedure for constructing covers (adjacent elements in this reduction) is given in [28]. However, while the block condition is a simple comparison of Young diagrams, the cover relation is not given in a simple form. (This is noteworthy, since a good understanding of the cover relation is useful for the next main objective — the determination of Cartan decomposition matrices.)

By a δ-dependent embedding of Λ^n in $\mathbb{R}^n$ (and implicitly Λ in $\mathbb{R}^{\mathbb{N}}$), in [29] the block result is recast in terms of orbits of a reflection group action. (This also allows the determination of some block results over other fields.) This property, while extremely useful, was *observed* rather than derived from the algebra, and so is not considered to be necessarily canonical. In the group action, adjacent elements are not necessarily in adjacent chambers, but (it is shown that?) they are related by involutive elements.

In [30] the directed graphs associated to the reductions of the connected components of the limit of block posets are explicitly considered. Let us call them block graphs. However, the definition appeals to the formal reduction, rather than an explicit construction, for the edge set.

Next the 'dominant dual graph' of the reflection group alcove geometry with respect to a certain maximal parabolic is introduced. [1]

Another graph defined (let us say, the orbit graph) has vertex set the dominant part of the group orbit of a given point. One then induces a partial order on this set from a partial order on $\mathbb{R}^{\mathbb{N}}$. Unlike this original partial order, the induced one has a transitive reduction, giving the graph edges. Again this is a formal construction.

[1]The role of the parabolic is not made explicit in these terms. NB Our statement in the paper of the definition is not quite right. See later.

By the geometrical block theorem of [29] the vertex sets of the block graph and orbit graph are in bijection. In [30] this is lifted to a graph isomorphism by noting that the poset structure is preserved (and implicitly that there is a unique transitive reduction).

17.6.2 Preliminary generalities

Let A be an algebra over a field and $1_A = \sum_{i=1}^{n} \epsilon_i$ a primitive orthogonal idempotent decomposition, ordered so that $\epsilon_1, ..., \epsilon_b$ are a complete set of representatives up to inner automorphism. Let $\epsilon = \sum_{i=1}^{b} \epsilon_i$ and $\epsilon^m = \sum_{i=1}^{m} \epsilon_i$ and recall that $\epsilon A \epsilon$ is then a *basic algebra* Morita equivalent to A. We have for any m

$$(\epsilon^m A \epsilon^m)^{op} \cong \mathrm{End}_A(A\epsilon^m)$$

(since this holds for any idempotent $e \in A$ by (8.6.11)). Thus $\epsilon^m A \epsilon^m$ and $\epsilon^{m'} A \epsilon^{m'}$ are Morita equivalent whenever $m, m' > b$.

Let Π_A denote an index set for simple modules of A. Thus $b = |\Pi_A|$. For $\lambda \in \Pi_A$ write $[\lambda]$ for the class of λ under the block relation. Write $\mathrm{Proj}^{\lambda}-$ for the projection functor onto the block $[\lambda]$ of $\lambda \in \Pi_A$.

17.6.3 Auslander: rep. thy. of small additive categories (as if rings)

Let C be a category and C' an *(isomorphism) dense* subcategory. Category C is *skeletally small* if it has a small dense subcategory.

Let C be an ab-category. A morphism a in $C(x, y)$ factors through object z if $a = bb'$ for some $b \in C(z, y)$ and $b' \in C(x, z)$. If $a'a = 1_x$ and $b'b = 1_y$ are factorizations through z such that $aa' + bb' = 1_z$ then we write $z \cong x \oplus y$. This defines a (not necessarily closed) *direct sum* on objects up to isomorphism. (Not necessarily closed since there may be no z playing this role for a given pair x, y.) If C is closed under direct sums we say it *has* direct sums.

Recall the following from §**??** (and originally from [4]).
If C, D are categories then (C, D) is the category of functors from C to D.
A category C is pre-additive (cf. an ab-category) if each $C(x, y)$ is an abelian group and category composition is correspondingly bilinear.
Category C is additive if every finite family of objects in C has a sum in C.

Suppose C is a skeletally small category. Auslander takes the *category of C-modules* to be the category $\mathrm{mod} - C := (C^{op}, \mathbf{Ab})$. This category has some analogous properties to categories of ring modules.

A C-module M is *finitely generated* if given $\{M_i\}_{i \in I}$ with each $M_i \in C$ and epimorphism $f : \coprod_{i \in I} M_i \to M$ (where $\coprod_i$ denotes repeated direct sum) there is a finite $J \subset I$ such that restriction $f : \coprod_{i \in J} M_i \to M$ is epi.

Auslander defines $\mathfrak{p}(C)$ as the subcategory of $\mathrm{mod} - C$ whose objects are finitely generated projective.

17.6.4 Towers of recollement

(**17.6.1**) A (mod-2) *tower of recollement* [27] (ToR) is a family of algebras $A_1 \subset A_2 \subset ...$ together with an idempotent e in each A_i such that $eA_ne \cong A_{n-2}$; satisfying various axioms. (A mod-m tower has $eAe \cong A_{n-m}$ instead.)

Write Ind$-$ for the induction functor for $A_n \subset A_{n+1}$ (any n); and Res$-$ for the restriction functor. Write Π_n for an index set for simple modules of A_n.

One axiom is that each A_i is quasihereditary. The quasihereditary axiom can be replaced by one of a number of similar axioms, but there should be a 'standard' module $\Delta_n(\lambda)$ for each $\lambda \in \Pi_n$ that is in *Brauer reciprocity* with simples $L_n(\lambda)$ and projectives $P_n(\lambda)$; with simple head, $\mathrm{head}\,(\Delta_n(\lambda)) = L_n(\lambda)$; and with $\mathrm{Res}\,\Delta_n(\lambda)$ Δ-filtered.

The globalisation functor

$$G_e- \;:=\; A_n e \otimes_{eA_ne} -$$

obeys

$$G_e\Delta_n(\lambda) \cong \Delta_{n+2}(\lambda)$$

and induces an embedding $\Pi_n \hookrightarrow \Pi_{n+2}$, and we take Π to be the disjoint union of the corresponding odd and even large-n limits. We assume here (see [27]) that there is a partition of Π that localises to the block partition on each Π_n.

Write $\mathrm{Ind}^\lambda- \;=\; \mathrm{Proj}^\lambda\mathrm{Ind}-$; and $\mathrm{Res}^\lambda- \;=\; \mathrm{Proj}^\lambda\mathrm{Res}-$. Write $\mathrm{supp}_n(\lambda)$ or $\mathrm{supp}_n^{\mathrm{Res}}(\lambda)$ for the set of standard filtration factors in $\mathrm{Res}\,\Delta_n(\lambda)$.

Note that $\mathrm{supp}_n(\lambda)$ stabilises in the limit of large n in the same way as Π_n (in fact it simply becomes constant, $\mathrm{supp}_n(\lambda) = \mathrm{supp}(\lambda)$, for sufficiently large n). The *category* A_n-mod stabilises similarly (albeit not necessarily as an algebra *module* category — we write $P(\lambda)$ for the *object* in the limit category, and so on — this may only correspond to a module in a suitable localisation).

(**17.6.2**) Consider $\lambda, \lambda' \in \Pi$ as above. Suppose there is a map between blocks $\theta : [\lambda] \to [\lambda']$ such that

$$\text{for each } \mu \in [\lambda] \qquad \mathrm{Ind}^{\lambda'}\Delta(\mu) = \Delta(\theta(\mu)) \qquad \text{and} \qquad \mathrm{Ind}^{\lambda}\Delta(\theta(\mu)) = \Delta(\mu).$$

Then λ, λ' are said to be *((R, I)-)translation equivalent* [30, §4]. (The relation of (R, I)-translation equivalence generates an equivalence relation, whose classes are called *((R, I)-)translation classes.*)

(**17.6.3**) Proposition. [30, Pr.4.2] *Suppose that $\lambda, \lambda' \in \Pi$ are translation equivalent. Then with $\mu \in [\lambda]$ and θ as above*

$$Ind^\lambda P(\theta(\mu)) = P(\mu)$$

(**17.6.4**) The example we have in mind is $A_n = B_n(\delta)$ over $\mathbb{C}$. Here $\Pi = \Lambda$ and $\mathrm{supp}(\lambda)$ is obtained by adding or removing a box in all possible ways (see later); and the blocks are obtained by embedding $\Pi = \Lambda$ in $\mathbb{R}^{\mathbb{N}}$ and considering orbits with respect to a reflection group action (with associated alcoves and so on) [28, 29].

The idea behind ToR is to play off between two or three adjoint pairs of functors moving around the tower – one pair which gives a global limit to the sequence of module categories and hence 'weights' and blocks, and another which 'smears' weights and so induces a geometry on the set of weights, such that the smear takes a simple form in combination with projection-onto-block... It is not forced that Induction/Restriction should be the smearing pair. There are potentially many possibilities... In [30] another idempotent restriction is used as well.

(**17.6.5**) In [30] it is shown that Brauer algebra translation classes extend through whole alcoves (of the reflection group action mentioned above); and then facets [30, Th.6.7]; and then through the least weights from blocks of lowest singularity [30, Pr.6.11] (which amounts to fundamental alcoves in the cases where these are non-empty, but corresponds to certain unions of facets in general, for the reflection group action in operation). In [**?**] translation equivalence is extended to adjacent weights of the same singularity. (The point is that this is enough to compute the decomposition matrices, as we shall see.)

(**17.6.6**) DO EXAMPLE

(**17.6.7**) MELD MARTIN'S SINGULARITY RESULTS TO CDM'S M.E. RESULTS.

(**17.6.8**) We continue with $e^2 = e \in A$. The functor $F_e : A - \mathrm{mod} \to \; eAe - \mathrm{mod}$ given by $F_eM = eM$ is exact, with G_e left adjoint to F_e (and hence right exact). Consider the adjunction property

$$\mathrm{Hom}_A(G_eM, {}_AN) \cong \mathrm{Hom}_{eAe}(M, eN)$$

Putting $N = G_eM$, since $eN = eG_eM \cong M$ we have

$$\mathrm{End}_A(G_eM) \cong \mathrm{End}_{eAe}(M).$$

(**17.6.9**) Of course

$$eAe \cong \bigoplus_{\lambda \in \Pi_{n-2}} P_{eAe}(\lambda)^{dim(L_{n-2}(\lambda))}$$

as a left-regular-module decomposition into indecomposable projectives, so setting $M = eAe$ and recalling $G_eP_{eAe}(\lambda) = P_A(\lambda)$ we have

$$\mathrm{End}_A\left(\bigoplus_{\lambda \in \Pi_{n-2}} P_A(\lambda)^{dim(L_{eAe}(\lambda))}\right) \cong \; \mathrm{End}_{eAe}(eAe) \; \cong \; (eAe)^{op}$$

Recall the indempotent decomposition $1_A = \sum_{i=1}^n \epsilon_i$. There will be a 'subset' of idempotents $I = \{\epsilon_i\}_{i \in B_e}$ (some B_e, possibly with multiplicities) such that

$$AI \; \cong G_eeAe \; \cong \; \bigoplus_{\lambda \in \Pi_{n-2}} P_A(\lambda)^{dim(L_{eAe}(\lambda))}$$

as left-modules. Whereupon

$$\mathrm{End}_A(AI) \cong \; (IAI)^{op}$$

...which we can relate of something embedded in $A - \mathrm{mod}$...

17.7 Appendix: Overview of following Chapters

A major aim of the next few chapters is to prove a Theorem determining the decomposition matrices of the Brauer algebras over $\mathbb{C}$. Our objective in this Section is to give an overview, by stating this Theorem. We will need some notation for this.

17.7.1 Blocks and the block graph $G_\delta(\lambda)$

(**17.7.1**) The δ-*charge* of a box b in a Young diagram (with content $c(b)$) is $chg(b) := \delta - 1 - 2c(b)$.

(**17.7.2**) For $\lambda \supset \mu \in \Lambda$ the skew λ/μ is a δ-*skew* if
(i) the boxes of λ/μ can be put into pairs such that the sum of δ-charges in each pair is zero;
(ii) if there is such a pairing of boxes in which each +1,-1 pair has the two boxes side-by-side, then the number of these side-by-side pairs is even.

(**17.7.3**) Fixing δ, we may define a relation on Λ, refining $\supset$, by $\lambda > \mu$ if λ/μ is a δ-skew.
Lemma. The relation $(\Lambda, >)$ is a poset, and has a transitive reduction (cover relation).

Considering this cover relation as a directed graph, the connected components are the *block graphs*. The λ-connected component is denoted $G_\delta(\lambda)$.

de:mibs1 (**17.7.4**) A *rim* is a connected skew Young diagram with no subset of shape (2^2). Fix δ and hence a diagonal δ-charge=0 line in every Young diagram. Let x be a point on this line. Define π_x as the rotation of the plane by π radians about x. Two non-intersecting rims are δ-*opposite* if there is a π_x that takes one rim into the other.

de:MiBS (**17.7.5**) For given δ, a δ-*pair* is a skew that is a δ-opposite pair of rims such that no row of the skew is fixed by the associated π-rotation.

(**17.7.6**) Proposition. *Fix δ. The δ-pairs are the edges of the block graphs $G_\delta(\lambda)$.*

17.7.2 Embedding the vertex set of $G_\delta(\lambda)$ in $\mathbb{R}^\mathbb{N}$

de:shift-embed1 (**17.7.7**) For $\delta \in \mathbb{R}$ define $\rho_\delta = -\frac{\delta}{2}(1,1,...) - (0,1,2,...) \in \mathbb{R}^\mathbb{N}$. Define $\mathsf{e}_\delta : \mathbb{R}^\mathbb{N} \hookrightarrow \mathbb{R}^\mathbb{N}$ by

$$\mathsf{e}_\delta(\lambda) \;=\; \lambda + \rho_\delta .$$

(**17.7.8**) Define a partial order $(\mathbb{R}^\mathbb{N}, \geq)$ by $v \geq w$ if $v_i \geq w_i$ for all i.

(**17.7.9**) A sequence $v \in \mathbb{R}^\mathbb{N}$ is *dominant* if it is strictly decreasing; and *strongly* decreasing if $v_i - v_{i+1} \geq 1$ for all i. We write A^+ for the set of strongly decreasing sequences.

Consider the magnitudes of terms in a sequence in A^+. We see that each magnitude occurs at most twice, i.e. in a sequence of form $(..., x, ..., -x, ...)$. We call such a $\pm x$ pairing a *doubleton*. Define a map $Reg : A^+ \to A^+$ such that $Reg(v)$ is obtained from v by removing the doubletons.

de:sing-set (**17.7.10**) For $\lambda \in \Lambda$ write $p_\delta(\lambda)$ for the set of pairs of rows $\{i, j\}$ such that $(\lambda + \rho_\delta)_j = -(\lambda + \rho_\delta)_i$ (i.e. $\mathsf{e}_\delta(\lambda)_j = -\mathsf{e}_\delta(\lambda)_i$). Write $s_\delta(\lambda)$ for the *singularity* of $\mathsf{e}_\delta(\lambda)$: $s_\delta(\lambda) = |p_\delta(\lambda)|$.

de: mag order (**17.7.11**) We say a sequence $v \in \mathbb{R}^\mathbb{N}$ is *regular* if no two terms have the same magnitude. Let $\mathbb{R}^{Reg}$ denote the set of regular sequences. Note that the terms in any $v \in \mathbb{R}^{Reg} \cap A^+$ have a well-defined *magnitude order*. That is, each term may be asigned a number giving its position in the list of terms ordered by increasing magnitude. For example $\frac{11}{2}$ is the 5-th term in the magnitude order of terms in $(\frac{11}{2}, \frac{9}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$. Define a map

$$o : \mathbb{R}^{Reg} \cap A^+ \;\to \mathbb{Z}^\mathbb{N}$$

as follows. In the i-th term, $o(v)_i$ of $o(v)$, the magnitude $|o(v)_i|$ is the position of v_i in the magnitude ordering of the set of numbers appearing in v. The sign of $o(v)_i$ is the sign of v_i, unless $v_i = 0$ in which case the sign is chosen so as to make an even number of positive terms.

de:Peven (**17.7.12**) Let $\mathtt{P}_{even}(\mathbb{N}), \mathtt{P}_{odd}(\mathbb{N}) \subset \mathtt{P}(\mathbb{N})$ denote the set of subsets of $\mathbb{N}$ of even (respectively odd) order. The *toggle map* between $\mathtt{P}_{even}(\mathbb{N})$ and $\mathtt{P}_{odd}(\mathbb{N})$ is given by toggling the presence of 1 so as to make an odd set even, or even set odd.

(**17.7.13**) Define

$$o_\delta : \Lambda \quad \to \quad P(\mathbb{N}) \tag{17.10}$$

by $\lambda \mapsto o(Reg(\mathtt{e}_\delta(\lambda)))|_+$ where $|_+$ signifies to restrict to the initial subsequence of positive terms (which may simply be recorded as a set); and finally to apply the toggle map if this subsequence contains an odd number of terms.
Note that $\mathtt{e}_\delta(\Lambda)$, $\mathtt{e}_{\delta'}(\Lambda)$ do not intersect, so given $\mathtt{e}_\delta(\lambda)$ we can *determine* δ and λ.

(**17.7.14**) Given δ and λ we define

$$o_\delta^\lambda : P_{even}(\mathbb{N}) \to \Lambda$$

as follows (indeed we could extend the domain to $P(\mathbb{N})$ by applying the toggle map to $P_{odd}(\mathbb{N})$). First construct $\mathtt{e}_\delta(\lambda)$. Note that this fixes the doubletons and (magnitudes of) singletons for its whole orbit, i.e. for every element of $o_\delta([\lambda]_\delta)$. We ignore the doubletons for a moment, and work out the magnitude order for the singletons. Note that the order in which the singletons can appear in a descending sequence is uniquely determined by their sign. Now for $v \in P_{even}(\mathbb{N})$ we give the positive sign to the corresponding singletons (in the magnitude order). Thus we have determined the singletons and their order in the sequence. The position of the doubletons is now forced, so the sequence $o_\delta(o_\delta^\lambda(v))$ is determined. But o_δ is invertible as already noted, so finally apply this inverse.

Alem:o-detla bij (**17.7.15**) Lemma. *Fix δ and λ. Then o_δ and o_δ^λ are mutual inverses on $[\lambda]_\delta \leftrightarrow P_{even}(\mathbb{N})$.* □

Ade:G_even (**17.7.16**) Define a directed graph, G_{even} with vertex set $\mathtt{P}_{even}(\mathbb{N})$ (we call these vertices *valley sets*); and labelled edges:

$$a \xrightarrow{\alpha} b \qquad \text{if} \qquad a \setminus b = \{\alpha\}, \;\; b \setminus a = \{\alpha+1\} \qquad (\alpha \in \mathbb{N})$$

$$a \xrightarrow{12} b \qquad \text{if} \qquad a \setminus b = \emptyset, \;\; b \setminus a = \{1,2\}$$

See Figure 19.11. There is a corresponding graph G_{odd} with vertices $\mathtt{P}_{odd}(\mathbb{N})$. The *toggle map* is readily seen to pass to a graph isomorphism (the edge labels 1 and 12 are interchanged).

17.7.3 Reflection group action on $\mathbb{R}^{\mathbb{N}}$

(**17.7.17**) Define reflection group actions on $\mathbb{R}^{\mathbb{N}}$:

$$(ij) : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., \lambda_j, ..., \lambda_i, ...)$$

$$(ij)_- : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., -\lambda_j, ..., -\lambda_i, ...)$$

Write $\mathcal{D}$ for the group generated by these (all $i < j$). Write $\mathcal{D}v$ for the orbit of a point $v \in \mathbb{R}^{\mathbb{N}}$ under the action of $\mathcal{D}$. Write $\mathcal{D}_+$ for the subgroup $\langle (ij) \rangle_{ij}$.

(**17.7.18**) For given δ, $w \in \mathcal{D}$ acts on Λ, via its action on $\mathsf{e}_\delta(\Lambda)$. We write $w.\lambda$ for this.

Ale:prod-com-ref (**17.7.19**) LEMMA. *If λ/μ is a δ-pair then*

$$\mathsf{e}_\delta(\lambda) \;=\; \left(\prod_{ij} (ij)_- \right) \; \mathsf{e}_\delta(\mu)$$

where the product is over pairs of rows in the skew, from the outer pair to the inner pair.

Note also that no subset of this product, applied to $\mathsf{e}_\delta(\mu)$, results in a dominant weight.

Proof. Compare the definitions of δ-pair (19.2.18), e_δ and $(ij)_-$. □

(**17.7.20**) LEMMA. *(I) The $\mathcal{D}$ action on λ includes a traversal of the block $[\lambda]_\delta$ for each λ. (II) This action intersects no other block.*

Proof. (I) Follows from Lemma 17.7.19. (II) Is proved in [29]. □

(**17.7.21**) Define $V(v) = Dv \cap A^+$. Define a directed graph $\mathbf{G}(v)$ with vertex set $V(v)$ by assigning an edge (t, u) if $(u - t)_i \geq 0$ for all i and this is a cover (i.e. (t, u) is not in the transitive closure of any other such pairs).

We write G_a for the dominant dual graph (as in **??**) of $\mathcal{D}/\mathcal{D}_+$.

Apr:gg3 (**17.7.22**) LEMMA. [30, Co.7.3] *For any regular v, i.e. lying within an alcove, the dominant part of its $\mathcal{D}$-orbit contains a point in each dominant alcove. Thus $\mathbf{G}(v) \cong G_a$.* □

A convenient example of a regular v is $\mathsf{e}_2(\emptyset)$. In light of the lemma we may use $\mathcal{D}\mathsf{e}_2(\emptyset)$ to label dominant alcoves, i.e. to set $G_a = \mathbf{G}(\mathsf{e}_2(\emptyset))$.

Ale:gg4 (**17.7.23**) LEMMA. [[30, Co.7.3] *et seq.*] *The map $\phi_+ : \mathcal{D}\mathsf{e}_2(\emptyset) \to \mathsf{P}_{even}(\mathbb{N})$ which discards all negative entries extends to a graph isomorphism $G_a \cong G_{even}$.* □

Ath:gg4 (**17.7.24**) THEOREM. *For all δ, λ we have graph isomorphisms*

$$G_\delta(\lambda) \overset{\mathsf{e}_\delta}{\to} \mathbf{G}(\lambda + \rho_\delta) \xrightarrow{Reg} \mathbf{G}(Reg(\lambda + \rho_\delta)) \overset{\sim}{\to} G_a \overset{=}{\to} \mathbf{G}(\mathsf{e}_2(\emptyset)) \overset{\phi_+}{\to} G_{even}.$$

In particular $o_\delta : G_\delta(\lambda) \overset{\sim}{\to} G_{even}$.

Proof. For the first isomorphim see [30, Pr.7.1]; the second is [30, Pr.7.2]; the third is [30, Co.7.3]; ϕ_+ follows from [30, Co.7.3] *et seq.*. □

17.7.4 Decomposition data: Hypercubical decomposition graphs

Ass:hyperDG

Ade:bb (**17.7.25**) Let $\mathbf{b} : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ denote the natural bijective map. Define $\mathbf{b}_\delta : \Lambda \to \{0,1\}^{\mathbb{N}}$ by $\mathbf{b}_\delta(\lambda) = \mathbf{b}(o_\delta(\lambda))$.

Ade: TL algo (**17.7.26**) Each binary sequence b has a partial TL-diagram $d(b)$ constructed as follows.
1. Draw a row of vertices, one for each entry in b (up to the last non-zero entry).
2. For each subsequence 01 draw an arc connecting the corresponding pair of vertices.

3. Consider the sequence obtained by ignoring paired vertices. For each subsequence 01 draw an arc connecting the corresponding pair of vertices.
4. Iterate 3 until termination.
5. Finally connect the remaining run of vertices binary-labelled 1 in adjacent pairs (if any) from the left. Leave the remaining vertices as singletons.

Ade:Gamma a (**17.7.27**) For $a \in P(\mathbb{N})$ we write Γ_a for the list of arcs (i.e. pairs) in $d(\mathbf{b}(a))$ corresponding to 01 subsequences, and an initial 11 subsequence (i.e. if there is one in the 12-position); and Γ^a for the list of all arcs. We write $\Gamma_{\delta,\lambda}$ for $\Gamma_{o_\delta(\lambda)}$, and Γ_δ^λ for $\Gamma^{o_\delta(\lambda)}$.

Ade:hyper DG (**17.7.28**) For S a finite set, let $\mathtt{h}_S$ denote the cover graph (the Hasse graph) of the subset partial order on $P(S)$. Two edges in $\mathtt{h}_S$ are *parallel* if they correspond to deleting the same element of S.

Ade:hyp (**17.7.29**) Each $a \in P(\mathbb{N})$ defines a hypercubical directed graph h^a isomorphic to $\mathtt{h}_{\Gamma^a}$, as follows. The vertices are binary sequences (considered as identified with elements of $P(\mathbb{N})$). The top sequence in h^a is the defining sequence $\mathbf{b}(a)$. There is an edge out of this corresponding to each completed arc in the TL-diagram $d(\mathbf{b}(a))$. The sequence at the other end of a given edge is obtained from the original by replacing $01 \to 10$ (or $11 \to 00$) at the ends of this arc. Indeed every edge parallel to this edge in the hypercube follows this transformation rule.

(**17.7.30**) We label each edge of the hypercube h^a (i.e. each direction) by the corresponding element $\{\alpha, \alpha'\} \in \Gamma^a$. That is, α, α' are the positions of the ends of the arc associated to this edge.

If label $\{\alpha, \alpha'\}$ has $\alpha' = \alpha + 1$ for an 01-arc, we may just label the edge by α. If $\{\alpha, \alpha'\} = \{1, 2\}$ for a 11-arc we may just label the edge by 12.

Ade:hypercub (**17.7.31**) Since fixing a block $[\lambda]_\delta$ establishes a bijection $o_\delta^\lambda : P_{even}(\mathbb{N}) \to [\lambda]_\delta$ the construction for h^a also defines a hypercubical directed graph $h_\delta(\lambda)$ for each pair $(\delta, \lambda) \in \mathbb{Z} \times \Lambda$, obtained by applying o_δ^λ to the vertices.

We write $h_\delta(\mu)_\nu = 1$ if ν appears in $h_\delta(\mu)$, and $= 0$ otherwise.

Chapter 18

General representation theory of the Brauer algebra

ch:BrauerII

We continue the study of Brauer algebra representation theory from Ch.17. We also continue with the notation from Ch.17.

As discussed in §17.5, the aim is to determine fundamental invariants of the representation theory of these algebras over fields. The individual Brauer algebras are amenable to various axiomatic approaches, such as π-modular representation theory [16]; and collectively they form a 'tower of recollement'. This holds more-or-less independently of the ground ring. Here we examine certain 'integral' modules of the π-modular approach, their roles in other axiomatisations, and their behaviour in the tower.

Contents

18.1 Initial filtration of the left regular module

Here k is a fixed but arbitrary commutative ring, and $\delta \in k$. For $n \in \mathbb{N}$ we set $n_0 = 0$ if n even and $n_0 = 1$ if n odd. We retain the notations of §17.3.2; but also define

$$\mathbb{B}(n,m) = k\mathbf{Br}(n,m), \qquad \mathbb{B}^{<l}(n,m) = k\mathbf{Br}^{<l}(n,m)$$

and so on.

de:regdec1 (**18.1.1**) Proposition. (I) For any ring k and $\delta \in k$, we have a sequence of $B_n(\delta)$-bimodules:

$$\mathbb{B}(n,n) = \mathbb{B}^{\leq n}(n,n) \supset \mathbb{B}^{\leq n-2}(n,n) \supset \mathbb{B}^{\leq n-4}(n,n) \supset ... \supset \mathbb{B}^{n_0}(n,n) \tag{18.1}$$ reg seq1

(II) There is a corresponding sequence of left-B_n-right-B_m-bimodules for any $\mathbb{B}(n,m)$.
(III) The i-th section of the sequence (18.1) has basis $\mathbf{Br}^{n-2i}(n,n)$, or more precisely

$$[\mathbf{Br}^{n-2i}(n,n)] \;:=\; \{[b] \;:=\; b + I \mid b \in \mathbf{Br}^{n-2i}(n,n);\; I = k\mathbf{Br}^{<n-2i}(n,n)\} \tag{18.2}$$ eq:delli

We call (18.1) the *p-sequence* (p for ‘pure’) of B_n-ideals.

Proof. Apply the bottleneck lemma for propagating number. □

papaebspecht$ (**18.1.2**) For any l, n define quotient algebra

$$B_n^l \;:= B_n / k\mathbf{Br}^{<l}(n,n)$$

In particular the first section in (18.1) is a quotient algebra, and obeys

$$B_n^n \;=\; k\mathbf{Br}^{\leq n}(n,n)/k\mathbf{Br}^{\leq n-2}(n,n) \;\cong\; kS_n$$

as a k-algebra. Thus each kS_n-module restricts to a B_n-module identical to it as a k-module, where the action of any diagram with fewer than n propagating lines is by 0.

de:bims (**18.1.3**) Hereafter we may sometimes write b for $[b]$ in (18.2), and hence shall understand $k\mathbf{Br}^{n-2i}(n,n)$ to be a B_n-bimodule by identification with the i-th section of the sequence (18.1). Similarly any $k\mathbf{Br}^l(n,l)$ $(l \leq n)$ is a left B_n- right B_l-bimodule.

Where k is clear we may simply write $\mathfrak{B}^l(n,l)$ for $k\mathbf{Br}^l(n,l)$ understood as a bimodule in this way. That is $\mathfrak{B}^l(n,l) \;=\; k[\mathbf{Br}^l(n,l)]$ by analogy with (18.2).

(**18.1.4**) Remark. The section $k[\mathbf{Br}^l(n,n)]$ is also an ideal of the quotient algebra B_n^l. (Example: $k[\mathbf{Br}^n(n,n)] \;=\; B_n^n$.)

lem:a Bn decomp

(**18.1.5**) Lemma. *For $n-2i=l$ we have a decomposition of this i-th section* as a left B_n-module*:*

$$k[\mathbf{Br}^l(n,n)] \;\cong\; \bigoplus_{w\in \mathbf{Br}^{1_l}(l,n)} k[\mathbf{Br}^l(n,l)]\; w \tag{18.3}$$

reg dec2

All the summands are isomorphic to $k[\mathbf{Br}^l(n,l)]$*.* □

(**18.1.6**) By the restriction $kS_l \subset B_l$, it will be evident that $\mathbf{Br}^l(n,l)$ gives a basis for a left-$B_n(\delta)$ right-kS_l bimodule, where the action on the left is via the category composition, quotienting by $k\mathbf{Br}^{\leq l-2}(n,l)$ as before, and on the right by the natural diagram composition:

$$[b]\sigma = (b+I)\sigma = b\sigma + I\sigma \mapsto [b\sigma]$$

Once again, where the intention is clear, we may abbreviate $k\mathbf{Br}^l(n,l)$ regarded as a bimodule in this way to $\mathfrak{B}^l(n,l)$.

pr:Brexact1

(**18.1.7**) Proposition. *Fix any ring k, and $n \geq l \in \mathbb{N}_0$. The free k-module $\mathfrak{B}^l(n,l) = k[\mathbf{Br}^l(n,l)]$, which is a left $B_n(\delta)$ right kS_l-bimodule, is a free right kS_l-module.*

The functor

$$\begin{aligned} \Phi^n \;:\; kS_l\!-\!mod \quad &\to \quad B_n(\delta)\!-\!mod && (18.4)\\ M \quad &\mapsto \quad \mathfrak{B}^l(n,l)\otimes_{kS_l} M && (18.5)\end{aligned}$$

is exact.

Similarly $\mathfrak{B}^l(l,n) = k\mathbf{Br}^l(l,n)$ is a right B_n-module and a projective left kS_l-module. The functor

$$\mathfrak{B}^l(l,n)\otimes_{B_n} - \;:\; B_n(\delta)\!-\!mod \;\to\; kS_l\!-\!mod$$

is right exact.

Proof. Noting that arcs on the n-vertex edge of a diagram play no role in the right S_l-action, we see that $\mathfrak{B}^l(n,l)$ is a direct sum of copies of the regular right kS_l-module. Now use Prop. (8.6.20) for exactness. The other case is standard. □

[1]

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[1]

(**18.1.8**) Example. Let us consider the result of composing the functors in the above. In case $l=0$, $m=2$, i.e. $k\mathbf{Br}^0(0,2)$, we have basis $\mathbf{Br}^0(0,2)=\{u'\}$. This is a left kS_0-module, where S_0 is the trivial group, and a right $B_2(\delta)$-module. Let us examine the composite functor

$$\underbrace{{}_{kS_0}k\mathbf{Br}^0(0,2)\otimes_{B_2} k\mathbf{Br}^0(2,0)}\otimes_{kS_0} -$$

The first thing is to examine the underbraced factor as a kS_0-bimodule, i.e. as a k-module. Since each factor is a free k-module with singleton basis, the module is spanned by $u\otimes_{B_2} u'$. That is, it is spanned by the equivalence class of the pair (u,u'). To construct this class we need the set of elements (b,m) of $B_n \times k\mathbf{Br}^0(2,0)$ such that $bm=u'$ (then $(u,u')\sim(ub,m)$). Note that $m=cu'$ for some scalar c, since $\{u'\}$ is a basis.

Let us consider the case in which $k\delta=0$. Then only scalar multiples of 1 act suitably on any such m. The class is simply elements of form $(cu, c^{-1}u')$ (c an invertible scalar).

Note that there is a multiplication map defined here: $(u,u')\mapsto uu' \in k\mathbf{Br}(0,0)$. In the case $k\delta=0$ we have $uu'=0$, so this is not a surjective map onto $k\mathbf{Br}(0,0)$. In other cases this map defines a bijection, so long as δ is invertible.

(**18.1.9**) According to [3, 19.10 Th.] tensor functors like this preserve direct sums.

18.2 Brauer Δ-modules

We now construct various interrelated classes of 'integral' Brauer algebra modules, and then examine their properties over certain fields. One class comes from applying the Φ^n functor to Specht modules. One class comes from suitably injecting symmetric group idempotents into B_n. One class from functors between Brauer algebra module categories.

18.2.1 Symmetric group Specht modules (a quick reminder)

ss:BSpSp

(**18.2.1**) Set $\Lambda_n = \{\lambda \vdash n\}$ and

$$\Lambda^n \ := \Lambda_n \cup \Lambda_{n-2} \cup \ldots \cup \Lambda_{0/1}$$

(**18.2.2**) For $\lambda \in \Lambda_n$ we write $\mathcal{S}(\lambda)$ for the $\mathbb{Z}S_n$ Specht module (as in Section 11.2.4, or for example

pa:def v_lambda [78]).

Recall from (11.2.30), or [78, Lem.7.1.4], that for each λ we may choose an element $v_\lambda \in kS_l$ such that

$$\mathcal{S}(\lambda) \cong kS_l v_\lambda.$$

and $v_\lambda \mathcal{S}(\nu) = 0$ unless $\lambda = \nu$.

(**18.2.3**) If k contains $\mathbb{Q}$ (or, for given n, an inverse to $n!$) then v_λ may be chosen a primitive idempotent. E.g. $\lambda = (2)$, $v_{(2)} = \frac{1}{2}(1+\sigma_1)$.

(**18.2.4**) Note that if $\lambda \vdash l$ then $\Phi^l(\mathcal{S}(\lambda))$ is the restriction noted in (18.1.2).

18.2.2 Brauer Δ-module constructions

We introduce certain Λ^n-indexed sets of B_n-modules defined by two different contructions starting from the symmetric group, and discuss equivalences between them. One construction uses the functors Φ^n. The other works more directly with the embedding of $S_n \hookrightarrow B_n$.

de:brauerspecht (**18.2.5**) For $\lambda \in \Lambda^n$ define B_n-module

$$\Delta_n(\lambda) \ := \ \Phi^n(\mathcal{S}(\lambda))$$

(**18.2.6**) More explicitly we have $\Delta_n(\lambda) = \ \mathfrak{B}^l(n,l) \otimes_{kS_l} \mathcal{S}(\lambda)$.

lem:eSpecht=0 (**18.2.7**) Lemma. *For a a Brauer diagram let $\#(a)$ be the number of propagating lines. Then*

$$a\Delta_n(\lambda) = 0 \qquad if \qquad \#(a) < |\lambda|$$

and indeed by construction $E_m \Delta_n(\lambda) = 0$ iff $\#(E_m) = n - 2m < |\lambda|$.

Proof. By construction. We have $a\mathfrak{B}^l(n,l) = 0$. □

pa:ind (**18.2.8**) Now recall the natural inclusion $kS_l \hookrightarrow B_l$. [2] Either regarding $k\mathbf{Br}^l(n,l)$ as a right B_l- or a right kS_l-module, we have for each $v \in kS_l$ a multiplication map $\circ$ giving $k(\circ(\mathbf{Br}^l(n,l) \times kS_l v)) =$

[2] We have a multiplication map $\circ$ giving $k(\circ(\mathbf{Br}(n,l) \times kS_l v_\lambda)) = k\mathbf{Br}(n,l)v_\lambda \subset k\mathbf{Br}(n,l)$. The image $k\mathbf{Br}(n,l)v_\lambda$ is also a left B_n-module. Indeed the subset $k\mathbf{Br}^{\leq l-2}(n,l)v_\lambda$ is also a left B_n-module. We may define a left B_n-module $k\mathbf{Br}(n,l)v_\lambda / k\mathbf{Br}^{\leq l-2}(n,l)v_\lambda$. Similarly,

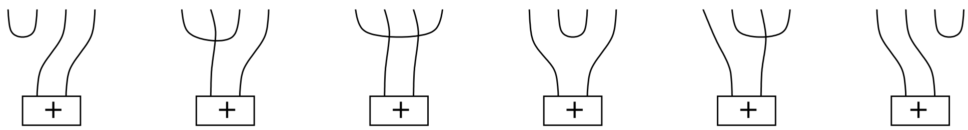


Figure 18.1: Basis for $D_4(2)$ (and $\Delta_4(2)$). fig:basis42

$k\mathbf{Br}^l(n,l)v \subset k\mathbf{Br}^l(n,l)$. Note that $k\mathbf{Br}^l(n,l)v$ is a left B_n-submodule of $k\mathbf{Br}^l(n,l)$. For each λ, with v_λ as in (18.2.2), define

$$D_n(\lambda) \;:=\; k\mathbf{Br}^l(n,l)v_\lambda \tag{18.6}$$ eq:vlambda

lem:Db1 (**18.2.9**) Lemma. *Let $\lambda \vdash l$ and let $b(\lambda)$ be a basis in $kS_l v_\lambda \subset kS_l$. Include this in B_l in the natural way, as above. Then $\circ$ (as above) restricts to an injective map on $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$, and the image*

$$b_{D_n(\lambda)} \;:=\; \circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda))$$

is a basis of $D_n(\lambda)$. □

Proof. An element in the image of the restriction of $\circ$, $\circ(d,x)$ say, is a linear combination of diagrams all with their non-propagating arcs in the same positions. These positions determine d, and removing all these arcs determines x, thus $\circ$ is reversible on such elements. Using (17.2), we have that $\circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda))$ spans $k\mathbf{Br}^l(n,l)v_\lambda$. □

(**18.2.10**) Example. A basis for $D_4(2)$ is given in Figure 18.1, where $+$ denotes the S_2 symmetrizer $1+(12)$. We shall see next that this also serves as a basis for $\Delta_4(2)$.

specht-basis (**18.2.11**) Proposition. *Let $\lambda \vdash l$ and let $b(\lambda)$ be a basis for $\mathcal{S}(\lambda)$. Then*

$$b_{\Delta_n(\lambda)} = \{a \otimes_{kS_l} b \;:\; (a,b) \in \mathbf{Br}^{1_l}(n,l) \times b(\lambda)\}$$

is a basis for $\Delta_n(\lambda)$.

Proof. The set $b_{\Delta_n(\lambda)}$ is a set of generators for $\Delta_n(\lambda)$ by (17.2), since factors in $\mathbf{Br}^l(l,l)$ may be moved to the right of the tensor product and 'absorbed' by $b(\lambda)$. By the same argument as for (18.2.9) the given set passes to a basis (of the image) under the multiplication map. It follows that this set is independent, so the module is k-free with this set as basis. □

Alternatively: Note that $\Delta_n = k[\mathbf{Br}^l(n,l)] \otimes_{kS_l} M$, for the given module M; and $k[\mathbf{Br}^l(n,l)]$ is a free right kS_l-module with basis $\mathbf{Br}^{1_l}(n,l)$ by (17.3.14) and (18.1.7). Since $A \otimes_A M \cong M$ it follows that as a k-space Δ_n is a sum of copies of M indexed by $\mathbf{Br}^{1_l}(n,l)$. □

specht-basis2 (**18.2.12**) Proposition. *Choose $b(\lambda) \subset kS_l \subset B_l$ as in (18.2.9). Then*
(i) $\Delta_n(\lambda) \cong D_n(\lambda)$.
(ii) The map defined on diagrams by $d \mapsto d \boxtimes (u')^{\boxtimes m}$ (with $m = (n-l)/2$), and extended k-linearly, takes $k\mathbf{Br}(n,l)$ to a left ideal in B_n. This map takes $k(\circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda)))$ to a left ideal of B_n^l isomorphic to $\Delta_n(\lambda)$ (as a left B_n^l-module).

Proof. (i) Follows immediately from (18.2.9) and (18.2.11). (ii) Holds since the map is an injection that has no effect on the left action. □

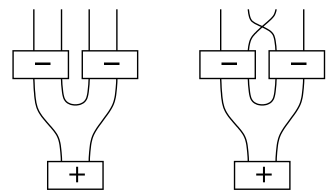


Figure 18.2: Basis for submodule of $\Delta_4(2)$ when $\delta = 2$. fig:basis422

18.2.3 Brauer Δ-module examples

(**18.2.13**) Example. The reader will readily confirm that when $\delta = 2$ there is a submodule of $\Delta_4(2)$ with basis given in Figure 18.2, where $-$ denotes the S_2 antisymmetrizer $1-(12)$. This submodule is isomorphic to $\Delta_4(2^2)$, that is there is a map

$$\Delta_4(2^2) \stackrel{\delta=2}{\to} \Delta_4(2) \tag{18.7}$$ eq:22map2

18.2.4 Simple head conditions for Δ-modules

(**18.2.14**) For $l > 0$, $v \in B_l$ and m such that $n = l + 2m$, define

$$E^v_m = (1_{2m} \boxtimes v)E_m \quad \in B_{l+2m}$$

See the top-lefthand corner of figure 18.3 for a picture of this.

(**18.2.15**) Let us write ψ_l for the natural ring homomorphism $\psi_l : B_n \to B^l_n$. Thus we have an exact functor $\mathrm{Res}_{\psi_l} : B^l_n-\mathrm{mod} \;\to\; B_n-\mathrm{mod}$ that takes module $M \mapsto M$.

We also have $\mathrm{Ind}_{\psi_l} : B_n-\mathrm{mod} \;\to\; B^l_n-\mathrm{mod}$ given by $N \mapsto B^l_n \otimes_{B_n} N$. (The behaviour of modules under this functor is less easy to predict.)

(**18.2.16**) Lemma. *For $\lambda \vdash l > 0$ and $m = (n-l)/2$ as above, $\Delta_n(\lambda)$ is also a B^l_n-module. Indeed*

$$\Delta_n(\lambda) \cong\; B^l_n E^{v_\lambda}_m \tag{18.8}$$ eq:Delxx

Meanwhile $\Delta_{2m}(\emptyset) \cong\; B_{2m}\overline{E}_m$.

(**18.2.17**) Exercise. What can we say about $B^{l'}_n E^{v_\lambda}_m$ for $l' < l$?

Regarded as B_n-modules we see that these modules are a succession of quotients of the (not necessarily indecomposable) module $B_n E^{v_\lambda}_m$; and that this latter module is projective over $\mathbb{Q}[\delta]$ (and hence over $\mathbb{C}$).

(**18.2.18**) If $\lambda \neq \nu \vdash l$ then $E^{v_\lambda}_m \Delta_n(\nu) = 0$. (To see this consider (18.8) and a corresponding mild variation of figure 18.3, where the vs above and below differ.)

pa:sh0 (**18.2.19**) Note from Figure 18.3 that for any $v \in B_l$ $(l > 0)$ there is a k-module isomorphism

$$E^v_m \; B_{l+2m} \; E^v_m \;\mapsto\; vB_lv$$

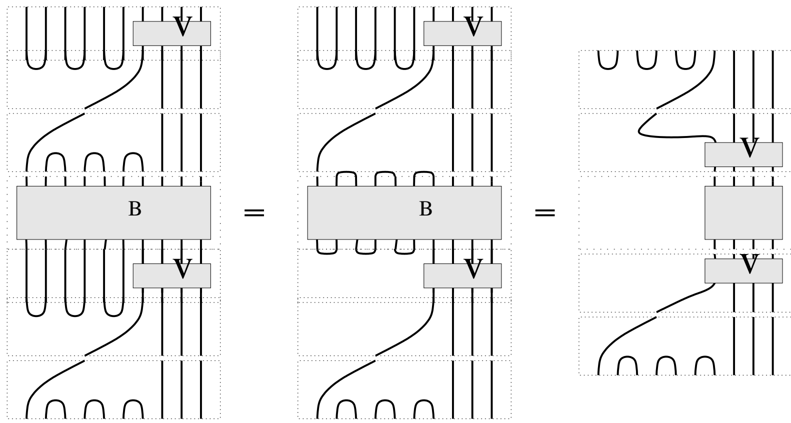


Figure 18.3: Diagramatic realisation of $E_m^v B_{l+2m} E_m^v$ fig:vEBvE

Thus if v is idempotent, then (I) so is E_m^v; (II) $E_m^v B_{l+2m} E_m^v$ and vB_lv are isomorphic k-algebras.

Suppose v is idempotent in kS_l. The ideals spanned by elements with support in diagrams with fewer than l propagating lines (i.e. by $E_m^v\ \mathbf{Br}^{l-2}(l+2m, l+2m)\ E_m^v$ and $v\mathbf{Br}^{l-2}(l,l)v$) are also isomorphic; and hence so are the corresponding quotient algebras, $E_m^v B_{l+2m}^l E_m^v$ and $vB_l^l v$. If $vB_l^l v$ is rank 1, then so is $E_m^v B_{l+2m}^l E_m^v$. Thus E_m^v is primitive in B_{l+2m}^l when v is primitive in $B_l^l = kS_l$.

lem:sh1 (**18.2.20**) Lemma. *(I) Suppose $\lambda \vdash l$, and v_λ in (18.6) may be chosen a primitive idempotent of kS_l (as if $k = \mathbb{C}$ or $k = \mathbb{Q}[\delta]$, for example). Then for $l > 0$, or $l = 0$ and δ a unit, $\Delta_n(\lambda)$ is indecomposable projective as a B_n^l-module, and hence has simple head.*

lem:sh2 *(II) If $n > 1$, $l = 0$ and $\delta = 0$ we have a surjection*

$$\Delta_n((2)) \to \Delta_n(\emptyset) \to 0$$

so that when $\Delta_n((2))$ has simple head, $\Delta_n(\emptyset)$ also has simple head.

Proof. (I) Consider (18.2.19). Now note (18.8).
The $l = 0$, δ a unit case is similar.
(II) Note that the basis of $\Delta_n((2))$ contains elements of the form $dv_{(2)}$ where $d \in \mathbf{Br}^2(4,2)$ (and $v_{(2)} = 1_2 + (12)$ in $k\mathbf{Br}^2(2,2)$). The map is given by $dv_{(2)} \mapsto du$. One readily checks that this is well-defined, and a surjection. □

We shall see shortly that:

pr:bgenerisimple (**18.2.21**) Lemma. *In particular, over the field of fractions $\mathbb{Q}(\delta)$, (I) every module $\Delta_n(\lambda)$ is simple; and (II) B_n is semisimple.*

Proof. See (18.4.16).

18.2.5 Brauer algebra representations: The base cases

de:base1

(**18.2.22**) We have $B_0(\delta) \cong B_1(\delta) \cong k$. For $B_2(\delta)$ we have $\Delta_2(\emptyset)$, $\Delta_2(2)$, $\Delta_2(1^2)$. These are each of rank 1, but are non-isomorphic if $k = \mathbb{C}$ as a field (or indeed if 2 is a unit in k), unless $\delta = 0$.

For $\delta = 0$ we have, over $\mathbb{C}$,

$$\Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset)$$

Thus we may regard $\Delta_2(2)$, $\Delta_2(1^2)$ as the inequivalent simple $B_2(0)$-modules. In this case $P_2(2)$ is the self-extension of $\Delta_2(2)$, while $P_2(1^2) = \Delta_2(1^2)$.

18.2.6 The case $k \supseteq \mathbb{Q}$

Note that all copies of $\Delta_n(\lambda)$ occur in a single p-section of B_n.

18.3 Δ-Filtration of projective modules

ss:BSfilt

18.3.1 Some character formulae

(**18.3.1**) Recall (e.g. from Lang [91, §III.8]) that if A is a finite dimensional algebra over a field k then associated to A-mod is the *Grothendieck group* $K(A-\text{mod})$ (or just $K(A)$) — the quotient of the free abelian group generated by isomorphism classes $[M]$ of A-modules by $[X] + [Z] - [Y] = 0$ whenever $0 \to X \to Y \to Z \to 0$ is a short exact sequence.

(**18.3.2**) Write χ_M for the image of module M in the Grothendieck group of B_n-mod. By (18.1.5)

$$\chi_{B_n} = \sum_{l=n_0}^{n} d_n^l \chi_{k\mathbf{Br}^l(n,l)}$$

for $d_n^l = |\mathbf{Br}^{1_l}(n,l)|$; and

$$\chi_{k\mathbf{Br}^l(n,l)} = \sum_{\lambda \vdash l} d_\lambda \chi_n(\lambda)$$

where $d_\lambda = rank_k(\mathcal{S}(\lambda))$ and $\chi_n(\lambda) = \chi_{\Delta_n(\lambda)}$. Altogether

$$\chi_{B_n} = \sum_{\lambda \in \Lambda^n} d_n^\lambda \chi_n(\lambda)$$

where $d_n^\lambda = rank_k(\Delta_n(\lambda))$.

18.3.2 General preliminaries

NB Issues with this section!! Probably we don't need it! It is stronger both specifically and in being more general than we need — we want that projectives have 'positive Δ-characters', and this is weaker than a filtration property! But the section is not perfect as it stands.

lem:inherit

(**18.3.3**) LEMMA. *Caveat emptor. Suppose $B = P \oplus Q$ is a decomposition of some R-module B. Then by (8.2.16) there is an idempotent e in $End_R(B)$ that projects onto P. That is, $P = eB$. For D an indecomposable submodule of B we see that the decomposition of D by $1_B = e + (1-e)$ must be trivial, hence D is a submodule either of P or Q.*

(This does not mean that every submodule isomorphic to D has this property. But then that seems to cast doubt on everything.)

Nonsense: Now suppose that D' is an indecomposable submodule of B/D. By the same argument — no! — this submodule also lies on one 'side' of $B/D = P/D \oplus Q/D$ (here we understand $Q/D = Q/(Q \cap D)$, noting that one of $P \cap D$ and $Q \cap D$ is zero). Thus we can iterate to pass a filtration of B by a set of indecomposable modules to a filtration of P (one simply omits some sections from the filtration — the omitted sections being the sections that go to form a filtration of Q).

(**18.3.4**) Suppose that A is an algebra and Γ, Γ' are sets of A-modules; and that

$$N = N_0 \supset N_1 \supset ... \supset N_l = 0$$

is a chain of A-modules such that every section N_i/N_{i+1} is isomorphic to some $M \in \Gamma$. Then we say, *N has a Γ-filtration.*

filtration tower

(**18.3.5**) PROPOSITION. *Suppose that A is an algebra and Γ, Γ' are sets of A-modules; and that module N has a Γ-filtration. Suppose also that every $M \in \Gamma$ has a Γ'-filtration. Then N has a Γ'-filtration.*

Proof. Suppose that $N_i/N_{i+1} \cong M_i = M^i_0 \supset M^i_1 \supset ... \supset M^i_r = 0$ is a chain corresponding to a Γ'-filtration of N_i/N_{i+1}. We write $f : N_i \to N_i/N_{i+1}$ for the quotient map. By Prop.8.3.31 there is a chain

$$N_{i-1} \supset N_i = f^{-1}M^i_0 \supset f^{-1}M^i_1 \supset ... \supset N_{i+1}$$

that is a (local) refinement of the initial chain, and that gives (locally) a Γ'-filtration. □

(**18.3.6**) For M a module, the shorthand notation

$$M \;=\; A_1//A_2//A_3//... \qquad \text{or} \qquad M \;=\; //_i A_i \tag{18.9}$$

section notation

shall indicate that M has a chain of submodules with sections $A_1, A_2, A_3, ...$ See also (18.6.11).

lem:Mfissummand

(**18.3.7**) LEMMA. *Let A be an algebra, f an idempotent, and M a bimodule. Then there are left-module maps $Mf \to M$ given by inclusion; and $M \to Mf$ given by $m \mapsto mf$. Indeed the sequence*

$$0 \to Mf \to M \to M(1-f) \to 0$$

is short-exact and split.

lem:gentry

(**18.3.8**) LEMMA. *Let A be a finite-dimenional algebra and suppose $0 \subset J_1 \subset J_2 \subset ... \subset J_k = A$ is a filtration by ideals. Let X be the set of indecomposable summands of all the left-modules J_i/J_{i-1} up to isomorphism. Then every projective left-module of A is filtered by X. Does this look the right level of generality? Maybe yes. Cf. dialogues/notes... elsewhere.*

Proof. It is enough to show for indecomposable projectives, and hence for modules of form Af where f is idempotent. For such a module we have a (possibly degenerate) filtration $0 \subseteq J_1 f \subseteq J_2 f \subseteq ... \subseteq J_k f = Af$ by assumption. Suppose $M \supset N$ are A-bimodules. We claim that there is a left-A-module map $(M/N)f \to Mf/Nf$ given by $\{mf + n : n \in N\} \mapsto \{mf + nf : n \in N\}$ (i.e. act on the set elementwise by f on the right), and that this is a left-A-module isomorphism. To see this note that (i) the image lies in Mf/Nf; (ii) this gives a vector space isomorphism; (iii) it is a left-A-module morphism (the map uses the action on the right which, by the bimodule property commutes with the action on the left).

Thus in particular $(J_i/J_{i-1})f \cong J_i f/J_{i-1} f$, so there is a sectioning of Af with sections isomorphic to modules $(J_i/J_{i-1})f$. By Lemma 18.3.7 $(J_i/J_{i-1})f$ is a sum of (some) direct summands of an indecomposable direct summand decomposition of J_i/J_{i-1}. And by the Krull–Schmidt Theorem for modules, and our working assumptions, every such decomposition is a sum from X. The Lemma now follows routinely using (6.21**??**), since a direct sum has a filtration by its summands. □

18.3.3 A Δ-filtration theorem

(**18.3.9**) *Remark:* See (11.2.38) for conditions on k for Specht filtrations of kS_l. In short, there is a filtration for any field k, but the multiplicities may not be unique unless charactersitic $p > 3$.

pr:projDfilt1 (**18.3.10**) THEOREM. (I) *Suppose k is such that ${}_{kS_l}kS_l$ is filtered by $\{\mathcal{S}(\lambda)\}_{\lambda\in\Lambda_l}$ for all $l \leq n$. Then, for any δ, the left regular module ${}_{B_n}B_n$ is filtered by $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$.*
(II) *In case $k = \mathbb{C}$, for any δ, projective B_n-modules are filtered by $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$.*

Proof. (I) By (18.1.1) the set $\{k\mathbf{Br}^l(n,l)\}_l$ gives (via the action therein) a left-B_n filtration of B_n. By Prop. 18.1.7 the functor Φ^n is exact, so each factor itself has a filtration by Δs under the stated condition. Specifically, if

$${}_{kS_l}kS_l \;\cong\; //_i\, \mathcal{S}(\lambda(i))$$

is a filtration (in the notation of (18.6.11)) then

$$\Phi^n{}_{kS_l}kS_l \;\cong\; //_i\, \Phi^n\mathcal{S}(\lambda(i))$$

is a filtration of $k\mathbf{Br}^l(n,l)$. We are done by Pr.(18.3.5).

(II) First note that $\mathbb{C}S_l$ is semisimple and hence filtered by $\{\mathcal{S}(\lambda)\}_{\lambda\in\Lambda_l}$. Secondly, the set $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$ are all indecomposable by the simple-head conditions (18.2.20)(I) and (18.2.20)(II) (whose hypotheses are met by $k = \mathbb{C}$).

Since $P_n(\lambda)$ is a direct summand of ${}_{B_n}B_n$, we may use CAVEAT!: Lemma 18.3.3 to deduce [3] that eventually $P_n(\lambda)$ will have a Δ-filtration inherited from that of B_n. □

(**18.3.11**) Consider further the case $k = \mathbb{C}$ (any δ). Note from () that in this case, regarded as a left-B_m-module, $k\mathbf{Br}^l(m,l)$ is a direct sum

$${}_{B_m}k\mathbf{Br}^l(m,l) = \oplus_{\lambda\vdash l} d_\lambda \Delta_m(\lambda)$$

(with multiplicities d_λ as indicated). Thus we have a filtration of form

$${}_{B_m}B_m \;=\; \oplus_{\lambda\vdash m} d_m^m d_\lambda \Delta_m(\lambda) \;//\; \oplus_{\lambda\vdash m-2}\, d_m^{m-2} d_\lambda \Delta_m(\lambda) \;//...$$

[3]comment demoted.

Note that this stretches to a filtration by Δ-modules, by any ordering of the summand modules in a section above. Since Δ-modules are indecomposable (Lemma 18.2.20) it follows -CAVEAT: clarify this!- that there is a filtration of each summand of ${}_{B_m}B_m$ obtained by deleting terms from such a stretched filtration.

In particular every indecomposable projective module P must have such a filtration. Thus for each P there is a λ such that $\Delta_m(\lambda)$ is the first term (with head agreeing with P); and subsequent terms $\Delta_m(\mu)$ having $|\mu| \leq |\lambda|$. This filtration would proceed next with any Δ-modules with $|\mu| = |\lambda|$; however no $\Delta_m(\mu)$ with $|\mu| = |\lambda|$ can appear in this filtration (else one such is next to $\Delta_m(\lambda)$ and there would be another such filtration in which they are reordered, (eventually) contradicting the simple head condition).
[4]

18.3.4 On simple modules, labelling and Brauer reciprocity

ss:Bosm

We have a useful set of Δ-modules indexed by Λ^n. What can we say about simple modules (and indecomposable projective modules) so far? And can we invoke Brauer reciprocity (essentially the $C = D^T D$ property of π-modular systems, as in §**??**) to relates these data?

(**18.3.12**) REMARK: we would like to assume that Δ-modules have simple head, but we have basically only shown this here for $k = \mathbb{C}$ (10.4.22 **??**) — and it is not always true (for example, Specht modules are special cases of Δ-modules, and these do not always have simple head, for example when the characteristic of k is $p = 2$ they are not always indecomposable). We might also be able to show it when the inflated Specht module has simple head. But I'm not sure we know when this happens (apart from p-regularly). And anyway we don't know about Specht filtrations in general... — One thing we could do in what follows is fix $k = \mathbb{C}$ and use δ in the notation instead of k... —

(**18.3.13**) Fixing k, δ, define
$$L^k_n(\lambda) = \text{ head}_k \, \Delta_n(\lambda)$$
(Note that the dependence on δ is left implicit in this notation. However if in particular $k = \mathbb{C}$, we may write instead $L^\delta(\lambda)$.) Note that $L^k_n(\lambda)$ is not necessarily simple.

(**18.3.14**) If $k = \mathbb{C}$ then by Prop.18.3.10 the head of every indecomposable projective is also the head of a Δ-module. Accordingly we have

m:simple over c1

(**18.3.15**) PROPOSITION. *If $k = \mathbb{C}$, the set $\{L^k_n(\lambda)\}_\lambda$ contains a complete set of simple modules for $B_n(\delta)$.* □

In fact we shall see that if $k = \mathbb{C}$ then this set *is* a complete set.

(**18.3.16**) REMARK. In fact the Proposition is true for much more general choice of k. We shall return to this point later.

(**18.3.17**) PROPOSITION. *CLAIM: Let k, δ, and λ be such that $\Delta_n(\lambda)$ is an (indecomposable) top factor in a Δ-filtration of some projective $B_n(\delta)$-module P. Let $P_n(\lambda)$ be an indecomposable summand of P containing this factor. (For now $P_n(\lambda)$ is just a local name. Later we will see that it is well-defined generally. Note that, up to isomorphism, only one such summand has the*

[4]comment demoted.

property of containing this factor.)
Then (i) every Δ-filtration of $P_n(\lambda)$ starts with $\Delta_n(\lambda)$ (up to isomorphism), and (ii) if $\Delta_n(\mu)$ lies below $\Delta_n(\mu')$ in all filtrations then $|\mu| < |\mu'|$.

Proof. By construction of $\Delta_n(\lambda)$ and the ${}_{B_n}B_n$ filtration.

??!!

(i)/(ii) idea is somehow to look at the original ideal chain for B_n as a chain of left-modules (and the refined chain with Δs in it). ...!

(**18.3.18**) Corollary: Since every other Δ-factor lies below the top $\Delta_n(\lambda)$ in $P_n(\lambda)$, none is $\Delta_n(\lambda)$, except in case $\delta = 0$, $n = 2$ where $\Delta_2(2) \cong \Delta_2(\emptyset)$.

18.4 Globalisation functors

ss:GG01

(**18.4.1**) Remark. Fixing commutative ring k and $\delta \in k$, we want a way to treat the representation theory of B_n for each n as far as possible simultaneously. One idea is to construct a large-n 'global' limit of the sequence of categories B_n-mod — *globalisation*.
We also want a way to regard Δ-modules (or something similar) as *canonical*, at least up to simple multiplicities — providing a basis for the Grothendieck group of B_n over any field k (and choice of δ); and so that projective modules have good filtration properties.

These wants are relatable. A good paradigm for this is Green [57, §6.6]. This treats a single pair of algebras B and eBe — but Brauer algebras provide a chain of such pairings. (Another similar, but more restricted, approach is Cox–Martin–Parker–Xi [27].)

The 'canonical module' want is related to the role of Specht modules (for S_n for fixed n) as 'generic irreducible modules'. In general this means: modules for an algebra B defined over a DVR K (in this S_n case containing $\mathbb{Z}$) that pass to simple modules over a suitable extension field K_0 of K. One then considers projective modules over some quotient field k of K that is of interest — these have preimages over K by *idempotent lifting* and hence have positive Specht characters by 'passing everything' to K_0. The Cartan decomposition matrix for B over k is then determined by, say, the Specht module decomposition numbers.

Now what about varying n?: Nominally there are a number of B_n-modules in level n that are 'images' of some level $l < n$ Specht module. We can inflate fully with Φ^n, or partially with some Φ (perhaps not even changing l) then continue with various G-functor-type steps up to n.

How are these modules related? What properties do they have?

18.4.1 Preliminaries: $\otimes$ versus category composition

ss:mumap11

lem:compose GG

(**18.4.2**) Lemma. *For $n \geq m \geq l$ the multiplication map defines a bimodule isomorphism*

$$\mathbb{B}(n,m) \otimes_{B_m} \mathbb{B}(m,l) \xrightarrow{\sim} \mathbb{B}(n,l)$$

unless δ is a non-unit and $l = 0$.

Proof. The case $l = m$ is trivial, so suppose $l < m$. In case $l > 0$ we have the bimodule isomorphism $k\mathbf{Br}(m,l) \cong B_m E_x$ from Lemma (17.4.5), where $x = (m-l)/2$. (In case δ a unit we may use $k\mathbf{Br}(m,0) \cong B_m E'_x$ with $E'_x = \prod_{i\ odd} \delta^{-1} U_i$ instead.) Thus

$$k\mathbf{Br}(n,m) \otimes_{B_m} k\mathbf{Br}(m,l) \cong k\mathbf{Br}(n,m) \otimes_{B_m} B_m E_x = k\mathbf{Br}(n,m)E_x \otimes_{B_m} E_x.$$

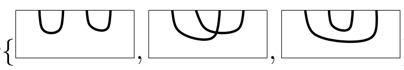
using that $E_x E_x = E_x$. The projection from here to $k\mathbf{Br}(n,m)E_x$ is clearly invertible and hence an isomorphism. Using Lemma 17.4.5 again we have $k\mathbf{Br}(n,m) \cong k\mathbf{Br}(n,n)E_{x'}$ ($x' = (n-m)/2$). Recall that this is a right B_m-module via the isomorphism $B_m \cong E_{x'}B_nE_{x'}$ that takes $d \mapsto E_{x'}(1_{2x'} \boxtimes d)E_{x'}$. In particular it takes $E_x \in B_m$ to $E_{x'}(1_{2x'} \boxtimes E_x)E_{x'}$. But one readily confirms (plugging in a suitable E_x for d in Figure 17.3 for example) that

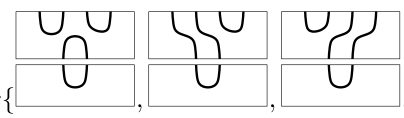
$$E_{x+x'} = E_{x'}(1_{2x'} \boxtimes E_x)E_{x'}$$

Thus $k\mathbf{Br}(n,m)E_x \cong k\mathbf{Br}(n,n)E_{x+x'}$. Finally [5] $k\mathbf{Br}(n,n)E_{x+x'} \cong k\mathbf{Br}(m,l)$ by using Lemma 17.4.5 once again. $\square$

(**18.4.3**) Remark. CLAIM: This can be strengthened to exlude only the case $\delta = 0$, $m = 2$ and $l = 0$ (see the following examples).

ex:n420 (**18.4.4**) Example. Case $n = 4$, $m = 2$, $l = 0$:

$k\mathbf{Br}(4,0) = k\{$  $\}$ (18.10) n420.01

$k\mathbf{Br}(4,2) \otimes_{B_2} k\mathbf{Br}(2,0) = k\{$  $, \ldots\}$ (18.11) n420.1

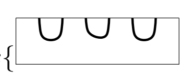
If δ is invertible then the first three diagrams as drawn in (18.11) are not independent: write the first one as $a \otimes u$ and the second as $a' \otimes u$, say, then

$$a \otimes_{B_2} u = a'U_1 \otimes_{B_2} u = a' \otimes_{B_2} U_1 u = \delta a' \otimes_{B_2} u$$

(note how U_1 both lies in B_2, and acts to reduce the propagating number of a'). A mirror-symmetrical argument relates the first and third diagrams. To construct the isomorphism one eliminates two of them, then maps the remaining one to the first diagram in (18.10); and so on.

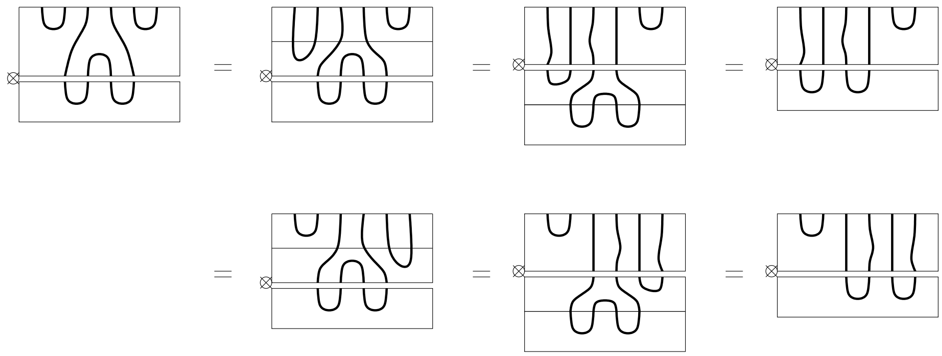
If $\delta = 0$ then the second and third diagrams as drawn in (18.11) are independent, and the isomorphism fails. The first diagram vanishes, so $\mathbb{B}(4,2) \otimes_{B_2} \mathbb{B}(2,0)$ is essentially two copies of $\mathbb{B}(4,0)$. The 'problem' here is that there is no non-zero idempotent in $\mathbb{B}^0(2,2)$ when $\delta = 0$ (i.e. no element of form $(1/\delta)U_1$). However every other $\mathbb{B}^{n-2}(n,n)$ contains an idempotent in all cases, so the problem is unique to this case, as we illustrate next.

eg:640 (**18.4.5**) Case $n = 6$, $m = 4$, $l = 0$:

$k\mathbf{Br}(6,0) = k\{$  $, \ldots\}$

[5] (RIGHT!!?!?!?!),

while $k\mathbf{Br}(6,4) \otimes_{B_4} k\mathbf{Br}(4,0)$ has elements of form



In this case one finds, by similar manipulations to those shown, that all the elements that would map to the first element of $\mathbf{Br}(6,0)$ drawn above if $\otimes$ were replaced by the multiplication map *are* equal (without any intermediate δ-factors appearing, and hence independently of δ). Thus in this case the isomorphism holds for all δ.

ss:GG1

18.4.2 G-functors

de:Gln (**18.4.6**) For $n+m$ even the k-space $k\mathbf{Br}(n,m)$ is a bimodule, so there is a functor

$$G^n_m \; : \; B_m\text{-mod} \quad \to \quad B_n\text{-mod} \tag{18.12}$$

$$M \quad \mapsto \quad \mathbb{B}(n,m) \otimes_{B_m} M \tag{18.13}$$

For given n, let us simply write

$$F- \; := \; G^{n-2}_n - \; = \mathbb{B}(n-2,n) \otimes_{B_n} - \; ;$$

and G for the functor $G^n_{n-2} = \mathbb{B}(n,n-2) \otimes_{B_{n-2}} - \,$.

claim:GS (**18.4.7**) CLAIM: For $n \geq l$ and $\lambda \vdash l$,

$$\Delta_n(\lambda) \; \cong \; G^n_l \Phi^l(\mathcal{S}(\lambda))$$

That is,

$$\Delta_n(\lambda) \quad \cong \quad \mathbb{B}(n,l) \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda)) \tag{18.14}$$

$$= \quad \mathbb{B}(n,l) \otimes_{B_l} \mathbb{B}^l(l,l) \otimes_{kS_l} \mathcal{S}(\lambda) \tag{18.15}$$

Proof. Our strategy is to construct a basis for $G^n_l \Phi^l(\mathcal{S}(\lambda))$ and compare it with the basis $b_{\Delta_n(\lambda)}$ for $\Delta_n(\lambda)$.

Let $b(\lambda)$ be as in (18.2.11). The B_l-module $\Phi^l(\mathcal{S}(\lambda))$ may be identified with $\mathcal{S}(\lambda)$ as a k-module. Thus the image of $\mathbf{Br}(n,l) \times b(\lambda)$ spans $\mathbb{B}(n,l) \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda))$ by construction (cf. the basis $b_{\Delta_n(\lambda)}$, which is in bijection with $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$). If $l = 0$ or 1 this set is a basis and we are done. We return to $l = 2$ shortly. If $l > 2$ we may proceed as follows. By Lemma 17.3.17, diagrams in $\mathbf{Br}^{<l}(n,l)$ can be expressed in the form dE_1d' for $d \in \mathbf{Br}^l(n,l)$ and $E_1, d' \in B_l$ (note that this requires $l \geq 3$). But $dE_1d' \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda)) = d \otimes_{B_l} E_1d'\Phi^l(\mathcal{S}(\lambda)) = 0$ since $E_1d'\Phi^l(\mathcal{S}(\lambda)) = E_1\Phi^l(\mathcal{S}(\lambda)) = 0$. Thus $\mathbf{Br}^l(n,l) \times b(\lambda)$ spans, and hence $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$, spans. Independence of the image under the multiplication map (as in (18.2.11)) ensures that this set is a basis.

Finally note that the given basis may be identified as a set with that for $\Delta_n(\lambda)$, and that the actions on the 'corresponding' basis elements are the same.

The case $l = 2$ is similar. One sees that elements of $\mathbf{Br}^0(n,2)$ can be expressed in the form dU_{12} where $d \in \mathbf{Br}(n,2)$ (note that this does not require any idempotency property), whereupon $dU_{12} \otimes_{B_2} \Phi^l(\mathcal{S}(\lambda)) = d \otimes_{B_2} U_{12}\Phi^l(\mathcal{S}(\lambda)) = 0$. One then argues as before. □

(**18.4.8**) Remark. CAUTIONARY TAIL: If δ is not a unit then

$$G_2^4 G_0^2 \ncong G_0^4$$

i.e. the underlying bimodules are not isomorphic (see (18.4.4)). In particular

$$G_2^4 \Delta_2(\emptyset) \ncong \Delta_4(\emptyset)$$

A quotient of B_n by an ideal contained in the radical is filtered by the Δ's excluding $\Delta_n(\emptyset)$, so the others span the Grothendieck group without $\Delta_n(\emptyset)$. [6] On the other hand, the others do not provide a filtration of projectives (see (**??**)).

pr:GG1 (**18.4.9**) Proposition. *For $\lambda \vdash l$ (and making $\mathcal{S}(\lambda)$ a B_l-module as in (18.1.2)), we have*

$$\Delta_{2m+l}(\lambda) \cong G^{\circ m}\Phi^l(\mathcal{S}(\lambda))$$

unless $l = 0$, $m = 2$ and $\delta = 0$.

Proof. Apply Lemma(18.4.2) to Lemma(18.4.7). □

18.4.3 Idempotent globalisation

pr:Fexact1 (**18.4.10**) Proposition. *Suppose either $n > 2$, or $n \geq 2$ and δ invertible in commutative ring k. Then*
(I) the free k-module $\mathbb{B}(n-2,n)$ is projective as a right B_n-module; and indeed there is an idempotent $e \in \mathbb{B}(n,n)$ such that

$$\mathbb{B}(n-2,n) \;\cong\; e\,(\mathbb{B}(n,n))$$

as a right B_n-module (for $n > 2$ take $e = U_{12}U_{23}$; for $n = 2$ take $e = \delta^{-1}U_{12}$).
(II) Functor $F : B_n\text{-mod} \to B_{n-2}\text{-mod}$ is exact; G is a right-exact left-adjoint/right-inverse to F.

Proof. (I) follows from Lemma 17.4.5. [7] For (II) use Prop. 8.6.20. □

---

[6] Exercise: EXPLAIN THIS.

[7] (We prove a left-handed version. The right-handed follows immediately.)
As a left module

$$B_3U_{12}U_{23} \cong B_3U_{23}U_{12} \cong k\mathbf{Br}(3,1)$$

(see also (17.9)), but $U_{12}U_{23}$ is idempotent, so $k\mathbf{Br}(3,1)$ is projective. The injection $i_{4,n} : \hom(3,1) \hookrightarrow \hom(n, n-2)$ $(n > 2)$ allows us to induce to $k\hom(n,n)\; i_4(\hom(3,1))$, which is therefore also left projective. This is a submodule of $k\hom(n,n-2)$ by construction; but considering for example the 'herniated' form of a diagram in

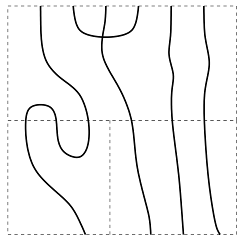
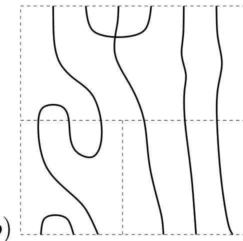
(a)  (b) 

Figure 18.4: Schematic for mapping $k\mathbf{Br}(6,4)$ into $k\mathbf{Br}(6,6)E_1$. fig:eB schematic

8

(**18.4.11**) In particular (unless $n = 2$ and $\delta = 0$) $B_{n-2}-$mod fully embeds in B_n-mod under G, and this embedding takes $\Delta_{n-2}(\lambda)$ to $\Delta_n(\lambda)$.

The embedding allows us to consider a formal limit module category (we take n odd and even together), from which all B_n-mod may be obtained by 'localisation' (action of the functor F).

---

hom$(n, n-2)$ as in figure 18.4(a), we deduce that every diagram appears in the submodule and hence

$$k\,\mathrm{hom}(n, n-2) \quad = \quad k\,\mathrm{hom}(n,n)\ \ i_{4,n}(\mathrm{hom}(3,1))$$

is left projective. The (left-handed version of the) claimed isomorphism is indicated in the passage to figure (b) above (in particular this shows that a suitable choice for e in case $n > 2$ is $e = U_{23}U_{12}$). In case δ invertible in k one sees directly that $k\,\mathrm{hom}(2,0)$ is left projective.

8

Lemma 18.1. *CLAIM: For $n-2 > 0$ and $i \in \mathbb{N}_0$*

$$k\mathbf{Br}(n-2,n) \otimes_{B_n} k\mathbf{Br}(n, n+2i) \cong k\mathbf{Br}(n-2, n+2i)$$

Proof. The case $i = 0$ is trivial. We do $i = 1$ and leave the rest as an exercise. By repeated use of (18.4.10)(I)

$$k\mathbf{Br}(n-2,n) \otimes_{B_n} k\mathbf{Br}(n,n+2) \cong ek\mathbf{Br}(n,n) \otimes_{B_n} e'k\mathbf{Br}(n+2,n+2)$$

$$\cong e \otimes_{B_n} ee'k\mathbf{Br}(n+2,n+2) \cong ee'k\mathbf{Br}(n+2,n+2) \cong k\mathbf{Br}(n-2,n+2)$$

(the last step uses a mild generalisation of (18.4.10)(I)).

18.4.4 Simple head(Δ) conditions revisited using G-functors

We have already seen in (18.2.20) that $\Delta_n(\lambda)$ has simple head if v_λ may be chosen a primitive idempotent of kS_l ($\lambda \vdash l$). We now give a different approach, using the G-functors (following Green[57, §6.2]).

(**18.4.12**) Remark. Note that $\mathcal{S}(\lambda)$ is, in general, one of various non-isomorphic integral lattices in the corresponding $\mathbb{Q}S_l$-module (the contravariant dual is not isomorphic in general, for example). In this Section we always pass, eventually, to a ring containing $\mathbb{Q}$, where these forms become isomorphic. However if we wanted to pass to char.$p > 0$ our choice of $\mathcal{S}(\lambda)$ as starting point for 'growing' by the G-functor would be potentially arbitrary, and would merit review. (Although these modules do often provide a complete set of simple heads for kS_l.)

(**18.4.13**) Lemma. *Let $n > l \geq 0$, with $n-l$ even.. If either $l > 0$ or $\delta \neq 0$ then functor $F' := G^l_n$ (the* direct *functor as in (18.4.6)) is right adjoint to G^n_l, and $F'G^n_l(M) \cong M$ for any B_l-module M.*

Proof. We shall bundle the proof in with the following.

(**18.4.14**) Lemma. *Let $\lambda \in \Lambda^n$, $l = |\lambda|$. Suppose k a field such that Specht module $\mathcal{S}(\lambda)$ is simple in kS_l. Then for $n = l + 2m$, for any $m \geq 0$, we have:*
(I) $\Delta_n(\lambda)$ has simple head. (Accordingly write $L(\lambda)$ for the simple head of $\Delta_n(\lambda)$ for each $\lambda \in \Lambda^n$ (N.B. we have not shown here, yet, that these are distinct over $\mathbb{C}$).)

If in addition $l > 0$ or $\delta \neq 0$:
(II) The maximal proper submodule $Q(\lambda)$ of $\Delta_n(\lambda)$ obeys $F'Q(\lambda) = 0$. That is, every simple factor $L(\mu)$ of $Q(\lambda)$ obeys $F'L(\mu) = 0$.

(III) Thus, if $k = \mathbb{C}$, every simple factor $L(\mu)$ of $Q(\lambda)$ obeys $|\mu| > l$. (Caveat: This last statement assumes the restricted label set $\Lambda^{n,0}$ in case $\delta = 0$.)

Proof. The case of $n = l$ is trivial, since $\Delta_l(\lambda) = \mathcal{S}(\lambda)$. Otherwise $n = l + 2m$ for some $m > 0$, and $\Delta_n(\lambda) = G^n_l \Delta_l(\lambda)$ by Lemma 18.4.7. The basic idea here is to check that G^n_l is a G-functor between $B_l \cong E_m B_{l+2m} E_m$ and B_n, and hence that we can use Lemma 13.4.4 (which says roughly that if N is simple and $M \subset G(N)$ then $F(M) = 0$) and Proposition 13.3 from §13.4 (themselves derived from Green [57, §6.2]).

In the cases in which $l = |\lambda| > 0$ then G^n_l is an idempotent globalisation by Lemma 17.4.5. Thus (I) follows immediately; and (II) follows once we check that F is the right adjoint. For (III) note from (**??**) that F' is also a left inverse to G^n_l, that is $F'(G^n_l(M)) \cong M$, and that every $\Delta_n(\mu)$ with $\mu \vdash l$ can be expressed as $G(M)$ for non-vanishing M. That is $F'(\Delta_n(\mu))$ is non-vanishing, and hence — by exactness of F' and (II) for μ (which holds since $k = \mathbb{C}$):

$$\begin{array}{c|l} \downarrow F' & 0 \longrightarrow Q(\mu) \longrightarrow \Delta(\mu) \longrightarrow L(\mu) \longrightarrow 0 \\ & \\ & 0 \longrightarrow F'Q(\mu) = 0 \longrightarrow F'\Delta(\mu) \xrightarrow{\sim} F'L(\mu) \longrightarrow 0 \end{array}$$

— the image of its simple head (simple by (I) for μ) is also non-vanishing.

In the case $l = 0$ and δ a unit the result follows similarly using a different idempotent to construct the G-functor.
In the other cases (δ a non-unit, i.e. $\delta = 0$, and $l = 0$) we have a surjection

$$\Delta_n((2)) \to \Delta_n(\emptyset) \to 0$$

□

(**18.4.15**) Example. Consider $n = 4$. We have $\Delta_4(\lambda)$ for $\lambda = (4), (3,1), (2,2), (3,1,1), (1^4), (2), (1^2), \emptyset$ and $L_4(\lambda)$ for the same partitions. The lemma states that $\Delta_4(4)$ (say) is simple; and any composition factor of $\Delta_4(2)$ (say) below the head must be one of $L(4), L(3,1), ..., L(1^4)$.

generic ss 2 (**18.4.16**) Note that Lemma(18.4.14) holds for $k = \mathbb{Q}[\delta]$. Using the Lemma, we can strengthen the result in this case to show:

Proposition. For $k = \mathbb{Q}[\delta]$ the Δ-modules are simple.
Outline proof. Each algebra $B_n(\delta)$ is isomorphic to its opposite, and there is a form on each $\Delta_n(\lambda)$ providing a map to the contravariant dual. One finds that the corresponding Gram matrix determinant is non-zero in $\mathbb{Z}[\delta]$, so this map is an isomorphism over $\mathbb{Q}[\delta]$. But from (18.4.14) this map must take the simple head to the unique copy of that simple in the contravariant dual (we claim that simples are contravariant self-dual here, by character considerations), which is the simple socle. That is, everything below the head is in the kernel of the map. Since there *is no kernel*, the Δ-modules are simple. Indeed they are a complete set of simples over $\mathbb{Q}[\delta]$. (Note that this means that each Brauer algebra has the basic set-up of a 'splitting modular system'.) □

(**18.4.17**) Let A be a f.d. algebra over a field and $\mathcal{M}$ a set of modules over A. For each order of this set, and order of the simple modules of A, we have a *simple decomposition matrix*. This records the composition multiplicities of the set in the obvious way.

Note that if a collection of modules over A has an upper unitriangularisable simple decomposition matrix, then it is a basis for the Grothendieck group.

(**18.4.18**) In summary we have that the set of Δ-modules (excluding $\Delta_n(\emptyset)$ if $\delta = 0$ and $n > 1$) is a basis for the Grothendieck group for B_n over $\mathbb{C}$.

18.4.5 Simple modules revisited using G-functors

We saw in Lemma(18.3.15) a set of simple B_n-modules over $\mathbb{C}$. Once again, G-functors provide a different approach.

pr:simp1 gen k (**18.4.19**) Proposition. *CLAIM: The set* $\{head\ (\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ...\}$ *contains a complete set of simple modules for* $B_n(\delta)$ *over any field* k*, for any* δ*.*

Proof. TO DO!

pr:simp1 (**18.4.20**) Proposition. *The set* $\{head\ (\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ...\}$ *is a complete set of simple modules for* $B_n(\delta)$ *over* $k = \mathbb{C}$ *for any* δ*.*

Proof. See (18.4.14) above. Here is a quick review: To show that head $(\Delta_n(\lambda))$ is simple, we apply Prop. 18.4.10 to Prop. 18.4.9 (recall: every $\Delta_n(\lambda)$ is $G^{\circ m}\mathcal{S}(\lambda)$ for some m). If M is any proper submodule of $\Delta_n(\lambda)$ then $F^{\circ m}M = 0$ (F and hence $F^{\circ m}$ is exact, and the Specht module is simple over $\mathbb{C}$); thus there is a unique maximal submodule Q_λ (say) of $\Delta_n(\lambda)$.

The only case not covered by this argument is $\Delta_{2m}^{\delta=0}(\emptyset)$ $(m>1)$. Here apply right exact functor G^{m-1} to

$$0 \to \Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset) \to 0 \tag{18.16}$$ eq:d=0 d iso

and use that $G^{m-1}\Delta_2(2)$ has simple head.

Completeness follows from Prop. 18.3.10. □

(**18.4.21**) However regarded as a list this construction may give rise to multiple entries, depending on k and δ. Over the complex field there is no overcount with $\delta \neq 0$, and with $\delta = 0$ just the element $\lambda = \emptyset$ should be excluded (as shown by the case treated above).
This completes task (1).

18.5 Induction and restriction for $B_n \hookrightarrow B_{n+1}$

ss:BrauerIndRes

(**18.5.1**) A 'branching theorem' tells us how to restrict ordinary irreducible representations from a group to a given subgroup — in particular from S_n to S_{n-1}. If L is an S_n module, James [74] uses the notation $L \downarrow S_{n-1}$ for restriction and $L \uparrow S_{n+1}$ for the corresponding induction. We have the following.

th:specht res (**18.5.2**) THEOREM. [74, Th.9.3] *When S_n Specht module $\mathcal{S}(\lambda)$ is defined over an arbitrary field, $\mathcal{S}(\lambda) \downarrow S_{n-1}$ has a filtration by Specht modules. The factors are as in the ordinary case: each $\mu \triangleleft \lambda$ occurs once. Factor $\mathcal{S}(\mu)$ occurs above $\mathcal{S}(\nu)$ if $\mu > \nu$ in the dominance order.* □

For given n, let Ind− and Res− denote the induction and restriction functors associated to the injection $B_n \hookrightarrow B_{n+1}$. Recall $\mu \triangleleft \lambda$ if μ is obtained from λ by removing one box.

(**18.5.3**) PROPOSITION. *Fix δ. For given n:*
(i) We may identify the functors $Res\, G-\ = Ind-$ on B_n-mod to B_{n+1}-mod.
(ii) Over the complex field we have short exact sequence

$$0 \to \bigoplus_{\mu \triangleleft \lambda} \Delta_{n+1}(\mu) \to Ind\, \Delta_n(\lambda) \to \bigoplus_{\mu \triangleright \lambda} \Delta_{n+1}(\mu) \to 0$$

(excluding case $n = 2$, $\lambda = \emptyset$ in case $\delta = 0$).

Proof. (i) Unpack the definitions. The 'disk bijection' of sets

$$\mathbf{Br}(n+2, n) \cong \mathbf{Br}(n+1, n+1) \qquad (\cong \mathbf{Br}(2n+2, 0))$$

induces a bimodule isomorphism ${}_{n+1}\mathbb{B}(n+2,n)_n \cong {}_{n+1}\mathbb{B}(n+1,n+1)_n$ so

$$\mathrm{Res}\, G-\ = \mathrm{Res}^{n+2}_{n+1}\mathbb{B}(n+2,n) \otimes_n -\ = {}_{n+1}\mathbb{B}(n+2,n) \otimes_n -\ \cong \mathbb{B}(n+1,n+1)_n \otimes_n -\ .$$

(ii) Note from (i) and Prop. 18.4.9 that it is enough to prove the equivalent result for restriction (the excluded case comes from the issue with $G = G_2^4$ in case $\delta = 0$ as noted in §18.4.2). Use the diagram notation above. Consider the restriction acting on the first n strings. We may separate the diagrams out into those for which the $n+1$-th string is propagating (which span a submodule, since action on the first n strings cannot change this property), and those for which it is not. The result follows by comparing with diagrams from the indicated terms in the sequence, using the induction and restriction rules for Specht modules from Th.18.5.2. □

(**18.5.4**) REMARK. CAUTIONARY TAIL: As noted, induction does not quite always work like this for $\Delta_n^{\delta=0}(\emptyset)$. (However this is not an issue. (TO BE SHOWN))

18.6 Characters and Δ-filtration factors over ℂ

de:d=0caveat (**18.6.1**) PROPOSITION. Over the complex field the modules $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$ are pairwise non-isomorphic, except precisely in the case $n = 2, \delta = 0$ in (18.2.22).

Proof. Let us compare $\Delta_n(\lambda)$, $\Delta_n(\mu)$. If $|\lambda| \neq |\mu|$, consider first $|\lambda| > |\mu| > 0$. Then we can choose m so that $2m = n - |\mu|$, so that, by Lemma 18.2.7, the trace of the E_m action distinguishes the two modules.
In case $\mu = \emptyset$, $n > 2$, E_m is not defined but $E_{m-1}\Delta_n(\emptyset) \neq 0$, so the only issue is with $|\lambda| = 2$. But it is easy to see that the module ranks do not agree in this case.

If $|\lambda| = |\mu|$ then use right-exactness of Φ' and the inequivalence of $\mathcal{S}(\lambda)$ and $\mathcal{S}(\mu)$ over $\mathbb{C}$. □

(**18.6.2**) Can we strengthen this to pairwise distinct images in the Grothendieck group? All of the distinctions are actually distinctions at the level of characters, except for the last one. However, since we are working over $\mathbb{C}$ there are idempotents in $\mathbb{C}S_{|\lambda|}$ that distinguish the various Specht modules (i.e. for each λ an idempotent that is nonvanishing only acting on the corresponding module). Can we elevate this to work in B_n with $n > |\lambda|$?

Probably yes. Try the product of commuting idempotents along the lines used for the partition algebra in [108].

pr:duniq (**18.6.3**) PROPOSITION. Suppose $\delta \neq 0$ (that is, $k = \mathbb{C}$ with $\delta \in \mathbb{C}^*$). Then (I) the heads $\{L_n^k(\lambda)\}_{\lambda\in\Lambda^n}$ are pairwise non-isomorphic (and a complete set of simples, as already noted).
(II) There is a unique expression for any character in terms of Δ-characters.

Proof. (I) Under these assumptions (i) Green's Lemmas about G (§13.4.2) hold, and (ii) the Δs are all constructible as 'inflations' $G(S)$ of distinct simple Specht modules S. Thus they have simple heads.

Now suppose some simple head $L_n^k(\lambda) = L_n^k(\lambda')$ — then $F(\Delta_n(\lambda)) = F(\Delta_n(\lambda'))$ since, by Lemma 13.4.4, the maximal submodule is killed by F. But $\mathcal{S}_\lambda = F(\Delta_n(\lambda))$ so we have $\mathcal{S}_\lambda = \mathcal{S}_{\lambda'}$ and hence $\lambda = \lambda'$. Thus these heads are distinct.
They are a complete set of simples by (**??**).
(II) By (**??**) the maximal submodule of $\Delta_n(\mu)$ only has composition factors from $\{L_n^k(\lambda)\}_{|\lambda|>|\mu|}$. This means that the unique expression for a character in terms of simple characters determines the expression in terms of Δ-characters, since the coefficients are related via a matrix that is (suitably ordered) lower unitriangular.

COULD SAY MORE EXPLICITLY?... □

(**18.6.4**) Prop.18.6.3 implies that for $\delta \neq 0$ the Δ-filtration multiplicities for projectives, denoted $(P_i : \Delta_n(\lambda))$, are also uniquely defined.

18.6.1 Aside on case $\delta = 0$

This is an interesting case as an example with a 'non-square' Δ-decomposition matrix.

(**18.6.5**) For the case $\delta = 0$, when $n = 2$ (there is no idempotent inflation from $n = 0$ ensuring pairwise non-isomorphism of Δ-modules, and indeed) the isomorphism (18.16) means that Δ-filtration multiplicities are not uniquely defined. (Specifically for $n = 2$, we could simply discard one of the isomorphic modules to make them so.)

(**18.6.6**) CLAIM: For $\delta = 0$ and $n \neq 2$, we can proceed as follows. We remove the overcount for simple modules by using the index set $\Lambda^{n,0}$. This determines a labelling scheme for projectives (according to their simple heads, for which we have not yet given a construction, but which are the heads of the correspondingly labelled Δ-modules). We claim $\Delta_n(\lambda)$ can be chosen as the top section of a Δ-filtration of $P_n(\lambda)$ (for each suitable λ), and then the non-isomorphism of Δs removes the abovementioned ambiguity and Δ-filtration multiplicities are well-defined.

Proof. ...

While, in the notation (18.9) for sections, the $n = 2$ ambiguity gives

$$P_2(2) = \Delta_2(2)//\Delta_2(\emptyset) = \Delta_2(2)//\Delta_2(2) = ... \qquad (\delta = 0)$$

(as already noted), the sectioning of projectives in the block of $\Delta_n(\emptyset)$ up to $\lambda \vdash 4$ is indicated for $n \geq 4$ by

$$P_4(2) = \Delta_4(2)//\Delta_4(\emptyset) \qquad P_4(31) = \Delta_4(31)//\Delta_4(2) \qquad (\delta = 0)$$

The $n = 4$ case is an easy direct calculation: We have

$$P_3^0(1) = \Delta_3^0(1)$$

(the module on the right is evidently projective) so by Prop. 18.5.3

$$\mathrm{Ind}\, P_3^0(1) \;=\; \mathrm{Ind}\, \Delta_3^0(1) \;\cong \Delta_4^0(2) + \Delta_4^0(1^2) + \Delta_4^0(\emptyset)$$

Since there are Δ-filtrations of this starting with either $\Delta_4^0(2)$ or $\Delta_4^0(1^2)$ (again by Prop. 18.5.3), we deduce that

$$\mathrm{Ind} P_3^0(1) = P_4^0(2) \oplus P_4^0(1^2)$$

(note that the other Δ-module is not projective). We shall see shortly that $\Delta_4^0(1^2)$ and $\Delta_4^0(\emptyset)$ are not in the same block, so we are done.

(**18.6.7**) In this sense
CLAIM: we may treat $\delta = 0$ as a degeneration of the more general case, and treat the multiplicities $(P_i : \Delta_n(\lambda))$ as uniquely defined throughout. We do this hereafter.

(**18.6.8**) Exercise: Check: What happens for $\Delta_6(2)$? Here the G functor is well-behaved and we may proceed as in the general case treated in the next section.

ss:chars

18.6.2 The main case

pr:FGD

(**18.6.9**) From Prop.18.4.9 recall (for $(\delta, \lambda, n) \neq (0, \emptyset, 2)$)

$$G\Delta_n(\lambda) = \Delta_{n+2}(\lambda) \tag{18.17}$$

eq:GD=D

Further (using Lemma 18.2.7 and the construction of F)

$$F\Delta_n(\lambda) = \begin{cases} \Delta_{n-2}(\lambda) & |\lambda| < n \\ 0 & |\lambda| = n \end{cases}$$

(**18.6.10**) By Prop. 18.4.20 every B_n-module character can be expressed as a not necessarily non-negative combination of Δ-characters:

$$\chi(M) = \sum_\lambda \alpha_\lambda(M)\, \chi(\Delta(\lambda)) \qquad (\alpha_\lambda(M) \in \mathbb{Z}) \tag{18.18}$$

eq:combi char

This expression is unique if $\delta \neq 0$. It is also unique if $\delta = 0$ and we replace Λ^n by $\Lambda^{n,0}$ (however, as already noted, in this case modules with Δ-filtrations may be given well-defined *filtration multiplicities* over Λ^n).

de:filtx

(**18.6.11**) *Notation:* For A a k-algebra and $N = N_0 \supset N_1 \supset ... \supset N_l = 0$ a chain of A-modules we may write $(N_i)_\supset$ for the chain; and (given this)

$$(S_i)_{//} \;=\; //_i S_i \;=\; S_0//S_1//S_2//...$$

for the corresponding list of sections $S_i = N_i/N_{i+1}$. We may write $N = //_i S_i$ as an abbreviation for this property of N, or $N = +_i S_i$ if the datum of chain *order* is being forgotten. Now suppose $\{Q_\mu\}_{\mu \in X}$ are a collection of inequivalent modules such that each $S_i \cong Q_{\mu(i)}$ for some $\mu(i) \in X$; and define $c_\mu = \#\{j | \; S_j \cong Q_\mu\}$. Then we may write

$$N \;=\; +_\mu \, c_\mu \, Q_\mu$$

Note that this does not preclude other Q-filtrations with different multiplicities, unless for example the Qs are a basis for the Grothendieck group.

pr:GM-M

(**18.6.12**) Proposition. *Let M be a $B_n(\delta)$-module over $\mathbb{C}$ with given δ ($\delta \neq 0$ if $n = 2$), such that GM has a Δ-filtration. Then M has a Δ-filtration and*

$$(GM : \Delta_{n+2}(\lambda)) = \begin{cases} (M : \Delta_n(\lambda)) & |\lambda| \leq n & \text{(I)} \\ 0 & |\lambda| = n+2 & \text{(II)} \end{cases}$$

Further, suppose that $GM = (N_i)_\supset$ is a Δ-filtration with section sequence $GM = //_i\Delta_{n+2}(\mu(i))$ (so that $(GM : \Delta_{n+2}(\lambda)) \;=\; c_\lambda = \#\{j \mid \mu(j) = \lambda\}$) then no $\mu(i) \vdash n+2$ and there is a section sequence $M = //_i\Delta_n(\mu(i))$.

Proof. Let $GM = N_0 \supset N_1 \supset ... \supset N_l = 0$ be a chain giving a Δ-filtration (with $\Delta_i^+ = \Delta_{n+2}(\mu(i)) = N_i/N_{i+1}$ the i-th Δ-factor, say). We have, by the exactness of F, that $M = FGM = FN_0 \supseteq FN_1 \supseteq ... \supseteq FN_l = 0$ is a sequence of B_n-modules, and for each i

$$0 \to FN_{i+1} \to FN_i \to \underbrace{F(N_i/N_{i+1})}_{=F\Delta_i^+} \to 0$$

has either $F\Delta_i^+ = \Delta_i$ or $F\Delta_i^+ = 0$, by (18.6.9). Thus removing any zero terms gives a Δ-filtration of M with the same multiplicities for all Δ_i such that $F\Delta_i^+ \neq 0$. So far we have proved (I).

It remains to show that there *are* no zero terms. For Δ-filtration of M of length $l = 1$ we may use (18.17). Before doing the general case let us do one more low rank case.

(**18.6.13**) EXAMPLE: For length 2 we may write $M = \Delta_a//\Delta_b$, that is

$$0 \to \Delta_b \to M \to \Delta_a \to 0$$

is exact. Applying right-exact functor G we have

$$X \to G\Delta_b \to GM \to G\Delta_a \to 0$$

and hence

$$X \to \Delta_b^+ \xrightarrow{f} GM \xrightarrow{g} \Delta_a^+ \to 0$$

where Δ_a^+ denotes the Δ-module in B_{n+2}-mod (we may omit the + hereafter). We don't know X ab initio, but the sequence is exact at GM so im f = ker g. Exactness at Δ_a^+ tells us that GM has a filtration with top section Δ_a^+ so we may write it as $GM = \Delta_a^+//...$, so the ellipsis is ker g. But the domain of f is Δ_b^+, so this is the biggest (as k-space) that im f can be. On the other hand we know by (I) that one of the factors in the ellipsis is Δ_b^+, so im f can be no smaller. That is, im $f = \Delta_b^+$ (so $X = 0$) and $GM = \Delta_a^+//\Delta_b^+$. That is, there are no factors Δ_i^+ such that $F\Delta_i^+ = 0$.

(**18.6.14**) Now suppose WLOG that $M = M_0 \supset M_1 \supset ... \supset M_m = 0$ is the chain derived from the chain for GM by applying F then deleting improper inclusions. We can write $M_i/M_{i+1} = \Delta_i$ and hence

$$M = \Delta_0//\Delta_1//\Delta_2//...//\Delta_{m-1}$$

That is, a collection of short exact sequences:

$$0 \to M_{i+1} \to M_i \to \Delta_i \to 0$$

Applying G to this we get exactness of $GM_{i+1} \to GM_i \to G\Delta_i \to 0$ which we can extend to exactness of

$$0 \to X_i \to GM_{i+1} \xrightarrow{\phi} GM_i \to G\Delta_i \to 0$$

for suitable X_i, i.e. for $X_i = \ker\phi$. Note that it is not assumed that GM_i has a Δ-filtration. However as a k-space we have $\dim_k GM_i + \dim_k X_i = \dim_k G\Delta_i + \dim_k GM_{i+1}$ (a case of the $\sum_{odd} = \sum_{even}$ generalisation of the rank-nullity theorem to longer exact sequences) so $\dim_k GM_i \leq \dim_k G\Delta_i + \dim_k GM_{i+1}$ (with equality only if we can take $X_i = 0$).

(**18.6.15**) CLAIM: Let M be *any* B_n-module with Δ-filtration of length m as above. Then for $0 \leq i < m$,

$$\dim_k GM_i \;\leq\; \Sigma_{j=i}^{m-1} \dim_k G\Delta_j \tag{18.19}$$

eq:dim bound1

(with equality only if we can take every $X_i = 0$).
PROOF: By induction on i. The base case is $i = m-1$. It is true since $GM_{m-1} = \Delta_{m-1}$. Now assuming case $i+1$ we get

$$\dim_k GM_i \;\leq\; \dim_k G\Delta_i + \dim_k GM_{i+1} \;\leq\; \dim_k G\Delta_i + \Sigma_{j=i+1}^{m-1} \dim_k G\Delta_j$$

Done.

Now the last ($i = 0$) case of (18.19) bounds $\dim_k GM_0 = \dim_k GM$ above. On the other hand we know from (I) that GM contains *at least* the corresponding Δ-filtration factors. Thus the bound is saturated, and it contains no other factors. □

(**18.6.16**) Remark. Note that the saturation of the bound above forces all the $X_i = 0$. This says that every GM_i constructed in this way has a Δ-filtration.

In fact one may use the machinery of quasi-heredity to provide an alternative to the above proof; and also to show that if M has a Δ-filtration then so does GM (i.e. M has a filtration if and only if GM has).

18.6.3 The n-independence of $(P(\lambda) : \Delta(\mu))$

(**18.6.17**) We continue working over $\mathbb{C}$, with fixed δ. By Prop.18.4.10 the functor G takes projectives to projectives. It also preserves indecomposability, so

$$GP_n(\lambda) = P_{n+2}(\lambda) \tag{18.20}$$ eq:GP=P

Combining (18.20) with Lemma 18.6.12, using the Δ-filtration property of projectives (Prop.18.3.10) we see the following.

th:D n indep (**18.6.18**) Theorem. *Let $k = \mathbb{C}$ and fix δ. For any suitable pair $\lambda, \mu \in \Lambda$ the multiplicity $(P_n(\lambda) : \Delta_n(\mu))$, once defined (i.e. for sufficiently large n), is independent of n. That is, there is a semiinfinite matrix D independent of n, with rows and columns indexed by Λ, such that*

$$(P_n(\lambda) : \Delta_n(\mu)) \;=\; D_{\lambda,\mu}$$

for any indecomposable projective $B_n(\delta)$-module $P_n(\lambda)$. Further

$$(P(\lambda) : \Delta(\lambda)) = 1$$

and otherwise

$$(P(\lambda) : \Delta(\mu)) = 0 \text{ if } |\mu| \geq |\lambda|$$

That is, the matrix D is lower unitriangularisable. □

From this we have

pr:proj1 (**18.6.19**) Proposition. *If P is a projective module containing $\Delta(\lambda)$ with multiplicity m and no $\Delta(\mu)$ with $|\mu| > |\lambda|$, then P contains $P(\lambda)$ as a direct summand with multiplicity m.* □

de:dominance (**18.6.20**) Here we define the *dominance* order on Λ as the order $(\Lambda, >)$ given by $\lambda > \mu$ if $|\lambda| > |\mu|$.

The induction functor takes projective modules to projective modules, and has a behaviour with regard to standard characters determined by Prop. (18.5.3). From this we see that

prppwprighj2 (**18.6.21**) Proposition. *For e_i a removable box of λ,*

$$Ind\, P_{\lambda - e_i} \cong P_\lambda \bigoplus ...$$

Proof. By Prop.18.6.19 a projective module is a sum of indecomposable projectives including all those with labels maximal in the *dominance* order of its standard factors. □

(**18.6.22**) Hereafter we may write simply $//_i \Delta_i$ in place of a module that has $//_i \Delta_i$ as a Δ-section sequence (note that the module does not define a unique sequence in general, and neither does the sequence define a unique module). Indeed we may write $//_i N_i$ for any section sequence.

pr: ind //

(**18.6.23**) We have

$$\mathrm{Ind} //_i \Delta_i \cong\ //_i \mathrm{Ind} \Delta_i$$

Proof. Using $\mathrm{Ind}- = \mathrm{Res}\, G$, then prop.18.6.12 [9], then the exactness of Res, we have

$$\mathrm{Ind} //_i \Delta_i = \mathrm{Res}\, G //_i \Delta_i = \mathrm{Res} //_i G\Delta_i = //_i \mathrm{Res}\, G\Delta_i = //_i \mathrm{Ind}\, \Delta_i$$

□

(**18.6.24**) We shall further treat the notation $//_i N_i$ as a kind of non-commutative semigroup, so that $(//_i N_i)(//_j M_j)$ makes sense, and so on.

[9] This does not immediately cover the non-quasiherditary case AFAICS. Let's leave it as an exercise.

Chapter 19

Complex representation theory of the Brauer algebra

ch:BrauerIII

In this Chapter we start by describing the blocks of the Brauer algebra over the complex field (largely following [28],[29]). Then we describe the decomposition matrices.

19.1 Blocks of $B_n(\delta)$

In general the precursor to describing the blocks of an artinian ring is to describe the set of simple modules $\{L_\mu\}_{\mu\in Y}$, on which the blocks are an equivalence relation. In the case of a collection of modular systems [**?**, 16, 57] with the same integral ring (and hence the same 'ordinary/generic' representations), however, it may be neater to describe the set of 'generic irreducible' modules $\{V_\lambda\}_{\lambda\in X}$. Since the lift of an indecomposable projective is a sum of V-modules one knows that the composition factors of the modular reduction of V_λ all lie in the same block (even if it is not indecomposable, for example). Thus there is a well-defined 'block relation' on the V-modules. One assumes that there is an injection $Y \hookrightarrow X$, so that a relation on X restricts to the block relation on Y.

Fix δ and recall idempotent $E_1 \in B_n$ ($n > 2$) and algebra isomorphisms $E_1 B_n E_1 \cong B_{n-2}$ (Lemma 17.4.7) and $B_n/B_nE_1B_n \cong kS_n$ ((18.1.2)). The first tells us that for each simple $B_n(\delta)$-module L_λ either $E_1L_\lambda = 0$ or E_1L_λ is a simple $B_{n-2}(\delta)$-module. The second tells us that the simples with $E_1L_\lambda = 0$ are indexed by simple kS_n-modules. These depend on k, but an index set for generic irreducibles is Λ_n.

In light of the above, or of 18.4.16, we may use (a subset of) the labels $\lambda \in \Lambda$ for simple modules of $B_n(\delta)$ over a given field k with $\delta \in k$. Thus we say $\mu, \lambda \in \Lambda$ in the same block if there are simples $L(\mu)$ and $L(\lambda)$ in the same block. If we are working over $k = \mathbb{C}$ (or over a given field), we also write this as $L(\mu) \sim^\delta L(\lambda)$, or indeed $\mu \sim^\delta \lambda$.

Finally recall from (18.6.18) that this relation does not depend on n, once λ, μ make sense as labels (i.e. for sufficiently large n).

We now turn to the problem of describing this relation explicitly.

ss:BnBlockI

19.1.1 Blocks I: actions of central elements on modules

Suppose z a central element of an algebra A. Recall from §1.4.6 or (**??**) that Schur's Lemma implies that z acts like a scalar multiple of a unipotent operator on any indecomposable module M, and as a scalar if M is derived from a 'generic irreducible'; so z acts like the same scalar on any two such indecomposable modules in the same block. Thus central element actions can be good for distinguishing blocks.

Recall from §11.4.2 that Jucys-Murphy's elements of $\mathbb{Z}S_n$ are:

$$R_i = (1i) + (2i) + ... + (i-1\ i) \qquad (1 < i \leq n)$$

Jucys-Murphy-Nazarov (JMN) elements for $B_n(\delta)$ (certain analogues of Jucys-Murphy elements for S_n) may be defined as follows [128].

(**19.1.1**) For $k = 1, 2, ..., n$ define

$$j_k = \sum_{l=1}^{k-1} (U_{lk} - (l, k)) \quad \in B_n(\delta)$$

(cf. Equation(17.1)). It is easy to check that

$$[j_k, B_{k-1}(\delta)] = 0$$

for all k, so for all k, k'

$$[j_k, j_{k'}] = 0$$

Hereafter we assume for simplicity that 2 has an inverse in k. Define Jucys-Murphy-Nazarov (JMN) elements

$$x_k = \frac{\delta - 1}{2} - j_k$$

It follows that

$$\left[\sum_{k=1}^{n} (x_k)^l,\ B_n(\delta) \right] = 0$$

for all $l = 1, 2, 3, ...$ In particular

$$T'_n = \sum_{k=1}^{n} j_k$$

is central in $B_n(\delta)$. So by Schur's Lemma T'_n acts like a scalar on each $\Delta_n(\lambda)$.

(**19.1.2**) For $\lambda \in \Lambda^n$ with $\lambda \vdash l$ let $2t = n - l \in 2\mathbb{N}_0$. Then define

$$\chi_n(\lambda) \ = \ t(\delta - 1) - \sum_{d \in \lambda} c(d) \ = \ \left(n(\delta - 1) - \sum_{d \in \lambda} ((\delta - 1) + 2c(d)) \right) /2$$

where the sum is over boxes in Young diagram λ and $c(d)$ is the ordinary *content* (column position - row position) of box d.

pr:bschur **(19.1.3)** PROPOSITION. *For all* $y \in \Delta_n(\lambda)$

$$T'_n \; y \;=\; \chi_n(\lambda)\, y$$

Proof. See (19.1.7) below (or [46]).

(19.1.4) EXAMPLE. With $n = 2$ we have $\lambda \in \{(2), (1^2), \emptyset\}$. Let I denote the ideal generated by U_{12}. Consider $\lambda = (2)$ (so $t = 0$). We have $e_{(2)} + I \in \Delta_2((2))$ and

$$T'_2(e_{(2)}+I) = (U_{12}-(12))(e_{(2)}+I) = (U_{12}-(12))((1+(12))+I) = -(1+(12))+I = -c((2))(e_{(2)}+I)$$

as required. Similarly we have $e_{(1^2)} + I \in \Delta_2((1^2))$ and

$$T'_2(e_{(1^2)}+I) = (U_{12}-(12))(e_{(1^2)}+I) = (U_{12}-(12))((1-(12))+I) = (1-(12))+I = -c((1^2))(e_{(1^2)}+I)$$

Meanwhile for $\lambda = \emptyset$ ($t = 1$) we have $U_{12} \in \Delta_2(\emptyset)$ and

$$T'_2 U_{12} = (U_{12} - (12))U_{12} = (\delta - 1)U_{12} = (t(\delta - 1) - c(\emptyset))U_{12}$$

(19.1.5) Again by Schur's Lemma we have that λ, μ are in different blocks unless this eigenvalue agrees:

$$\lambda \sim \mu \qquad \Rightarrow \qquad \chi_n(\lambda) = \chi_n(\mu) \tag{19.1}$$ T'ev

Note the direction of the implication. This T' eigenvalue condition is not sufficient, as the following example shows.

(19.1.6) EXAMPLE. With reference to the example above we see, therefore, that (2), $\emptyset$ are in different blocks over $\mathbb{C}$ unless, possibly, $\delta = 2$; and that (1^2), $\emptyset$ are in different blocks unless, possibly, $\delta = 0$. We shall see shortly that (2), $\emptyset$ are indeed in the same block when $\delta = 2$, for all n. But now compare the actions of another central element in case $n = 2$, the element $(U_{12} + (12))$,:

$$(U_{12} + (12))\; (e_{(1^2)} + I) \;\;=\;\; (U_{12} + (12))((1 - (12)) + I) = -(1 - (12)) + I = -(e_{(1^2)} + I)$$

and

$$(U_{12} + (12))U_{12} = (\delta + 1)U_{12}$$

We see that the scalars no longer agree, thus (1^2), $\emptyset$ are in different blocks, when $\delta = 0 \in \mathbb{C}$.

proof:bschur **(19.1.7)** A proof of Prop.19.1.3 is as follows.
First we recall Murphy's elements of S_n:

$$R_i = (1i) + (2i) + ... + (i - 1\; i) \qquad (1 < i \leq n)$$

Murphy [127] (see also, e.g. Green–Diaconis [38]) computed the representation matrices for these elements in Young's seminormal form of each Specht module $\mathcal{S}_\lambda$:

$$\rho_\lambda(R_i) = \; diag(c_1(i), c_2(i), ...)$$

where $c_l(i)$ is the content of the box containing i in the l-th standard Young tableau of shape λ (in some chosen total order of tableau, which will not be important to us).

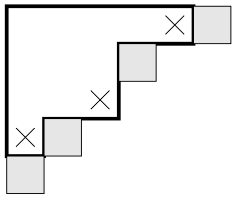


Figure 19.1: Schematic for the Δ-induction rule. fig:induce001

Since $\sum_i R_i$ is central it acts like a scalar, so we only need the first diagonal entry to determine this scalar. This is, then, the sum of the contents of all the boxes in the 'first' (or indeed any) standard tableau of shape λ. That is

$$\rho_\lambda(\sum_i R_i) = \ (\sum_{b\in\lambda} c(b))I$$

where the sum is simply over the contents of boxes of λ; and I is the identity matrix.

Now consider the action of T'_n on an element of $\Delta_n(\lambda)$. For example consider the element of form $X = \cup^{\otimes t} \otimes w$ where w is a basis element of $\mathcal{S}_\lambda$. We are only interested in the summands of T'_n that take X to itself (since we know it acts like a scalar). These are U_{12}, U_{34} and so on (t summands); together with the corresponding $-(12)$, $-(34)$ and so on, altogether making a contribution of $t(\delta-1)$; and then various $-(ij)$s that act on the factor w. The $-(ij)$s that act diagonally here correspond to (minus) a Murphy $\sum_i R_i$ for the kS_l-subalgebra that acts non-trivially only on the w. Thus they act like $-\sum_{b\in\lambda} c(b)$. Altogether we have

$$T'_n X \ = \ \left(t(\delta-1) \ - \ \sum_b c(b) \right) X$$

as required. □

19.1.2 Easy Lemmas and the DWH Lemma

Consider Figure 19.1, which recalls schematically the addition and removal of boxes from a given Young diagram. We note from this the following.

le:little block **(19.1.8)** LEMMA. *If e_i, e_j are two distinct addable boxes to $\lambda \in \Lambda$ then they do not have the same content.*

Thus, over $\mathbb{C}$, for any δ, we have $\chi_n(\lambda+e_i) \neq \chi_n(\lambda+e_j)$ and hence the two diagrams are in different blocks.

We have the same rule for removable boxes. We can also see that for each $\lambda + e_i$ there is *at most* one $\lambda - e_j$ such that $\chi_n(\lambda+e_i) = \chi_n(\lambda-e_j)$, and that this j, if any, will depend on δ.

(19.1.9) Note that

$$\chi(\lambda,\mu) := \chi_n(\lambda) - \chi_n(\mu)$$

does not depend on n (although it does depend on δ).

There is an important Lemma of DHW [46, Th.3.4] (see also [28, Th.4.4]):

le:DWH **(19.1.10)** LEMMA. *[46, 28] Fix δ. There is a map $\Delta_n(\lambda+e_i) \to \Delta_n(\lambda-e_j)$ if and only if the T' eigenvalue condition is satisfied (i.e. $\chi(\lambda+e_i, \lambda-e_j)=0$) and $\lambda+e_i/\lambda-e_j$ is not of shape (1^2).*

Proof. Outline: The eigenvalue condition is necessary for a map by (19.1). See the source references for sufficiency. □

Let us call the condition { $\chi(\lambda+e_i, \lambda-e_j)=0$ and $\lambda+e_i/\lambda-e_j$ not of shape (1^2) } the *short*-condition.

pr:nonsplit **(19.1.11)** PROPOSITION. Fix δ and let $\lambda, \lambda+e_i, \lambda-e_j \in \Lambda$. Then for all suitable n,
(i) $\dim \mathrm{Hom}(\Delta_{n+1}(\lambda+e_i), \mathrm{Ind}\Delta_n(\lambda))=1$;
(ii) Consider the short exact sequence

$$0 \to \Delta_{n+1}(\lambda-e_j) \to \mathrm{Proj}\,\mathrm{Ind}\Delta_n(\lambda) \to \Delta_{n+1}(\lambda+e_i) \to 0$$

where Proj$-$ denotes projection onto the union of not-necessarily distinct blocks in question. This SES is non-split if and only if $\chi(\lambda+e_i, \lambda-e_j)=0$ and $\lambda+e_i/\lambda-e_j$ is not of shape (1^2).

Proof. Evidently if $\Delta_{n+1}(\lambda+e_i)$ (or indeed any other summand) is the only element of its block in the filtration of $\mathrm{Ind}\Delta_n(\lambda)$ then we have $\dim \mathrm{Hom}=1$. The other possibility is that there is a $\Delta_{n+1}(\lambda-e_j)$ in the same block.

We take (i) and (ii) together, and proceed by induction on $|\lambda|$. Assume then that $\dim \mathrm{Hom}(\Delta_n(\mu+e_i), \mathrm{Ind}\Delta_{n-1}(\mu))=1$ for all μ with $|\mu|<|\lambda|$ for all $\mu+e_i \in \Lambda$, for every possible n (this is trivially true for $\mu=\emptyset$ for example, since $\mathrm{Ind}\Delta(\emptyset)=\Delta((1))$; and automatically true in general when there is no block partner to $\mu+e_i$ in the induction). Then in particular we assume that $\dim \mathrm{Hom}(\Delta_n(\lambda), \mathrm{Ind}\Delta_{n-1}(\lambda-e_j))=1$ for any valid e_j.

Using $\mathrm{Ind} \cong \mathrm{Res}G$ and Frobenius reciprocity respectively we have for any $\mu, \mu+e_i$ and any suitable n:

$$\mathrm{Hom}(\Delta_n(\mu+e_i), \mathrm{Ind}\Delta_{n-1}(\mu)) \cong \mathrm{Hom}(\Delta_n(\mu+e_i), \mathrm{Res}G\Delta_{n-1}(\mu)) \cong \mathrm{Hom}(\mathrm{Ind}\Delta_n(\mu+e_i), G\Delta_{n-1}(\mu))$$

In particular

$$\mathrm{Hom}(\Delta_n(\lambda), \mathrm{Ind}\Delta_{n-1}(\lambda-e_i)) \cong \mathrm{Hom}(\mathrm{Ind}\Delta_n(\lambda), \Delta_{n+1}(\lambda-e_i)) \tag{19.2}$$ eq:GFR

Thus by (19.2) and the inductive assumption we have

$$\dim \mathrm{Hom}(\underbrace{\mathrm{Ind}\Delta_n(\lambda)}_{\Delta_{n+1}(\lambda-e_j)+\Delta_{n+1}(\lambda+e_i)+\ldots}, \Delta_{n+1}(\lambda-e_j))=1.$$

If the SES above does not split then the head of $\mathrm{Proj}\mathrm{Ind}\Delta_n(\lambda)$ is the head of the factor $\Delta_{n+1}(\lambda+e_i)$. Thus our map here must factor through a map $\Delta_n(\lambda+e_i) \to \Delta_n(\lambda-e_j)$ (indeed $\dim \mathrm{Hom}(\Delta_n(\lambda+e_i), \Delta_n(\lambda-e_j))=1$). But by Lemma 19.1.10 this implies the short-condition. In other words, SES non-split implies short-condition. If the SES does split then there is an isomorphism present, and there can be no futher map $\Delta_n(\lambda+e_i) \to \Delta_n(\lambda-e_j)$, so the short-condition does not hold.

We conclude that (ii) holds. From (ii), using Lemma 19.1.10 again, we have that

$$\dim \mathrm{Hom}(\Delta_{n+1}(\lambda+e_i), \mathrm{Ind}\Delta_n(\lambda))=1$$

i.e, we have (i). This completes the inductive step. □

(**19.1.12**) Remark. We showed earlier that $\lambda = \emptyset$ and (1^2) are always in different blocks, irrespective of $\chi(\emptyset, (1^2))$,. However we have not here shown the following claim: that $\lambda + e_i, \lambda - e_j$ with $\lambda + e_i / \lambda - e_j$ of shape (1^2) are in different blocks, even when $\chi(\lambda + e_i, \lambda - e_j) = 0$, in general. (Even though we have quoted Lemma 19.1.10 showing that there is no map between the corresponding Δ-modules.) However, this claim is also true. See [28] for a proof in involving Littlewood–Richardson coefficients.

19.2 Blocks II: δ-balanced pairs of Young diagrams

Section 19.1.1 gives a general indication of how it is that the block structure of B_n comes to depend on the relative content of the labelling Young diagrams. See [28] for complete details.

It will be convenient now to cast the appropriate content condition for blocks in various forms.

Aside on the original definition of 'δ-balanced pairs' in [28]

From [28] Defn.4.7 we have a definition of the term '*δ-balanced* pairs' of Young diagrams (we use inverted commas here since we shall employ, at least implicitly, the version on conjugate partitions, possibly without further comment). This is effectively as follows.

(**19.2.1**) Fix δ. From [28], a pair $\mu \subset \lambda \in \Lambda$ are 'δ-balanced' (call the relation $(\Lambda, \triangleleft)$) if
(i) there exists a pairing of boxes in λ/μ such that the content of each pair sums to $1-\delta$; and
(ii) if δ even and the boxes in λ/μ with content $-\delta/2$ and $(2-\delta)/2$ can be paired up in columns, then the number of such columns is even.

For example, the pair $\boxed{0}\boxed{1} \supset \emptyset$ is '0-balanced' in [28]. (Note that the *transposed* skew is a column and so does not have a fixed row in π-rotation, and so is a δ-pair in the sense of (19.2.18).)

le:balanced orig1

(**19.2.2**) Finally, *a general pair λ, μ are 'δ-balanced'* if each is 'δ-balanced' with $\lambda \cap \mu$.

Note what this construction is saying. The $\mu \subset \lambda$ part of the definition is an antisymmetric relation on Λ — although the ordered aspect is not stressed. One can show that it is transitive (cf. (19.2.7)). The general part extends this not by taking RST closure ('μ, λ are balanced if there is an undirected path between') but by connections $\mu \sim \lambda$ requiring a certain specified (maximum 2-step) path between. In fact this *agrees* with RST closure, but this is not obvious (it is, nominally, potentially less connective). Agreement follows if we can show that if λ, μ comparable then $\lambda \cap \mu$ is a GLB in $(\Lambda, \triangleleft)$. If it is a LB it is clearly maximal, so it remains to show that $\lambda \cap \mu \triangleleft \lambda, \mu$.

(**19.2.3**) In [28] Def.6.1 we then have, for $\lambda \supset \mu$ a δ-balanced pair, a definition (in effect) of a *maximal δ-balanced subpartition* μ^i (say) of λ between μ and λ, as a subpartition of λ that is maximal (with respect to the subpartition order) among those δ-balanced with λ and containing μ. Note that the role of μ in this formulation is simply to ensure that λ *has* a δ-balanced subpartition. (In [28] the formulation is slightly more involved, leading to an algorithmic construction for μ^i which makes use of μ.)

It will be useful next to develop a more direct constructive approach to these objects.

19.2.1 Towards a constructive treatment: δ-charge and δ-skew

(**19.2.4**) Here we define the *δ-charge* of a box b in a Young diagram as

$$chg(b) \; := \; \delta - 1 - 2c(b)$$

(in the literature this is sometimes called *δ-conjugate-charge*). For λ a Young diagram or skew

$$chg(\lambda) \; := \; \sum_{b \in \lambda} chg(b)$$

(a)

1	−1

(b)

−1	−3
1	−1
3	1

Figure 19.2: Examples of skews with charges. Case (a) is not a δ-skew. `fig:examples-charge-pairs`

Note that the T' eigenvalue condition (19.1) now becomes

$$\chi_n(\lambda) - \chi_n(\mu) \;=\; chg(\lambda^T) - chg(\mu^T) \;= 0$$

Thus $\mu \subset \lambda$ is in the same block only if $chg(\lambda^T/\mu^T) = 0$ (and in fact only if λ^T/μ^T contains $\pm$charge pairs). For example, with $\delta = 2$ the skew $(2^2)/(1^2)$ contains ± 1, so potentially (and in fact, as we already found in (18.7)) we have $(2^2) \sim^{\delta=2} (2)$.

Remark: Just as for content, the lines of constant δ-charge run parallel to the main diagonal. The key difference from content is that the line of δ-charge 0 for given δ is no longer (unless $\delta = 1$) the main diagonal itself. That is, the δ-charge-0 main diagonal is shifted from the ordinary main diagonal of the Young diagram. (Indeed for δ even there are no boxes with charge 0, so the charge 0 line lies 'between' diagonal runs of charge +1 and charge -1 boxes.)

(**19.2.5**) A pair $\lambda \supset \mu \in \Lambda$ is δ-*flat* if
(i) the boxes of λ/μ can be put into pairs such that the sum of δ-charges in each pair is zero;
(ii) if there is such a pairing of boxes in which each +1,-1 pair has the two boxes side-by-side, then the number of these side-by-side pairs is even.

If, for given δ, a pair $\lambda \supset \mu$ is δ-flat we shall say that skew λ/μ is a δ-skew.

(**19.2.6**) Examples: Consider the skews with charges in Figure 19.2. Case (a) is not a δ-skew. Case (b) is a δ-skew (there is no pairing in which every +1,-1 pair is side-by-side).

`lem:pst1` (**19.2.7**) Fixing δ, we may define a relation $(\Lambda, \leq_\delta)$, refining $(\Lambda, \subseteq)$, by $\mu <_\delta \lambda$ if λ/μ is a δ-skew.

LEMMA. The relation $(\Lambda, \leq_\delta)$ is a poset.

Proof. It will be clear that this relation is antisymmetric. We can see that it is also transitive as follows. Condition (i) is clearly transitive. Now suppose $\lambda >_\delta \mu >_\delta \nu$ (i.e. $\mu <_\delta \lambda$ and so on) and that λ/ν has a pairing of boxes with each +1,-1 pair side-by-side — we need to show that the number of these is even. First note that this precludes the existence of a non-side-by-side -1 in μ/ν, and hence in λ/μ. But then both of these are 'even' and we are done. □ [1]

1

(**19.2.8**) LEMMA. *(I) Each connected component of $(\Lambda, \leq_\delta)$ has a unique minimum element. (II) Any distinct pair of form $\lambda + e_i, \lambda + e_j$ are in different components.*

Proof. (II) Suppose $\mu, \nu <_\delta \lambda$. Then λ/μ and λ/ν are both δ-skews.
...
proBABLY NEED δ-pair to prove THIS!!! (see [100] at the mo).

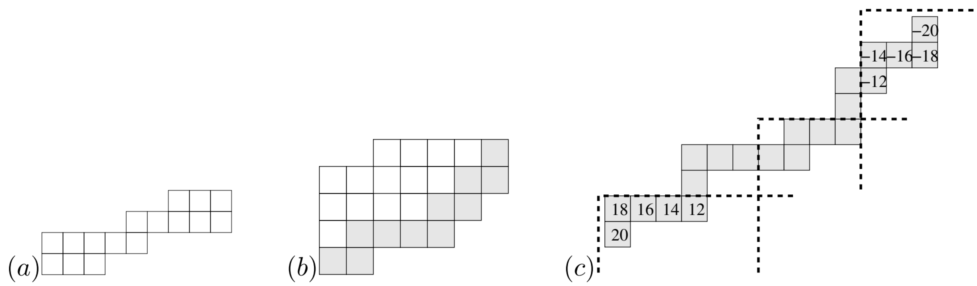


Figure 19.3: (a) Concave skew. (b) Skew outer rim construction. (c) Some removable subrims of
fig:outerrim1 a rim.

19.2.2 Sections and rims of Young diagrams

de:mibs01 (**19.2.9**) Let $\lambda \supset \mu^1 \supset ... \supset \mu^l$ be any chain of Young diagrams. Then the skew μ^i/μ^{i+1} is a *skew-section* of this chain.

(**19.2.10**) Fix δ. Suppose there is a chain $\lambda \supset \mu^1 \supset ... \supset \mu^l = \mu$ with each skew-section a δ-skew. Then we say a δ-skew λ/μ has a δ-*section* μ^i/μ^{i+1}.

(**19.2.11**) The *width at content (or charge)* c of a skew is the number of boxes of content c. The *(maximum) width* of a skew is the maximum number of boxes of any given content (or charge).

(**19.2.12**) We say that a skew is *concave* if the sequence of widths from left to right has a (not necessarily consecutive) subsequence 212.
Example: Fig.19.3(a).

(**19.2.13**) A *rim* is a skew shape that is a chain, i.e. a connected skew with no (2^2) subset. (Equivalently a rim is a connected skew shape of width 1.)

(**19.2.14**) For a connected skew $\gamma = \lambda/\nu$ define the *outer rim* $\Omega(\gamma)$ as the collection of the bottom-right-most box of each content present. (NB this is also the bottom-right-most box of each charge present, for any δ.)

An example of a skew with outer rim shown shaded is given in figure 19.3.

19.2.3 A constructive treatment: π-rotations and δ-pairs

de:mibs1 (**19.2.15**) Two non-intersecting rims are δ-*opposite* if there is a point x on the δ-charge-0 diagonal (see e.g. Figure 19.5) and a rotation by π radians of the plane about x, denoted π_x say, that takes one rim into the other.
(Evidently this rotation is the same as reflection in the vertical line defined by the point of rotation; followed by reflection in the horizontal line defined by this point.)

(**19.2.16**) Note that any such π-rotation is necessarily about a point positioned as shown in one of the cases in Figure 19.4.

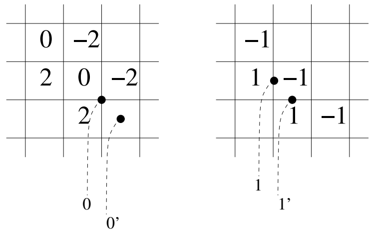


Figure 19.4: Possible π-rotation points. fig:pipoints

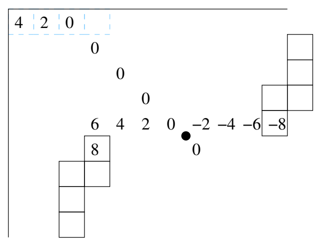


Figure 19.5: A π-rotation of rims in case $\delta = 5$. fig:rimrotat1

pa:ex pairs (**19.2.17**) Note further that such a rotation has the effect of exchanging boxes in specific pairs, that are $\pm$charge pairs. Example (rotation of rims about the black dot shown): Figure 19.5. In this case the position of the charge-0 diagonal corresponds to $\delta = 5$.

de:MiBS (**19.2.18**) For given δ, a δ-*pair* is a skew that is a δ-opposite pair of rims such that no row of the skew is fixed by the associated π-rotation.

Examples: See Figure 19.6. (Remark: Note from Figure 19.6(iii) that there may be more than one way to realise a δ-pair as a δ-opposite pair of rims.)

(**19.2.19**) LEMMA. *Every δ-pair is a δ-skew.* $\square$

19.2.4 Conditions for a width-1 δ-skew to have a section

le:or rem (**19.2.20**) LEMMA. Let b_1, b_2 be boxes in a rim γ. Then the subrim from b_1 to b_2 is removable if either of the following (equivalent) conditions are satisfied.
(i) Consider traversing the rim starting from the bottom-left box, facing to the right. Then b_1 is between a right and a left turn (or before the first left-turn) and b_2 is between a left and a right-turn.
(ii) The intersection of the subrim with the set of the removable and the co-removable boxes of γ is an alternating sequence starting and ending with removable boxes; or bounded by a removable

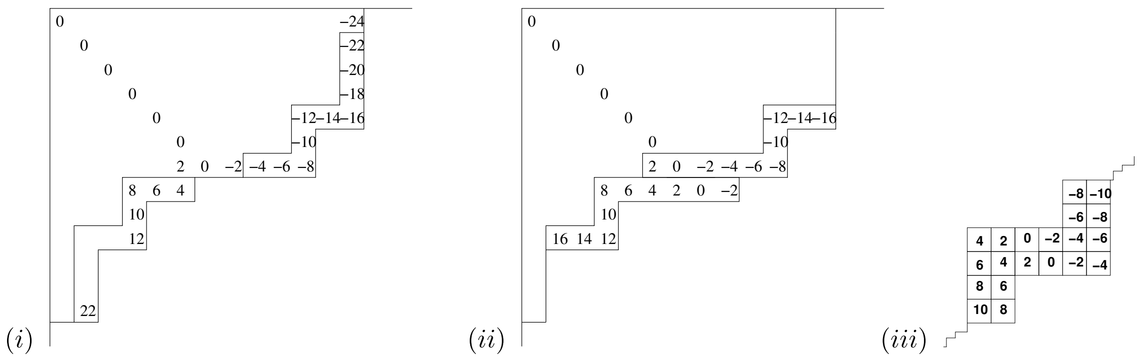


Figure 19.6: Examples of δ-pairs. fig:minskew eg

box at one end and by the end of the rim at the other.
(Note that there is an obvious corresponding co-removable subrim condition.)

Proof. Outline: Consider Fig.19.3(b). □

Example: in Fig. 19.3 the first two boxes are removable; the first three are not.
Remark: Of course there is a corresponding Lemma to (19.2.20)(i) got by *reflecting* in the main diagonal, i.e. with bottom-left and top-right exchanged and so on.

le:iw-1=mibs (**19.2.21**) Lemma. *A width-1 δ-skew either has a section or is a δ-pair.*

Proof. Outline: Let γ be a width-1 δ-skew. There is a rotation taking the outermost $\pm$charges into each other. Looking from these outermost charges in, we claim that the first point of departure from rotational symmetry results in a removable or co-removable section. To see this (schematically) consider Fig.19.3(c), and in particular the departure from symmetry at charge ± 12. It remains to show that no row is fixed — but a fixed row would force a single side-by-side +1,-1, contradicting the δ-skew condition. □
[2] [3]

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[2]ALTERNATIVELY:

(**19.2.22**) Note that a rim γ has a subset of boxes where it changes direction (we will include the outer two boxes in this). We may order these boxes as a sequence $\mathfrak{p}(\gamma)$ by the order they appear in traversing the rim from the bottom-left. Then (excluding the case of the singleton 'rim') they alternate between the list of removable and co-removable boxes of the rim, starting with a removable one if the first row is only length-1.

(**19.2.23**) Fixing δ we can define a certain sequence $\mathfrak{p}_\delta(\gamma)$ from $\mathfrak{p}(\gamma)$ as follows. First write out the sequence of charges in $\mathfrak{p}(\gamma)$. Then bracket each number corresponding to a co-removable box (thus alternate numbers are bracketed, and the bracket arrangement is determined by whether or not the first number is bracketed).

(**19.2.24**) CLAIM: A width-1 δ-skew γ has a section unless it is fixed under a rotation (or, equivalently, the elements of the sequence $\mathfrak{p}_\delta(\gamma)$ can all be paired: $\{(x), -x\}$).

Proof: Let $(x), -y$ be the outermost unmatched pair in the sequence, so we have $..., z, (x), ..., -y, (-z), ...$ If $x > y > 0$ then $[x, -] \cup [-, -x]$ is co-removable and hence a section. Other cases are similar. □

[3]OLD: (**19.2.25**) A box b together with a sequence $(s_1, s_2, ..., s_l)$ from $\mathbb{N}$ (s_1 can be zero) defines a rim set as

le:w1 (**19.2.26**) LEMMA. *If a δ-skew is not connected then it has a section of width 1.*

Proof. It is enough to consider a two component skew. Call the components $\gamma_{\pm}$. Consider the outer rim of γ_+. This visits each positive charge in the skew once, and is removable. The corresponding rim of γ_- has a balancing set of charges, and is also removable, hence together they are a section. □

le:concave (**19.2.27**) LEMMA. *If the width of a skew goes from 2 to 1 in traversing from left to right then the outer rim has a removable subrim at that point (up to and including the last width 2 charge). The same holds from right to left.*

Thus a concave δ-skew has a removable width-1 section.

19.2.5 Connections and properties of δ-skews and δ-pairs

le:w2 (**19.2.28**) LEMMA. *A δ-skew has a section of width at most 2.*

Proof. We construct such a section. (By (19.2.26) and (19.2.27) it is enough to consider the connected non-concave case.) Consider the part of the outer rim from the positive end to the last box in the set of consecutive diagonals, moving in the -ve direction from the charge-0 diagonal, with width ≥ 2 (NB there may be none such, in which case stop at $+1$). This rim visits all charges in this range and is removable. Remove this and consider, for what remains, the outer rim from the -ve end to the corresponding stopping point on the +ve side (NB this point is reachable since the original width to here, as it were, was ≥ 2, by the balance condition). This rim is removable, so the combined skew is removable from the original; and the two rims are a δ-skew by construction, with a possible matching of $\pm$-charges such that each pair intersects both rims. □

For later use we will again call the components of this contruction γ_+ and γ_-. We will call such a skew *convex*.

(**19.2.29**) LEMMA. *A convex δ-skew has a section or is a δ-pair.*

Proof. The idea is similar to (19.2.21). First note that a convex skew is determined by the outer rim, together with the datum of where the width-2 part starts — since the remainder, what we can call the *inner rim*, is simply a parallel copy of the same rim, shifted inward. This is illustrated in Fig.19.7. Note also that the outer rim itself is largely made up of $\pm$charge pairs, and only fails to be a section if it has either a single charge-0 box or a side-by-side ± 1 pair. (The same holds for the inner rim.) It follows that we can remove a front section from the outer rim if one presents itself, precisely in the way of Lemma 19.2.21. Fig.19.7 provides an example of this — the dashed lines indicate the section. Note on the other hand that the conditions (from Lemma 19.2.21) for a co-removable section are not generally directly relevant here, since the presence of the inner rim prevents any co-removal. (Recall that it is these two sets of conditions together which force the rotation invariance property in the width-1 case.) However we can co-remove a back section from the *inner* rim if the co-removable section conditions are satisfied. Since the paths of the inner and outer rims are parallel, the overall effect is that there is a section unless every turn in the positive

follows. First draw a row of length s_1 starting from b, to b' say (if $s_1 = 0$ then $b = b'$); then a column of length s_2 starting from b'; and so on.

For each rim define a sequence of its boxes as follows. First number the boxes from the bottom-left. Then $s_1 = b_1$, $s_2 = b_l$

A rim is determined by its sequence of turning points, as follows. If we fix δ then an integer fixes a box up to

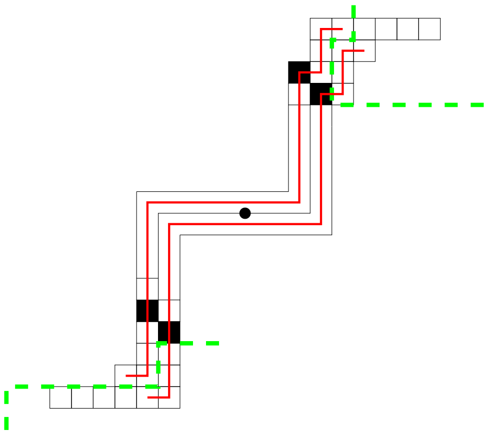


Figure 19.7: A convex δ-skew with a section. The dot is the rotation point for the outer charges. The red lines show the outer and inner rims.

fig:Matchpath4

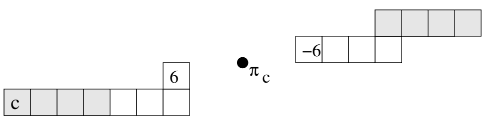


Figure 19.8: A δ-skew of width 1 with c-connected component of $\gamma_+ \cap \pi_c(\gamma_-)$ shaded. fig:width1-1

charge part is matched by one in the negative part — i.e. if the original skew is rotation invariant, and hence a δ-pair.

□

co:BScontainsMiBS (**19.2.30**) Corollary. Every δ-skew has a δ-pair section and hence a δ-pair filtration.

Proof. It is enough to consider the skews described in the proofs of (19.2.26) and (19.2.28). That is, it is enough to consider the disconnected-width-1 and convex cases. We have shown that these all either have sections or are δ-pairs. Since any sectioning reduces the size of components to be considered, we are done by iteration (all rank-2 cases are obviously δ-pairs).

□

Note that this does not say that "no δ-pair has a proper section". However we shall see in (19.2.34) that this is also true.

le:o12 (**19.2.31**) Lemma. *Possible π-rotation points for a δ-pair are of the forms shown in Figure 19.4. In case-$0'$ there can be no intersection of the skew with the row or column containing the point. In case-1 there can be no intersection of the skew with the row containing the point. Hence in either of these cases the skew is disconnected.* □

le:o2 (**19.2.32**) Lemma. *(Pinning Lemma) Let π_x be a rotation as above, and b, b' two boxes comparable in the light-cone order (5.7.6), then*

$$b' > b \qquad \Rightarrow \qquad \pi_x(b) > \pi_x(b')$$

□

19.2.6 The graph $G_\delta(\lambda)$

(**19.2.33**) Fix δ. Define a relation $(\Lambda, \leftarrow^\delta)$ by $\mu \leftarrow^\delta \lambda$ if λ/μ is a δ-pair. Define $(\Lambda, \leq^\delta)$ as the partial order that is the reflexive transitive closure of this relation.

pr:cov1 (**19.2.34**) PROPOSITION. *Fix δ.*
(I) If $\mu \subset \lambda \in \Lambda$ and λ/μ a δ-pair, then there is no $\mu \subset \mu' \subset \lambda$ such that μ'/μ is a δ-pair.
(II) The relation $(\Lambda, \leftarrow^\delta)$ is the cover of the partial order $(\Lambda, \leq^\delta)$.
(III) The relations $(\Lambda, \leq^\delta)$ and $(\Lambda, \leq_\delta)$ coincide.

Proof. (I): Let π_0 be a π-rotation fixing λ/μ and suppose (for a contradiction) that a rotation π_γ fixes $\gamma = \mu'/\mu \subset \lambda/\mu$.

The positive charge part of λ/μ is connected, so there exists a $b' \in \lambda/\mu'$ (i.e. in λ/μ and not in γ) adjacent to some $b \in \gamma$. Thus $\pi_0(b')$ lies in λ/μ adjacent to $\pi_0(b)$. Since λ/μ' is a skew over μ', we have $b' \not\leq b$ and hence (since adjacent) $b' > b$. Thus $\pi_0(b) > \pi_0(b')$ by Lemma 19.2.32.

Suppose for a moment that $\pi_0 = \pi_\gamma$ (i.e. they are rotations about the same point). Then $\pi_0(b') < \pi_\gamma(b)$, contradicting that γ is a skew over μ. Thus $\pi_0 \neq \pi_\gamma$.

Now, since $\pi_0 \neq \pi_\gamma$, π_0 fixes no pair $b, \pi_\gamma(b)$ in γ. That is, for each $b \in \gamma$ the boxes $b, \pi_\gamma(b), \pi_0(b), \pi_0(\pi_\gamma(b))$ are four distinct boxes in λ/μ. Observe also that b and $\pi_0(\pi_\gamma(b))$ have the same charge (and so lie in the same diagonal). Note that no charge can occur more than once in a rim or (therefore) more than twice in any MiBS. Thus for example no charge appears more than once in γ, while all the charges appearing in γ appear twice in λ/μ. Thus in particular λ/μ is connected. Note that the rotation point of π_0 is necessarily half a box down and to the right of π_γ. It then follows from Lemma 19.2.31 that γ_+ and γ_- are disconnected from each other.

Let c be the lowest charge box in γ_+. The box $\pi_0(\pi_\gamma(c))$ is below and to the right of it. Thus there is a box of λ/μ to its immediate right — call it d. There cannot be a box of λ/μ above c (since γ is a skew over μ), but there must be another box of λ/μ in the same diagonal as d (since λ/μ is connected), so the second box of λ/μ in the same diagonal as d would have to be to the right of $\pi_0(\pi_\gamma(c))$. But the π_0 image of *this* is to the left of $\pi_\gamma(c) \in \gamma$, contradicting the γ skew over μ property.

Claim (II) follows from (I) since $\mu \subset \lambda$ is a necessary condition for $\mu <^\delta \lambda$ so any failure of the δ-pair relation to be a transitive reduction implies the existence of a μ' contradicting (I).

Claim (III) now follows. □
[4]

(**19.2.36**) Let $G_\delta(\lambda)$ be the λ-connected component of $(\Lambda, \leftarrow^\delta)$. The relation $G_\delta(\lambda)$ may thus be thought of as a directed acyclic graph.

---

[4]NOT NEEDED NOW?:

(**19.2.35**) REMARK. Continue with λ/μ a δ-pair as above. We want also to rule out the possibility of an intermediate μ' such that λ/μ' is a δ-skew (hence showing that every δ-pair λ/μ corresponds to a MaBS: a pair λ, μ with μ a maximal δ-balanced subpartition of λ). If λ/μ is disconnected then each charge in it occurs exactly once, and a single π-rotation is forced, once and for all, for each charge pair. Thus the same rotation would match the charge pairs in any balanced subset — i.e. it would have the properties of a δ-pair, and a contradiction arises as above. What about the case of λ/μ connected? ...

...anyway, eventually we get there.

(In light of Prop. 19.3.4 we shall call $G_\delta(\lambda)$ the *block graph* for the block of λ.)
See Figure 19.9 for an example (the edge labels will be explained in §19.5.2).

19.3 Brauer algebra Δ-module maps and block relations

(**19.3.1**) Remark. For $\lambda \supset \mu$, μ is a maximal δ-balanced subpartition of λ (as defined in [28]) if and only if λ^T/μ^T is a δ-pair.

The following will be clear from the definition of δ-charge: If λ^T/μ^T is a δ-pair then λ, μ are 'δ-balanced' in the sense of [28]. For this μ to be a maximal δ-balanced subpartition of λ we still need to show that there is no 'δ-balanced' partition between the two. This is shown in Prop.19.2.34.

th:bmap (**19.3.2**) Theorem. *[28, Theorem 6.5] Fix δ. If λ/μ is a δ-pair then for $B_n(\delta)$ over $\mathbb{C}$*

$$Hom_{B_n(\delta)}(\Delta_n^\delta(\lambda^T), \Delta_n^\delta(\mu^T)) \neq 0$$

Proof. Noting the formulation in [28, Theorem 6.5], it is ETS that the definitions of δ-pair and MaBS are equivalent up to transposition of the partitions. It is straightforward to show that every balanced skew in the sense of [28] contains a (transposed) δ-pair (see (19.2.30)). Equivalence then follows from (19.2.34). □

(**19.3.3**) Write $\Lambda^{\sim^\delta} = (\Lambda, \sim^\delta)$ for the RST closure of the partial order $(\Lambda, <^\delta)$. Write $[\lambda]_\delta$ for the $\Lambda^{\sim^\delta}$-class of $\lambda \in \Lambda$.

pr:block rel (**19.3.4**) Proposition. *The relation $\Lambda^{\sim^\delta}$ is the (transposed) block relation for $B_n(\delta)$ over $\mathbb{C}$.*

Outline proof: We see from Prop. 19.1.3 (after some further work — see [28]) that the blocks are no bigger; and from Theorem 19.3.2 that they are no smaller. □

(**19.3.5**) In light of Prop. 19.3.4 we shall call $G_\delta(\lambda)$ the *block graph* for the block of λ.

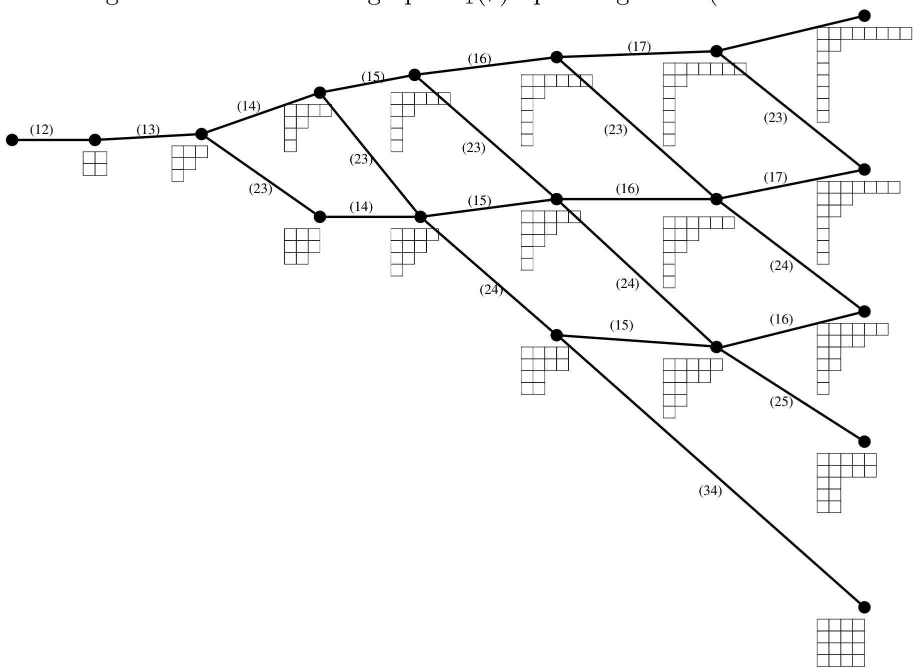
(**19.3.6**) In light of Prop.19.2.34 we could also refer to a δ-pair as a *minimal δ-balanced skew.*

19.4 On the block graph $G_\delta(\lambda)$

It is useful to give an alternative characterisation of the block graph $G_\delta(\lambda)$ which further emphasises geometrical aspects of the block condition.

(**19.4.1**) Why *is there* a geometrical formulation? Suppose λ/μ a δ-pair. Note that if we suspend, for intermediate steps, the dominance requirement (the requirement to work with partitions rather than arbitrary compositions) then we can pass to λ from μ by a sequence of transformations on pairs of rows. Each transformation extends two rows: adding part of one row, and the corresponding opposite charges in the other row. The no-row-fixed condition of (19.2.18) ensures that it is always pairs of rows (as opposed to a single row) that are involved. For each row in question one takes the leading edge of the row in μ and performs the two reflections mentioned in (19.2.15). The vertical reflection (i.e., in the horizontal) simply swaps the two rows. The other reflection takes this leading edge as far beyond the charge-0 diagonal as it was short of it beforehand. From these observationss it will be evident that the transformation $\mu \to \lambda$ can be reformulated geometrically. We shall describe this formulation in detail in (19.4.3) et seq..

Figure 19.9: The block graph $G_1(\emptyset)$ up to degree 16 (drawn directed left to right). regu1



(**19.4.2**) Remark. Alternatively λ can be built from μ by a sequence of transformations manipulating *columns* in pairs. The difference is firstly that, unless we transpose, the intermediate stages are neither partitions nor compositions (they are 'transpose compositions'); and secondly that it is possible in some cases to require a manipulation on a single column, rather than a pair; and thirdly that the no-row-fixed condition must still be imposed. In light of this we use here the rows-in-pairs version.
Exercise: How can we impose that row-fixed transformations do not arise in this formulation?

19.4.1 Embedding the vertex set of $G_\delta(\lambda)$ in $\mathbb{R}^{\mathbb{N}}$

de:shift-embed1 (**19.4.3**) For $\delta \in \mathbb{R}$ define

$$\rho_\delta = -\frac{\delta}{2}(1, 1, ...) - (0, 1, 2, ...) \in \mathbb{R}^{\mathbb{N}}$$

Define $\mathbb{Z}^f$ as the subset of finitary elements of $\mathbb{Z}^{\mathbb{N}}$. Define

$$\mathsf{e}_\delta : \mathbb{Z}^f \hookrightarrow \mathbb{R}^{\mathbb{N}} \tag{19.3}$$
$$\lambda \mapsto \lambda + \rho_\delta \tag{19.4}$$

x

In other words, since $\Lambda \hookrightarrow \mathbb{Z}^f$, we have embedded our index set Λ into a Euclidean space. Thus our blocks now correspond to collections of points in this space.

Examples:

$$\mathsf{e}_2(\emptyset) = (0,0,0,0,...) - (1,1,1,1,...) - (0,1,2,3,...) \;=\; (-1,-2,-3,-4,...)$$

$$\mathsf{e}_{-1}(2) = (2,0,0,0,...) - \frac{-1}{2}(1,1,1,1,...) - (0,1,2,3,...)$$
$$= \;(2,0,0,0,...) + (\frac{1}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},...) \;=\; (\frac{5}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},...)$$

Note that if $\delta \in \mathbb{Z}$ and $\lambda \in \Lambda$ then $\mathsf{e}_\delta(\lambda)$ is either in $\mathbb{Z}^\mathbb{N}$ or half-integral (i.e. in $\mathbb{Z}^\mathbb{N} + w$) in the sense of §5.8.2 para.(5.8.11).

19.4.2 Reflection group $\mathcal{D}$ acting on $\mathbb{R}^\mathbb{N}$

(**19.4.4**) Consider (from 5.2.1, 5.2.18) the reflection group actions on $\mathbb{R}^\mathbb{N}$

$$(ij) : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., \lambda_j, ..., \lambda_i, ...)$$

$$(ij)_- : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., -\lambda_j, ..., -\lambda_i, ...)$$

Write $\mathcal{D}$ for the group generated by these (all $i < j$); and $\mathcal{D}_+$ for the subgroup $\langle (ij) \rangle_{ij}$. Write $\mathbb{H}_\mathcal{D}$ and $\mathbb{H}_+$ for the corresponding closed sets of reflection hyperplanes. Define subsets $S_{\mathcal{D}_+} := \{(i\; i{+}1) : i \in \mathbb{N}\}$ and $S_\mathcal{D} := S_{\mathcal{D}_+} \cup \{(12)_-\}$. Write $\mathcal{D}v$ for the orbit of a point $v \in \mathbb{R}^\mathbb{N}$ under the action of $\mathcal{D}$.

de:D-alcove (**19.4.5**) A $\mathcal{D}$-*alcove* (or here simply an *alcove*) is a connected component of $\mathbb{R}^\mathbb{N} \setminus \cup_{H \in \mathbb{H}_\mathcal{D}} H$. A *chamber* is a connected component of $\mathbb{R}^\mathbb{N} \setminus \cup_{H \in \mathbb{H}_+} H$.

Note that there is a chamber consisting of the strictly descending sequences; and that $\mathsf{e}_\delta(\Lambda)$ is a distinct subset of this chamber for each $\delta \in \mathbb{Z}$ (indeed for each $\delta \in \mathbb{R}$). We denote this chamber by C_0. We also call it the *'dominant' chamber* (borrowing from the somewhat analogous use of the term in Lie theory).

Write $\mathcal{A}$ for the set of alcoves and $\mathcal{A}^+$ for the subset of alcoves in C_0. Choose the 'fundamental' alcove $a_0 \in \mathcal{A}^+$ as the one containing $v_- := (-1,-2,-3,...)$.

de:Dregular (**19.4.6**) A point $v \in \mathbb{R}^\mathbb{N}$ is $\mathcal{D}$-*regular* (or just *regular*) if, for $w \in \mathcal{D}$, $wv = v$ implies $w = 1$. That is, v is regular if it does not lie on any reflection hyperplane of $\mathcal{D}$. Equivalently a sequence $v \in \mathbb{R}^\mathbb{N}$ is *regular* if no two terms have the same magnitude.

Let $\mathbb{R}^{Reg}$ denote the set of regular sequences. Thus $\mathbb{R}^{Reg} = \mathbb{R}^N \setminus \cup_{H \in \mathbb{H}_\mathcal{D}} H$.

de:alphamap (**19.4.7**) If $v \in \mathbb{R}^\mathbb{N}$ is $\mathcal{D}$-regular then it lies in an alcove. Define

$$\alpha : \mathbb{R}^N \setminus \cup_{H \in \mathbb{H}_\mathcal{D}} H \;\to \mathcal{A}$$

such that $\alpha(v)$ is the alcove in which v lies.

Clearly if $w \in \mathcal{D}$ then $w\alpha(v) = \alpha(wv) \in \mathcal{A}$.

(**19.4.8**) There are obviously a lot of strongly descending (in the sense of (19.4.11)) and even integral points in every dominant alcove. For example

$$\alpha^{-1}(\alpha(v_-)) = \{(-1,-2,-3,...),(1,-2,-3,...),(0,-2,-3,...),(-2,-3,-4,...),...\}$$

gives a partial list of points in the fundamental alcove. Note that the second and third examples here are 'connected' to the first (in the sense that their difference has finite support, say), while the fourth example is not. Some more strongly descending (but half-integral or non-integral) examples from this alcove:

$$\{..., (-1/2, -3/2, -5/2, ...), (1/2, -3/2, -5/2, ...), \\ (-.1, -1.1, -2.1, ...), (.1, -1.1, -2.1, ...), \quad (.49, -.51, -1.51, ...), ...\}$$

In this alcove the lowest magnitude entry in a sequence v is the first entry; but this first entry is not necessarily negative. We can generalise this observation as follows.

(**19.4.9**) Of course an alcove is convex (albeit unbounded), so if v, v' are in the same alcove so is every point on the line between them. Suppose $|v_i| < |v_{i+1}|$ but $|v'_i| > |v'_{i+1}|$. Then somewhere on the line between we have v'' with $|v''_i| = |v''_{i+1}|$, so that v'' lies on a reflection hyperplane and v, v' are on different sides of this hyperplane (and hence not in the same alcove). It follows that if v is in the same alcove as v' then the magnitude order of terms is the same. In particular if v is in the fundamental alcove a_0 then the magnitudes of terms are increasing.

Note that it follows that $v \in a_0$ has at most one non-negative term.

(**19.4.10**) Remark. It might be helpful (for comparison) to condider also the group of reflections generated by $(1)_-$ and the (ij)'s — as we did in §5.2.2. Here there is a hyperplane between $(-1, -2, -3, ...)$ and $(1, -2, -3, ...)$.

19.4.3 Constructing graph morphisms for $G_\delta(\lambda)$: combinatorial approach

In §19.5 we will give an alcove geometric perspective on the block graphs. This geometric perspective is not canonical, however, so we also develop a 'vanilla' combinatorial approach. On the other hand there are several steps so, alongside the formal construction, we shall also indicate what can be considered to be happening from an 'alcove geometric perspective'.

de:dom1 (**19.4.11**) Note that all the image points in $\mathsf{e}_\delta(\Lambda) \subset \mathbb{R}^{\mathbb{N}}$ are strictly decreasing/descending sequences, and so lie in the chamber C_0. We called such sequences *dominant*. Indeed all the image points $\mathsf{e}_\delta(\Lambda)$ are *strongly* descending sequences, meaning that $v_i - v_{i+1} \geq 1$ for all i. We write $A^+ \subset C_0$ for the set of strongly descending sequences.

(**19.4.12**) Remark. The given definition of A^+ will serve our purposes (to extract the dominant part from a full orbit in the form $\mathcal{D}v \cap A^+$; and to desribe the domain of the o-map, and ...?), but is not canonical. What can we say about it? Could we use the stronger condition II: $v_i - v_{i+1} \in \mathbb{N}_+$ instead? Could we use C_0 (descending) instead? Consider $\mathsf{e}_\delta^{-1}(\mathcal{D}\mathsf{e}_\delta(\lambda))$ for $\lambda \in \Lambda$ and various choices of δ. Taking $\mathcal{D}\mathsf{e}_\delta(\lambda)$ to be the finitary orbit then the inverse image of the orbit lies in $\mathbb{Z}^f$ so long as δ is integer. If $\delta = .2$ (say) then $\mathsf{e}_\delta(\emptyset) = (-.1, -1.1, -2.1, ...)$ which is strongly descending, indeed the gaps $v_i - v_{i+1}$ are positive integral. But $(12)_-(-.1, -1.1, -2.1, ...) = (1.1, .1, -2.1...)$ so the gaps are no longer all integral, thus the property II is not preserved by $\mathcal{D}$; and $(1.1, .1, ...) - (-.1, -1.1, ...) = (1.2, 1.2, 0, 0, ...)$ so the formal geometrical 'orbit' includes 'weights' that make no sense for us. More generally

$$((12)_- - 1)(a, b, c, ...) = (-b, -a, c, ...) - (a, b, c, ...) = (-a - b, -a - b, 0, ...)$$

In practice we need only deal separately with $\delta \in \mathbb{Z}$.

(**19.4.13**) Considering for a moment the magnitudes of terms in a sequence in A^+, we see that each magnitude occurs at most twice, i.e. in a sequence of form $(..., x, ..., -x, ...)$. We call such a $\pm x$ pairing a *doubleton*. Define a map

$$Reg : A^+ \to A^+$$

such that $Reg(v)$ is obtained from v by removing the doubletons.

For example

$$Reg(1, -1, -3, -4, -5, -6, ...) = (-3, -4, -5, -6, ...)$$

(note in this case that the input is $\mathsf{e}_2((2,1))$ while the output is $\mathsf{e}_6(\emptyset)$, that is, the *Reg* map can increase δ);

$$Reg(4, 3, 1, 0, -1, -5, -6, ...) = (4, 3, 0, -5, -6, ...)$$

de:sing-set

(**19.4.14**) Fixing δ, for $\lambda \in \Lambda$ write $p_\delta(\lambda)$ for the set of pairs of rows $\{i, j\}$ such that $(\lambda + \rho_\delta)_j = -(\lambda + \rho_\delta)_i$ (i.e. $\mathsf{e}_\delta(\lambda)_j = -\mathsf{e}_\delta(\lambda)_i$).

This $p_\delta(\lambda)$ gives the set of hyperplanes $(ij)_-$ upon which $\mathsf{e}_\delta(\lambda)$ lies. The set $p_\delta(\lambda)$ is clearly not an invariant of the $\mathcal{D}$-orbit of $\mathsf{e}_\delta(\lambda)$, but $|p_\delta(\lambda)|$ is an invariant of the part of that orbit that lies in C_0. (Proof: For any sequence $v \in \mathbb{R}^{\mathbb{N}}$ the set of *magnitudes* of entries appearing in v, and the multilicities of these magnitudes, is an invariant of $\mathcal{D}v$. In particular, if there is a doubleton in a *descending* sequence v then it must be a $\pm x$ pair, as above.)

Furthermore, given (δ, λ) the magnitudes and multiplicities of entries appearing in the corresponding sequence $\mathsf{e}_\delta(\lambda)$ is determined, and hence determined for the whole orbit. Supposing then that we have some descending v in this orbit (i.e in the intersection of the orbit with C_0), its image $Reg(v)$ has all the singeltons but no doubletons. However since we know the magnitudes of the doubletons, there is a unique way of replacing these in to $Reg(v)$ which keeps the sequence in C_0. Thus we can recover v.

It follows that *Reg* restricts to a bijection between $\mathcal{D}\mathsf{e}_\delta(\lambda) \cap C_0$ and the corresponding partial orbit $\mathcal{D}Reg(v) \cap C_0$ for the regular sequence $Reg(v)$. (Proof: we have shown that *Reg* is injective on this set, so it remains to show that it is surjective onto the target — but this is clear, since appropriate doubletons can be added to *any* descending sequence with the given collection of singleton magnitudes, and this is the inverse map to *Reg*.)

(**19.4.15**) Remark:
1. Note that the target orbit is not necessarily the orbit for any $\lambda' \in \Lambda$ under e_δ (we need a different δ).
2. All the regular orbits are isomorphic via the bijection they each have with the orbit of alcoves. Thus we have here overall bijections between *every* pair of orbits, regular or otherwise.
3. This (infinite set) bijection is straightforward. What is less clear is that the associated graph structure is preserved (as we shall see).

Write $s_\delta(\lambda)$ for the *singularity* of $\mathsf{e}_\delta(\lambda)$:

$$s_\delta(\lambda) \ := \ |p_\delta(\lambda)|$$

de: mag order

(**19.4.16**) Note that the terms in any $v \in \mathbb{R}^{Reg} \cap A^+$ have a well-defined *magnitude order*. That is, each term may be asigned a number giving its position in the list of terms ordered by increasing magnitude. For example $\frac{11}{2}$ is the 5-th term in the magnitude order of terms in $(\frac{11}{2}, \frac{9}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$.

Define a map
$$o : \mathbb{R}^{Reg} \cap A^+ \ \to \mathbb{Z}^{\mathbb{N}}$$
as follows. In the i-th term, $o(v)_i$ of $o(v)$, the magnitude $|o(v)_i|$ is the position of v_i in the magnitude ordering of the set of numbers appearing in v. The sign of $o(v)_i$ is the sign of v_i, unless $v_i = 0$ in which case the sign is chosen so as to make an even number of positive terms.
(Remark: this sign choice is simply for definiteness. The definition of the function we eventually use (constructed next) will make it independent of this convention.)

Example:
$$o(\frac{11}{2}, \frac{9}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...) \ = \ (5, 4, 2, 1, -3, -6, -7, ...)$$

(**19.4.17**) Note that $o(\mathbb{R}^{Reg} \cap A^+)$ is the set of descending signed permutations of v_-.

Define a 'toggle' map τ' on this set, onto the subset of permutations with an even number of positive terms, by toggling the sign of ± 1 as necessary. Example $\tau'(1, -2, -3, -4, ...) = (-1, -2, -3, -4, ...)$.

We claim that v and $\tau' v$ are always in the same alcove. (Proof: we can continuously deform $1 \rightsquigarrow -1$ without passing through a singular point, regardless of the signs of other terms.)

pa:special k (**19.4.18**) Note that if v is a descending signed permutation of $\mathtt{e}_2(\emptyset) = (-1, -2, -3, ...)$, such as $(3, 1, -2, -4, ...)$, then the magnitude order is obtained simply by ignoring the signs. Thus in this case $o(v) = v$.

de:Peven (**19.4.19**) Let $\mathtt{P}_{even}(\mathbb{N}), \mathtt{P}_{odd}(\mathbb{N}) \subset \mathtt{P}(\mathbb{N})$ denote the set of subsets of $\mathbb{N}$ of even (respectively odd) order. The *toggle map* $\tau : \mathtt{P}_{odd}(\mathbb{N}) \to \mathtt{P}_{even}(\mathbb{N})$ is given by toggling the presence of 1 so as to make an odd set even, or even set odd.

(**19.4.20**) Note that $o(v)$ is strongly descending and indeed integral descending, and regular. For u integral descending we define $\phi_+(u) \in \mathtt{P}(\mathbb{N})$ as the set obtained by first taking the set of positive terms in u and then, if this set contains an odd number of terms, applying the toggle map.

We also use the notation $u|_+ \ := \ \phi_+(u)$. E.g., $(5, 4, 2, 1, -3, -6, -7, ...)|_+ = \{1, 2, 4, 5\}$.

(**19.4.21**) CLAIM: If v regular then $\alpha(v) = \alpha(o(v))$. (or nearly) That is, $o(v)$ tells us which alcove v is in, in a $\mathtt{P}_{even}(\mathbb{N})$ labelling.

Proof: EXERCISE!?!?!?!

(**19.4.22**) Define
$$\begin{aligned} o_\delta : \Lambda \ &\to \ P(\mathbb{N}) \\ \lambda \ &\mapsto \ o(Reg(\mathtt{e}_\delta(\lambda)))|_+ \end{aligned} \tag{19.5}$$

Examples: $\emptyset \overset{\mathtt{e}_2}{\mapsto} \ \mathtt{e}_2(\emptyset) = (-1, -2, -3, ...) \ \overset{\phi_+}{\mapsto} \ \emptyset$

$(1) \overset{\mathtt{e}_2}{\mapsto} \ \mathtt{e}_2(1) = \ (\ 0, -2, -3, ...) \ \overset{o}{\mapsto} \ (-1, -2, -3, ...) \ \overset{\phi_+}{\mapsto} \ \emptyset$

$(3, 3) \overset{\mathtt{e}_2}{\mapsto} \ (\ 2, 1, -3, -4, ...) \mapsto \{1, 2\}$

Note that the image sets $\mathtt{e}_\delta(\Lambda)$ do not intersect (as δ varies), so given $\mathtt{e}_\delta(\lambda)$ we can *determine* δ and λ. In the case above we have $\delta = 2$. In the next case we have $\delta = 0$:

$(3, 3, 3, 1) \overset{\mathtt{e}_0}{\mapsto} (3, 2, 1, -2, -4, -5, ...) \overset{Reg}{\mapsto} (3, 1, -4, -5, ...) \overset{o}{\mapsto} (2, 1, -3, -4, ...) \overset{\phi_+}{\mapsto} \{1, 2\}$

$(4, 3, 3, 1) \overset{\mathtt{e}_0}{\mapsto} (4, 2, 1, -2, -4, -5, -6, ...) \overset{Reg}{\mapsto} (1, -5, -6, ...) \overset{o}{\mapsto} (1, -2, -3, ...) \mapsto \{1\} \overset{toggle}{\mapsto} \emptyset$

(**19.4.23**) Given δ and λ we define
$$o_\delta^\lambda : P_{even}(\mathbb{N}) \to \Lambda$$
as follows (indeed we could extend the domain to $P(\mathbb{N})$ by applying the toggle map to $P_{odd}(\mathbb{N})$). First construct $\mathtt{e}_\delta(\lambda)$. Note that this fixes the doubletons and (magnitudes of) singletons for its whole orbit, i.e. for every element of $o_\delta([\lambda]_\delta)$. We ignore the doubletons for a moment, and work out the magnitude order for the singletons. Note that the order in which the singletons can appear in a descending sequence is uniquely determined by their sign. Now for $v \in P_{even}(\mathbb{N})$ we give the positive sign to the corresponding singletons (in the magnitude order). Thus we have determined the singletons and their order in the sequence. The position of the doubletons is now forced, so the sequence $o_\delta(o_\delta^\lambda(v))$ is determined. But o_δ is invertible as already noted, so finally apply this inverse.

Example: $o_{-1}^{(2)}(\{1,2,4,5\})$:
From an example above we see that the doubletons of $\mathtt{e}_{-1}(2)$ are just $\{5/2, -5/2\}$. The singletons have magnitudes $\{1/2, 3/2, 7/2, 9/2, 11/2, ...\}$ — where we have written them out in the magnitude order.
For $v = \{1,2,4,5\}$ we give $+$ signs to $1/2$, $3/2$, $9/2$ and $11/2$ and the remaining singletons are negative. Thus
$$o_{-1}^{(2)}(\{1,2,4,5\}) = (\frac{11}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-5}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$$

lem:o-detla bij (**19.4.24**) LEMMA. *Fix δ and λ. Then o_δ and o_δ^λ are mutual inverses on $[\lambda]_\delta \leftrightarrow P_{even}(\mathbb{N})$.* $\square$

19.4.4 The graph G_{even}

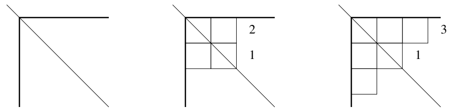
(**19.4.25**) For $\alpha < \beta \in \mathbb{N}$ and $a, b \in \mathtt{P}(\mathbb{N})$ write
$$a \overset{\alpha\beta}{\ggg} b \qquad \text{if} \qquad a \setminus b = \{\alpha\}, \;\; b \setminus a = \{\beta\}$$
$$a \overset{\underline{\alpha\beta}}{\ggg} b \qquad \text{if} \qquad a \setminus b = \{\}, \;\; b \setminus a = \{\alpha, \beta\}$$

NOW RELATE THIS TO RIGHT ACTION ON COSETS —

de:G_even (**19.4.26**) Define a directed graph, G_{even} with vertex set $\mathtt{P}_{even}(\mathbb{N})$ (we call these vertices *valley sets*); and labelled edges:
$$a \xrightarrow{\alpha} b \qquad \text{if} \qquad a \setminus b = \{\alpha\}, \;\; b \setminus a = \{\alpha + 1\} \qquad (\alpha \in \mathbb{N})$$
$$a \xrightarrow{\underline{12}} b \qquad \text{if} \qquad a \setminus b = \emptyset, \;\; b \setminus a = \{1, 2\}$$
See Figure 19.11.

There is a corresponding graph G_{odd} with vertices given by subsets of $\mathbb{N}$ of odd order. The *toggle map* between the vertex sets is readily seen to pass to a graph isomorphism (the edge labels 1 and 12 are interchanged).

(**19.4.27**) We shall shortly construct an isomorphism $G_\delta(\lambda) \cong G_{even}$ for each δ, λ. For now we note that the case $G_1(\emptyset)$ takes a relatively simple form. The vertex map, $o_1 : [\emptyset]_1 \to P_{even}(\mathbb{N})$, may be pictured as follows. First draw the main diagonal on the Young diagram, as in these three examples from $[\emptyset]_1$:



then count the number of boxes wholly or partly to the right of the diagonal in each row, and write down the subset of these numbers that are positive. Thus our examples become $\emptyset - \{2,1\} - \{3,1\} \cdots$. More of this case can be computed from $G_1(\emptyset)$ as shown in Figure 19.9.

LEMMA. The vertex map o_1 passes to an isomorphism $G_1(\emptyset) \cong G_{even}$.

Proof. Comparing with (19.2.18) we readily see that ...

th:graph 01

(**19.4.28**) THEOREM. *For each δ, λ, the map o_δ passes to an isomorphism*

$$G_\delta(\lambda) \cong G_{even}$$

th:graph isom0

(via G_o and the toggle map in case $o_\delta(\lambda)$ of odd order).

By Lemma 19.4.24 we have that o_δ restricts to a bijection on vertex sets, with o_δ^λ the inverse map. The next few paragraphs build up to a proof (in (19.5.21)) of the graph isomorphism.

(**19.4.29**) PROPOSITION. *Fix a block-pair (δ, λ). If (v, w) is an edge in G_{even} with label α then the corresponding pair $(\mu, \lambda) = (o_\delta^\lambda(v), o_\delta^\lambda(w))$ gives λ/μ a δ-pair.*

THIS IS JUST A RESTATEMENT, AN UNPACKING, OF PART OF THEOREM (19.4.28).

19.5 Graph isomorphisms, via geometrical considerations

ss:gigc

ss:geom3

19.5.1 Dual graphs and alcove geometry

Here we recall some general properties of reflectiong groups from §5.3 *et seq*, and from Humphreys [68].

ss:alc geom

(**19.5.1**) Fix a Euclidean space V. Let $\mathbb{H}$ be a set of hyperplanes $H_t \subset V$ (as in Humphreys [68, §1.16]). Write $W_{\mathbb{H}}$ for the group generated by reflections in these hyperplanes; and $\bar{\mathbb{H}}$ for the closure of $\mathbb{H}$ under $W_{\mathbb{H}}$.

(**19.5.2**) The $\mathbb{H}$-*singularity* of a point in V is the number of hyperplanes $H \in \mathbb{H}$ it intersects.

Write $\mathcal{A}_{\mathbb{H}}$ for the set of connected components of $V \setminus \cup_{H \in \mathbb{H}} H$. That is, $a \in \mathcal{A}_{\mathbb{H}}$ is a zero-singularity-connected component.

A hyperplane H is a *wall* of $a \in \mathcal{A}_{\mathbb{H}}$ if a is *not* in $\mathcal{A}_{\mathbb{H} \setminus H}$. That is, H is a wall of a if its euclidean-space closure $\bar{a}$ intersects H in a defining subset of H (i.e. if the linear span of $\bar{a} \cap H$ is H). Write $\mathbb{H}_a$ for the set of walls of a. Pair $a, b \in \mathcal{A}_{\mathbb{H}}$ are *adjacent* if they have a common wall.

If $\mathbb{H} = \bar{\mathbb{H}}$ then, matching hyperplanes to the reflections they generate, we have that $(W_{\mathbb{H}}, \mathbb{H}_a)$ is a Coxeter system, for any $a \in \mathcal{A}_{\mathbb{H}}$. That is, $\mathbb{H}_a$ is a simple system for $W_{\mathbb{H}}$.

(**19.5.3**) Given a choice of a 'root' $a_0 \in \mathcal{A}_{\mathbb{H}}$, we may associate a directed *dual graph* $D(\mathbb{H})$ to $(V, \mathbb{H}, a_0)$ as follows. Graph $D(\mathbb{H})$ has vertex set $\mathcal{A}_{\mathbb{H}}$; and an edge $a \to b$ whenever a, b are adjacent, with common wall H say, and it is a that lies on the same side of H as a_0. [5]

The maximal open subset of $\bar{a} \cap H$ in H (which is the subset of $\bar{a} \cap H$ of singularity 1) is an example of a *facet* in the *alcove geometry* of the pair $(V, \mathbb{H})$. More generally a facet of an alcove a is the intersection of $\bar{a}$ with the intersection of some subset of $\mathbb{H}$ (excluding the intersection of any larger subset).

pa:HVup

(**19.5.4**) Consider a closed set $\mathbb{H} = \bar{\mathbb{H}}$ of hyperplanes and a closed subset $\mathbb{H}_+$. We call the elements of $\mathcal{A} = \mathcal{A}_{\mathbb{H}}$ *alcoves*, and the elements of $\mathcal{C} = \mathcal{A}_{\mathbb{H}_+}$ *chambers*. Thus a chamber C defines a subset $\mathbb{H}_C$ of $\mathbb{H}_+$ by analogy with $\mathbb{H}_a$. Let a_0 be a root alcove. We assume that $\mathbb{H}_+$ is generated by a subset $\mathbb{H}_{C_0}$ of $\mathbb{H}_{a_0}$, where C_0 is a 'fundamental' chamber of $\mathbb{H}_+$. The *dominant dual graph* $D(\mathbb{H}/\mathbb{H}_+)$ is the intersection of the dual graph with the fundamental chamber for $\mathbb{H}_+$ (we may also write $D(W_{\mathbb{H}}/W_{\mathbb{H}_+})$ for this graph). As for $D(\mathbb{H})$ we make the graph rooted and directed by choosing a_0.

For example Figure 19.10 shows the dominant dual graph for affine-A_2 (generated by the hyperplanes 1,2 and $3'$ shown) over the subset corresponding to A_2 (generated by the hyperplanes 1 and 2).

CHECK THE REST!!!—

Note that v is regular in the sense of (19.4.16) iff it is $\mathcal{D}$-regular.

(**19.5.5**) We chose the fundamental chamber of $\mathcal{D}_+$ to be the chamber of descending sequences, so that (writing S_{C_0} for the set of reflections matching $\mathbb{H}_{C_0}$) $S_{C_0} = S_{\mathcal{D}_+}$ We chose the root alcove a_0 as the one containing $v_- := (-1, -2, -3, ...)$, so that $S_{a_0} = S_{\mathcal{D}}$ Write G_a for $D(\mathcal{D}/\mathcal{D}_+)$.
(A graph isomorphic to this, using a notation we shall explain shortly, is shown in Figure 19.11.)

19.5.2 Group $\mathcal{D}$ action on $\mathsf{e}_\delta(\Lambda)$

ss:Dact

(**19.5.6**) Now fix δ and consider how some $w \in \mathcal{D}$ acts on $\mathbb{Z}^f$, and hence Λ, via its action on $\mathsf{e}_\delta(\mathbb{Z}^f)$. We write $w.\lambda$ for this:

$$w.\lambda \;:=\; \mathsf{e}_\delta^{-1}(w\mathsf{e}_\delta(\lambda))$$

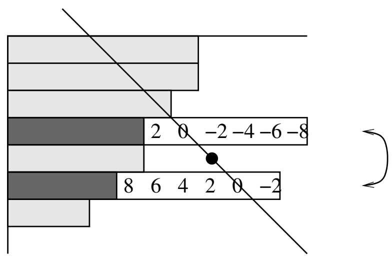
The action fixes $\mathbb{Z}^f$, but the restriction to Λ does not lie inside Λ in general. For $\lambda \in \Lambda$ it modifies two rows: in case (ij) this breaks the descending property, so cannot lie in Λ; in case $(ij)_-$ it extends or contracts both rows, so sometimes lies in Λ.

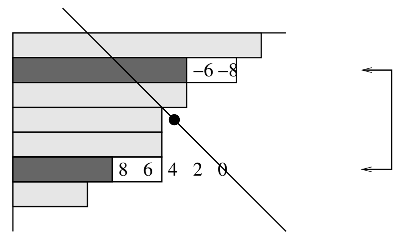
(**19.5.7**) Example. Fix δ. Here is what happens when $(ij)_-$ is applied to λ in this way in a couple of cases. Consider the 'leading edge' (i.e. the right-hand edge) of the partition shape λ in rows-i, j. A rotation by a point on the 0-charge diagonal half way between these two rows takes each leading edge component into the *other* row. The change $\lambda \to (ij)_-.\lambda$ is that the new i, j leading edge altogether is given by these two rotation images.

In each example the given λ is shaded — with the rows i, j heavily shaded. The effect in the

[5]NB the definition of an edge as stated in [30] §7 (in the definition of **Alc**) does not quite work, since it includes any pair a, b with $\bar{a} \cap \bar{b} \neq \emptyset$. Such intersections do not define a unique hyperplane in general.

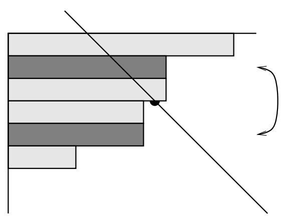
first two cases is to extend these rows as shown (the extensions are not shaded):

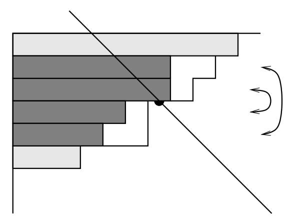




In the first case the outcome is not a partition. In both cases the *change* is by a collection of boxes in each changed row such that pairs of opposite charges match up between the rows. That is, the effect is indeed as if the *leading edge* of the row has been *rotated* about a point on the δ-diagonal half-way between the two rows, to form part of the leading edge of the new shape.

In the next case the reflection has no effect:





In the final example we use two commuting reflections to make a $w.\lambda$ such that $w.\lambda/\lambda$ is a δ-pair.

de:rotref1 (**19.5.8**) Suppose we have a δ-pair λ/μ, fixed by a rotation π_x. The rotation on the δ-pair naturally groups the rows involved into pairs (i, j), such that if some box b in the δ-pair is in one of these rows then its image $\pi_x(b)$ is in the other. That is, each such pair of rows lies symmetrically above and below the horizontal line defined by the reflection point. Note that if there is another box involved in row-i then its rotation partner is again in row-j. One can see from this and the example that there is a reflection $(ij)_-$ that achieves the part of the overall change (between the two partitions involved) that intersects rows i, j.

(**19.5.9**) Let λ/μ be a δ-pair, with w a product of commuting reflections as above such that $w.\mu = \lambda$ (so that $w.\lambda = \mu$ also). Consider $\lambda - e_i \in \Lambda$ with $e_i \in \lambda/\mu$. If we apply the same w, and in particular the same $(ij)_-$ factor from w, to $\lambda - e_i$, then row-j of the outcome will be one box longer: $w.(\lambda - e_i) = \mu + e_j$, where $e_j = \pi_x(e_i)$ (by a mild abuse of notation). This is the effect of applying the *same rotation* to the 'leading edge' of $\lambda - e_i$ as one could consider to have been applied to λ.

— ENLARGE on the dominance of $w.(\lambda - e_i)$: want to say removability of e_i ensures addability of e_j... ...right??? —

(**19.5.10**) Note that $\mathcal{D}$ does not preserve the image $\mathsf{e}_\delta(\Lambda)$, for any δ, but (it is routine to show that in a suitable sense)

$$\text{orbit} \cap \text{`dominant'} = \text{block}$$

le:prod-com-ref (**19.5.11**) LEMMA. *If λ/μ is a δ-pair then $\mathsf{e}_\delta(\lambda)$ can be obtained from $\mathsf{e}_\delta(\mu)$ by a sequence of one or more transformations by $(ij)_-$s, extending rows in pairs of δ-balanced part-rows. Specifically*

$$\mathsf{e}_\delta(\lambda) \;=\; w_{\lambda/\mu}\,\mathsf{e}_\delta(\mu), \qquad \textit{with} \quad w_{\lambda/\mu} \;=\; \left(\prod_{ij}(ij)_-\right)$$

where the product is over pairs of rows in the skew λ/μ, from the outer pair to the inner pair.

Note also that no subset of this product, applied to $\mathsf{e}_\delta(\mu)$, results in a dominant weight.

Proof. Compare the definitions of δ-pair (19.2.18), e_δ and $(ij)_-$. □

(**19.5.12**) REMARK. On the other hand there may be a substitute $w' \in \mathcal{D}$ say, for $w_{\lambda/\mu}$ with *more* $(ij)_-$ factors (since there may be reflections $(kl)_-$ that fix $\mathsf{e}_\delta(\lambda)$).

th:orbit-block (**19.5.13**) LEMMA. *Fix δ and consider $\lambda \in \Lambda$. (I) $\mathcal{D}.\lambda \cap \Lambda \supset [\lambda]_\delta$.*
(II) $\mathcal{D}.\lambda$ intersects no other block.

Proof. (I) Follows from Lemma 19.5.11. (II) Is proved in [29]. □

pa:regu1-key (**19.5.14**) In the example in Figure 19.9 the edge label (ij) indicates that the required group element is $(ij)_-$ (in this case they are all simple reflections).

19.5.3 The graph isomorphism

(**19.5.15**) Define a partial order $(\mathbb{R}^{\mathbb{N}}, \geq)$ by $v \geq w$ if $v_i \geq w_i$ for all i. (Write $v > w$ if $v \geq w$ and $v \neq w$.)

de:Vv (**19.5.16**) For $v \in \mathbb{R}^{\mathbb{N}}$ define

$$V(v) = \mathcal{D}v \cap A^+$$

Define a directed graph $\mathbf{G}(v)$ with vertex set $V(v)$ by assigning an edge (t, u) if $(u-t)_i \geq 0$ for all i and this is a cover (i.e. (t, u) is not in the transitive closure of any other such pairs).

pr:gg1 (**19.5.17**) PROPOSITION. *For $\lambda \in \Lambda$ the map e_δ restricts to a bijection $[\lambda]_\delta \to V(\lambda + \rho_\delta)$; and this bijection extends to a graph isomorphism*

$$\mathsf{e}_\delta : G_\delta(\lambda) \xrightarrow{\sim} \mathbf{G}(\lambda + \rho_\delta).$$

Proof: By (19.5.13) (Th.5.2 of [29]) we have that e_δ defines a bijection between $[\lambda]_\delta$ and $V(\lambda+\rho_\delta)$. Note that $\mu \subset \nu \in \Lambda$ if and only if $\mathsf{e}_\delta(\mu) < \mathsf{e}_\delta(\nu)$. Thus, restricting this to $[\lambda]_\delta$, the graphs are (by (19.2.34) and (19.5.16) respectively) covers of isomorphic partial orders. These covers thus agree on arbitrarily large finite sub-orders, and hence agree. □

Note that v is regular if and only if every sequence in $\mathcal{D}v$ is regular.

pr:gg2 (**19.5.18**) PROPOSITION. *For $v \in A^+$ the map Reg restricts to a bijection $V(v) \to V(Reg(v))$; and this bijection extends to a graph isomorphism*

$$\mathbf{G}(v) \cong \mathbf{G}(Reg(v))$$

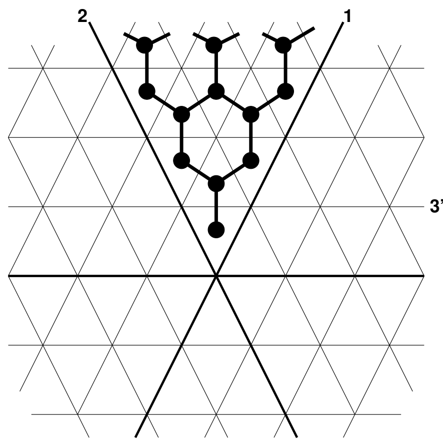


Figure 19.10: Example of a dominant dual graph: case $\hat{A}_2/A_2$ A2dual

Proof: The set of doubletons is an invariant of the elements of $V(v)$, and there is a unique way of adding these into an element of $V(Reg(v))$ that keeps the sequence decreasing. Thus the restriction of *Reg* here has an inverse, i.e. the set map is a bijection. Now suppose $t, u \in A^+$ and $a \in \mathbb{R}$ such that

$$s = (t_1, t_2, ..., t_i, a, t_{i+1}, ...) \qquad s' = (u_1, u_2, ..., u_j, a, u_{j+1}, ...)$$

are in A^+. Then $t < u$ if and only if $s < s'$. The *Reg* map can be built from pairs of such moves, so $t < u$ if and only if $Reg(t) < Reg(u)$, which establishes the graph isomorphism. □

pr:gg3 (**19.5.19**) LEMMA. *For any regular v, i.e. lying within an alcove, we have a bijection $\alpha : V(v) \to \mathcal{A}^+$ and a graph isomorphism $\alpha : \mathbf{G}(v) \stackrel{\sim}{\to} G_a$.*

Proof. By 19.5.17 and the construction of $G_\delta(\lambda)$ we know that no two elements of $V(v)$ lie in the same alcove, so α is injective. By the definition of α, if $w \in \mathcal{D}$ such that $wv \in V(v)$ then $w\alpha(v) = \alpha(wv) \in \mathcal{A}^+$. But there exists a traversal of $\mathcal{A}^+$ by $\mathcal{D}$ (see e.g. Humphreys), so α is surjective.

A convenient example of a regular v is $\mathsf{e}_2(\emptyset)$. In light of the bijection we may use the orbit $\mathcal{D}\mathsf{e}_2(\emptyset)$ and in particular $V(\mathsf{e}_2(\emptyset))$ to label dominant alcoves, and hence vertices in G_a. That is, we shall consider this to be an identification of the set of alcoves with $V(\mathsf{e}_2(\emptyset))$.

IT remains to show graph isom. ...

Recall that $\mathbf{G}(v)$ is the transitive reduction of the restriction of $(A^+, \leq)$ to $V(v)$. It is clear that the edges of G_a generate an isomorphic partial order on the set of dominant alcoves.

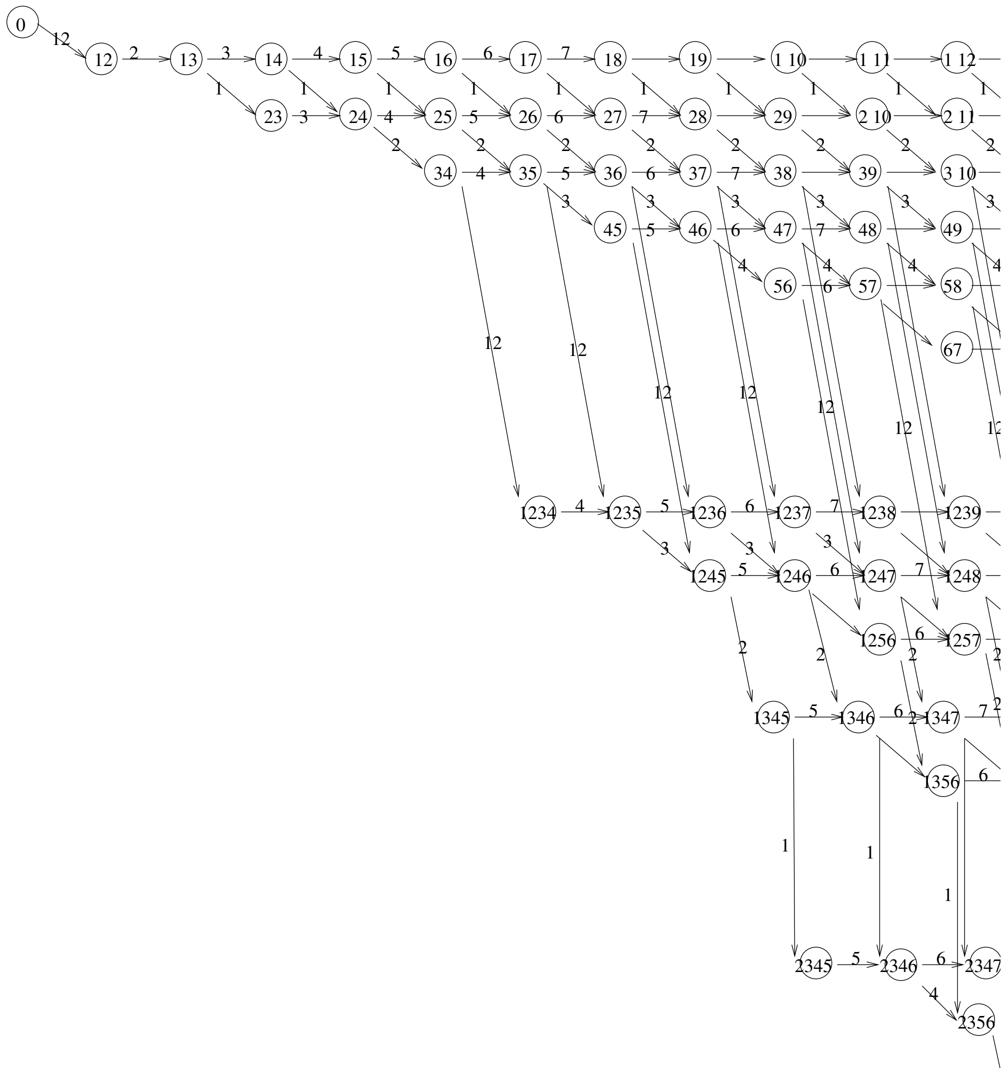
FINISH ME!

pr:gg4 (**19.5.20**) LEMMA. *The map $\phi_+ : V(\mathsf{e}_2(\emptyset)) \to \mathsf{P}(\mathbb{N})$ is injective with image $\mathsf{P}_{even}(\mathbb{N})$. This extends to a graph isomorphism $G_a \cong G_{even}$.*

Proof. We identify G_a with $\mathbf{G}(\mathsf{e}_2(\emptyset))$ as above. The 'inverse' set map from $\mathsf{P}_{even}(\mathbb{N})$ to sequences is clear. Now note that every element of $\mathcal{D}\mathsf{e}_2(\emptyset)$ has an even number of positive terms.

fig:valley graph

Figure 19.11: The beginning of the graph G_{even}, with edge labels as in (**??**). (Vertex labels have been written in an obvious shorthand.) See also Fig.19.22.



Recall that $\mathbf{G}(v)$ is the transitive reduction of $(V(v), \leq)$. In our case, for example:

$$(7,3,-1,-2,-4,-5,-6,-8,-9,...) \leq (9,6,3,2,-1,-4,-5,-7,-8,...)$$

Note that it is sufficient to check the componentwise inequalities on the positions where not both are negative. Indeed it is sufficient to compare the truncations of the sequences to their positive terms. This induces an obvious partial order on $\mathtt{P}_{even}(\mathbb{N})$, writing out the subsets in descending order and comparing in the same way. We then require only to show that G_{even} is the transitive reduction of this order. This follows easily from the definition (19.4.26). $\square$

th:gg4 (**19.5.21**) THEOREM. *For all* δ, λ

$$G_\delta(\lambda) \cong G_a \cong G_{even}$$

In particular $o_\delta : G_\delta(\lambda) \xrightarrow{\sim} G_{even}$ *is an isomorphism.*

Proof. By (19.5.17), (19.5.18), (19.5.19) and (19.5.20) we have isomorphisms $G_\delta(\lambda) \xrightarrow{\mathsf{e}_\delta} \mathbf{G}(\lambda + \rho_\delta) \xrightarrow{Reg} \mathbf{G}(Reg(\lambda+\rho_\delta)) \xrightarrow{\sim} G_a \xrightarrow{\equiv} \mathbf{G}(\mathsf{e}_2(\emptyset)) \xrightarrow{\phi_+} G_{even}$. $\square$

This is a remarkable result, since the right hand side does not depend on λ or even δ.

19.5.4 Aside on alternative reflection group actions on Λ

In §19.8 we move ('translate') representation theoretic data between blocks in a way that is evocative of a similar operation in Lie theory, where alcove geometry is a powerful organisational tool. This translation is particularly simple between blocks having representatives in the same facet. In our case the alcove geometry we have described is useful, but not canonical. Thus the notion of 'same facet' is not the correct characterisation, or at least it is only part of it. As a result we have to treat some not-same-facet same-facet-like cases. This Section is an effort to recast (enough) not-same-facet cases as same-facet, by attaching a different reflection group action.

We could embed $\lambda \in \Lambda$ (regarded as the index set for Δ-modules in the way we currently use it!!) in $\mathbb{R}^{\mathbb{N}}$ as λ^T. The effect of this on the original Young diagram would be that reflections would act on columns rather than rows. Viewed in isolation this is certainly a less useful embedding than the one we use (it does not cleanly describe blocks for example!). Is it any use as a partner to it? AND SO WHAT???

19.6 The decomposition matrix theorem

The main tool in the *statement* of Theorem 19.6.12 is a combinatorial construction (in §19.6.1) which the Theorem then equates to the decomposition matrix. The Theorem statement itself is in §19.6.2, and the remainder of this Chapter is then concerned with the proof.

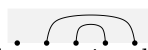
19.6.1 Decomposition data: Hypercubical decomposition graphs

ss:hyperDG

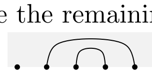
de:bb (**19.6.1**) Let $\mathbf{b} : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ denote the natural bijective map to binary sequences. For example:

$$\mathbf{b} : \{1,3,5,6\} \mapsto 101011$$

(if set a is finite we omit the open string of 0s on the right of $\mathbf{b}(a)$).
Define $\mathbf{b}_\delta : \Lambda \to \{0,1\}^{\mathbb{N}}$ by $\mathbf{b}_\delta(\lambda) = \mathbf{b}(o_\delta(\lambda))$.

(**19.6.2**) A generalisation of Brauer diagrams is to allow singleton vertices. A vertex pairing in such a diagram *covers* a vertex if the pair lie either side of it. A *TL-diagram* (TL as in Temperley–Lieb) is here a diagram drawn in the positive quadrant of the plane, consisting of a collection of vertices drawn on the horizontal part of the boundary (countable by the natural numbering from left to right); together with a collection of *non-crossing* arcs drawn in the positive quadrant, each terminating in two of the vertices, such that no vertex terminates more than one arc, and no arc covers a singleton vertex. An example is: 
It will be convenient to label each arc by the associated pair of numbered vertices.
Remark. As with a Brauer diagram, it is the vertex pairings (and here singletons) rather than the precise routes of the arcs that are important.

de: TL algo (**19.6.3**) Each binary sequence b has a TL-diagram $d(b)$ constructed as follows.
1. Draw a row of vertices, one for each entry in b (up to the last non-zero entry).
2. For each binary subsequence 01 draw an arc connecting the corresponding vertices.
3. Consider the sequence obtained by ignoring the vertices paired in 2. For each subsequence 01 draw an arc connecting these vertices (it will be evident that this can be done without crossing).
4. Iterate this process until termination (it will be evident that it terminates, since the sequence is getting shorter).
5. Note that this process terminates either in the empty sequence or in a sequence of 1s then 0s (either run possibly empty). Finally connect the run of vertices binary-labelled 1 in adjacent pairs (if any) from the left. Leave the remaining vertices as singletons.

Example: $d(10011)$ =  A number of examples are shown in Figure 19.12.

de:Gamma a (**19.6.4**) For $a \in P(\mathbb{N})$ we write Γ_a for the list of arcs (i.e. pairs) in $d(\mathbf{b}(a))$ corresponding to 01 subsequences, and an initial 11 subsequence (i.e. if there is one in the 12-position); and Γ^a for the list of all arcs.

Example,

$$\Gamma_{1356} = \{\{2,3\},\{4,5\}\} \qquad\qquad \Gamma^{1356} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

(NB we may write 1356 for $a = \{1,3,5,6\}$, and so on, where no ambiguity arises). See Figure 19.12 for more examples. We may write $\Gamma_{\delta,\lambda}$ for $\Gamma_{o_\delta(\lambda)}$, and Γ^λ_δ for $\Gamma^{o_\delta(\lambda)}$.

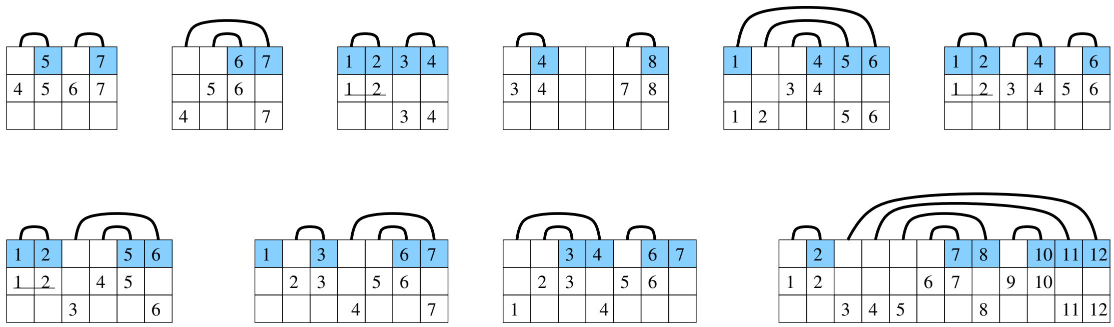


Figure 19.12: Examples for the map from sequences to TL-diagrams, and to sets of pairs Γ_a and Γ^a. In each case the top row of boxes gives $a \in P(\mathbb{N})$ and the sequence $\mathbf{b}(a)$ (by shading 1s). The second row indicates the set of pairs of numbers Γ_a extracted from the TL construction. The third row shows the further pairs added to obtain the set Γ^a.

algo-check11

de:hyper DG (**19.6.5**) For S a finite set, let $\mathtt{h}_S$ denote the cover graph (the Hasse graph) of the subset partial order on $P(S)$. (For definiteness we might draw the vertex corresponding to the empty set at the top; and vertices corresponding to single element subsets of S directly below this; and so on, with the vertex corresponding to S at the bottom.)

Note that $\mathtt{h}_S$ can be 'drawn' as a hypercube in $\mathbb{R}^{|S|}$ in an obvious way. Two edges in $\mathtt{h}_S$ are *parallel* if they correspond to adding/deleting the same element of S. The edges coming out of the top vertex are called *shoulder* edges, and every edge is parallel to one of these.
There is an obvious association with the notion of the (geometrical) hypercube or hypercuboid, i.e. the $\{0,1\}$-span of any linearly independent collection of vectors in a space. The notion of parallel edges comes from this.

A *hypercubical directed graph* is a rooted directed graph isomorphic to $\mathtt{h}_S$ for some set S.

de:hyp (**19.6.6**) Each $a \in P(\mathbb{N})$ defines a hypercubical directed graph h^a isomorphic to $\mathtt{h}_{\Gamma^a}$, as follows. The vertices are binary sequences (these should be considered as identified with elements of $P(\mathbb{N})$ by the bijection $\mathbf{b}$, but it is convenient to treat them as binary sequences for the construction). Firstly a defines a TL-diagram $d(\mathbf{b}(a))$. The top sequence in h^a is the defining sequence $\mathbf{b}(a)$. There is an edge out of this corresponding to each completed arc in the TL-diagram $d(\mathbf{b}(a))$. The sequence at the other end of a given edge is obtained from the original by replacing $01 \to 10$ (or $11 \to 00$) at the ends of this arc. Indeed every parallel edge in the hypercube follows this transformation rule.

There is an example in Figure 19.13 (and an example starting from given δ and λ in Section 19.6.3).

(**19.6.7**) We label each edge of the hypercube h^a (i.e. each direction) by the corresponding element $\{\alpha, \alpha'\} \in \Gamma^a$. That is, α, α' are the positions of the ends of the arc associated to this edge.

If label $\{\alpha, \alpha'\}$ has $\alpha' = \alpha + 1$ for an 01-arc, we may just label the edge by α. If $\{\alpha, \alpha'\} = \{1, 2\}$ for a 11-arc we may just label the edge by 12.
(Note that these are the edges associated to Γ_a.)

rem:gammaa Remark. Note that the α-edges and 12-edges in h^a in particular then coincide with edges of G_{even}, although other edges do not.

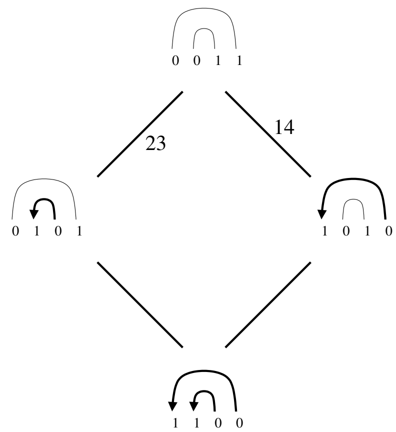


Figure 19.13: Hypercube $h^{3\,4}$ (showing the TL arcs used in the construction). fig:seg

(**19.6.8**) Note from the construction that these hypercubes are multiplicity-free. That is, no two vertices have the same label.

de:hypercub (**19.6.9**) Since fixing a block $[\lambda]_\delta$ establishes a bijection $o_\delta^\lambda : P_{even}(\mathbb{N}) \to [\lambda]_\delta$ the construction for h^a also defines a hypercubical directed graph $h_\delta(\mu)$ for each pair $(\delta, \mu) \in \mathbb{Z} \times \Lambda$, obtained by applying o_δ^μ to the vertices.

That is, abusing notation slightly, $h_\delta(\mu)$ is such that

$$o_\delta(h_\delta(\mu)) = h^{o_\delta(\mu)}$$

pa:isMiBS (**19.6.10**) Lemma. *If a vertex of some hypercube $h_\delta(\tau)$ is $\mathbf{b}_\delta(\lambda)$ for some λ, then a vertex beneath it down an α or 12-edge is $\mathbf{b}_\delta(\mu)$ for some μ a maximal δ-balanced subpartition of λ.*

Proof. This follows from the hypercube construction, the remark in (19.6.7), and the isomorphism $G_\delta(\lambda) \cong G_{even}$ (Theorem 19.4.28).

para:decomp data (**19.6.11**) Note that we have assigned a hypercube to each appropriate binary sequence and hence to each vertex of G_{even}. Thus for any given block $[\lambda]_\delta$ we have asigned a hypercube to each partition in the block. The vertices in this hypercube then correspond to partitions in the same block (the defining one, together with one of each of some collection below the defining one). In this way we can use the hypercubes to determine, for each δ, a matrix (of almost all 0s, and some 1s), with rows and columns labelled by partitions. The 1's in any given row are given by the vertices of the hypercube associated to the partition labelling that row.
In light of this interpretation we shall write $h_\delta(\mu)_\nu = 1$ if ν appears in $h_\delta(\mu)$, and $= 0$ otherwise. We will see in Theorem 19.6.12 that the resultant matrix gives our block decomposition matrix.

It will also be useful to consider an intermediate encoding, between the hypercube and the constant matrix row, in which we record the *depth* i of each entry in the hypercube, by writing v^i (v a formal parameter) instead of 1 in the appropriate position. (Thus this polynomial version evaluates to the decomposition matrix at $v = 1$.) The first few vertices of this form are shown in Figure 19.14, using the $P(\mathbb{N})$ labelling scheme.

	0	12	13	14	23	15	24	16	25	34	17	26	35	18	27	36	45	1234	19	28	37	46	1235	1X	29	38	47	56	1236	1245	1 11	2 10	39	48	57	1237	1246	1345	1 12
0	1																																						
12	v	1																																					
13		v	1																																				
14			v	1																																			
23			v		1																																		
15				v		1																																	
24			v^2	v	v		1																																
16						v		1																															
25				v^2		v	v		1																														
34		v^2	v				v			1																													
17								v			1																												
26						v^2		v	v			1																											
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36									v^2			v	v			1																							
45					v^2		v						v				1																						
1234	v^2	v								v								1																					
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37												v^2			v	v					1																		
46													v^2			v	v					1																	
1235										v^2			v					v					1																
1X																			v					1															
29														v^2					v	v					1														
38															v^2					v	v					1													
47																v^2					v	v					1												
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1 11																								v							1								
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39																				v^2					v	v							1						
48																					v^2					v	v							1					
57																						v^2					v	v							1				
1237																v^2					v								v							1			
1246													v^3			v^2	v^2					v	v^2						v	v							1		
1345									v^2				v																	v								1	
1 12																															v								1

Figure 19.14: Table encoding array of polynomials in the G_{even} labelling scheme (every non-zero polynomial is of form v^i).

fig: big pKL

ss:decomp thy

19.6.2 The main Theorem

When δ is fixed, we may use the abbreviation: $P_\lambda \;:=\; P_n^\delta(\lambda^T) = P_n^\delta(\lambda)'$, and similarly for Δ_λ.

th:decomposi

(**19.6.12**) THEOREM. *For each $\delta \in \mathbb{Z}$ and $\lambda \in \Lambda$, the hypercube $h_\delta(\lambda)$ (as defined in (19.6.9)) gives the λ-th row of the (δ, λ)-block of the matrix D of Δ-filtration multiplicities of indecomposable projective modules for $B_n(\delta)$ over $\mathbb{C}$ (any n). That is*

$$(P_n^\delta(\lambda)' : \Delta_n^\delta(\mu)') = h_\delta(\lambda)_\mu$$

for all μ, for all $n \geq |\lambda|$; or equivalently

$$P_n^\delta(\lambda)' = \; \mathop{+}_\mu \, h_\delta(\lambda)_\mu \; \Delta_n^\delta(\mu)'$$

(Recall we omit $\lambda = \emptyset$ in case $\delta = 0$.)

(**19.6.13**) REMARK. (With the $\delta = 0$ caveat Specht and standard modules coincide and we may interpret the above either as Specht characters, as required for the Cartan decomposition matrix; or as multiplicities in standard filtrations.)

Proof. We prove for a fixed but arbitrary δ, working by induction on n. The base cases are $n = 0, 1$, which are trivial (and $n = 2$ for $\delta = 0$, which is straightforward). Let $\mathbb{P}[\lambda]$ be the proposition that $(P_m^\delta(\lambda)' : \Delta_m^\delta(\mu)') = h_\delta(\lambda)_\mu$ for all μ, with $m = |\lambda|$. (Note that $\mathbb{P}[\lambda]$ implies the same result for all $m > |\lambda|$ by (18.6.18)). We assume that the theorem holds up to level $n - 1$, and consider $\lambda \vdash n$ (for $|\lambda| < n$ the result at level n holds by (18.6.18) and the inductive assumption).

The λ-th row of D encodes the standard content of projective module P_λ. We apply the 'translation' functor $\mathrm{Proj}_\lambda\mathrm{Ind}-$ to a suitable $P_{\lambda - e_i}$ in level $n - 1$ (with content known by the inductive assumption), and use Prop.(18.6.21): $\mathrm{Proj}_\lambda\mathrm{Ind}\; P_{\lambda - e_i} \;\cong\; P_\lambda \bigoplus Q$ (some Q depending on i and λ). Thus the main challenge is to determine this Q. In general this can be complicated, but we will show that there is always a choice of $\lambda - e_i$ which makes it tractable.

Note that if λ is at the bottom of its block then the claim is trivially true. If λ is not at the bottom of its block then (by construction) the binary sequence $\mathbf{b}_\delta(\lambda)$ has at least one 01 (or initial 11) subsequence. Thus we can choose e_i to be a removable box from the skew associated to the edge α (say) of $h_\delta(\lambda)$ corresponding to one such subsequence. We write $\mu = \alpha\lambda$ for the partition at the other end of this edge, so the skew is $\lambda/\mu = \lambda/\alpha\lambda$. Note that, by (19.6.10), this skew is a δ-pair.

Before starting the induction, in §19.8, we will need to establish some Lemmas.

ss:hypex

19.6.3 Hypercubical decomposition graphs: examples

(**19.6.14**) Here is a concrete example of $h_\delta(\lambda)$ with $\delta = 2$. We take $\lambda = (7, 7, 6, 5, 3, 2)$ so

$$\mathsf{e}_2(\lambda) \;=\; \lambda + \rho_2 = (6, 5, 3, 1, -2, -4, -7, -8, ...)$$

giving $o_2(\lambda) = \{1, 3, 5, 6\}$ and hence $\Gamma_\delta^\lambda = \{\{2, 3\}, \{4, 5\}, \{1, 6\}\}$. The specific hypercube (with integer partitions at the vertices) is thus (a) in Figure 19.15. In the figure we have recorded both the α-action and the specific reflection group action required to achieve it on each edge (for the shoulder layer). The version in (b) shows the G_{even} vertex labels. The version in (c) shows

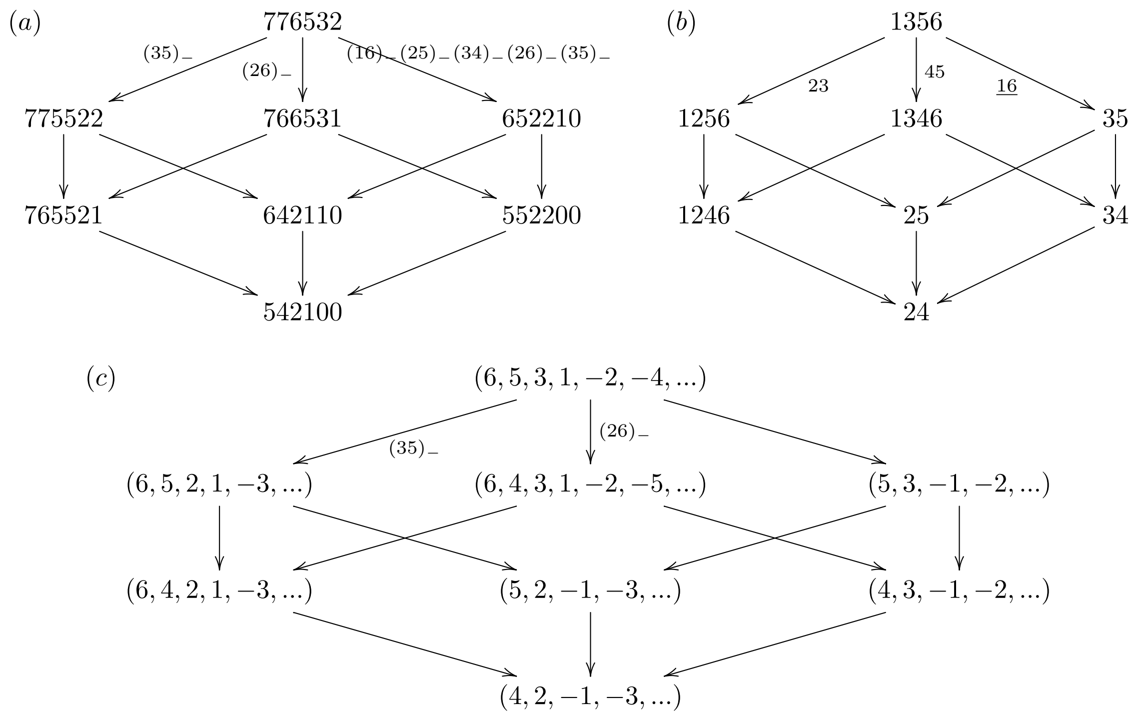


fig:below: Figure 19.15: Three labellings of the same hypercube in case $\delta = 2$: (a) partition labelling; (b) $P(\mathbb{N})$ labelling; (c) descending sequence labelling.

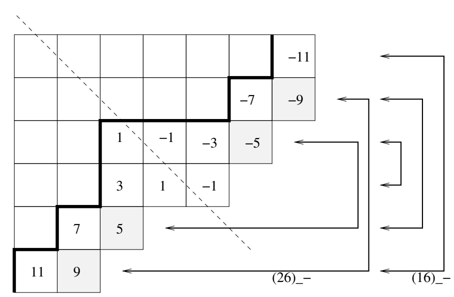


Figure 19.16: Explicit reflections on $\lambda = 776532$ in case $\delta = 2$. fig:fol

the ρ_δ-shifted vertex labels. Figure 19.16 shows the explicit reflections and composite reflection in the shoulder. Note that the composite can be built as five dominance preserving but not all commuting reflections.

(**19.6.15**) Keeping the same δ, λ, now consider $\lambda - e_i$ in case $i = 4$.

This gives $(7,7,6,4,3,2) \overset{e_2}{\leadsto} (6,5,3,0,-2,-4,-7,...) \overset{o_2}{\leadsto} \{1,3,5,6\}$ (by the toggle rule). This means that the hypercube $h_\delta(\lambda - e_i)$ is isomorphic [6] to that for λ above, so in particular the α-actions (the formal edge labels) are the same. Note also that the specific reflections (realising these α-actions) in the shoulder of $h_\delta(\lambda - e_i)$ are the same as for λ.

REMARK. We show in Section **??** that so long as e_i does not 'separate' a MiBS (in the sense of (19.7.5)) this holds true in general. That is the hypercubes are isomorphic and the reflections needed to move through the hypercube are the same.

A more complicated example is given in Figure 20.1.

19.6.4 Hypercubical decomposition graphs: tools

We conclude this Section with some tools for manipulating the hypercubes h^a and $h_\delta(\lambda)$, that we shall need later.

de:bump (**19.6.16**) Let $b = (b_1, b_2, ...)$ be a binary sequence, and α a natural number. Then $\hat{\alpha}b$ is the sequence obtained from b by inserting 01 in the $\alpha, \alpha+1$ positions (i.e. so that this pair become the elements in the α and $\alpha+1$ positions in the sequence, with any terms at or above these positions in b bumped two places further up in $\hat{\alpha}b$):

$$\hat{\alpha}(b_1, b_2, ...) = (b_1, b_2, ..., b_{\alpha-1}, 0, 1, b_\alpha, ...)$$

Similarly $\check{\alpha}b$ is the sequence obtained from b by inserting 10 in the $\alpha, \alpha+1$ positions.
Examples: $\hat{2}01 = 0011$, $\check{2}01 = 0101$.

de:ai act (**19.6.17**) Let h be a hypercube considered 'geometrically' as in (19.6.5). That is, h is the $\{0,1\}$-span of any linearly independent collection of vectors. Let α be a vector outside the span of h (or an operator that can otherwise be considered to shift all the vertices of h by the same amount in a new direction). Then by αh we mean the translate of h determined by α, and by $(1,\alpha)h$ we mean the new hypercube which contains h and a translate of h by α together with the edges in the α direction.

(**19.6.18**) If h is a hypercube whose vertices are binary sequences, all of which have 01 (or all 11) in the $\alpha, \alpha+1$ positions, then αh is the hypercube defined from h by modifying this $01 \to 10$ (respectively $11 \to 00$). In this case $(1,\alpha)h$ is the 'hypercubical union' of h and αh.

(**19.6.19**) Suppose there is a bumped sequence $\hat{\alpha}\mathbf{b}_\delta(\lambda)$. Then by $\hat{\alpha}h_\delta(\lambda)$ we understand the corresponding vertex-modified hypercube (insert 01 at the same position in every vertex binary sequence, and modify any edge labels affected by this bump accordingly — the partition forms of the vertices are obtained from the $P(\mathbb{N})$ forms by applying o_δ^λ, just as for the definition of $h_\delta(\lambda)$ itself). Note that $\hat{\alpha}h_\delta(\lambda)$ is not a hypercube of form $h_\delta(\mu)$ (in fact it is a subgraph of somesuch). Similarly define $\check{\alpha}h_\delta(\lambda)$ (and note that $\check{\alpha}h_\delta(\lambda) = \alpha\hat{\alpha}h_\delta(\lambda)$). Note that $\check{\alpha}h_\delta(\lambda)$ is another hypercube not of form $h_\delta(\mu)$. However we have the following.

[6] when do we spell out what this means?

LEMMA.

$$(1,\alpha)\hat{\alpha}h_\delta(\lambda) \cong h_\delta(\mu) \qquad \text{for any } \mu \text{ such that } \mathbf{b}_\delta(\mu) = \hat{\alpha}\mathbf{b}_\delta(\lambda) \tag{19.6}$$

eq:hyp lemma

Here $\cong$ means that $o_\delta((1,\alpha)\hat{\alpha}h_\delta(\lambda)) = o_\delta(h_\delta(\mu))$.

This is simply a restatement of part of the definition (19.6.6), that will be useful later.

Proof. The difference between $\hat{\alpha}\mathbf{b}_\delta(\lambda)$ and $\mathbf{b}_\delta(\lambda)$ is such that the TL-diagram of the former has an extra arc, so Γ^μ_δ has an extra element. Hypercube $h_\delta(\mu)$ thus has a 'half' which is like $h_\delta(\lambda)$ but with the $\hat{\alpha}$ bump, and a half like $h_\delta(\lambda)$ but with the $\check{\alpha}$ bump (i.e. the vertices on which $01 \to 10$ has been applied at this arc). $\square$

19.7 Embedding properties of δ-blocks in Λ

In this section we consider how the block graphs embed in $\mathbb{R}^{\mathbb{N}}$ and hence how the embeddings of the different block graphs relate to each other. The branching rule result (18.5.3) means, loosely speaking, that the usual *metrical* structure on $\mathbb{R}^{\mathbb{N}}$ has relevance in our representation theory. This, together with the embedding results we develop here, will allow us to pass information between blocks.

(**19.7.1**) Notations:

de:alpha lambda When δ is fixed, for $w \in \mathcal{D}$ and $\lambda \in \Lambda$, we write $w.\lambda$ for the μ such that $w\mathtt{e}_\delta(\lambda) = \mathtt{e}_\delta(\mu)$.

If a is a vertex of G_{even} and α is the label on an edge into (respectively, out of) a, then we write $\overline{\alpha}a$ (resp. $\underline{\alpha}a$) for the vertex at the other end.

(Note that there is at most one edge with label α out of, or indeed touching, each vertex.)

In case δ is fixed, if λ is a vertex of some $G_\delta(\mu)$, and α is the label inherited from G_{even} on an edge into λ, then we write $\overline{\alpha}\lambda$ for the vertex at the other end.

Where no ambiguity arises, we may write simple $\alpha\lambda$ for $\overline{\alpha}\lambda$.

Note that, for given δ, Lemma (19.5.11) associates a specific involutive $w \in \mathcal{D}$ to each such pair $(\lambda, \overline{\alpha}\lambda)$, such that $w.\lambda = \overline{\alpha}\lambda$.

Suppose b is a binary sequence. Then for $\alpha \in \mathbb{N}$, if the α-th and α+1-th terms in the sequence are 01, then $\overline{\alpha}b$ denotes *toggling* these terms, i.e. taking 01 to 10.

le:nutbolt1 LEMMA. CLAIM: Fix δ. For $\lambda \in \Lambda$, note that the hypercube shoulder subset $\Gamma_{\delta,\lambda}$ coincides with the set of labels on *'descending' edges out of* λ (i.e. G_{even}-directed edges into λ) in $G_\delta(\lambda)$. Thus if we take $\alpha \in \Gamma_{\delta,\lambda}$ then $\lambda/\alpha\lambda$ is a δ-pair. (Although other elements in the hypercube shoulder set $\Gamma^{\delta,\lambda}$ will not give a δ-pair.) Furthermore $\mathbf{b}_\delta(\alpha\lambda) = \overline{\alpha}\mathbf{b}_\delta(\lambda)$.

Proof. Only the last identity needs proof. This follows since by construction $\mathbf{b}(\overline{\alpha}a) = \overline{\alpha}\mathbf{b}(a)$ for $a = o_\delta(\lambda)$ (or indeed for any suitable a).

(**19.7.2**) Consider the isomorphism

$$\begin{tikzcd}
\mathtt{f}_{\lambda,\lambda'} : G_\delta(\lambda) \arrow[rr, "\longrightarrow", phantom] \arrow[dr, "o_\delta"] & & G_\delta(\lambda'), \\
& G_{even} \arrow[ur, "o_\delta^{\lambda'}"'] &
\end{tikzcd} \tag{19.7}$$ eq:gif01

implicit in Theorem 19.5.21, between any pair of block graphs. In particular this map defines a pairing of each vertex v in $G_\delta(\lambda)$ with the corresponding vertex $\mathtt{f}_{\lambda,\lambda'}(v)$ in $G_\delta(\lambda')$.

A pair of block graphs is *adjacent* if they have the same singularity, and every such pair of vertices $(v, \mathtt{f}_{\lambda,\lambda'}(v))$ is adjacent as a pair of partitions.

(**19.7.3**) REMARK. Fix δ. If λ, λ' are adjacent partitions such that their images $\mathtt{e}_\delta(\lambda), \mathtt{e}_\delta(\lambda')$ lie in the same $\mathcal{D}$-facet (in the alcove geometric sense of (19.5.1)) then the corresponding pair of graphs are adjacent, since the same reflection group elements serve to traverse these graphs (i.e. a sequence of reflections taking λ to μ, say, will take λ' to the isomorphic image $\mathtt{f}_{\lambda,\lambda'}(\mu)$), and reflection group elements preserve adjacency of partitions. We shall need to show adjacency of a more general pairing of graphs.

de:fie (**19.7.4**) For given λ, if $\lambda' = \lambda - e_i$ in (19.7), for some i, we write $\mathtt{f}_{\lambda,\lambda'} = \mathtt{f}_i$, and hence also

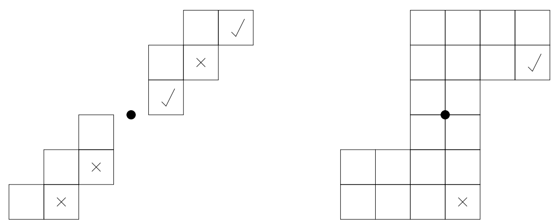
$$\mathtt{f}_i : [\lambda]_\delta \;\rightarrow\; [\lambda - e_i]_\delta$$

for the restriction of the graph isomorphism $\mathtt{f}_{\lambda,\lambda'}$ to vertices.

de:separates (**19.7.5**) Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i. Suppose that $\lambda/\alpha\lambda$ is a δ-pair containing e_i. Write π_α for the π-reflection fixing this δ-pair. Then
(I) Note that (by construction) $\pi_\alpha(e_i)$ is an addable box of $\alpha\lambda$.
(II) If $\lambda/\alpha\lambda \setminus \{e_i, \pi_\alpha(e_i)\}$ is not a δ-pair (of $\lambda - e_i$) we say that e_i *separates* $\lambda/\alpha\lambda$.

(**19.7.6**) REMARK. We avoid using the notion of separation *per se* in the proof of Theorem 19.6.12. Instead we will use a method of choosing a removable box from a δ-pair which, while not canonical, serves our purpose. (However, the point is that the chosen box does not separate.)

Examples: crosses show boxes that separate; ticks show boxes that do not:

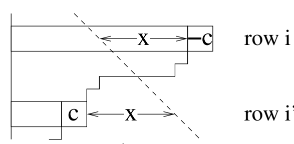


le:ch3 (**19.7.7**) LEMMA. *(Charge-row lemma) Fix any δ. If a row i of partition λ ends in a box with charge c we have*

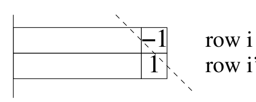
$$(\lambda + \rho_\delta)_i = -\frac{c}{2} + \frac{1}{2}$$

le:combinME-alt1 (**19.7.8**) LEMMA. *Fix δ. If $\lambda, \lambda - e_i \in \Lambda$ and $\lambda/\lambda - e_i - e_{i'}$ a δ-pair then $s_\delta(\lambda - e_i) = s_\delta(\lambda) + 1$.*

Proof. The relevant part of λ is as here:



or here:



Removing the box with charge $\boxed{\text{-c}}$ means rows i, i' become a singular pair in $\lambda - e_i$, where they were not before, so $p_\delta(\lambda - e_i) \neq p_\delta(\lambda)$. Thus $s_\delta(\lambda - e_i) = s_\delta(\lambda) + 1$ unless we also *lost* a singular pair i, i''. This would have to be with $i'' = i' + 1$ directly under $\boxed{\text{c}}$, but this cannot happen since $\boxed{\text{c}}$ is removable. (The argument in the $c = 1$ case is similar.)

We have (for some $x \geq 0$):

$$\begin{array}{rlccccc}
\mathsf{e}_\delta(\lambda) & = (\lambda_1 - \frac{\delta}{2}, \ldots, & \overbrace{x+1}^{i-th} & , \ldots, & \overbrace{-x}^{i'-th} & , \ldots) \\
\mathsf{e}_\delta(\lambda - e_i) & = (\lambda_1 - \frac{\delta}{2}, \ldots, & x & , \ldots, & -x & , \ldots) \\
\mathsf{e}_\delta(\alpha\lambda) = \mathsf{e}_\delta(\lambda - e_i - e_{i'}) & = (\lambda_1 - \frac{\delta}{2}, \ldots, & x & , \ldots, & -x-1 & , \ldots)
\end{array}$$

$\square$

19.7.1 The Relatively–regular–step Lemma

(**19.7.9**) REMARK. Fix δ, and hence an embedding $\mathtt{e}_\delta$ of the index set Λ into $\mathbb{R}^{\mathbb{N}}$ (on which $\mathcal{D}$ then acts). For each λ, $o_\delta(\lambda)$ determines its *position* in the block. Thus (I) if $\lambda, \lambda - e_i$ are regular in the same alcove then $o_\delta(\lambda) = o_\delta(\lambda - e_i)$. On the other hand it is not possible to make such a small step and jump entirely from one alcove to another. Thus (II) if $\lambda, \lambda - e_i$ are both regular, they are in the same alcove.

The argument for (I) applies if $\lambda, \lambda - e_i$ are in the same facet. However it is possible to change to a co-regular (i.e. equal singularity) but distinct facet in one step. It is not so clear that this new facet corresonds to the same block position. It turns out, though, that it does. Here we show this.

le:combinME-alt (**19.7.10**) LEMMA. *Fix δ. If $\lambda, \lambda - e_i \in \Lambda$ and $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $o_\delta(\lambda) = o_\delta(\lambda - e_i)$.*

Proof. Write x for $(\lambda + \rho_\delta)_i$. Thus, for some $y < x - 1$:

$$\lambda + \rho_\delta \sim (..., \underbrace{x}_{i}, y, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., \underbrace{x-1}_{i}, y, ...) \tag{19.8}$$ eq:xx-1y

If $p_\delta(\lambda) = p_\delta(\lambda - e_i)$ then one can readily check that the changed row i appears in the magnitude order in both cases, and in the same position. In case $x = 1/2$ there is a sign change, but by the toggle rule $o_\delta(-)$ remains unchanged. If $p_\delta(\lambda) \neq p_\delta(\lambda - e_i)$ then from (19.8) we see firstly that $-x$ occurs in $\lambda + \rho_\delta$ and $1 - x$ occurs in $\lambda + \rho_\delta - e_i$ (for if neither occurs then p_δ does not change between them; while if only one occurs then s_δ changes);
it follows immediately that $1 - x, -x$ occur (and are adjacent) in both;
secondly, $y < x - 1$ so $x - 1$ does not occur in $\lambda + \rho_\delta$.

In computing o_δ we discount the $\pm x$ pair in $\lambda + \rho_\delta$ and the $\pm(x-1)$ pair in $\lambda + \rho_\delta - e_i$. The discrepancy is thus now a $1 - x$ in $\lambda + \rho_\delta$ compared to a $-x$ in $\lambda + \rho_\delta - e_i$. But if $1 - x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta$ then $-x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta - e_i$, with all else equal, so o_δ is unchanged. □

lem:fh (**19.7.11**) LEMMA. *If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $\mathtt{f}_i(h_\delta(\lambda)) = h_\delta(\lambda - e_i)$. In particular $h_\delta(\lambda)_\mu = h_\delta(\lambda - e_i)_{\mathtt{f}_i(\mu)}$ for all μ.*

Proof. By Prop. 19.7.10 $o_\delta(\lambda) = o_\delta(\lambda - e_i)$, so these hypercubes pass to the same $h^a = h^{o_\delta(\lambda)}$ by (19.6.9). Thus $h_\delta(\lambda - e_i) = o_\delta^{\lambda - e_i}(o_\delta(h_\delta(\lambda)))$. But $\mathtt{f}_i(-) = o_\delta^{\lambda - e_i}(o_\delta(-))$ by definition. □

19.7.2 The Reflection Lemmas

le:do1 (**19.7.12**) LEMMA. *Fix δ and suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before. Suppose λ has an edge down in $G_\delta(\lambda)$ labelled α (i.e. $\alpha \in \Gamma_{\delta,\lambda}$); and let $w = w_{\lambda/\alpha\lambda} \in \mathcal{D}$ be the product of commuting reflections such that $w\mathtt{e}_\delta(\lambda) = \mathtt{e}_\delta(\alpha\lambda)$, as in Lemma (19.5.11). Then*
(I) $w\mathtt{e}_\delta(\lambda - e_i)$ is dominant (i.e. $w.(\lambda - e_i) \in \Lambda$);
(II) $w\mathtt{e}_\delta(\lambda - e_i) = \mathtt{e}_\delta(\alpha(\lambda - e_i))$;
(III) $\alpha(\lambda - e_i) \overset{\triangleleft}{\triangleright} \alpha\lambda$ (i.e. $\alpha(\lambda - e_i)$, $\alpha\lambda$ are adjacent).

Proof. (I) We split into two cases:
(A) If e_i intersects $\lambda/\alpha\lambda$ then $\pi_\alpha(e_i)$ (the rotation image of the box at the end of row-i, call it e_j) is addable to $\alpha\lambda$ as noted in (19.7.5)(I). [7]

Since w is a product of commuting reflections achieving the same effect on $\alpha\lambda$ as adding the skew $\lambda/\alpha\lambda$, we would like to claim that row-i and row-j — the row containing $\pi_\alpha(e_i)$ — are paired in w (WOULD WE?! IS IT TRUE? DOES IT MATTER?! — see (19.5.8)) so that the effect of w on $\lambda - e_i$, where row-i is one box shorter, is indeed to make row-j one box longer.

This all seems to hang on the nature of $w_{\lambda/\alpha\lambda}$.

...

That is $\mathsf{e}_\delta(\alpha\lambda + \pi_\alpha(e_i)) = \mathsf{e}_\delta(\alpha\lambda) + e_j = w\mathsf{e}_\delta(\lambda - e_i)$ is dominant.
(B) If e_i does not intersect $\lambda/\alpha\lambda$ then $w\mathsf{e}_\delta(\lambda-e_i)$ is the same as $w\mathsf{e}_\delta(\lambda)$ everywhere except in row-i: $w\mathsf{e}_\delta(\lambda - e_i) = w\mathsf{e}_\delta(\lambda) - e_i$ (since w acts non-trivially only on the rows involved in $\lambda/\alpha\lambda$). Since $\lambda - e_i$ is dominant (i.e. a partition), $\lambda_i > \lambda_{i+1}$, but $(\alpha\lambda)_i = \lambda_i$ in this case, and $(\alpha\lambda)_{i+1} \leq \lambda_{i+1}$ (since $\alpha\lambda \subset \lambda$), so $(\alpha\lambda)_i > (\alpha\lambda)_{i+1}$, so $\alpha\lambda - e_i$ is dominant, so $\mathsf{e}_\delta(\alpha\lambda - e_i) = \mathsf{e}_\delta(\alpha\lambda) - e_i = w\mathsf{e}_\delta(\lambda) - e_i = w\mathsf{e}_\delta(\lambda - e_i)$ is dominant.

(II) Note that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ by Lemma 19.7.10, so there is an α-edge out of $(\lambda - e_i)$ in $G_\delta(\lambda-e_i)$, and so $\alpha(\lambda-e_i)$ makes sense. Indeed $o_\delta(\alpha(\lambda-e_i)) = o_\delta(\alpha\lambda) = \alpha o_\delta(\lambda)$. Since $w\mathsf{e}_\delta(\lambda-e_i)$ is dominant (by (I)) in the $\mathcal{D}$-orbit of $\lambda-e_i$ there is some $\mu \in [\lambda-e_i]_\delta$ such that $w\mathsf{e}_\delta(\lambda-e_i) = \mathsf{e}_\delta(\mu)$. Since $w\mathsf{e}_\delta(\lambda - e_i) = \mathsf{e}_\delta(\mu)$ is adjacent to $w\mathsf{e}_\delta(\lambda) = \mathsf{e}_\delta(\alpha\lambda)$ and has the same singularity, then by Lemma (19.7.10) (applied appropriately) $o_\delta(\mu) = o_\delta(\alpha\lambda)$. That is, $\mu = \alpha(\lambda - e_i)$.

(III) Follows immediately from (II). □

le:do2 **(19.7.13)** LEMMA. *Fix δ. Suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before, and $\underline{\alpha}\lambda/\lambda$ is MiBS (i.e. α is an edge up from λ). Then there is a reflection group element w among those obeying $w.\lambda = \underline{\alpha}\lambda$ (and $w.\underline{\alpha}\lambda = \lambda$) such that (I) $w.(\lambda - e_i)$ is dominant; whereupon (II) $w.(\lambda - e_i) = \underline{\alpha}(\lambda - e_i)$, and so (III) $\underline{\alpha}\lambda, \underline{\alpha}(\lambda - e_i)$ are adjacent.*

Proof. Suppose that there *is* a w such that $w.\lambda = \underline{\alpha}\lambda$ and $w.(\lambda-e_i)$ is dominant. Then $w.(\lambda-e_i)$ is $\mu \in [\lambda-e_i]_\delta$ adjacent to $w.\lambda = \underline{\alpha}\lambda$ with the same singularity, hence the same o_δ by Lemma (19.7.10). Thus it is enough to show that there is a suitable w such that $w.(\lambda - e_i)$ is dominant. (It is worth noting that the condition $w.\lambda = \underline{\alpha}\lambda$ does not necessarily uniquely determine $w \in \mathcal{D}$.)
[8]

We must consider the cases: (A) the i-th row of λ is affected by w, that is, e_i lies 'behind' the skew (i.e. it's image under the π-rotation π_α that fixes $\underline{\alpha}\lambda/\lambda$ extends some row of the skew); or (B) the i-th row is not moved by w.

(A) In this case the failure of dominance (if any) would have to be that the image of e_i under the π-rotation broke dominance, i.e. extended beyond the row above it.

Suppose e_i is behind other than the last row of the skew. Then there is a box of the skew immediately to its right and one immediately below it. The π-rotation images of these are behind and above the image of e_i, so $w.(\lambda - e_i)$ is dominant.

[7] This argument is one of the reasons for giving a direct geometrical characterisation of δ-pair $\lambda/\alpha\lambda$.

[8] — DO WE NEED ALL OF THE NEXT BIT ??? —

Given that $w.\lambda$, $\lambda - e_i$ are dominant, any *failure* of dominance of $w.(\lambda - e_i)$ must either involve the i-th row in $w.(\lambda - e_i)$ itself being shorter than row-$i+1$ (i.e. row-$i+1$ intersects the MiBS; row i does or does not???); or a row with which row-i is paired in w (j, say) being longer than row-$j-1$ in $w.(\lambda - e_i)$.

— DO WE NEED THE BIT ABOVE HERE???? —

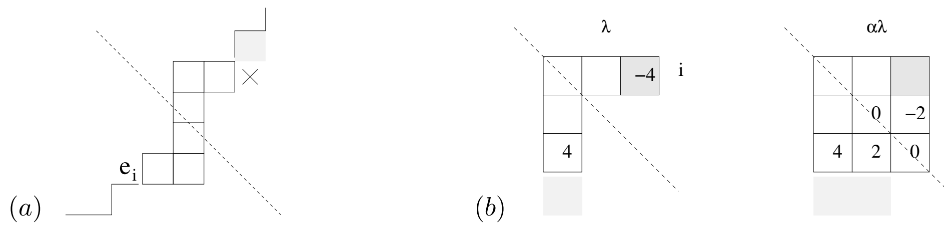


Figure 19.17: Examples (a), (b) for Lemma 19.7.13. fig:ab12

On the other hand, suppose e_i is behind the last row of the skew. For example see Figure 19.17(a) (the box $\pi_\alpha(e_i)$ is marked $\times$). Here $w.(\lambda - e_i)$ is dominant unless the box above $\pi_\alpha(e_i)$ is missing from λ. But if this is missing then this row and the i-row are a singular pair in $\lambda - e_i$. Neither row can be in a singular pair in λ so this contradicts the hypothesis.

(B) If the i-th row of λ is not moved by w then any failure would have to be one of two possibilities. Either (i) at row-i: w does not involve row-i at all, so $(w.(\lambda - e_i))_i = (w.\lambda)_i - 1$, while the skew $\underline{\alpha}\lambda/\lambda$ includes a box directly under e_i. Or (ii) that w contains a factor $(i\, i')_-$ (say) that fixes λ but not $\lambda - e_i$, and that row-i' of $w.(\lambda - e_i)$ is 'too long'.

But in case (i) a δ-balanced box to e_i given by $\pi_\alpha(e_i)$ is directly to the left of the skew, and we have a setup something like Figure 19.17(b) (the δ-balanced box is the box marked 4). If there is no box below the $\pi_\alpha(e_i)$ in λ (the lightly shaded space in the Figure — let us call this row i') then row-i is not in a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$, thus $s_\delta(\lambda) \neq s_\delta(\lambda - e_i)$ so we can exclude this.

If there is a box below the $\pi_\alpha(e_i)$ in λ then this row-i' and row-i are a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$. In this case, a w not involving row-i could produce a non-dominant $w.(\lambda - e_i)$. However, in this case a w which has a factor $(i\, i')_-$ acting on the i-th and undrawn i'-row has the same effect on λ as one which does not, so we can use this instead (any viable w will serve the purpose of preserving adjacency), and consider ourselves to be in case (ii). The effect of the 'redundant' factor on $\lambda - e_i$ is to restore the box e_i and to add a box in the undrawn i'-row (the second lightly shaded box in the Figure). But this $w.(\lambda - e_i)$ is dominant (thus eliminating the last possibility for non-dominance) so long as the added box in row-i' is under a box added in the original skew — i.e. an added box in row-$(i' - 1)$. To see that this is the case, recall firstly that w acts on row-$(i + 1)$, and indeed there is a box (x say) below e_i in $\underline{\alpha}\lambda/\lambda$ by assumption. Thus $\underline{\alpha}\lambda/\lambda$ also contains a balance partner to x, i.e. a box with the opposite δ-charge. One such box would be $y = \pi_\alpha(x)$ to the right of $\pi_\alpha(e_i)$, and in fact no other box on this diagonal is a candidate (the box above $\pi_\alpha(e_i)$ is in λ, and hence so are all those to the NW of it). But if y is in $\underline{\alpha}\lambda/\lambda$ (and hence in $w.(\lambda - e_i)/(\lambda - e_i)$, which agrees with $\underline{\alpha}\lambda/\lambda$ on rows $i + 1$ to $i' - 1$) then the extra box in row-i' does not break dominance. □

19.7.3 The Embedding Theorem and the Proj$_\lambda$Ind$-$ functor

th:embed **(19.7.14)** THEOREM. *(Embedding Theorem)* *If* $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ *then* $G_\delta(\lambda)$ *is adjacent to* $G_\delta(\lambda - e_i)$.

Proof. Since the (undirected) block graph is connected all the necessary adjacent pairings follow using Lemmas 19.7.12 and 19.7.13 as appropriate to traverse the graph from λ. □

le:ijx (**19.7.15**) LEMMA. *Fix δ. No distinct pair $\lambda, \lambda - e_i + e_j \in \Lambda$ are in the same block. That is, no distinct pair $\lambda + e_i, \lambda + e_j \in \Lambda$ are in the same block.*

Proof. Such a pair cannot meet the charge-pair form of the balance condition (see Defn.4.7/Cor.4.8 in [28], or (19.2.2)), since the skew $(\lambda + e_i)/\lambda = (\lambda + e_i)/((\lambda + e_i) \cap (\lambda + e_j))$ has rank 1. □

le:ei11.1 (**19.7.16**) LEMMA. *Fix δ. For $\lambda, \lambda - e_i \in \Lambda$, if $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then for $n \geq |\lambda|$:*

$$\mathit{Proj}_\lambda \mathit{Ind}\, \Delta_{n-1}(\lambda - e_i)' = \Delta_n(\lambda)'$$

$$\mathit{Proj}_{\lambda - e_i} \mathit{Ind}\, \Delta_n(\lambda)' = \Delta_{n+1}(\lambda - e_i)' \tag{19.9}$$ eq:back.1

Proof. For any ν, Prop.18.5.3 gives $\mathrm{Ind}\, \Delta(\nu)' = \left(+_j \Delta(\nu + e_j)'\right) + \left(+_k \Delta(\nu - e_k)'\right)$ in the notation of §18.6.2. For $\nu = \lambda - e_i$, one of these summands is $\Delta(\lambda)'$. Other summands are of form $\lambda - e_i + e_j$, or $\lambda - e_i - e_k$. By Lemma (19.7.15) the former cases are not in $[\lambda]_\delta$, and since $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ we may use Lemma 19.7.8 to exclude the latter also. This proves the first identity. The other case is similar. □

An almost trivial corollary is the following.

le:ei11 (**19.7.17**) LEMMA. *Fix δ. If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then for any pair $(\mu, f_i(\mu)) \in [\lambda]_\delta \times [\lambda - e_i]_\delta$*

$$\mathit{Proj}_\lambda \mathit{Ind}\, \Delta_{n-1}(f_i(\mu))' = \Delta_n(\mu)'$$

$$\mathit{Proj}_{f_i(\lambda)} \mathit{Ind}\, \Delta_n(\mu)' = \Delta_{n+1}(f_i(\mu))' \tag{19.10}$$ eq:back

Proof. Note that the pair $(\mu, f_i(\mu))$ are adjacent by Theorem 19.7.14, i.e. they are a pair of form $\{\nu, \nu - e_j\}$ (in some order). They also have the same singularity, since singularity is a block invariant. Similarly $\mathrm{Proj}_\mu = \mathrm{Proj}_\lambda$ and $\mathrm{Proj}_{f_i(\mu)} = \mathrm{Proj}_{f_i(\lambda)}$. We can now use Lemma 19.7.16 with $\nu, \nu - e_j$ replacing $\lambda, \lambda - e_i$. □

(**19.7.18**) REMARK. In the language of [30] this says that if $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $\lambda, \lambda - e_i$ are *translation equivalent*.

19.7.4 The generic projective lemma

pr:gen0001 (**19.7.19**) PROPOSITION. [9] *Fix δ. For $\lambda, \lambda - e_i \in \Lambda$, if $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then this data fixes a block-graph isomorphism $f_i : [\lambda]_\delta \to [\lambda - e_i]_\delta$ as defined in (19.7.4) and*

$$(P_\lambda : \Delta_\mu) = (P_{\lambda - e_i} : \Delta_{f_i(\mu)}) \qquad \forall \mu \in [\lambda]_\delta$$

[9] [30] establishes Morita equivalences between certain blocks, in principle, and hence this result in some cases. However (i) the cases of Morita equivalence may be too rare to be useful; and (ii) as far as decomposition matrices are concerned the philosophy in [30], as written, is to reduce to a problem 'in alcoves' (or in some other least-singular block). Our proof here, however, *relies* on moving systematically between singularities.

Proof: By Prop. 18.6.21, $\mathrm{Proj}_\lambda \mathrm{Ind}\, P_{\lambda-e_i} \;=\; P_\lambda \oplus Q$ with $Q = \mathrm{Proj}_\lambda Q$ some projective, possibly zero. Each sequence $\mu(1), \mu(2), \ldots$ of elements from $[\lambda]_\delta$ passes to a sequence $f_i(\mu(1)), f_i(\mu(2)), \ldots$ of elements from $[\lambda - e_i]_\delta$ and so there is such a sequence for which $\Delta_{f_i(\mu(1))}, \Delta_{f_i(\mu(2))}, \ldots$ is a Δ-filtration series for $P_{\lambda-e_i}$. We write this as

$$P_{\lambda-e_i} = \;//_j\; \Delta_{f_i(\mu(j))}$$

NB we already showed [10] that this holds for any relevant n. Thus

$$\mathrm{Ind}P_{\lambda-e_i} \stackrel{(1)}{=} \mathrm{Res}GP_{\lambda-e_i} \;=\; \mathrm{Res}P_{\lambda-e_i} \;=\; \mathrm{Res}//_j\; \Delta_{f_i(\mu(j))} \;=\; //_j\; \mathrm{Res}\Delta_{f_i(\mu(j))}$$

$$= \;//_j\; \mathrm{Res}G\Delta_{f_i(\mu(j))} \;=\; //_j\; \mathrm{Ind}\Delta_{f_i(\mu(j))}$$

that is

$$P_\lambda \oplus Q \;=\; \mathrm{Proj}_\lambda \mathrm{Ind}\, P_{\lambda-e_i} \stackrel{(1)}{=} \;//_j\; \mathrm{Proj}_\lambda \mathrm{Ind}\Delta_{f_i(\mu(j))} \stackrel{(2)}{=} \;//_j\; \Delta_{\mu(j)}$$

where step-(1) uses Prop. 18.5.3(i), §18.6.2 and the exactness of Res$-$ and $\mathrm{Proj}_\lambda-$; and step-(2) uses Lemma 19.7.17.

Comparing with the decomposition of P_λ asserted in the Proposition, we see that it only remains to show that $Q = 0$.

By (19.10) each Δ-factor in $P_\lambda \oplus Q$, derived from Δ_ν in $P_{\lambda-e_i}$ say, maps back to Δ_ν under $\mathrm{Proj}_{\lambda-e_i}\mathrm{Ind}-$. Thus $\lambda - e_i$ is among these νs, and is dominant among them. But $P_{\lambda-e_i}$ is the only projective with this content, so

$$\mathrm{Proj}_{\lambda-e_i}\mathrm{Ind}\,(P_\lambda \oplus Q) \;=\; P_{\lambda-e_i}$$

Since this is not split, $Q = 0$. □

[10] ASSERTED RECENTLY!

19.7.5 Properties of δ-pairs and rim-end removable boxes

Examples of δ-pairs are shown in Figure 19.6.

(**19.7.20**) We will say that a skew is *boxy* if every box in it lies within a (2^2)-shape that also lies within the skew.

(**19.7.21**) In our case (skews of form $\lambda/\alpha\lambda$), the boxy skews are those in which the pair of rims fully overlap (i.e. run side-by-side). Thus in our case boxy skews have a terminal (2^2)-shape at each end, in which the largest magnitude charges reside. Note that since no (2^2)-shape has a removable box of largest magnitude charge, neither does a boxy skew (on the other hand every such shape has a removable box of next-largest magnitude, and one can see that the largest of these is removable at one end of the boxy skew or the other).

If a minimal skew is neither of form (1)+(1) nor boxy we shall say that it is generic.

(**19.7.22**) Lemma. *Let $\lambda/\mu = \lambda/\alpha\lambda$ be a minimal δ-balanced skew. Then there are a pair of boxes in the skew of greatest magnitude charge. In case the skew is of shape (1)+(1) both of these are removable; in the boxy cases (such as (2^2)) neither are removable (but precisely one of the next-largest is removable); and otherwise precisely one of them is removable.*

Proof. All statements are (by now) clear except the last. For this note that if both were removable this would contradict that $\alpha\lambda$ is MBS, since removing just this pair from λ would give a larger BS; while if neither were removable then again this would contradict the MBS property, since removing the complement (i.e. the boxes in $\lambda/\alpha\lambda$ *not* in this pair) would give a larger BS. □

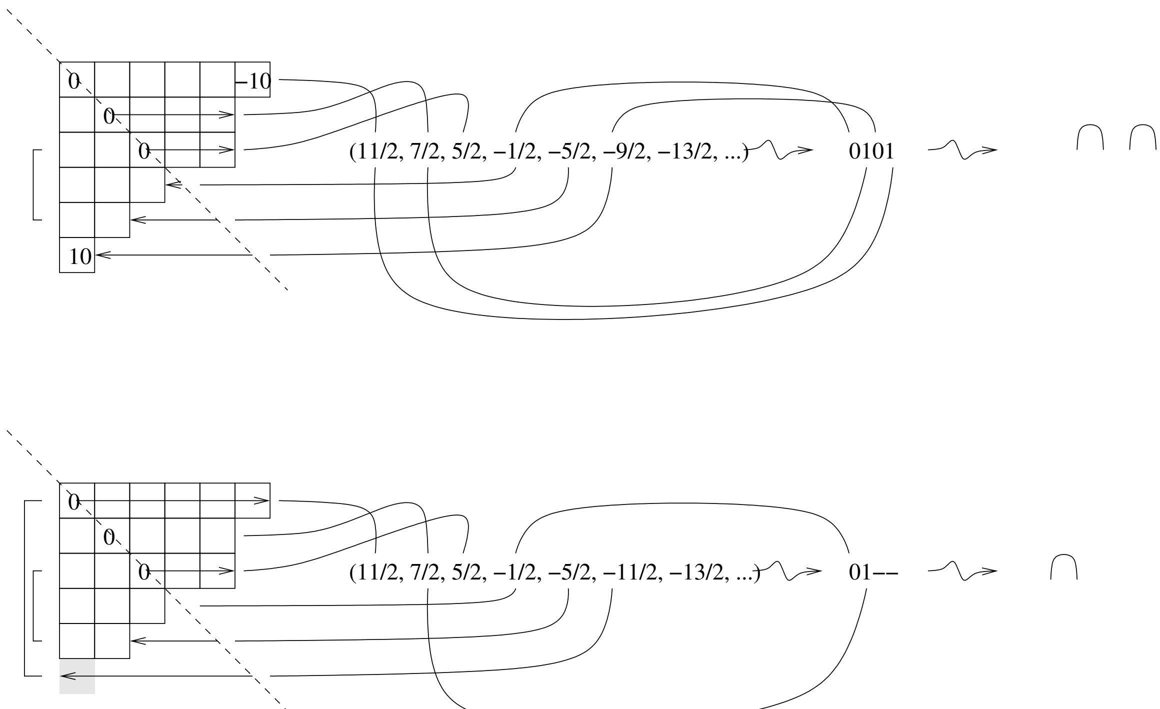
(**19.7.23**) A *rim-end removable box* in $\lambda/\alpha\lambda$ is a box of largest magnitude charge among the removable boxes of $\lambda/\alpha\lambda$.

The justification for this term is that such a box always lies at the end of a rim in some decomposition of the skew into rims (noting that there is not always a *unique* decomposition of a skew into rims).

exa:get (**19.7.24**) Example: The rim-end removable boxes (as labelled by charge) in Figure 19.6 are (i) 22; (ii) -16; (iii) 8. (For $\delta = 1$ example (i) is, in greater detail,

$$\lambda + \rho_1 = (25/2, 23/2, 21/2, 19/2, 17/2, 11/2, 9/2, -3/2, -9/2, -11/2, -17/2, -19/2, -21/2, ...)$$

which is five-fold singular (in the sense of (19.4.14)), giving $o_1(\lambda) = \{2, 3\}$ for its valley set.)



epos01 Figure 19.18: Young diagram, (case $\delta = 1$) descending sequence, binary sequence and TL diagram for two cases, λ, $\lambda - e_i$, illustrating a step up in singularity.

19.7.6 The singularity lemma

prsing (**19.7.25**) PROPOSITION. *Fix δ. For $\lambda \in \Lambda$ not minimal in its block pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. If $|\lambda/\alpha\lambda| \neq 2$ then $s_\delta(\lambda - e_i) = s_\delta(\lambda)$.*

Proof. Here $\lambda/\alpha\lambda$ is a pair of rims each of length at least 2, so every case is one of the following:
(i) the upper end of a rim ends in a row of length greater than 1 (as in Example 19.7.24(ii));
(ii) the lower end ends in a column of length greater than 1 (as in Example 19.7.24(i));
(iii) the skew is 'boxy' (the upper end contains a shape (2^2), as in Example 19.7.24(iii)).

(i) Suppose the upper end of a rim ends in a row (of length greater than 1). Then, if $\lambda/\alpha\lambda$ is not boxy, the end box of this row (with charge $-x$ say, in row i) is removable, but its balance partner in the skew is not (else the skew is not minimal). (Cf. the upper rim in Example 19.7.24(ii), which ends in $-x = -16$.) It follows that singularity is unchanged on removing the end-box $-x$ in row i, since this row becoming part of a singular pair while ending in $-x + 2$ would imply a pair partner row ending in x, already present in λ. Such an x cannot be the balance partner of the original $-x$ since it must lie at the end of a row, but then it would lie SE of the x in the skew, as in figure 19.19(a) — a contradiction. (One easily checks, for example using the figure again, that λ cannot have a row ending in $x + 2$, so no singular pair is *lost* in removing $-x$.)

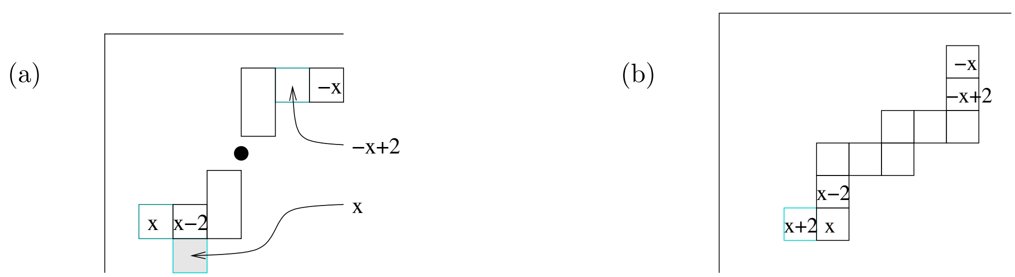


Figure 19.19: (a) illustration of $\lambda/\alpha\lambda$ for (19.7.25) proof case (i). (b) illustration for case (ii). fig:singlem

Note here that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ (cf. 19.7.10), indeed we remain in the same facet.

(ii) If the lower end of a rim ends in a column (of length greater than 1), then the end-box of this column (charge x, say) is removable. (Cf. the lower rim in Example 19.7.24(i), which ends in $x = 22$.) After removal, the new end-box charge is $x + 2$ (if the row is not now empty). In λ the pair of rows ending in x and $-x + 2$ (the box below the balance partner $-x$ to x) as in figure 19.19(b) are a singular pair. In $\lambda - e_i$ the x is lost and hence so is this singular pair, but the row now ends in $x + 2$, which forms a new singular pair with the $-x$. So this time $\lambda - e_i$ lies on one *different* hyperplane to λ, but *overall* singularity is unchanged. (In Example 19.7.24(i) the change from $(\lambda + \rho_\delta)_i$ is $-21/2 \to -23/2$.)

Since singular pairs of rows are ignored in computing $o_\delta(\lambda)$, the valley sequence is also still unchanged, as per 19.7.10.

(iii) For the boxy cases there are a couple of analogous variations. The largest magnitude charge removable box is either at the end of a row (the lower row at the upper end of the skew); or at

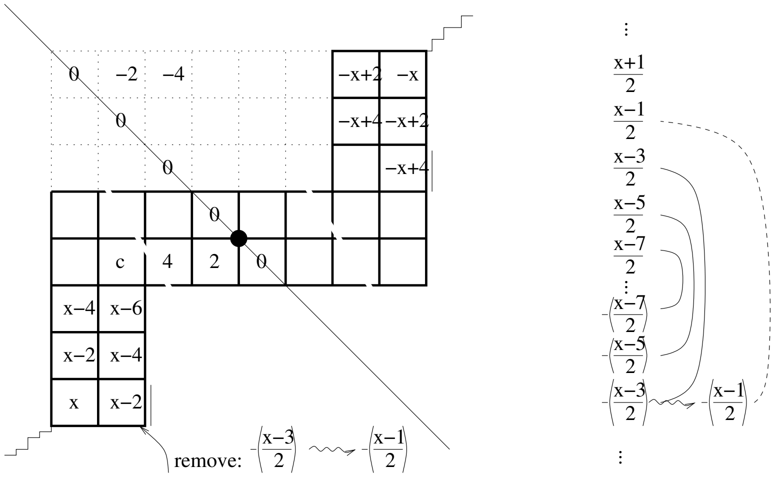
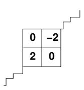
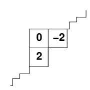


Figure 19.20: Illustration of $\lambda/\alpha\lambda$ in case (iii) showing charges c and $\mathtt{e}_\delta(\lambda) \rightsquigarrow \mathtt{e}_\delta(\lambda - e_i)$. $\boxed{\texttt{fig:boxy-gen1}}$

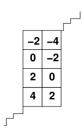
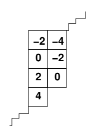
the end of a column at the lower end (as in Example 19.7.24(iii)). The proofs are also analogous. For the lower-end-removable case see Figure 19.20. We omit further details, in favour of a couple of representative examples.

Examples: In case $\lambda/\alpha\lambda$ is of shape (2^2) we have

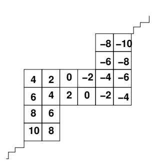
 $\overset{\mathtt{e}_\delta}{\mapsto} (..., 3/2, 1/2, \leq -5/2, ...) \quad \rightsquigarrow$  $\mapsto (..., 3/2, -1/2, \leq -5/2, ...)$

which shows that the singularity does not change. (By the toggle rule these both, as far as shown, have valley set $\{1, 2\}$.)

In case $\lambda/\alpha\lambda$ is of shape (2^4), which is of 'column-end' type, we have

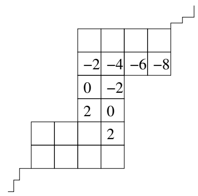
 $\mapsto (..., 5/2, 3/2, 1/2, -1/2, \leq -7/2, ...) \quad \rightsquigarrow$  $\mapsto (..., 5/2, 3/2, 1/2, -3/2, \leq -7/2, ...)$

which shows that the singularity does not change, although the wall does. (The valley set does not change — as far as shown it is $\{1, 2\}$ again.) In the case (3^2), which is of row-end type, we have (similarly embedded, in general) $(3^2) \mapsto (..., 2, 1, \leq -3, ...) \rightsquigarrow (32) \mapsto (..., 2, 0, \leq -3, ...)$ which has the same singularity (and wall set). A typical boxy skew of column-end type is

 $\mapsto (..., 11/2, 9/2, 7/2, 5/2, -5/2, -7/2, \leq -13/2, ...)$

Removing the removable 8 here changes $-7/2 \to -9/2$, giving the same singularity (different wall). (And once again the same valley set.)

A final example with no change in singularity (on removing the row-end -8 box):



$$\mapsto (..., 11/2, \underbrace{9/2}_{\rightsquigarrow 7/2}, 3/2, 1/2, -1/2, -3/2, \leq -13/2, ...)$$

As noted, the general argument is much the same as for the generic cases. $\square$

(**19.7.26**) REMARKS: In the simplest case this proposition looks like regular (or relatively regular) alcove geometry. If the 'distance' (more properly the skew) between a balanced pair of weights is minimal (a rank 2 skew), then the step off one of them (λ say), towards the other, must lie on the reflection wall. While if they are further apart, a single step away from λ towards the other will lie in the same alcove as λ (or at least it will be possible to take a step in the same facet as λ).

However we see that in general the proposition deals with more complex 'singularity non-changing' cases, in which the single step *does not* stay in the same facet.

19.8 Proof of The Decomposition Matrix Theorem

ss:pf

We will show (still with fixed but arbitrary δ) that if $\lambda \in \Lambda$ is not at the bottom of its block then there is a $\lambda - e_i \in \Lambda$ such that $\mathbb{P}[\lambda - e_i] \Rightarrow \mathbb{P}[\lambda]$.

19.8.1 The generic inductive-step lemma

(**19.8.1**) PROPOSITION. *Fix δ. For $\lambda \in \Lambda$ not at the bottom of its block, pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in skew $\lambda/\alpha\lambda$. If $|\lambda/\alpha\lambda| \neq 2$ then $\mathbb{P}[\lambda - e_i] \Rightarrow \mathbb{P}[\lambda]$.*

Proof: If $|\lambda/\alpha\lambda| \neq 2$ then $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ by (19.7.25). Thus by Prop. 19.7.19 we have $(P_\lambda : \Delta_\mu) = (P_{\lambda - e_i} : \Delta_{f_i(\mu)}) \quad \forall \mu \in [\lambda]_\delta$. Now use Lemma 19.7.10 and the definition of $h_\delta(\lambda)$, which determines [11] (see Lemma 19.7.11) that $h_\delta(f_i(\lambda)) = f_i(h_\delta(\lambda))$. $\square$

19.8.2 The rank-2 inductive-step lemma

The remaining cases needed to move between level $n-1$ and n are those manifesting skews of rank 2, i.e. of form (1)+(1), including (1^2). We start with some 'combinatorial' (i.e. representation theory independent) properties relating hypercubes $h_\delta(\lambda)$ and $h_\delta(\lambda - e_i)$, then use these and (19.1.11) etc. to prove the inductive step.
[12]

pr:rank2I (**19.8.2**) PROPOSITION. *Fix δ. For $\lambda \in \Lambda$, pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. If $|\lambda/\alpha\lambda| = 2$ then we have the following.*
(I) For some $x \geq 0$ and some i', either:

$$\begin{array}{rlccccc}
\mathsf{e}_\delta(\lambda) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & \overbrace{x+1}^{i-th} & , ..., & \overbrace{-x}^{i'-th} & , ...) \\
\mathsf{e}_\delta(\lambda - e_i) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & x & , ..., & -x & , ...) \\
\mathsf{e}_\delta(\alpha\lambda) = \mathsf{e}_\delta(\lambda - e_i - e_{i'}) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & x & , ..., & -x-1 & , ...)
\end{array} \qquad (19.11)$$

eq:3ki

if e_i above $e_{i'}$, or else:

$$\begin{array}{rlccccc}
\mathsf{e}_\delta(\lambda) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & \overbrace{x}^{i'-th} & , ..., & \overbrace{-x+1}^{i-th} & , ...) \\
\mathsf{e}_\delta(\lambda - e_i) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & x & , ..., & -x & , ...) \\
\mathsf{e}_\delta(\alpha\lambda) = \mathsf{e}_\delta(\lambda - e_i - e_{i'}) & = (\lambda_1 - \frac{\delta}{2}, \lambda_2 - \frac{\delta}{2} - 1, ..., & x-1 & , ..., & -x & , ...)
\end{array} \qquad (19.12)$$

eq:3ki'

(The second case can be chosen without loss of generality, since it includes the case $\lambda/\alpha\lambda = (1^2)$; but the first cannot.)
(Ia) The sequence $\mathbf{b}_\delta(\lambda) = \hat{\alpha}\mathbf{b}_\delta(\lambda - e_i)$ (as defined in (19.6.16)).
(Ia') The sequence $\mathbf{b}_\delta(\alpha\lambda) = \check{\alpha}\mathbf{b}_\delta(\lambda - e_i)$ (i.e. $\mathbf{b}_\delta(\alpha\lambda)$ differs from $\mathbf{b}_\delta(\lambda - e_i)$ by insertion of subsequence 10 in the α position).
(Ib) The hypercube

$$h_\delta(\lambda) \;\cong\; (1, \alpha)\hat{\alpha} h_\delta(\lambda - e_i)$$

[11] ENLARGE ON THIS!
[12] REMARK: PART I of the following is not in the induction and should be promoted...
...IN ORDER to sort this out we need to tidy up the bump notation... DO THIS NOW.

(i.e. $h_\delta(\lambda)$ has increased 'hypercube dimension' by +1 compared to $h_\delta(\lambda - e_i)$).

Proof: (I,Ia,Ia') As shown in the proof of Lemma 19.7.8 (or Prop. 19.7.25; or see below), removing e_i from λ here makes row-i part of a singular pair with the row ending in the box with opposite charge (and does not affect any other singular pairing). Thus $\mathbf{b}_\delta(\lambda - e_i)$ differs from $\mathbf{b}_\delta(\lambda)$ in that a pair which contributed an 01 subsequence in the latter (note the 0,1 are adjacent since row-i only changes length by 1) does not contribute to the valley sequence in the former — i.e. $\mathbf{b}_\delta(\lambda - e_i)$ differs by the removal of this 01 sequence. (Figure 19.18 serves as an example here.) It remains to confirm the *position* of the removal.

Altogether the bracketed pair of positions in (19.11,19.12) contribute an 01 (resp. 10) in the binary sequence $\mathbf{b}_\delta(\lambda)$ (resp. $\mathbf{b}_\delta(\alpha\lambda)$). By definition of $\alpha\lambda$ (see (19.7.1)) this subsequence is in the $\alpha, \alpha+1$ position (coming from the magnitude order, as defined in (19.4.16), of singleton terms in $\lambda + \rho_\delta$). In $\mathsf{e}_\delta(\lambda - e_i)$ the $x, -x$ are a singular pair, so do not appear in the magnitude order — to obtain its binary representation from that of λ one deletes the 01 binary pair. Thus

$$\mathbf{b}_\delta(\lambda) = \hat{\alpha}\mathbf{b}_\delta(\lambda - e_i) \tag{19.13}$$ eq:b from

Claim (Ib) now follows from (19.6) in Lemma (19.6.17). □

lem:mu **(19.8.3)** LEMMA. *Fix δ. For $\lambda \in \Lambda$, pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. If $|\lambda/\alpha\lambda| = 2$ then for $\mu \in h_\delta(\lambda - e_i)$, (I) the pair $\mu+ = \hat{\alpha}\mu$ and $\mu- = \check{\alpha}\mu$ are in $h_\delta(\lambda)$, and (II) both are adjacent to μ as partitions. Indeed (III) for some j, j' and x either (in case row j above row j', i.e. $j < j'$):*

$$\begin{array}{rlcccc} \mathsf{e}_\delta(\hat{\alpha}\mu) & = (..., & \overbrace{x+1}^{j-th} & , ..., & \overbrace{-x}^{j'-th} & , ...) \\ \mathsf{e}_\delta(\mu) & = (..., & x & , ..., & -x & , ...) \\ \mathsf{e}_\delta(\alpha\hat{\alpha}\mu) = \mathsf{e}_\delta(\check{\alpha}\mu) & = (..., & x & , ..., & -x-1 & , ...) \end{array} \tag{19.14}$$ eq:3kin

or else:

$$\begin{array}{rlcccc} \mathsf{e}_\delta(\hat{\alpha}\mu) & = (..., & \overbrace{x}^{j'-th} & , ..., & \overbrace{-x+1}^{j-th} & , ...) \\ \mathsf{e}_\delta(\mu) & = (..., & x & , ..., & -x & , ...) \\ \mathsf{e}_\delta(\alpha\hat{\alpha}\mu) = \mathsf{e}_\delta(\check{\alpha}\mu) & = (..., & x-1 & , ..., & -x & , ...) \end{array} \tag{19.15}$$ eq:3kin'

pa:h isom unpack **(19.8.4)** *Proof.* The poset isomorphism $(1,\alpha)\hat{\alpha}\ h_\delta(\lambda - e_i) \ \xrightarrow{\sim}\ h_\delta(\lambda)$ of Prop. 19.8.2 can be considered to relate $h_\delta(\lambda - e_i)$ to $h_\delta(\lambda)$, in the sense that $\lambda - e_i \in h_\delta(\lambda - e_i)$ is taken to the pair $(1,\alpha)\hat{\alpha}(\lambda - e_i) = \{\hat{\alpha}(\lambda - e_i), \check{\alpha}(\lambda - e_i)\} = \{\lambda, \alpha\lambda\} = \{\lambda, \lambda - e_i - e_{i'}\}$. For general $\mu \in h_\delta(\lambda - e_i)$ the pair $\mu+ = \hat{\alpha}\mu$ and $\mu- = \check{\alpha}\mu$ can similarly be considered to be engendered by $(1,\alpha)\hat{\alpha}$ acting on μ. Thus these two are vertices in $h_\delta(\lambda)$ and we have established (I).

This notation is convenient in the binary sequence realisation, but what is the significance of the hypercube isomorphism when these labels are viewed as *partitions* (or descending sequences)?

First note from the proof of 19.8.2 that there are i, i' such that

$$\mathsf{e}_\delta(\lambda) = (i, i')_-\ \mathsf{e}_\delta(\alpha\lambda)$$

while $(i,i')_-$ fixes $\mathsf{e}_\delta(\lambda - e_i)$.

For general $\mu \in h_\delta(\lambda - e_i)$ the descending sequence $\mathsf{e}_\delta(\mu)$ may be obtained from $\mathsf{e}_\delta(\lambda - e_i)$ by action of some *not necessarily involutive* $w \in \mathcal{D}$. The binary sequence $\mathbf{b}_\delta(\mu)$ is related to $\mathbf{b}_\delta(\lambda - e_i)$ by some *not necessarily consecutive* subsequences 01 (arc endpoints in the associated TL diagram) being replaced by 10 (or 11 by 00).

Now, by construction of $\hat{\alpha}\mu$, μ can be obtained from $\mu+ = \hat{\alpha}\mu$ (say) by deleting a pair 01 in the $\alpha, \alpha+1$ position in $\mathbf{b}_\delta(\mu+)$ to obtain $\mathbf{b}_\delta(\mu)$. The question is: what does this process look like in the partition realisation $\mathsf{e}_\delta(\mu) \rightsquigarrow \mathsf{e}_\delta(\hat{\alpha}\mu)$?

For $\mathbf{b}_\delta(\lambda)$ in particular (the case $\mu+ = (\lambda - e_i)+$, as it were) we have seen in (19.11) (say) that this 01 pair in the $\alpha, \alpha+1$ position correspond to $x+1, -x$ in the i, i' positions in $\mathsf{e}_\delta(\lambda)$ (some x, i, i') as in (19.11). This becomes $x, -x$ in $\mathsf{e}_\delta(\lambda - e_i)$ — which in turn corresponds to $\mathsf{e}_\delta(\lambda - e_i)$ lying on the $(i,i')_-$-reflection hyperplane (other $\mathsf{e}_\delta(\mu)$ will not lie on this $(i,i')_-$-hyperplane, but rather some reflection group image of it).

Every $\mathsf{e}_\delta(\lambda')$ throughout the λ-block is a signed permutation of $\mathsf{e}_\delta(\lambda')$, by Theorem 19.5.13. In particular, two sequences related by a simple reflection differ only in the corresponding pair of points. In particular then, by construction the pair of terms in the $\alpha, \alpha+1$ position in $\mathbf{b}_\delta(\mu\pm)$ correspond to $x+1, -x$ (respectively $x, -x-1$) in *some* pair of positions in $\mathsf{e}_\delta(\mu\pm)$. So for each $\mu = w.(\lambda - e_i)$ the pair $\mu\pm$ are adjacent to it, images of each other on either side of a wall that it lies on (that is the image of the $(i,i')_-$-wall under w). $\square$

(**19.8.5**) Proposition. *Fix δ. For $\lambda \in \Lambda$, pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. If $|\lambda/\alpha\lambda| = 2$ then we have the following.*
(I) $Proj_\lambda Ind\, P_{\lambda-e_i} = P_\lambda$.
(II) $\mathbb{P}[\lambda - e_i] \Rightarrow \mathbb{P}[\lambda]$.

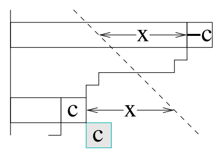
Proof: (I) Recall from the Δ-module branching rule Prop.(18.5.3)(ii) that under the present assumptions

$$\text{Proj}_\lambda \text{Ind}\Delta_{\lambda-e_i} = \Delta_\lambda + \Delta_{\lambda-e_i-e_{i'}} = \Delta_\lambda + \Delta_{\alpha\lambda} \tag{19.16}$$ p-dd

This is a non-split sum, by [28, Lem.4.10] (or by our Prop.19.1.11).

Recall from 18.6.18 that $P_{\lambda-e_i}$ contains $\Delta_{\lambda-e_i}$ and then no Δ_ν with $|\nu| > |\lambda - e_i| - 2$. Thus $\text{Proj}_\lambda \text{Ind}\, P_{\lambda-e_i}$ contains Δ_λ and then no Δ_ν with $|\nu| > |\lambda| - 2$, since by 19.7.15 there is no $\lambda - e_i + e_j \sim^\delta \lambda$ (with $j \neq i$).

It follows that the only way to get a Δ_μ with $\mu > \lambda - e_i$ in $\text{Ind}(\text{Proj}_\lambda \text{Ind}\, P_{\lambda-e_i})$ is adding a box to λ (i.e. from the $\text{Ind}\,\Delta_\lambda$ part). But we *claim* there is no $\lambda + e_j \sim^\delta \lambda - e_i$ here.
Proof: If $\lambda - e_i, \lambda + e_j$ in the same block then $(\lambda + e_j)/(\lambda - e_i)$ is a balanced skew, hence consisting of two boxes of balanced charge. One of these, however, would also have to be in the skew $\lambda/\alpha\lambda$. This skew looks like the marked pair of boxes in the schematic for λ here:



Suppose the common box is the $-c$. Then the other one (the $+e_j$ box) in $(\lambda + e_j)/(\lambda - e_i)$ must be a charge c, for balance. Any such box must be SE of the c in $\lambda/\alpha\lambda$ (since lines of constant charge run NW to SE). But since that c in λ is removable, it is not possible to add a box SE of it, as indicated in the figure. Supposing the common box is the c leads to a similar contradiction. Done.

In this case then $\text{Proj}_{\lambda-e_i}\text{Ind}\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}$ is a projective containing no Δ_μ with $\mu > \lambda - e_i$, and hence whose dominant Δ-content (in the sense of **??**) includes two copies of $\Delta_{\lambda-e_i}$ (one from each of the summands on the right of (19.16)).

Hence, by (18.6.19),

$$\text{Proj}_{\lambda-e_i}\text{Ind}(\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}) = P_{\lambda-e_i} \oplus P_{\lambda-e_i} \oplus Q$$

(some Q, possibly zero).

On the other hand by (19.7.15) every Δ-factor of $P_{\lambda-e_i}$ engenders at most two Δ-factors in $\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}$, and hence (by **??**) at most two factors in $\text{Proj}_{\lambda-e_i}\text{Ind}(\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i})$. Thus

$$\text{Proj}_{\lambda-e_i}\text{Ind}(\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}) = P_{\lambda-e_i} \oplus P_{\lambda-e_i}.$$

Since the section (19.16) in $\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}$ contributes to both summands and is non-split, it follows that $\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i}$ is non-split, and hence

$$\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i} = P_\lambda$$

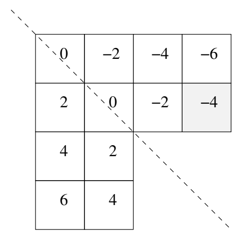
(II) It remains to show that here $\mathbb{P}[\lambda - e_i]$ implies $(\text{Proj}_\lambda\text{Ind}P_{\lambda-e_i} : \Delta_-) = h_\delta(\lambda)$.

So, we start with the assumption: $(P_{\lambda-e_i} : \Delta_-) = h_\delta(\lambda - e_i)$. By the last para of (19.8.4) we see that for each Δ_μ occuring in this decomposition we have $\mathrm{Proj}_\lambda \mathrm{Ind}\Delta_\mu = \Delta_{\mu+} + \Delta_{\mu-}$ for some pair $\mu+, \mu- \in [\lambda]_\delta$, each differing from μ in one box. Since $\mu+, \mu- \in [\lambda]_\delta$, the terms of $\mathtt{e}_\delta(\mu\pm)$ are a signed perm of those of $\mathtt{e}_\delta(\lambda)$, and hence (since they agree with $\mathtt{e}_\delta(\mu)$ except in one term) they must agree with $\mathtt{e}_\delta(\mu)$ except for $x, -x \rightsquigarrow x+1, -x$ (up to sign), so they are the descending sequences in $[\lambda]_\delta$ corresponding to $\hat{\alpha}\mathbf{b}_\delta(\mu)$ and $\check{\alpha}\mathbf{b}_\delta(\mu)$. [13] That the collection thus engendered is $h_\delta(\lambda)$ now follows from 19.8.4.
□

Up to some minor variations to deal with skews containing (2^2) (see **??**), this completes the main inductive step for the Theorem. □

19.8.3 Example for the rank-2 inductive step

(**19.8.6**) Example: $\delta = 1$, computing for $\lambda = 4422$ via $\lambda - e_2 = 4322$. We have



In particular $\mathtt{e}_1(4322) = (7/2, 3/2, -1/2, -3/2, ...)$ so $o_1(4322) = \mathrm{toggle}(\{2\}) = \{1, 2\}$. By the inductive hypothesis we have

$$(P_{4322} : \Delta_-) \quad = \quad h_1(4322)_-$$

Here, in the style of Fig.19.15, the hypercube may be given in any of the following forms:

$$h_1(4322) = \begin{tikzcd} 4322 \arrow[dr, "(14)_-(23)_-"] & \\ & 221 \end{tikzcd} \cong \begin{tikzcd} 12 \arrow[dr, "\underline{12}"'] & \\ & \emptyset \end{tikzcd} \cong \begin{tikzcd} 11 \arrow[dr] & \\ & 00 \end{tikzcd} \cong \begin{tikzcd} 01 \arrow[dr] & \\ & 10 \end{tikzcd}$$

---

[13] NOT NEEDED:
Since $[\lambda - e_i]_\delta$ is a strictly more singular orbit than $[\lambda]_\delta$ the reflection group elements moving through $h_\delta(\lambda - e_i)$ will also serve to move the pair $\lambda, \alpha\lambda$ through these pairs $\mu+, \mu-$, thus they remain adjacent above and below μ (not necessarily in that order).
For $\lambda - e_i$ itself we have seen in (19.13) that $\mathbf{b}_\delta(\lambda - e_i)$ gives $\mathbf{b}_\delta(\lambda)$ and $\mathbf{b}_\delta(\alpha\lambda)$ by inserting 01 (respectively 10) in the α position. For other $\mu \in h_\delta(\lambda - e_i)$, note that the relevant singular pair of rows in $\lambda - e_i$, while not contributing to the magnitude order (since they are singular) are formally permuted (in the $\mathcal{D}$-action sense) along with the rest of the rows, in the collection of reflection group actions that traverse $h_\delta(\lambda - e_i)$. Thus they (jointly) maintain a formal position in the magnitude order, between two terms that are properly consecutive in this order. The difference between μ and μ_+, μ_- is that in these one of the pair is extended by 1, or contracted by one. Thus the singularity is broken, and the pair appear properly in the order, between the given two terms, and hence bumping up the larger of the two. Since μ is just a permutation of $\lambda - e_i$ (as far as the magnitudes are concerned), the position of the pair in the magnitude order, and hence the position of the bump in the binary representation, is the same as for $\lambda - e_i$.

where the last is the untoggled binary representation. The edge label in the first (partition labelling) form is an element w of $\mathcal{D}$ such that $w\mathsf{e}_1(4322) = \mathsf{e}_1(221)$, i.e.

$$(14)_-(23)_-(7/2, 3/2, -1/2, -3/2, ...) = (3/2, 1/2, -3/2, -7/2, ...)$$

This is an involution (as per Prop.19.5.11). Note however that we could have applied $(14)_-(23)_-(24)_- = (13)_-(14)_-(23)_-$ to the same effect, since $(24)_-$ fixes the first vector (we say it is on the $(24)_-$-wall), and $(13)_-$ fixes the second.

Translating $\mathsf{e}_1(4322)$ off the $(24)_-$-wall with $\mathrm{Proj}_\lambda \mathrm{Ind}-$ we get

$$4322 + 221 \to (4422 + 4321) + (321 + 22). \tag{19.17}$$

eq:4322

In binary this corresponds to $01 \to 0{*}{*}1 \to 0101 + 0011$ and $10 \to 1{*}{*}0 \to 1100 + 1010$. These four sequences therefore encode the content of a projective containing P_{4422} (but which is non-split by the arguments in the proof).

We see that the Theorem is verified in this case by comparing (19.17) with:

$$h_1(4422) = \begin{tikzcd} & 4422 \arrow[dl, "(24)_-"'] \arrow[dr, "(14)_-(23)_-(24)_-"] & \\ 4321 \arrow[dr] & & 321 \arrow[dl] \\ & 22 & \end{tikzcd} \cong \begin{tikzcd} & 34 \arrow[dl, "23"'] \arrow[dr, "14"] & \\ 24 \arrow[dr] & & 13 \arrow[dl] \\ & 12 & \end{tikzcd} \cong \begin{tikzcd} & 0011 \arrow[dl] \arrow[dr] & \\ 0101 \arrow[dr] & & 101 \arrow[dl] \\ & 11 & \end{tikzcd}$$

(**19.8.7**) One or two further remarks on $h_1(4422)$ are in order:

The other $\mathcal{D}$ actions traversing the hypercube edges are:

$$\underbrace{(14)_-(23)_-(24)_-}_{=(13)_-(14)_-(23)_-} \underbrace{(7/2, 3/2, -1/2, -5/2, -9/2, ...)}_{\mathsf{e}_1(4321)} = \underbrace{(3/2, 1/2, -5/2, -7/2, -9/2, ...)}_{\mathsf{e}_1(22)}$$

and

$$(13)_- \underbrace{(5/2, 1/2, -3/2, -7/2, -9/2, ...)}_{\mathsf{e}_1(321)} = \underbrace{(3/2, 1/2, -5/2, -7/2, -9/2, ...)}_{\mathsf{e}_1(22)}$$

In a suitable sense the SE moves are the same as the SE move (which could be) used in the 4322 hypercube, while the SW moves would fix the 4322 hypercube.

Note

$$(14)_-(23)_- \underbrace{(7/2, 5/2, -1/2, -3/2, ...)}_{\mathsf{e}_1(4422)} = \underbrace{(3/2, 1/2, -5/2, -7/2, ...)}_{\mathsf{e}_1(22)}$$

$$(14)_-(23)_- \underbrace{(7/2, 3/2, -1/2, -5/2, ...)}_{\mathsf{e}_1(4321)} = \underbrace{(5/2, 1/2, -3/2, -7/2, ...)}_{\mathsf{e}_1(321)}$$

so $(14)_-(23)_-$ moves the elements in pairs through the hypercube.

(**19.8.8**) Note that the composite $\mathcal{D}$ element $(14)_-(23)_-(24)_-$ (or $(13)_-(14)_-(23)_-$) in the shoulder of $h_1(4422)$, which corresponds to an element of Γ^{34} but not Γ_{34}, is not involutive; while $(24)_-$, which corresponds to an element of Γ_{34}, is involutive by Prop.19.5.11.

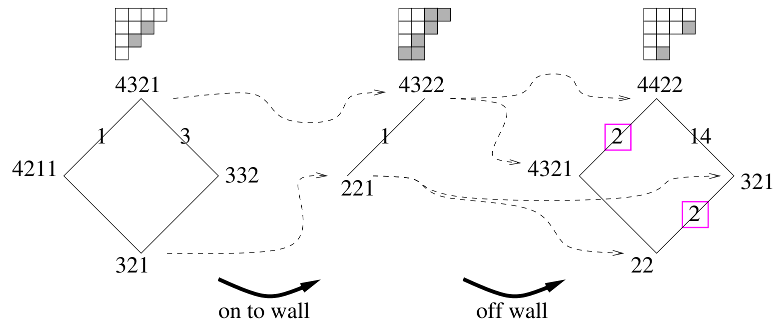


Figure 19.21: On and off the wall in our $\delta = 1$ example. `fig:wall-on-offd1`

Note from the binary realisation that the insertion of a binary pair in the α position ($\alpha = 2$), and action of α on it (in the sense of **??**), modifies the labels of $h_1(4322)$ and extends it by a new generating direction (labelled by α), precisely as required to produce $h_1(4422)$.

(**19.8.9**) In §20.1.1 we will draw parallels with the recursive computation of Kazhdan–Lusztig polynomials. To this end we note the following. We have

$$
h_1(4321) = \begin{tikzcd} & 4321 \arrow[dl, "(23)_-"'] \arrow[dr, "(14)_-"] & \\ 4211 \arrow[dr] & & 332 \arrow[dl] \\ & 321 & \end{tikzcd} \cong \begin{tikzcd} & 24 \arrow[dl, "12"'] \arrow[dr, "34"] & \\ 14 \arrow[dr] & & 23 \arrow[dl] \\ & 13 & \end{tikzcd} \cong \begin{tikzcd} & 0101 \arrow[dl] \arrow[dr] & \\ 1001 \arrow[dr] & & 011 \arrow[dl] \\ & 101 & \end{tikzcd}
$$

The route to 4422 from 4321 is along a 2-edge (i.e. right-multiplication by (23)). This kills the middle two terms (they do not have such an edge "in the dominant region"). Thus the KL recursion gives copies of 4321 and 321 plus their 2-edge images, which agrees with $h_1(4422)$.

(**19.8.10**) Fig. 19.21 shows this $\delta = 1$ example following 4321 up onto and then off the relevant wall. We compare this with the KL recursion (or its on-off-wall version). Going onto the wall we actually lose elements here, since there is nothing in the 221 block near 4211 or 332. There is no reason to suppose that the 221 block is adjacent to the 332 block, and indeed it is not. (Something also happens to the polynomial exponent.) Going off the wall we get a doubling *'as ever'*.

Representation theoretically we can ask what the collapse of the 4321 'diamond' to the 4322 pair corresponds to. It is important to note that **this question is academic**, since in our approach we do not get the structure of 4322 from 4321, we get it from (say) 3322.

19.9 Some remarks on the block graph

19.9.1 Yet more

The block graph G_e (or isomorphically G_o [**?**]) is an interesting thing. Here are some features.

le:m101 **Lemma 19.1.** *Inserting sequence 01 at any point in a valley sequence $b(\lambda)$ increases the number of arcs in the TL diagram $\mathcal{T}(\lambda)$ by one.*

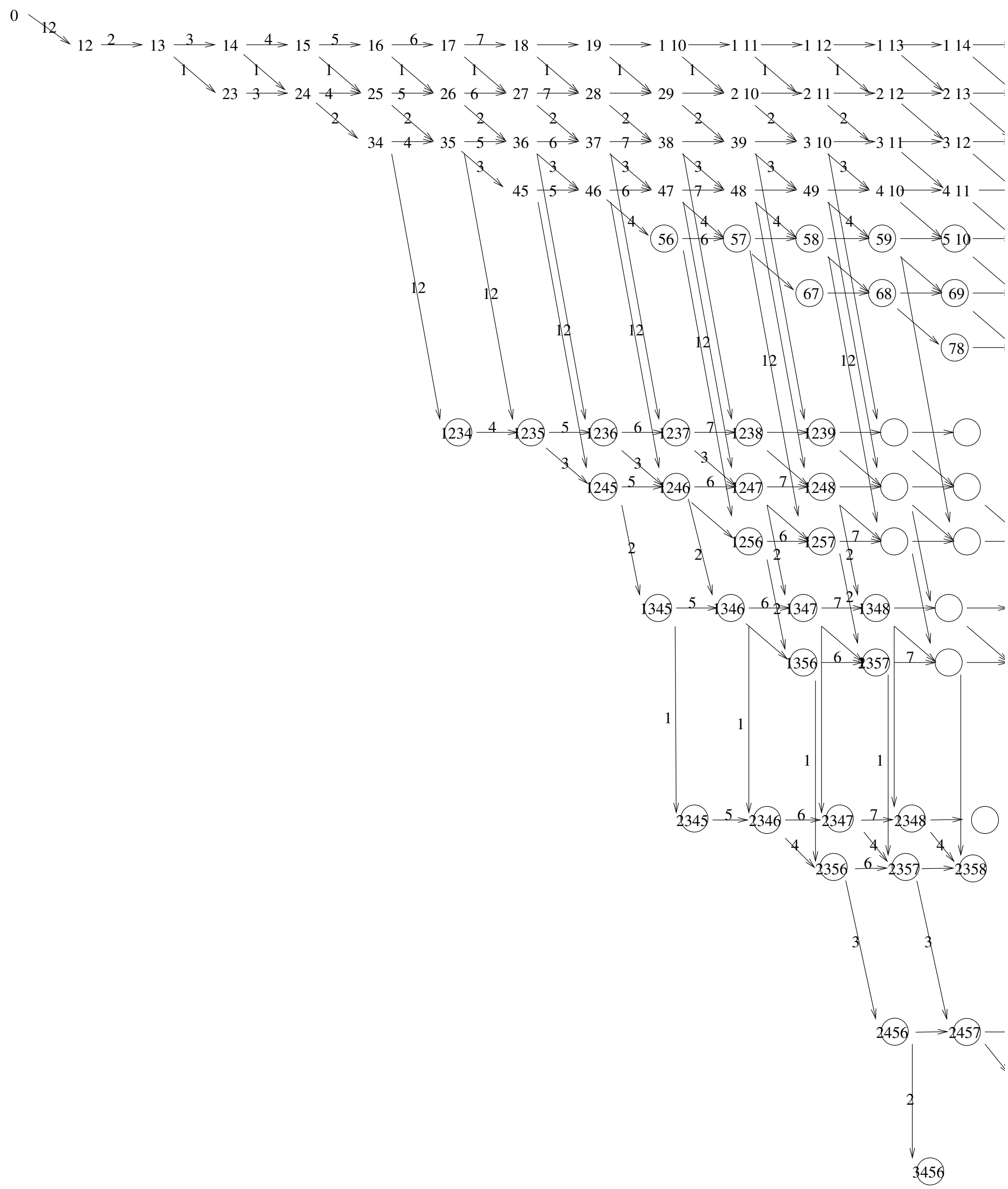
Proof. Note that the algorithm will remove this subsequence, and add an arc, at the first iteration; returning us to the original sequence for the next iteration.

(**19.9.1**) By Lemma **??** every vertex in G_e is the top of an ascending hypercubical subgraph, which includes all its incoming edges. We define the *dimension* of a vertex λ as the maximum number of incoming edges of any vertex in any ascending path from the root to λ.

le:m102 **Lemma 19.2.** *The number of arcs in the TL diagram $\mathcal{T}(\lambda)$ is equal to the dimension of λ (and to $|\Gamma^\lambda|$).*

Proof. Moving up an edge we either make $00 \to 11$ at the beginning of the sequence $b(\lambda)$, or change some $10 \to 01$. It is routine to check that neither move reduces the number of arcs. If we pass to a vertex of dimension one-greater than before, each incident edge corresponds to a different pair transforming in one of these two ways. Each of these pairs separately gives rise to an arc.

Figure 19.22: The beginning of the graph G_{even}, with edge labels. fig:valley graph1



Chapter 20

Properties of Brauer block graphs

ch:BrauerIV

In §20.1 we talk about connections between our Theorem 19.6.12 and the theory of Kazhdan–Lusztig polynomials (as in §5.2). In §20.5 we consider how to describe the set consisting of the lowest weight taken from each block. In §20.6 we look at connections between different δ values.

20.1 Kazhdan–Lusztig polynomials revisited

ss:pKLpr

20.1.1 Overview

See also §5.2. We recall some notation from there.

Let $\mathbb{H} \supset \mathbb{H}_+$ be closed sets of hyperplanes in space V, defining reflection groups $W' = W_{\mathbb{H}}$ and $W_+ = W_{\mathbb{H}_+}$ respectively. Let $a_0 \in \mathcal{C}_{\mathbb{H}}$ and $C_0 \in \mathcal{C}_{\mathbb{H}_+}$ such that $(W', S') = (W_{\mathbb{H}}, \mathbb{H}_{a_0})$ is a Coxeter system and $(W_+, S_+) = (W_{\mathbb{H}_+}, \mathbb{H}_{C_0})$ a parabolic subsystem, as in (19.5.3). Write $\mathcal{A}^+ \subset \mathcal{C}_{\mathbb{H}}$ for the alcoves in C_0.

Background: parabolic Kazhdan–Lusztig polynomials

ss:pKLp

Here we explain where the idea for the hypercubical decomposition graphs in (19.6.1) comes from.

Associated to each Coxeter system W' and parabolic W, acting as reflection groups on a suitable space, is an alcove geometry on that space. For each such pair W', W there is, therefore, an array $P = P(W'/W)$ of Kazhdan–Lusztig polynomials — one for each ordered pair of alcoves. (Deodhar's recursive formula [37] computes these polynomials in principle. However it generally tells us very little about them in practice.) These polynomials are of interest from a number of points of view. For example they are often important in representation theory (see [142, 121] and references therein). So, with the reflection group pair $\mathcal{D}/\mathcal{D}_+$ manifesting itself in Brauer algebra block theory (as we have seen in §19.5.1), one is motivated to compute them in this case.

20.1.2 The recursion for $P(W'/W)$

ss:pKLp def2

See §5.6.1 for the general case.

20.2 The reflection group action $\mathcal{D}$ on $\mathbb{R}^{\mathbb{N}}$

S:D/A

Define $v_- = o_2(\emptyset) = -(1, 2, 3, ...) \in \mathbb{R}^{\mathbb{N}}$. In Section 19.5.1 we chose the alcove containing v_- as a_0, i.e. as C', for the reflection group $\mathcal{D}$. (We shall refer to $\mathcal{D}v_-$ as the *fully-regular* orbit.) Thus our choice of C' corresponds to choosing $S_{\mathcal{D}} = \{(12)_-,\ (i\ i{+}1)\}_{i\in\mathbb{N}}$ for the Coxeter generating set of $\mathcal{D}$.

(**20.2.1**) The orbit $\mathcal{D}v_- \supset V(v_-) = o_2^{\emptyset}(\mathtt{P}_{even}(\mathbb{N}))$. This orbit consists in the set of *co-even permutations* (signed permutations of v_- with an even number of positive terms). By (5.3.9) this orbit (and hence each of the others) is isomorphic, via the left action of $\mathcal{D}$ upon it, to the (limit) regular representation. It is easy to check that the action we are using is the left-regular action. By (5.3.15) it is the associated right action that we need to determine in order to compute (5.9). This commuting right action corresponds to signed permutations of the *entries* in the sequence, rather than signed permutations of the *positions*. For example

$$(4, 3, -1, -2, -5, ...)(45) = (5, 3, -1, -2, -4, ...)$$

(**20.2.2**) Via the isomorphism between $V(v_-)$ and $P_{even}(\mathbb{N})$ we understand left- and right-actions of $w \in \mathcal{D}$ on any $a \subset \mathbb{N}$ (noting that wa, respectively aw, is not necessarily expressible in $P(\mathbb{N})$, since it is not necessarily dominant). When aw *is* dominant we shall see now that the right-action transformation $a \to aw$ is expressible in a simple form in $P(\mathbb{N})$ which facilitates computation of the pKLps. Let G_e denote the simple relabelling of G_{alc} from $P(\mathbb{N})$ using the above isomorphism. (We shall shortly be able to identify G_e with G_{even}.) The following crucial result is routine to show.

th:crux

(**20.2.3**) Theorem. *Let $a \subset \mathbb{N}$ of even degree. Then there exists an edge $(a, a(\alpha, \alpha+1))$ in G_e iff $a \cap \{\alpha, \alpha+1\} = \{\alpha\}$, whereupon $a(\alpha, \alpha+1) \cap \{\alpha, \alpha+1\} = \{\alpha+1\}$; and an edge $(a, a(12)_-)$ in G_e iff $a \cap \{1, 2\} = \emptyset$, whereupon $a(12)_- \cap \{1, 2\} = \{1, 2\}$. Every edge is one of these types.* □

That is, we may associate edge labels corresponding to the right-action in G_e, taken from the Coxeter generating set $S_{\mathcal{D}}$ (as required by (5.3.9)). To streamline still further we may write simply α as 'right-action' label for edges of form $(\lambda, \lambda(\alpha, \alpha{+}1))$ and 12 for $(\lambda, \lambda(12)_-)$. This makes explicit the identification with G_{even}. See Figure 19.11.

(**20.2.4**) Remark. The left-action labels are of course different in this regard. Only elements of form $(ij)_-$ preserve dominance.

A convenient summary of the above is as follows (when we speak of an edge orbit on G_{even} we shall mean the orbit induced by the graph isomorphism with G_{alc} from the edge orbit thereon):

th:big edge

(**20.2.5**) Theorem. *Two edges in G_{even} pass to G_{alc} edges in the same $\mathcal{D}$-orbit (up to direction) if and only if they have the same label.* □

20.3 Solving the polynomial recursion for $P(\mathcal{D}/\mathcal{D}_+)$

To give an indication of the nature of the data set, note that a table of the first few parabolic Kazhdan–Lusztig polynomials is encoded in Figure 19.14 (these first few may even be computed by brute force if desired). Now we solve the recursion (5.9) in closed form.

20.3.1 Hypercubes h^a revisited

As noted in Theorem 20.2.3, the right-action of $S_{\mathcal{D}} \subset \mathcal{D}$ takes a simple form when between 'dominant' elements (elements expressable as $a \subset \mathbb{N}$). For α, a such that $a(\alpha, \alpha+1)$ is dominant we define

$$\langle\alpha\rangle a \;:=\; a(\alpha, \alpha{+}1).$$

I.e. 'operator' $\langle\alpha\rangle-$ is defined only for the appropriate domain. (Because the underlying descending sequences consist first of positive terms of descending magnitude, and then negative terms of ascending magnitude, we call $a \subset \mathbb{N}$ a *valley set*, and $\langle\alpha\rangle$ a valley edge operator.)

(**20.3.1**) We generalise $\langle\alpha\rangle$ as follows. Operator $\langle ij\rangle$ has action defined in case one of i, j is in a, and swaps it for the other (i.e. swaps the side of the valley that each of i, j are on). Example:

$$\langle 36\rangle 56 = 35$$

(Thus $\langle\alpha\rangle = \langle\alpha \;\; \alpha{+}1\rangle$. NB, We write αa for $\langle\alpha \;\; \alpha{+}1\rangle a$ where no ambiguity arises.)
Where defined, each $\langle ij\rangle$ acts involutively; and, where defined, takes a to $\langle ij\rangle a$ comparable to a in the G_{even} order. Each $\langle ij\rangle$ has the same effect on the given a as some (strictly descending (or ascending)) sequence of $\langle\alpha\rangle$ edge operators. In our example

$$56 \xrightarrow{4} 46 \xrightarrow{3} 36 \xrightarrow{5} 35$$

Operator $\langle\underline{ij}\rangle$ has action defined in case both or neither of i, j are in a, and toggles this state. Example:

$$\langle\underline{16}\rangle 1456 = 45$$

which expands, for example, as

$$1456 \xrightarrow{3} 1356 \xrightarrow{2} 1256 \xrightarrow{4} 1246 \xrightarrow{5} 1245 \xrightarrow{12} 45$$

(**20.3.2**) Remark. Let v be the fully-regular (FR) image of $a \in P(\mathbb{N})$, that is, let $v = o_2^{\emptyset}(a)$, with $a \in \mathtt{P}(\mathbb{N})$ such that $\langle ij\rangle a$ is defined. Unless $j = i+1$ it does *not* follow that the fully-regular image of $\langle ij\rangle a$ is given by the right-action of (i, j) on v. Note, for example, that the underlying descending sequence of $\langle ij\rangle a$ is *not* in general a pair permutation of that of a.

(**20.3.3**) Let S be a set of $\langle ij\rangle$'s, and $a \in P(\mathbb{N})$. If for each subset $S' \subseteq S$ the elements of S' may be applied to a in any order to obtain the same set, and this set lies below a in G_{even}, then the *dominant hypercube* $hh(a, S)$ is the digraph consisting of this collection of sets (vertices) and edges. By construction we have

$$h^a = hh(a, \Gamma^a)$$

(with the understanding that if $\{i, j\}$ appears in Γ^a and is a subset of a then the edge operator is $\langle\underline{ij}\rangle$). See Figure 20.1 for an example. Write $\mathcal{T}(a) = d(\mathbf{b}(a))$.

`lem:TL1` (**20.3.4**) Lemma. *Suppose $\{\alpha, \alpha{+}1\} \in \Gamma_a$ (so $\langle\alpha\rangle a < a$). Let $\{\alpha\} \cup X$, $\{\alpha{+}1\} \cup Y$ be parts in $\mathcal{T}(\langle\alpha\rangle a)$ (X, Y could contain a vertex or be empty). Then $\mathcal{T}(a)$ differs from $\mathcal{T}(\langle\alpha\rangle a)$ in that these parts are replaced by $\{\alpha, \alpha{+}1\}$, $X \cup Y$ ($X \cup Y$ may be empty).*

Proof: It is clear that $\{\alpha, \alpha{+}1\}$ is in $\mathcal{T}(a)$, so it remains to consider X, Y; and to show that all other pairs agree between $\mathcal{T}(a)$ and $\mathcal{T}(\langle\alpha\rangle a)$. If $X \cup Y = \emptyset$ then $\alpha, \alpha{+}1$ singletons in $\langle\alpha\rangle a$ and

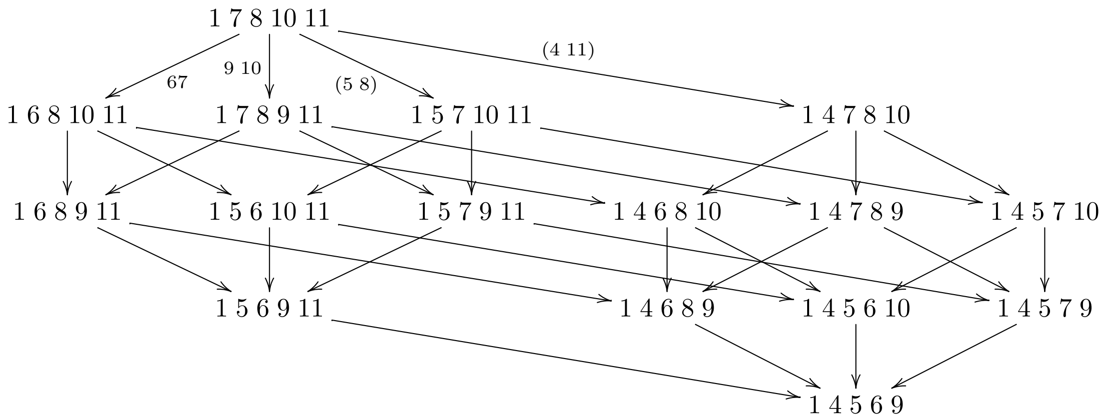
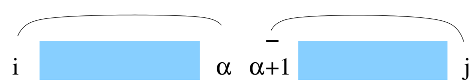
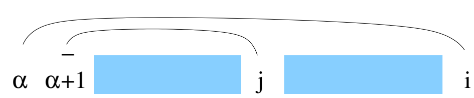


Figure 20.1: The hypercube $h^{1\ 7\ 8\ 10\ 11}$. fig: big h

there are no pairs bridging over them, so no other pair is changed between $\langle\alpha\rangle a$ and a.
If $X = \{i\}$, $Y = \{j\}$ say, then $j \in \langle\alpha\rangle a$ (since $\alpha{+}1 \notin \langle\alpha\rangle a$ by construction). Suppose $j > \alpha{+}1$ and $i < \alpha$. Then we are in a situation like



By construction there are no 11 pairs in the i, α or $\alpha{+}1, j$ intervals. The algorithm for extracting the sequences in the shaded regions will thus operate in the same way for each sequence. In a the algorithm generates a pair at $\alpha, \alpha{+}1$ as already noted, so we may pass to an iteration where these and both shaded parts have been dealt with. Vertex i is not involved in a pair from below (else it would be in $\langle\alpha\rangle a$), and $j \in a$, so we get a pair $\{i, j\}$ as required.
Suppose $j > \alpha{+}1$ and $i > j$. Then we are in a situation like



The same argument goes through until noting that $\alpha, i \in \langle\alpha\rangle a$, so that there is an even number of 1s in the remainder sequence (algorithm stage 5) left of α. This even property still holds for a, so j is not involved in a pair from below. Again we have the required outcome.
The other cases are similar. $\square$

20.3.2 Kazhdan–Lusztig polynomials for $\mathcal{D}/\mathcal{D}_+$

We continue to use labels $a \subset \mathbb{N}$ for alcoves. Thus the rows (and columns) of $P(\mathcal{D}/\mathcal{D}^+)$ may be indexed by these labels. That is, there is a polynomial $p_a(b) = p_{a,b}$, in the formal variable v, for each pair $a, b \in P_{even}(\mathbb{N})$. We write $p_a = \{p_{a,b}\}_{b \in P_{even}(\mathbb{N})}$ for the complete row. Following (19.6.11)

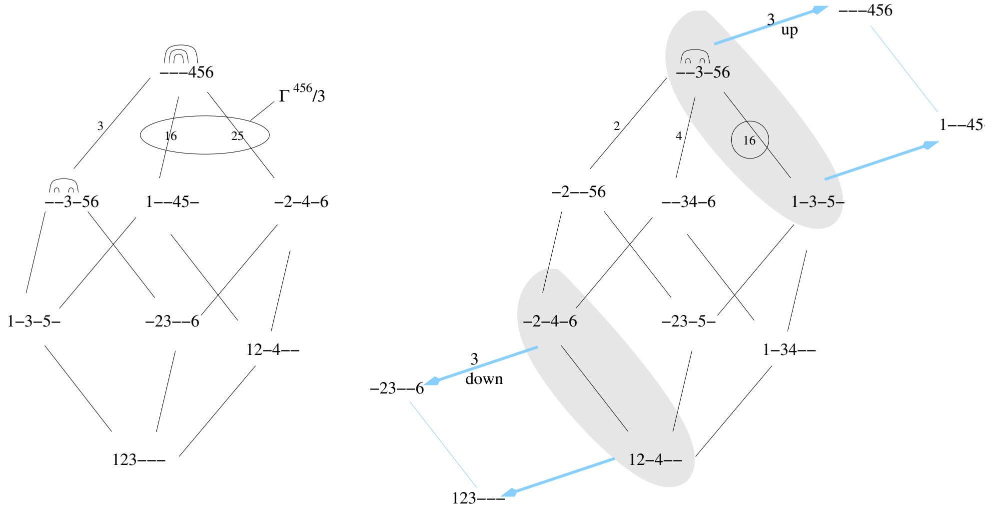


Figure 20.2: Example to show verification that h^{456} is obtained from h^{356} by the KL recursion. Shading indicates the set $\langle\alpha\rangle^2 h^{356}$ of vertices in $h^{356} = h^{\langle\alpha\rangle 456}$ on α-walls. Arrows indicate the set $\langle\alpha\rangle h^{356}$ of images under reflection in these walls. The shaded and image vertices make up h^{456}. One checks that the recursion correctly changes polynomials. For example for 124 we have $v^3 \to v^2$
fig:egg456 since this vertex is 'left behind' in the reflection *down* to 123.

we define polynomial h^a_b by

$$h^a_b = \begin{cases} v^i & \text{if } b \text{ appears in hypercube } h^a \text{ at depth } i; \\ 0 & \text{if } b \text{ does not appear in } h^a. \end{cases}$$

(**20.3.5**) THEOREM. *Let $a, b \subset \mathbb{N}$ label alcoves. Then $p_{a,b} = h^a_b$.*

Proof: We work by induction on the graph order on a. We can then get p_a by looking at $p_{\langle\alpha\rangle a}$ (given by $h^{\langle\alpha\rangle a}$ by the inductive assumption), where α labels one of the edges in the 'shoulder' of the hypercube associated to Γ_a.

Specifically, by the definition of P in Section 5.6.1, Theorem 20.2.5, and the inductive assumption we need to determine all the dominant α images of vertices in $h^{\langle\alpha\rangle a}$.

(**20.3.6**) NB, our strategy amounts to showing that the KL recursion (5.9), suitably applied, takes $h^{\langle\alpha\rangle a}$ to h^{α}.

Example: Let us compare p_{456} with h^{456}, under the assumption that $p_{356} \sim h^{356}$. See Fig.20.2 to apply the KL recursion to p_{356} via an equivalent procedure on h^{356}.

(**20.3.7**) For any α and $b \in P(\mathbb{N})$ let

$$\langle\alpha\rangle h^b := \{\langle\alpha\rangle c \mid c \in h^b;\ \langle\alpha\rangle c \text{ defined}\}$$

For example $\langle\alpha\rangle h^{\langle\alpha\rangle a} \ni a$ since $\langle\alpha\rangle\langle\alpha\rangle a = a$. Similarly let $\langle\alpha\rangle^2 h^b = \{c \mid c \in h^b;\ \langle\alpha\rangle c \text{ defined}\}$.

Thus $\langle\alpha\rangle^2 h^b$ is the subset of vertices of h^b with an α-edge; and $\langle\alpha\rangle h^b$ is the collection of vertices at the other ends of these edges.

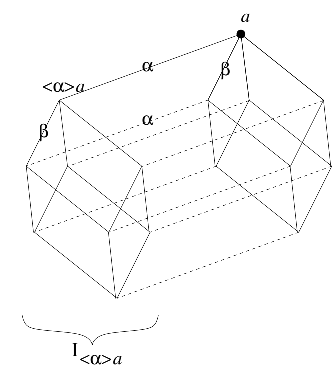
$h^a =$ 

Figure 20.3: fig:hypercub1a

Note that there is a map $\zeta : \langle\alpha\rangle h^{\langle\alpha\rangle a} \to \langle\alpha\rangle^2 h^{\langle\alpha\rangle a}$ given by $c \mapsto \langle\alpha\rangle c$, and that this is a bijection between disjoint sets.

By Section 5.6.1 (equation(5.9)) and Theorem 20.2.5, $p_{a,c} \neq 0$ for alcove c if there is a c' in $p_{\langle\alpha\rangle a}$ that, as a vertex of G_{even}, has an edge labelled α attached to it, and either $c = c'$ or $c = \langle\alpha\rangle c'$ (there is a subtraction to perform after equation(5.9), but we shall see that all such are null). The vertices occuring in p_a will thus be those in $\langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$. Note that by the bijection ζ and the inductive hypothesis every alcove appears in at most one way, and hence that every polynomial will be of form v^i. We need to check that this set of vertices in p_a agrees with those of h^a, and that they aquire the right powers via this identification.

For any $b \in P(\mathbb{N})$ define

$$\Gamma^b \setminus \alpha = \Gamma^b \setminus \{\alpha, \alpha{+}1\} \qquad \text{and} \qquad \Gamma^b(\alpha) = \{\{i,j\} \in \Gamma^b \mid \{i,j\} \cap \{\alpha, \alpha{+}1\} = \emptyset\}$$

Example: the circled sets in Fig.20.2 are $\Gamma^{356} \setminus 3$ and $\Gamma^{356}(3)$ respectively.

Consider the 'ideal' $I_{\langle\alpha\rangle a}$ with vertices $c \leq \langle\alpha\rangle a$ in hypercube h^a (e.g. Fig.20.3). Note that this sub-hypercube has shoulder $\Gamma^a \setminus \alpha$; that is

$$I_{\langle\alpha\rangle a} = hh(\langle\alpha\rangle a, \Gamma^a \setminus \alpha) \tag{20.1}$$ use1

and that the quotient of h^a by this ideal has the same shoulder set. Note also that this quotient $h^a / I_{\langle\alpha\rangle a}$ consists of the images under α of the vertices in $I_{\langle\alpha\rangle a}$, as exemplified in Figure 20.3.

It follows from Lemma 20.3.4 that $\Gamma^a \setminus \alpha$ agrees with the set $\Gamma^{\langle\alpha\rangle a}(\alpha)$ of pairs in $\Gamma^{\langle\alpha\rangle a}$ that do not intersect α or $\alpha + 1$, *except* that if there are pairs α, i and $\alpha{+}1, j$ in $\Gamma^{\langle\alpha\rangle a}$ then there will be a pair i, j in $\Gamma^a \setminus \alpha$ (that obviously does not appear in $\Gamma^{\langle\alpha\rangle a}$):

$$\Gamma^{\langle\alpha\rangle a}(\alpha) \;\subseteq\; \Gamma^a \setminus \alpha \tag{20.2}$$ use2

From (20.1) and (20.2) we have that $hh(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ is a subgraph of $I_{\langle\alpha\rangle a}$ and hence of h^a (albeit one layer down from the 'head'), and also of $h^{\langle\alpha\rangle a}$.

As noted, all the vertices in the subgraph $hh(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ of $h^{\langle\alpha\rangle a}$ have α-images (and these images are above in the graph order). Thus all these vertices and images appear in p_a

(by the inductive assumption $p_{\langle\alpha\rangle a} \equiv h^{\langle\alpha\rangle a}$ and the constructive definition of p_a from $p_{\langle\alpha\rangle a}$). The power of v for each image vertex is inherited from the original vertex (for example $p_a(a) = p_a(\langle\alpha\rangle\langle\alpha\rangle a) = p_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = v^0$), while the power of v for the original vertex is raised by 1 (example: $p_a(\langle\alpha\rangle a) = vp_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = vv^0 = v^1$). We see, therefore, that *all these vertices have the correct exponent.*
The other vertices in the shoulder of $h^{\langle\alpha\rangle a}$ (the ones, if any, at the end of edges of form α, i and $\alpha{+}1, j$) do not have α-images. Thus we have agreement between h^a and $p_a \sim \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$ except for the ideal generated by $\langle ij\rangle a$ as above (if any) in h^a on the one hand; and the possible descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ that *do* have α-images on the other.

It remains to show that these contributions match up (with the correct powers).

If there is no such $\langle ij\rangle a$ then one can show that there are not descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ with α-images and we are done. So let us suppose there is $\langle ij\rangle a$ in h^a. Note that for our a we have

$$\langle ij\rangle a = \langle\alpha, i\rangle\langle\alpha{+}1, j\rangle\langle\alpha\rangle a \tag{20.3}$$

eq:cool

See Figure 20.4 for a representative example of this. We have there

$$a = \{1,5,8,10,11,12\} \stackrel{\langle 45\rangle}{\to} 1\ 4\ 8\ 10\ 11\ 12 \stackrel{\langle 5\ 12\rangle}{\to} 1\ 4\ 5\ 8\ 10\ 11 \stackrel{\langle 3\ 4\rangle}{\to} 1\ 3\ 5\ 8\ 10\ 11 = \langle 3\ 12\rangle a$$

A similar version works for $\langle \underline{ij}\rangle$ operators.

The $\langle ij\rangle a$ in $h^{\langle\alpha\rangle a}$ is in level $i = 2$ by (20.3), and has a hypercube $hh(\langle ij\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ below it. All the elements of this hypercube have α-images, since $\langle\alpha\rangle, \langle ij\rangle$ commute. Note for example that $\langle ij\rangle a$ itself has an α-image (although $\langle ij\rangle a$ is below $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$, which does not have an α-image, in the *graph* order), and that its α-image $\langle\alpha\rangle\langle ij\rangle a$ is *below* it in the graph order. The other labels in the ideal behave similarly. Thus the polynomials asigned by Equation(5.9) to the relevant part of $p_a \sim \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$ are, for v^i the relevant polynomial from $p_{\langle\alpha\rangle a}$, v^i (for the α-image) and v^{i-1} (the vertex 'left behind') respectively. The -1 compensates for the fact that the vertex appears in $h^{\langle\alpha\rangle a}$ one layer lower than in h^a (where it appears in the shoulder in the case of $\langle ij\rangle a$ itself for example), so subject to the working assumptions we verify $p_a \equiv h^a$.

Note finally that this -1 increment only occurs for the vertex $\langle ij\rangle a$ and those below it, and thus for polynomials v^i with exponent $i \geq 2$. Thus we never have an increment of form $v^1 \to v^{1-1} = v^0$ (which would incur a subtraction in the polynomial construction). □

20.4 Related notes and open problems

An application of this work is as a base for corresponding calculations over fields of finite characteristic (cf. [29, §6]).

A *physically* motivated application is in computing eigenvectors of the Young matrix (the adjacency matrix of the Young graph [87]), which are involved in quantum spin chain computations (see e.g. [20]).

We note that formal connections between parabolic Kazhdan–Lusztig polynomials and Brauer algebra decomposition matrices can be constructed in principle by other approaches [129]. However such formal approaches do not give the specific decomposition numbers.

Finally we note that [47] includes formulations of Kazhdan–Lusztig polynomials related to the $\mathcal{D}/\mathcal{D}_+$ case, considered from a different perspective.

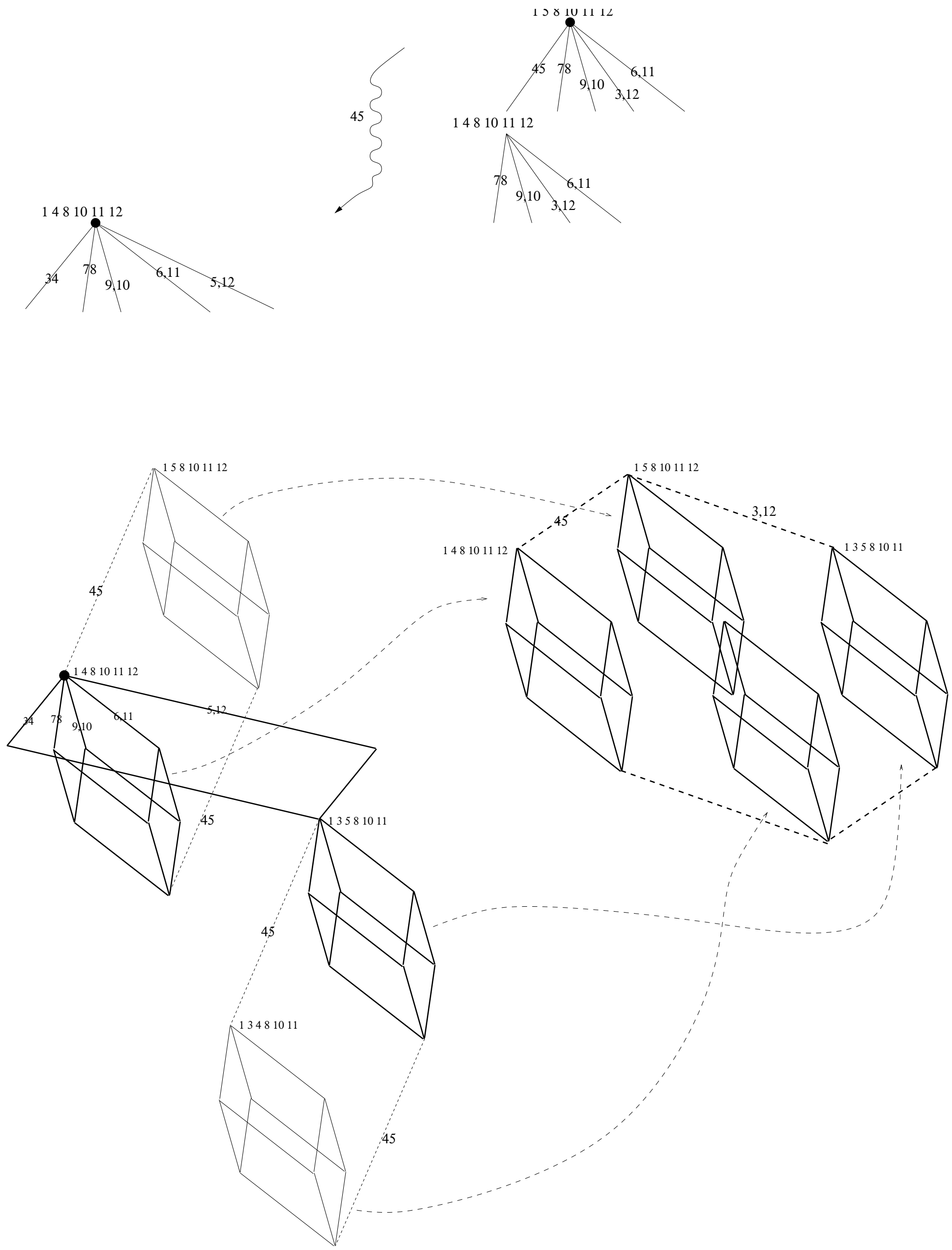


fig:eg158

Figure 20.4: A representative example. The top part of the figure shows the shoulder of h^a for $a = \{1, 5, 8, 10, 11, 12\}$; and the shoulder of the ideal below $\langle 45\rangle a = \{1, 4, 8, 10, 11, 12\}$ within h^a. Immediately below-left of this is the shoulder of $h^{\langle 45\rangle a}$ itself (showing that this hypercube is bigger than the corresponding ideal within h^a). The bottom-left part of the figure shows all of the vertices in $h^{\langle 45\rangle a}$ that have $\langle 45\rangle$ images (and a couple which do not, that are relevant for the construction); together with a representation of those images. The bottom-right part shows how all these vertices may be collected together to constitute the vertices of h^a.

20.5 Block labelling weights

ss:blab weights

How can we tell when a weight is the lowest weight in its block? There is a characterisation in [28].

20.6 Changing δ

ss:changeQQ

Vague Claim: Take a minimal weight and add a minimal row to the top or column to the left. The new weight is often minimal...

Chapter 21

More Brauer algebra modules

ch:BrauerV

Here we describe explicit actions of the Brauer algebras on simple modules with labels in the regular sector (regular in the relative sense we have introduced in §**??**). Given the correspondence between the regular sector and the appropriate N-allowed part of the Young graph (§**??**), this amounts to a review and geometrical recasting of the appropriate parts of Ram–Wenzl [134] (in particular Theorem 2.4 therein).

The layout is roughly as follows. We make use of the Ram–Wenzl/Leduc–Ram representations [134, 93] (which make sense for non-integral δ, or $\delta >> n$) of the corresponding Brauer–Specht module. This in turn makes heavy use of work of El-Samra—King [**?**], so we start by reviewing the relevant parts of that work. We also need to review various bits of S_n representation theory, that play an important role. Then we recall the relevant result of Leduc–Ram; and finally 'truncate' and specialise this to give the Brauer action on our simple bases.

21.1 King's polynomials and other results

ss:KingPoly

In light of the invariant theory connection (§17.1), the dimensions of irreducible representations of the classical Lie groups $SO(N)$, $Sp(N)$ and $O(N)$ are relevant to Brauer algebra representation theory. In particular, fixing a suitable irreducible label (λ, say), this then makes sense for all sufficiently large N, and formulae for these dimensions as functions of N are relevant. Specifically, these formulae give the multiplicities in tensor space of irreducible Brauer algebra modules (whose dimensions stabilize for given λ — become independent of N — for sufficiently large N):

$$N^n = \sum_{N\text{-allowed } \lambda} \mathrm{Dim}_{O(N)}(\lambda)\ \ \mathrm{Dim}_{B_n(N)}(\lambda)$$

The dimensions of irreducible representations of the classical Lie groups can be computed from the Weyl character formula, as already noted in §**??**. A particularly useful formulation is treated in El-Samra–King [139]. We summarize some of their results here.

(**21.1.1**) Firstly recall the *Frobenius hook characterisation* of a Young diagram as the list of pairs of arm and leg lengths associated to the boxes down the main diagonal (not counting the diagonal box itself in either case):

$$\lambda = \begin{pmatrix} \lambda_1 - 1 & \lambda_2 - 2 & \dots & \lambda_i - i & \dots & \lambda_r - r \\ \lambda'_1 - 1 & \lambda'_2 - 2 & \dots & \lambda'_i - i & \dots & \lambda'_r - r \end{pmatrix}$$

where (r, r) is the last diagonal box in λ. Note that for $i \leq r$, $\lambda_i - i$ (resp. $-(\lambda'_i - i)$) is the content of the last box in the i-th row (resp. column) of λ.

(**21.1.2**) There is a fairly simple formula (from the WCF) for the polynomial

$$\mathcal{K}_\lambda(N) = \mathrm{Dim}_{O(N)}(\lambda)$$

in N that evaluates to the dimension of the $O(N)$-module with label a hook $\lambda = (a+1, 1^b) = \begin{pmatrix} a \\ b \end{pmatrix}$ (where this makes sense — i.e. for sufficiently large N). We have

$$\mathcal{K}_{\begin{pmatrix} a \\ b \end{pmatrix}}(N) = \frac{1}{a!b!(a+b+1)} \frac{(N+2a)}{(N-1+a-b)} \frac{(N-1+a)!}{(N-2-b)!}$$

$$= \frac{1}{a!b!(a+b+1)} (N+2a) \frac{(N-1+a)(N-1+a-1)\dots(N-1-b)}{(N-1+a-b)}$$

(**21.1.3**) Then in general we have from El-Samra–King:

$$\mathrm{Dim}_{O(N)}(\lambda) = \left| \mathrm{Dim}_{O(N)} \begin{pmatrix} a_i \\ b_j \end{pmatrix} \right|$$

$$= \begin{vmatrix} \mathrm{Dim}_{O(N)} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} & \mathrm{Dim}_{O(N)} \begin{pmatrix} a_1 \\ b_2 \end{pmatrix} & \dots \\ \vdots & \ddots & \\ \mathrm{Dim}_{O(N)} \begin{pmatrix} a_r \\ b_1 \end{pmatrix} & \dots & \mathrm{Dim}_{O(N)} \begin{pmatrix} a_r \\ b_r \end{pmatrix} \end{vmatrix}$$

El-Samra–King then use a lemma of Cauchy to compute the determinant.

Unfortunatelyt (if unsurprsingly) the resultant general formula of El-Samra–King (or similarly Leduc-Ram) is not quite as clean as the hook cases. We shall need some preparation to state it.

(**21.1.4**) For each box (i, j) in a Young diagram λ we define (following [139]; see also Leduc–Ram)

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & if\ i \leq j \\ -\lambda'_i - \lambda'_j + i + j - 1 & if\ i > j \end{cases}$$

We also write $h(i, j)$ for the usual hook length.

Figure 21.1 has some examples for $d(i, j)$. Cf. The combinatoric associated to the Weyl Character Formula as in (24.3).

Figure 21.2 draws the Bratteli diagram with associated King polynomials, indicating some of the N-allowed subgraphs.

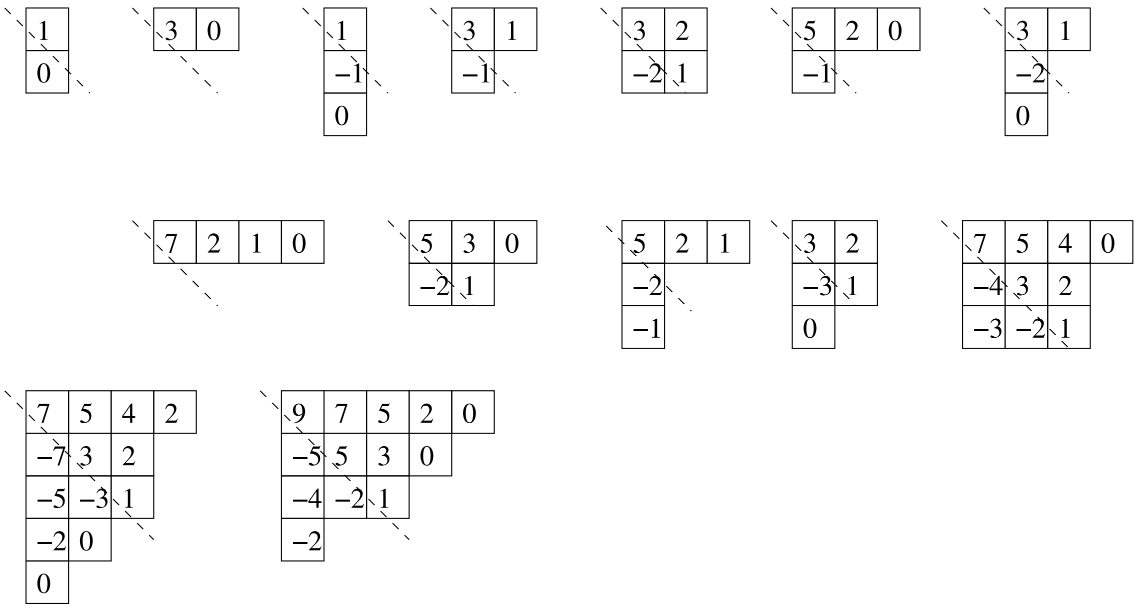


Figure 21.1: Examples of $d(i,j)$. JOBS: DO some more. fig:king-length2

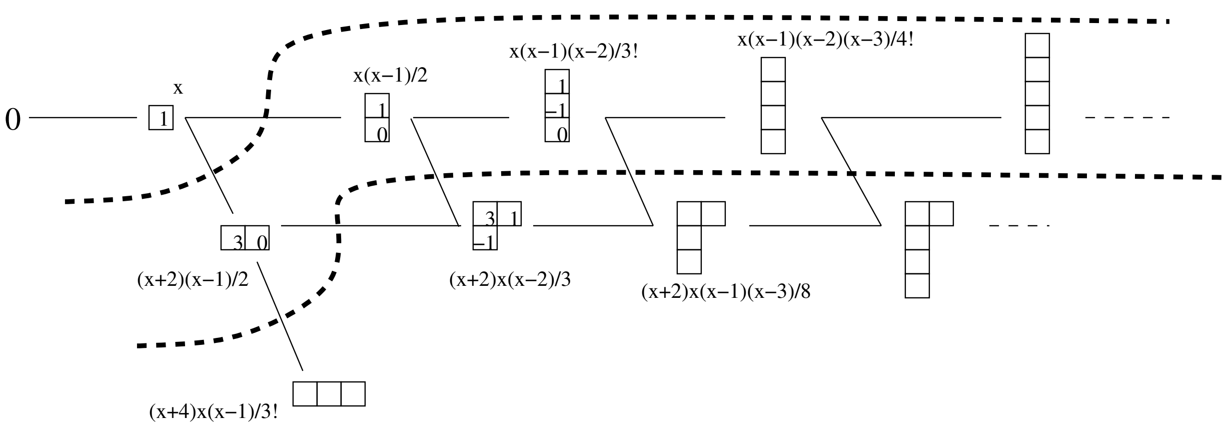


Figure 21.2: King Polynomials on the Young graph. JOBS: DO some more. fig:0123res-poly

(**21.1.5**) Let λ be a partition. Define the King (or El-Samra–King) polynomial [139]

$$P_\lambda(x) = \prod_{(i,j)\in\lambda} \frac{x-1+d(i,j)}{h(i,j)}$$

For example, $P_\emptyset = 1$, $P_\square = x$,

$$P_{(1^2)} = \frac{x(x-1)}{2}, \quad P_{(2)} = \frac{(x+2)(x-1)}{2}$$

$$P_{(1^3)} = \frac{x(x-2)(x-1)}{3!}, \quad P_{(2,1)} = \frac{(x+2)x(x-2)}{3}, \quad P_{(3)} = \frac{(x+4)x(x-1)}{3!}$$

$$P_{(1^4)} = \frac{x(x-3)(x-2)(x-1)}{4!}, \quad P_{(21^2)} = \frac{(x+2)x(x-3)(x-1)}{4.2}, \quad ...$$

Let V^λ denote the λ-irreducible representation of $SO(2r+1)$. Then

$$P_\lambda(2r+1) = \dim V^\lambda$$

In short, $\mathcal{K}_\lambda(N) = P_\lambda(N)$.

21.2 Connection between King polynomials and D/A alcove geometry

There is a useful connection between King polynomials and alcove geometry.

(**21.2.1**) First we recast P_λ slightly. Ignoring the denominator (which is easy to compute from the dimension of the corresponding S_n irreducible (and which will in any case be irrelevant for our work over the complex field)) we can encode the King polynomial P_λ as the multiplicity of each root. The roots are given by a subset of $\mathbb{Z}$, so we end up with an element K_λ of $Hom(\mathbb{Z}, \mathbb{N}_0)$ of finite support. Then:

$$P_\lambda \;=\; \frac{\prod_{a\in\mathbb{Z}}(x-a)^{K_\lambda(a)}}{\prod hook\ lengths}$$

(**21.2.2**) Now consider, for each $x \in \mathbb{Z}$, the dominant weight $e_x(\lambda)$:

$$e_x(\lambda) = \lambda - (0,1,2,3,...) - \frac{x}{2}(1,1,1,1,...)$$

This has a degree of singularity relative to $\rho_x = e_x(0)$, which degree lies in $\mathbb{N}_0$. Running over $x \in \mathbb{Z}$, this then defines another element $X_\lambda \in Hom(\mathbb{Z}, \mathbb{N}_0)$.

That is, to compute X_λ, for each x we count the number of singular pairs in $\rho_x = e_x(0)$ (easy), and compare with the number in $e_x(\lambda)$.

JOB: CHECK DEFINITIONS AND GIVE EXAMPLES.

ASIDE: The definition says that $X_\lambda(a) > 0$ when $e_x(\lambda)$ lies on more $x = a$-walls than $e_x(0)$. There must be an ij-plane we can look in where $e_x(0)$ does not lie on the D-hyperplane (i.e. the $(ij)_-$-hyperplane), but $e_x(\lambda)$ does.

thmalamKing1 (**21.2.3**) THEOREM. *For* $\lambda \vdash n < 18$

$$K_\lambda = X_{\lambda'}$$

Conjecture: This holds for all n.

Proof. /Remarks: I have checked about 10,000 cases in gap, which covers the claim. (It may be easy to prove the conjecture — I have not tried.) The basis for the conjecture is that King polynomials (give dimensions of $O(N)$-modules and) determine the mixing between states in Leduc–Ram's orthogonal form for *generic* Brauer Specht modules. If walks in the fundamental D/A-alcove decouple for given x (a direct analogy of the Hecke algebra case) then this implies a connection between vanishing of K_λ and contact of $e_x(\lambda)$ with a relevant wall. The conjecture is the simplest form of such connection, and holds in very many cases.

One approach to a proof would be to consider the changes on each side in the event of adding a box.

21.3 Leduc–Ram representations of Brauer algebras and other results

One should look at §11.5.1 before proceeding.

Fix a tower of semisimple algebras, and hence a corresponding Bratteli diagram. Leduc–Ram [93] use the notation Ω^m for the set of pairs of paths of length m from the *root* of a Bratteli diagram to the same vertex (various assumptions, supported by the Bratteli diagram $\mathfrak{B}(B)$ as in (**??**), are needed). For k a field they define a k-algebra A_k with basis E_{ST} $((S,T) \in \Omega^m)$ and

$$E_{ST}E_{PQ} = \delta_{TP}E_{SQ}$$

This is called a *path algebra*. The point is that it is a multimatrix algebra isomorphic (on elementary combinatorial grounds) to the m-th algebra in the tower defining the Bratteli diagram.

Here we describe the Leduc–Ram action of the 'generic' Brauer algebra on the walk basis of the path algebra. To do this we need to introduce various functions, used in giving the matrix elements of generators. As for the symmetric group, mixing terms are restricted to walks whose difference is described by single diamond pairs.

21.3.1 Brauer diamonds

Note that for walks on $\mathfrak{B}(B)$, each vertex $s(m)$ is a Young diagram and steps in S may be: (++) = add-box, add-box; (+-) = add-box, remove-box; (-+), or (- -).

Here we shall consider S, S to constitute an (improper) diamond pair (at $m-1 = i$ say), in the sense of (11.5.2). If S is (++) (resp. (- -)) then so is any T such that S, T form a diamond pair. If S is (+-), or (-+), then T may be either of these.

(**21.3.1**) For S, T a (not necessarily proper) diamond pair define

$$\diamond^B_{m-1}(S,T) = \begin{cases} \pm(s(m)_k - k - t(m-1)_l + l) & \text{if } t(m-1) = s(m-2) \pm \epsilon_l \\ & \text{and } s(m) = s(m-1) \pm \epsilon_k \\ \pm(x + t(m-1)_l - l + s(m)_k - k) & \text{if } t(m-1) = s(m-2) \mp \epsilon_l \\ & \text{and } s(m) = s(m-1) \pm \epsilon_k \end{cases}$$

(**21.3.2**) Consider the ++ case for a moment:

$$S \;=\; (s(m-2), s(m-1), s(m)) \;=\; (\lambda, \lambda+\epsilon_j, \lambda+\epsilon_j+\epsilon_k)$$

for some λ, j, k. Then $s(m)_k - k$ is the content of the last added box in the sequence $S = (s(m-2), s(m-1), s(m))$; while $t(m-1)_l + l$ (some l) is the content of the first added box in $T = (s(m-2), t(m-1), s(m))$.

Note that if $t(m-1) \neq s(m-1)$ — the non-diagonal case — then $l = k$. (Interesting that j plays no apparent role here!) In the diagonal case $l = j$.

(**21.3.3**) Note that if $S \neq T$ then $k = l$. (Right?!)

(**21.3.4**) Example. Let us consider the 'diagonal' case $\diamond^B_{m-1}(S, S)$. For example we have $S = (\emptyset, \square, \emptyset)$, which is in case $2-$ with $t(m-1) = s(m-1) = \square = \emptyset + \epsilon_1$ and $k = 1$ and $l = 1$, giving

$$\diamond^B_{m-1}(S, S) = -(x + \square_1 - 1 + \emptyset_1 - 1\ ) = -(x\ - 1).$$

Meanwhile $S = (\square, \emptyset, \square)$, is in case $2+$ with $t(m-1) = s(m-1) = \emptyset = \square - \epsilon_1$ and $k = 1$ and $l = 1$, giving

$$\diamond^B_{m-1}(S, S) = (x + \emptyset_1 - 1 + \square_1 - 1\ ) = x\ - 1.$$

(**21.3.5**) More:

$$\diamond^B(\emptyset \square \begin{smallmatrix}\square\\ \square\end{smallmatrix}, \emptyset \square \begin{smallmatrix}\square\\ \square\end{smallmatrix}) = +(\begin{smallmatrix}\square\\ \square\end{smallmatrix}_2 - 2 - \square_1 + 1) = -1$$

$$\diamond^B(\emptyset \square \square\!\square, \emptyset \square \square\!\square) = +(\square\!\square_1 - 1 - \square_1 + 1) = 1$$

Note that when we step up-up like this we get the *content* of the last added box in S minus the content of the first added box in T.

(**21.3.6**) Thus, when S is up-up $\diamond^B(S, S)$ *is the hook length between the two added boxes.*
("If we step down-down we get minus this result." — what does this mean?!)
Note that in general (unlike our examples) there are two orders in which the two boxes can be added, giving a pair S, T (indeed $S, \sigma_i(S)$, in a suitable local sense — keeping in mind that S is not actually a tableau in general). Note that the hook lengths are equal and opposite.

(**21.3.7**) We shall also need

$$\diamond^B(\square \underbrace{\emptyset \square}_{k=1}, \underbrace{\square \begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{l=2} \square) = (\square_1 - 1 - \begin{smallmatrix}\square\\ \square\end{smallmatrix}_2 + 2) = 1$$

$$\diamond^B(\square \underbrace{\emptyset \square}_{k=1}, \underbrace{\square \square\!\square}_{l=1} \square) = (\square_1 - 1 - \square\!\square_1 + 1) = -1$$

Note that here S, T cooperate to make ++ cases. The statement about content applies precisely as stated before. Swapping the roles of S, T the last case becomes a $--$ case:

$$\diamond^B(\square \underbrace{\square\!\square \square}_{k=1}, \underbrace{\square \emptyset}_{l=1}\ \square) = -(\square_1 - 1 - \emptyset_1 + 1) = -1$$

Note that the overall effect was not to change the answer. The content statement here...

(**21.3.8**) More $+-$ cases:

$$\diamond^B(\square\underbrace{\yng(1,1)\square}_{k=2},\underbrace{\square\yng(1,1)}_{l=2}\square) = -(x+\square_2 - 2 + \yng(1,1)_2 - 2) = -(x-3)$$

$$\diamond^B(\square\underbrace{\yng(2)\square}_{k=1},\underbrace{\square\yng(2)}_{l=1}\square) = -(x+\square_1 - 1 + \yng(2)_1 - 1) = -(x+1)$$

$$\diamond^B(\square\underbrace{\yng(1,1)\square}_{k=2},\underbrace{\square\yng(2)}_{l=1}\square) = -(x+\square_2 - 2 + \yng(2)_1 - 1) = -(x-1)$$

Note this last one is $+-$.

(**21.3.9**) $-+$ Cases with $s(m-2) \neq s(m)$ but of same degree:

$$\diamond^B(\yng(1,1)\underbrace{\square\yng(2)}_{k=1},\underbrace{\yng(1,1)\square}_{l=2}\yng(2)) = (x+\square_2 - 2 + \yng(2)_1 - 1) = x-1$$

$++$:

$$\diamond^B(\yng(1,1)\underbrace{\square\yng(2)}_{k=1},\underbrace{\yng(1,1)\yng(2,1)}_{l=1}\yng(2)) = (\yng(2)_1 - 1 - \yng(2,1)_1 + 1) = 0$$

(although it is not clear if any such case is ever used).

(**21.3.10**) Just as we can think of the $++$ cases in terms of content, and hooks, we can also think of them in terms of *charge* — let us say x-charge (since we compute a difference of charges, the xs cancel). If we think of the $-+$ cases as *adding charges*, then the two types are unified. (N.B. One must take care of an overall factor of 2.)

(**21.3.11**) What is the relationship between $\diamond^B$ and King polynomials?

21.3.2 Leduc–Ram representations

(**21.3.12**) The Brauer diagram algebra is generated by the symmetric group Coxeter generators $\langle\sigma_i\rangle_i$ and the TL diagram generators $\{e_i\}$ (indeed e_1 is enough).

(**21.3.13**) Theorem. (Leduc–Ram) [93, (6.22)] *There is an identification of the Brauer algebras with the path algebras of $\mathfrak{B}(B)$. The identification is:*

$$\sigma_i = \sum_{(S,T)\in\Omega_{i-1}^{i+1}} (\sigma_i)_{ST} E_{ST}$$

(the sum is over pairs of paths that differ in at most the indicated position) where

$$(\sigma_i)_{SS} = \begin{cases} \frac{1}{\diamond_i^B(S,S)} & \text{if } s(i-1) \neq s(i+1) \\ \frac{1}{\diamond_i^B(S,S)}\left(1 - \frac{P_{s(i)}(x)}{P_{s(i-1)}(x)}\right) & o/w \end{cases}$$

$$(\sigma_i)_{ST} = \begin{cases} \sqrt{\frac{(\diamond_i^B(S,S)-1)(\diamond_i^B(S,S)+1)}{\diamond_i^B(S,S)^2}} & \textit{if } s(i-1) \neq s(i+1) \\ -\frac{1}{\diamond_i^B(S,T)} \left(\frac{\sqrt{P_{s(i)}(x)P_{t(i)}(x)}}{P_{s(i-1)}(x)} \right) & o/w \end{cases}$$

Similarly

$$(e_i)_{ST} = \begin{cases} \frac{\sqrt{P_{s(i)}(x)P_{t(i)}(x)}}{P_{s(i-1)}(x)} & \textit{if } s(i-1) = s(i+1) \\ 0 & \textit{if } s(i-1) \neq s(i+1) \end{cases}$$

Note that with $i = 1$ we have $s(i-1) = \emptyset$, thus only diagonal terms for sequences S beginning $\emptyset\square\emptyset$ are nonzero in the corresponding representation of e_1.

(**21.3.14**) Note that the action decouples into parts corresponding to sequences ending at a specific λ. Following Leduc–Ram we call the corresponding B_n-representation Π^λ.

(**21.3.15**) Note that the bit of this construction which looks directly like the collection of bases for Young's SNF (i.e. the cases $\lambda \vdash n$) looks (at least superficially) directly like Young's SNF, with $\diamond^B$s replacing hook lengths in some way...

(**21.3.16**) Shortly we shall look for specialisations of x that lead to manifest block diagonalisations of these representations. Recalling that we only need e_1 and the S_n part to generate B_n-representations; and noting the form of the matrices for e_1, indeed, we see that we can restrict attention to the S_n action to do this.
(Aside: Note that the matrix for e_i is at least as sparse (in the obvious sense) as σ_i, for any i.)

(**21.3.17**) Note that specialisation to $x = 0$ is hopeless here. Presumably we need to embed in a suitable generalisation, like the MWL generalisation of the SNF, to make this work.

Examples

There are several cases worked out explicitly in Leduc–Ram [93], including the following. We reproduce them here is slightly more detail, so we can chat about the aspects that interest us.

(**21.3.18**) Consider the $\lambda = \square$ representation of $B_3(x)$ (note that this illustrates most aspects of the construction, but not the 'off-diagonal hook' part — i.e. the off-diagonal ++ part). We have

$$\Pi^\square(\sigma_1) = \begin{array}{cc} \begin{pmatrix} \frac{1}{\diamond^B(\emptyset\square\emptyset,\emptyset\square\emptyset)}(1-\frac{P_\square}{P_\emptyset}) & 0 & 0 \\ 0 & \frac{1}{\diamond^B(\emptyset\square\substack{\square\\\square},\emptyset\square\substack{\square\\\square})} & 0 \\ 0 & 0 & \frac{1}{\diamond^B(\emptyset\square\square\!\square,\emptyset\square\square\!\square)} \end{pmatrix} & \begin{array}{l} \emptyset\square\emptyset\square \\ \emptyset\square\substack{\square\\\square}\square \\ \emptyset\square\square\!\square\square \end{array} \end{array}$$

$$= \begin{pmatrix} \frac{1}{1-x}(1-\frac{x}{1}) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Assuming for the moment that all $\diamond^B$s are S,T-swap symmetric we get:

$$\Pi^{\square}(\sigma_2) = \begin{pmatrix} \frac{1}{\diamond^B(\square\emptyset\square,\square\emptyset\square)}(1-\frac{P_\emptyset}{P_\square}) & -\frac{1}{\diamond^B(\square\emptyset\square,\square\substack{\square\\\square}\square)}\frac{\sqrt{P_\emptyset P_{\substack{\square\\\square}}}}{P_\square} & -\frac{1}{\diamond^B(\square\emptyset\square,\square\square\!\square\square)}\frac{\sqrt{P_\emptyset P_{\square\!\square}}}{P_\square} \\ -\frac{1}{\diamond^B(\square\emptyset\square,\square\substack{\square\\\square}\square)}\frac{\sqrt{P_\emptyset P_{\substack{\square\\\square}}}}{P_\square} & \frac{1}{\diamond^B(\square\substack{\square\\\square}\square,\square\substack{\square\\\square}\square)}(1-\frac{P_{\substack{\square\\\square}}}{P_\square}) & -\frac{1}{\diamond^B(\square\substack{\square\\\square}\square,\square\square\!\square\square)}\frac{\sqrt{P_{\substack{\square\\\square}} P_{\square\!\square}}}{P_\square} \\ -\frac{1}{\diamond^B(\square\emptyset\square,\square\square\!\square\square)}\frac{\sqrt{P_\emptyset P_{\square\!\square}}}{P_\square} & -\frac{1}{\diamond^B(\square\substack{\square\\\square}\square,\square\square\!\square\square)}\frac{\sqrt{P_{\substack{\square\\\square}} P_{\square\!\square}}}{P_\square} & \frac{1}{\diamond^B(\square\square\!\square\square,\square\square\!\square\square)}(1-\frac{P_{\square\!\square}}{P_\square}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{x-1}(1-\frac{1}{x}) & \frac{-\sqrt{x(x-1)/2}}{x} & \frac{\sqrt{(x+2)(x-1)/2}}{x} \\ & \frac{1}{-(x-3)}(1-\frac{x(x-1)/2}{x}) & -\frac{1}{-(x-1)}\frac{\sqrt{x(x+2)(x-1)^2/4}}{x} \\ & & \frac{1}{-(x+1)}(1-\frac{(x+2)(x-1)/2}{x}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{x} & -\frac{\sqrt{x-1}}{\sqrt{2}\sqrt{x}} & \frac{\sqrt{x-1}\sqrt{x+2}}{\sqrt{2}\,x} \\ -\frac{\sqrt{x-1}}{\sqrt{2}\sqrt{x}} & \frac{1}{2} & \frac{\sqrt{x+2}}{2\sqrt{x}} \\ \frac{\sqrt{x-1}\sqrt{x+2}}{\sqrt{2}\,x} & \frac{\sqrt{x+2}}{2\sqrt{x}} & \frac{x-2}{2\,x} \end{pmatrix}$$

(**21.3.19**) Aside on checking this representation: To work out $(\Pi^{\square}(\sigma_2))^2$ we need the dot products of the rows (still assuming a symmetric matrix). Firstly row-1.row-1:

$$\frac{1}{x^2}(1+(x(x-1)/2)+((x+2)(x-1)/2)) = \frac{1+x^2+(-x/2)+x+(-x/2)-2/2}{x^2} = 1$$

The remaining conditions for a representation can be checked using (say) Maxima[1].

(**21.3.20**) What specialisations of the parameter x make sense here? Clearly not $x=0$. All others make sense. But dramatic things happen at $x=1$, where the state $\emptyset\square\emptyset\square$ decouples from the rest. And at $x=-2$, where the first two decouple from the third.

(**21.3.21**) Next consider the $\lambda=(2,1)$ representation of $B_3(x)$. We have

$$\Pi^{(2,1)}(\sigma_1) = \begin{array}{cc} \begin{pmatrix} \frac{1}{\diamond^B(\emptyset\square\substack{\square\\\square},\emptyset\square\substack{\square\\\square})} & 0 \\ 0 & \frac{1}{\diamond^B(\emptyset\square\square\!\square,\emptyset\square\square\!\square)} \end{pmatrix} & \begin{array}{l} \emptyset\square\substack{\square\\\square}\substack{\square\square\\\square} \\ \emptyset\square\square\!\square\substack{\square\square\\\square} \end{array} \end{array} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Pi^{(2,1)}(\sigma_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{-2} \end{pmatrix}$$

and $\Pi^{(2,1)}(e_1)=0$. Note that this coincides with Young's OF.

We are more interested here in cases where the coupling of states involving $\diamond^B$ depends on x. What would be the lowest rank such?

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[1] I have done this for the S_3 part so far. See my "defns.mac".

(**21.3.22**) Case $n = 4$, $\lambda = \yng(2)$:

$$\Pi^{\yng(2)}(\sigma_2) = \left(\begin{array}{ccc|ccc} & \Pi^{\yng(1)}(\sigma_2) & & & 0 & \\ \hline 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

(the top left block as above);

$$\Pi^{\yng(2)}(\sigma_3) = \begin{array}{c} \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\diamond^B(\yng(1,1)\yng(1)\yng(2),\yng(1,1)\yng(1)\yng(2))} & 0 & \sqrt{\frac{\diamond^B(\yng(1,1)\yng(1)\yng(2),\yng(1,1)\yng(1)\yng(2))^2-1}{\diamond^B(\yng(1,1)\yng(1)\yng(2),\yng(1,1)\yng(1)\yng(2))^2}} & 0 & 0 \\ 0 & 0 & * & 0 & * & \frac{\sqrt{P_{\yng(1)}P_{(3)}}}{P_{\yng(2)}} \\ 0 & \sqrt{\frac{\diamond^B(\yng(1,1)\yng(1)\yng(2),\yng(1,1)\yng(1)\yng(2))^2-1}{\diamond^B(\yng(1,1)\yng(1)\yng(2),\yng(1,1)\yng(1)\yng(2))^2}} & 0 & \frac{1}{\diamond^B(\yng(1,1)\yng(2,1)\yng(2),\yng(1,1)\yng(2,1)\yng(2))} & 0 & 0 \\ 0 & 0 & * & 0 & * & \frac{-1}{-x}\frac{\sqrt{P_{\yng(2,1)}P_{(3)}}}{P_{\yng(2)}} \\ 0 & 0 & * & 0 & * & * \end{array}\right) \end{array} \begin{array}{c} \emptyset\,\yng(1)\,\emptyset\,\yng(1)\,\yng(2) \\ \emptyset\,\yng(1)\,\yng(1,1)\,\yng(1)\,\yng(2) \\ \emptyset\,\yng(1)\,\yng(2)\,\yng(1)\,\yng(2) \\ \emptyset\,\yng(1)\,\yng(1,1)\,\yng(2,1)\,\yng(2) \\ \emptyset\,\yng(1)\,\yng(2)\,\yng(2,1)\,\yng(2) \\ \emptyset\,\yng(1)\,\yng(2)\,(3)\,\yng(2) \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{x-1} & 0 & \sqrt{\frac{(x-1)^2-1}{(x-1)^2}} & 0 & 0 \\ 0 & 0 & * & 0 & * & * \\ 0 & \sqrt{\frac{(x-1)^2-1}{(x-1)^2}} & 0 & \frac{-1}{x-1} & 0 & 0 \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & * & 0 & * & * \end{pmatrix}$$

What decouplings can we see here?

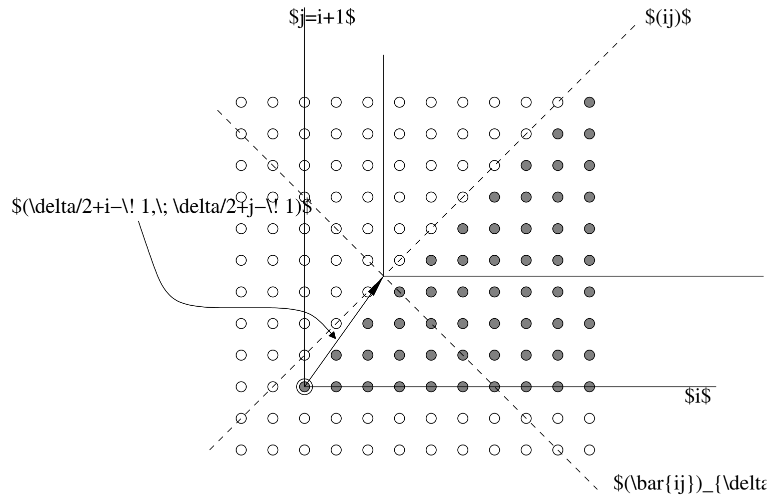
There is of course something for the vanishing of $P_{(3)} \propto (x+4)x(x-1)$, corresponding to the map from $(4) \to (2)$ when $x = -4$ (one readily checks that $(4)/(2)$ is x-balanced in the sense of CDM-I when $x = -4$).

For the specialisation $x = 2$ note that the basis walks that touch the partition $(2,1)$, with $P_{(2,1)} = (x+2)x(x-2)/3$, decouple from the rest (one checks that $(2^2)/(2)$ is 2-balanced in the sense of CDM-I).

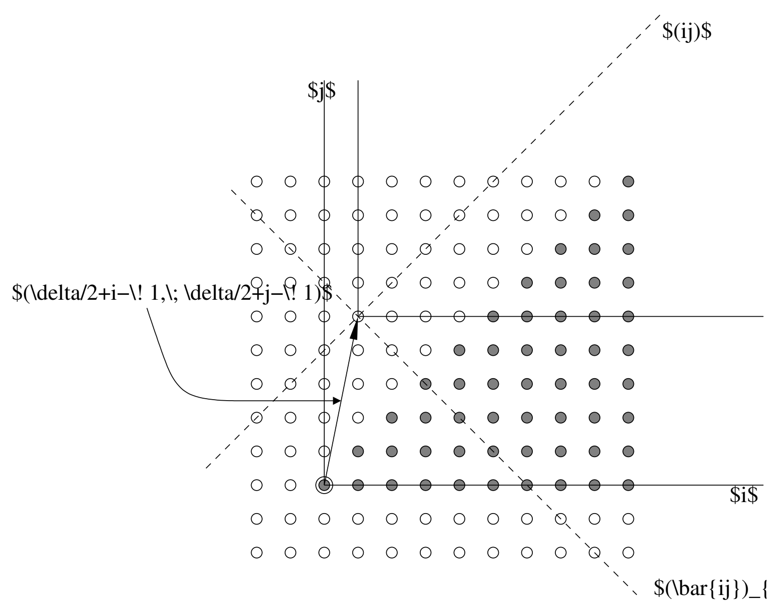
Similar use of δ-balance points to $x = 0$. This specialisation is apparently quite badly defined here, corresponding to the fact that the labelling weight is singular.

21.3.3 Geometrical realisation

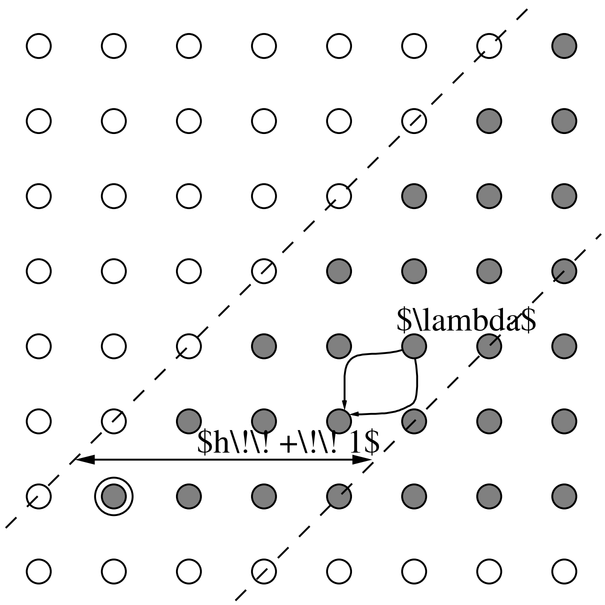
Consider the usual CDM embedding $e_\delta : \Lambda \to \mathbb{R}^{\mathbb{N}}$ given by $e_\delta(\lambda) = \lambda - \rho - \delta/2(1,1,...)$. In particular $e_\delta(\emptyset) = (-\delta/2, -1-\delta/2, ...)$. For any δ such embedding takes Λ into the A-dominant (descending) region. See Figure 21.3. (No two images $e_\delta(\Lambda)$ intesect, but their projections onto the ij-plane intersect — every one being essentially of the form shown.)



xcd2 Figure 21.3: Projection of $\mathbb{R}^{\mathbb{N}}$ onto the ij-plane (in case $j = i+1$), showing reflection hyperplanes (this example has $(\delta/2+i-1, \delta/2+j-1) = (5/2, 7/2)$, so $i = 7/2 - \delta/2$; thus $\delta = 5$, $i = 1$, $j = 2$, say; or equivalently $\delta = -3$, $i = 5$, $j = 6$). Fibres containing A-dominant (i.e. descending) integral weights are shaded.



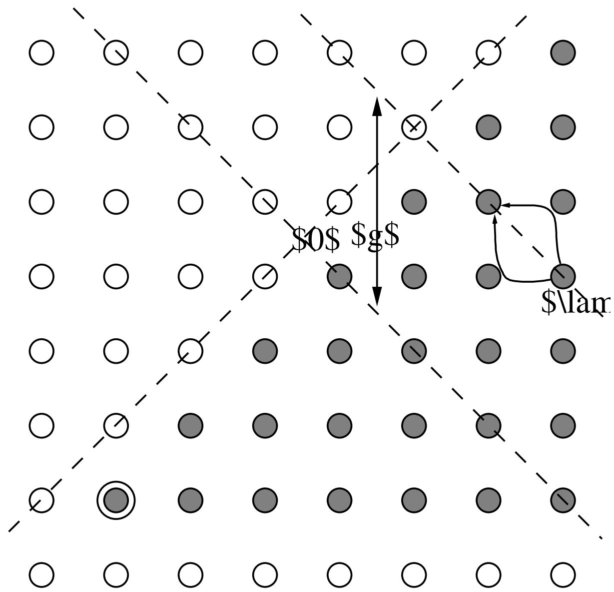
xcd Figure 21.4: Projection of $\mathbb{R}^{\mathbb{N}}$ onto the ij-plane, showing reflection hyperplanes (this example has $(\delta/2+i-1, \delta/2+j-1) = (1, 5)$, so $i = 2 - \delta/2$ and $j = i+4$; thus $\delta = 2$, $i = 1$, $j = 5$, say; or equivalently $\delta = -4$, $i = 4$, $j = 9$). Fibres containing A-dominant integral weights are shaded.



$$\leadsto \frac{1}{h}\begin{pmatrix} 1 & \sqrt{h^2-1} \\ \sqrt{h^2-1} & -1 \end{pmatrix}$$

fig:general h

Figure 21.5: The action of σ_k mixing between two walks differing only in the k-th position, stepping from λ to $\lambda - e_j - e_i$ as shown (i.e. differing locally by reflection in a suitable affine A-wall), is as indicated.



$$\rightsquigarrow \frac{1}{g}\begin{pmatrix} 1 & \sqrt{g^2-1} \\ \sqrt{g^2-1} & -1 \end{pmatrix}$$

fig:general g

Figure 21.6: The action of σ_k mixing between two walks differing only in the k-th position, stepping from λ to $\lambda - e_j + e_i$ as shown (i.e. differing locally by reflection in a suitable affine D-wall), is as indicated.

(**21.3.23**) Caveat: This embedding leads to the CDM geometric linkage principle, but one must use λ^t for λ.

(**21.3.24**) An *integral walk* in $\mathbb{R}^{\mathbb{N}}$ is a walk where every step is an axial unit vector (i.e. $\pm\mathfrak{e}_i$ for some i). Thus an integral walk starting in $\mathbb{Z}^{\mathbb{N}}$ remains in $\mathbb{Z}^{\mathbb{N}}$. The projection of such a walk is a walk in $\mathbb{Z}^2$ where each step is either an axial unit vector or null (corresponding to an axial unit vector step in an orthogonal direction). It follows trivially that any walk of two steps can be represented in an ij-plane with no null steps.

(**21.3.25**) We claim that sets of walks of length n from v to w provide bases for B_{n-1}-modules. HOW!? Any such set with $n = 2$ from v to v is infinite... $0 \to \pm\mathfrak{e}_i \to 0$ for $i \in \mathbb{N}$. What is the action of σ_1, U_1? Can we abstract this from LR by thinking about the mixing for $\lambda \to \lambda \pm \mathfrak{e}_i \to \lambda$ where λ is large-generic, so allowing add and subtract in all rows...? Or do we want to ban steps that are non-dominant beyond a certain extent?

To start with maybe we should ban non-dominant steps altogether...

See Figure 21.4 for more general j. See Figures 21.5,21.6 for the LR matrix entries determined in the geometric setting.

de:g (**21.3.26**) In LR, if $s(i-1) \neq s(i+1)$ the non-S_n-like case is:

$$\begin{tikzcd}
& \overbrace{\lambda+e_h}^{\cdots} \arrow[dr] & \\
s(i-1)=\lambda \arrow[ur] \arrow[dr] & & s(i+1)=\lambda+e_h-e_j \\
& s(i)=\lambda-e_j \arrow[ur] &
\end{tikzcd}
\qquad \rightsquigarrow \quad \frac{\sqrt{\diamond^B(S,S)^2-1}}{|\diamond^B(S,S)|}$$

Here $\diamond^B(S,S)$ is

$$\diamond^B(\lambda, \underbrace{\lambda-e_j, \lambda+e_h-e_j}_{k=h}, \underbrace{\lambda, \lambda-e_j}_{l=j}, \lambda+e_h-e_j) = +(x+\underbrace{(\lambda-e_j)_h-h}_{\lambda_h-h}+(\lambda_j-j))$$

On the other hand, the setup is essentially as depicted, in the CDM *transpose* setting $\lambda \mapsto e_\delta(\lambda^t)$, in Figure 21.6 (strictly speaking the direction of walks is reversed, and the given labels i, j in the figure are not compatible and should be ignored here!). We can see from Figure 21.5 that the $\diamond^B(S,S)$ in that case has a very simple geometrical expression, determined by the distance from the A-reflection wall. We will show something very similar here.

We say that a weight is *X-irregular*, for some $X \in \mathbb{Z}$, if its e_δ image lies on the D-reflection wall for this alternative δ value. Thus every weight is X-irregular for some X. For example we have from the figure that $(\lambda+e_h)^t$ ($(\lambda+e_h)$ transpose) is $(\delta+g+1)$-irregular. From this we know that the column (i.e. the transposed row) containing e_h, column λ_h+1, which has length h, is in a reflection-fixed pair with some other column — indeed this must be the column containing the e_j in $\lambda+e_h$, since $(\lambda+e_h-e_j)^t$ is $(\delta+g+1)$-regular. Thus we have (with $N=\delta$):

$$-((\lambda_h+1)-h) - \frac{(\delta-2)+g+1}{2} \;=\; -(-(\lambda_j-j) - \frac{(\delta-2)+g+1}{2})$$

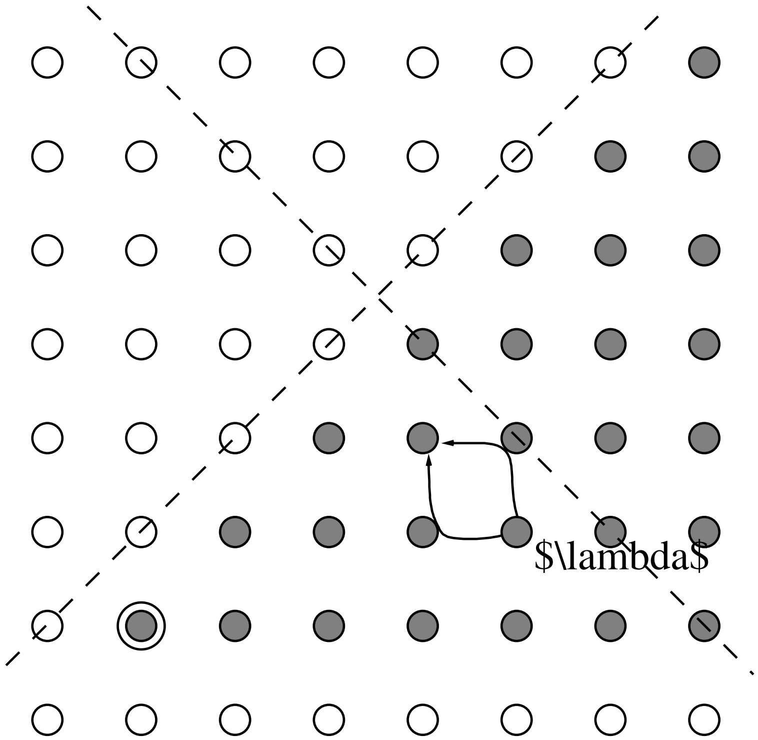


Figure 21.7: The mixing between the two walks from λ to $\lambda + e_j - e_i$ shown is zero, since $\diamond^B_{\lambda+e_j}(\delta) = 0$.

xcd3d

giving

$$\lambda_h - h + \lambda_j - j + ((\delta - 2) + g + 1) + 1 = 0$$

Thus $\diamond^B(S, S) = g$.

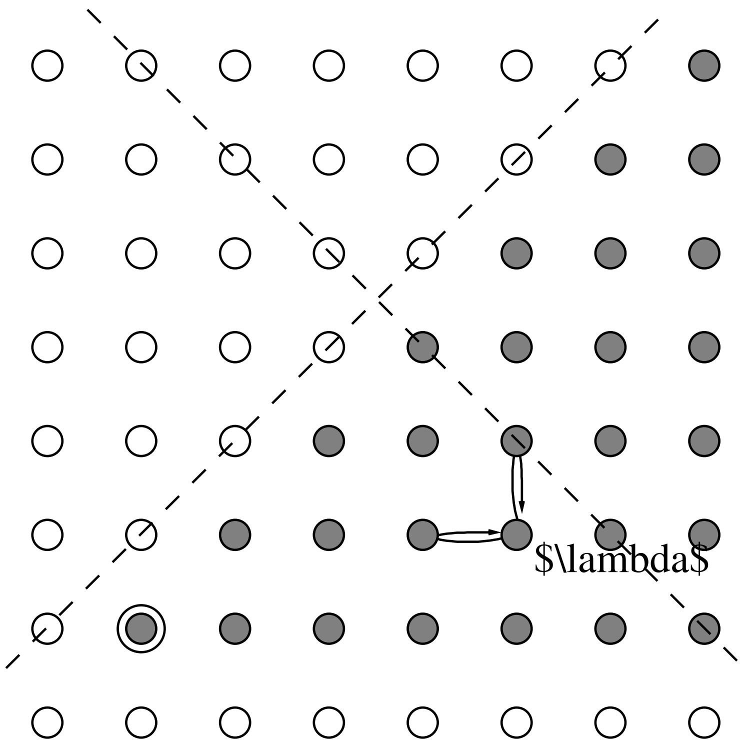


Figure 21.8: The mixing between the two walks from λ to λ shown is zero, since $P_{\lambda+e_j}(\delta) = 0$. $\boxed{\texttt{xcd3}}$

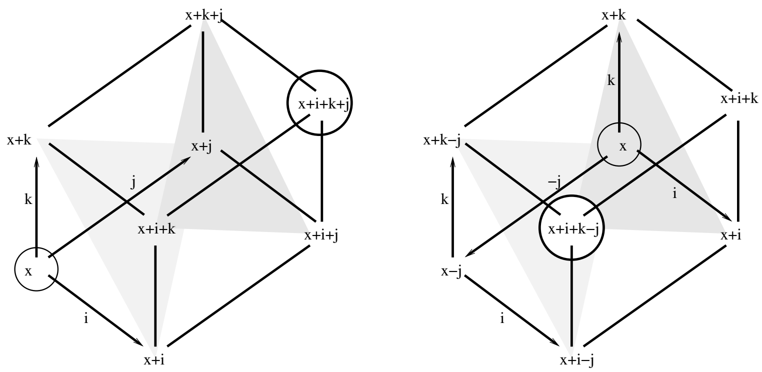


Figure 21.9: Paths of length three on $\mathbb{Z}^{\mathbb{N}}$ from x to $x+i+j+k$ and to $x+i+k-j$. yb1

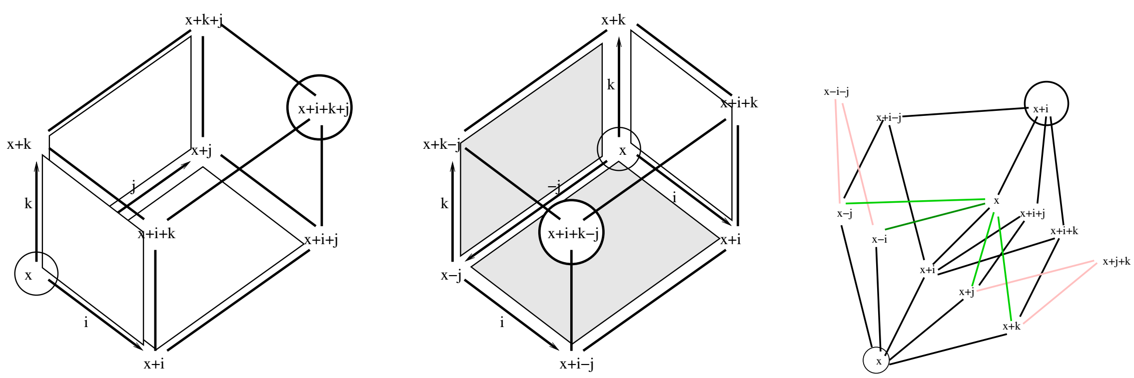


Figure 21.10: Paths of length three on $\mathbb{Z}^{\mathbb{N}}$ from x to $x+i+j+k$, to $x+i+k-j$ and to $x+i$. Examples of case-1 diamonds are shown white, case-2 diamonds are shaded. yb21

21.4 On 'untruncating' Leduc–Ram representations

LR representations have a basis of up-down walks on the Young graph. That is, a step corresponds to adding or removing a box. The representation of a given generator is described by giving the mixings between walks differing in a certain position, and the validity of the representation (held up against a presentation on these generators) can be checked by looking at walks differing in two positions.

The precise scope for adding or removing a box depends on the starting Young diagram (we cannot add a box to the lower of two rows of equal length, and so on). However 'generically' Young diagram rows are not of equal length and a box can be added or removed in any position. In most cases then, the suite of walks needed for checking form the possible paths on a cube from one vertex to the opposite vertex (with the edges labelled by the rows where boxes are moved). See Fig.21.9 and 21.10.

Essentially the same check works in infinitely many cases, so our idea is to replace the Young graph with unrestricted walks on our underlying lattice. (And hence construct infinite-dimensional modules — 'Verma modules' or 'Young modules', depending on ones perspective.) The new feature is that there is one type of case to check where a walk step may be taken in any 'addable/removable' direction — in our case this is, then, any of *countably many* directions, indexed by $\mathbb{N}$. This is illustrated by Fig.21.10(c).

The exercise is to make this work.

21.5 Truncating Leduc–Ram representations (old version!)

ss:main5

See figures 21.8,21.7.

(**21.5.1**) Fix any $N \in \mathbb{Z}$ [2] and λ an N-admissible partition. We *claim*
(I) that for fixed $N \in \mathbb{Z}$ the restriction of the LR representation Π^λ to N-regular walks (under the D/A-isomorphism) gives a well-defined representation of $B_n(N)$ by setting $x = N$.
(II) that this is the simple head of the Brauer–Specht representation with the same label.

We prove this (modulo the proof of our earlier lemma (21.2.3)) in a way that emphasises the geometry. One could also follow Ram–Wenzl [134].

The proof of (I) requires two things.
(1) That all the matrix entries in this sector are well-defined in the specialisation;
(2) that the off-block-diagonal matrix entries (coupling between the regular sector and its comlement) are all zero.

(2): note that there are two types of potentially non-zero coupling terms. Those involving King polynomials, as happens if $s(i-1) = s(i+1)$, with $s(i)^t$ N-regular and $t(i)^t$ non-regular — in which case $P_{t(i)}$, and hence the coupling, is zero by Theorem 21.2.3:

$$\begin{tikzcd}
 & \overbrace{\lambda + e_j}^{irreg.} \arrow[dr] & \\
s(i-1) = \lambda \arrow[ur] \arrow[dr] & & s(i+1) = \lambda \\
 & s(i) = \lambda - e_j \arrow[ur] &
\end{tikzcd}
\qquad \rightsquigarrow \quad \frac{\sqrt{P_{\lambda - e_j} P_{\lambda + e_j}}}{P_\lambda} = 0$$

where *irreg.* means that the transpose partition is N-irregular in our relative sense; and those

[2] or do we need: nonnegative, or negative even.

involving $\diamond^B$s, as happens if $s(i-1) \neq s(i+1)$:

$$\begin{tikzcd}
 & \overbrace{\lambda + e_h}^{irreg.} \arrow[dr] & & \rightsquigarrow \quad \dfrac{\sqrt{\diamond^B(S,S)^2 - 1}}{|\diamond^B(S,S)|} \\
s(i-1) = \lambda \arrow[ur] \arrow[dr] & & s(i+1) = \lambda + e_h - e_j & \\
 & s(i) = \lambda - e_j \arrow[ur] & &
\end{tikzcd}$$

Here $\diamond^B(S,S)$ is

$$\diamond^B(\lambda, \underbrace{\lambda - e_j, \lambda + e_h - e_j}_{k=h}, \underbrace{\lambda, \lambda - e_j}_{l=j}, \lambda + e_h - e_j) = +(x + \underbrace{(\lambda - e_j)_h - h}_{\lambda_h - h} + (\lambda_j - j))$$

On the other hand, since $(\lambda + e_h)^t$ (transpose) is N-irregular we know that the column (i.e. the transposed row) containing e_h, column $\lambda_h + 1$, which has length h, is in a reflection-fixed pair with some other column — indeed this must be the column containing the e_j in $\lambda + e_h$, since $(\lambda + e_h - e_j)^t$ is N-regular. Thus we have (with $N = \delta$):

$$-((\lambda_h + 1) - h) - \frac{\delta - 2}{2} \;=\; -(-(\lambda_j - j) - \frac{\delta - 2}{2})$$

giving

$$\lambda_h - h + \lambda_j - j + (\delta - 2) + 1 = 0$$

Thus $\diamond^B(S,S)^2 - 1 = 0$ as required.
SIGNS ETC. NEED CHECKING!

(1) We need to show that none of the $\diamond^B$ or P_- denominators evaluate to zero in this sector. The $s(i-1) = s(i+1)$ cases are covered by Theorem 21.2.3, which tells us that none of the 'lower' polynomials vanish in this specialisation. The other cases are essentially covered by arithmetic like that in the proof of (2). For suppose $S = ..., \lambda, \lambda - e_j, \lambda + e_h - e_j, ...$ as before, but suppose now

$$\diamond^B(S,S) = +(x + \underbrace{(\lambda - e_j)_h - h}_{\lambda_h - h} + (\lambda_j - j)) = 0$$

This gives $g = 0$ (from (21.3.26)) which says that $e_\delta(\lambda^t)$ is on a D-reflection wall, so λ^t is singular (and more singular than $(\lambda - e_j)^t$). We have thus shown that here $\diamond^B(S,S) = 0$ only if we step off (the transpose of) a relatively singular weight. But this does not happen in the relatively regular sector by definition.

...CHECK THIS AGAIN!

21.6 JOBS

Traces

BMW
simple restriction rules
what IS connection between LR and outer products?

Chapter 22

Example: the Temperley–Lieb algebra again

22.1 More on categories of modules

22.1.1 More fun with F and G functors

ss:TLFG2

We continue here with the assumptions of §13.4.

innerprod **Proposition 22.1.** *Suppose that A possesses an involutive antiautomorphism that fixes e. If $N \in eAe-$mod simple then $G(N)$ has at most one contravariant form on it. If such a form exists then its rank is the rank of the head of $G(N)$ (and this head is contravariant self-dual).*

Proof: Write a^t for the image of $a \in A$ under the antiautomorphism (so $e^t = e$). Write L for the head of $G(N)$, so that $eL \neq 0$. Thus L does not appear below the head of $G(N)$. L^o is isomorphic to L as a vector space, and the action of e on it is given by the action of e^t on L^*. Thus $eL^o \neq 0$. Thus neither L nor L^o appears anywhere except possibly in the socle of $G(N)^o$. If $L \not\cong L^o$ it follows that there is no map from $G(N)$ to $G(N)^o$. If $L \cong L^o$ then $G(N)$ satisfies the assumptions of proposition 10.1.20, so we may invoke proposition 10.1.34. □

Theorem 13.6 is a powerful result. Its power is ameliorated somewhat by the failure of left-exactness in G. This motivates us to learn more about G.

returny **Lemma 22.2.** *Suppose $S_1 \overset{\psi}{\hookrightarrow} S_2$ an inclusion of left eAe-ideals, and the multiplication map associated to $G(S_1)$ is an isomorphism. Then $G(S_1) \overset{G(\psi)}{\hookrightarrow} G(S_2)$ is an injection, i.e. G behaves as if left exact.*

Proof: We have

$$\begin{array}{ccc} S_1 & \stackrel{\psi}{\hookrightarrow} & S_2 \\ \downarrow G & & \downarrow G \\ Ae \otimes_{eAe} S_1 & \stackrel{G(\psi)}{\to} & Ae \otimes_{eAe} S_2 \\ \downarrow \mu_1 & & \downarrow \mu_2 \\ AS_1 & \stackrel{\psi_A}{\hookrightarrow} & AS_2 \end{array}$$

Recall that $G(\psi)(ae \otimes s) = ae \otimes \psi(s) = ae \otimes s$ (keeping in mind that such a map may have a kernel, in principle — i.e. the expression $ae \otimes s$ on the right may not be identified with the $ae \otimes s$ on the left, since the one lies in $G(S_2)$ while the other lies in $G(S_1)$, and G is not left exact); and that $\mu_i(ae \otimes s) = aes$. The map ψ_A is simply the inclusion of a left A-subideal. We thus have $\psi_A(\mu_1(ae \otimes s)) = \psi_A(aes) = aes$ and $\mu_2(G(\psi)(ae \otimes s)) = \mu_2(ae \otimes s) = aes$, so that $\psi_A \circ \mu_1 = \mu_2 \circ G(\psi)$. That is, the bottom square in our diagram commutes. But if μ_1 is an isomorphism then both factors in $\psi_A \circ \mu_1$ have trivial kernel, so it is an injection, and hence so is $\mu_2 \circ G(\psi)$. Thus $G(\psi)$ is an injection. $\square$

Suppose that S is a left sub-eAe-module of eAe (i.e. a left ideal), then there is a multiplication map

$$\begin{array}{rcl} \mu : Ae \otimes_{eAe} S & \to & AeS \\ ae \otimes s & \mapsto & aes \end{array}$$

(in the rest of this section, μ applied to a tensor product of this form will always be the appropriate multiplication map).

The surjection μ need not be an injection in general. However, suppose that there are $f, g \in eAe$ such that $S = eAef$ and $fgf = f$. (Such an f is said to satisfy the *return condition*. We call a left eAe-ideal S of form $eAef$ with f satisfying the return condition a *return ideal*.) Then there is a map $\nu : AeS \to Ae \otimes_{eAe} S$ given by

$$\nu(x) = x \otimes gf$$

so that $\mu(\nu(x)) = xgf = x$ and $\nu(\mu(a \otimes s)) = \nu(as) = as \otimes gf = a \otimes sgf = a \otimes s$. Therefore

lemmin **Proposition 22.3.** *If $S = eAef$ is a left ideal of eAe generated by $f \in eAe$ such that $fgf = f$ for some $g \in eAe$, then the multiplication map μ is an isomorphism*

$$G(S) \cong AeS = AS = Af$$

(NB, μ and its inverse are given explicitly). In particular the set inclusion of S in AS passes to an injection ν of S into $G(S)$. This is not an algebra-module map, but if D is a linearly independent set in S then it is linearly independent in AS and $\nu(D)$ is in $G(S)$.

Note that fg is idempotent, so

$$S = eAef \twoheadrightarrow eAefg$$

is a surjective map onto a projective eAe-module.

22.1.2 Saturated towers

stuff HERE SUPRESSED!...

See §9.5 for notes on quasi-heredity.

22.1.3 Quasi-heredity of planar diagram algebras

stuff HERE SUPRESSED!

Part VI

Lie

Chapter 23

Lie groups

23.1 Introduction (to algebraic groups etc)

References: King [**?**], Serre [141], Fulton–Harris [50], Murnaghan [126], Carter [21].

There are many references on Lie groups. Our agenda here is to address aspects of their exposition that inform the appearance of geometry in representation theory (both here and more generally). We start with a very brief overview of algebraic groups in general, and then of the representation theory of the Lie algebra sl_n. Details are given later.

(**23.1.1**) Let k be an algebraically closed field. An affine algebraic group is an affine variety over k (as in (3.3.6)) that is a group, where multiplication and inverse are both morphisms of varieties. For example $SL_n(k) = \{(a_{ij}) \in k^{n^2} \mid \det A = 1\}$ is affine algebraic. Every such group is isomorphic to a (Zariski) closed subgroup of $GL_n(k)$ for some n. Such a group is *connected* if it is connected as a topological space. An element of $GL_n(k)$ is *unipotent* if all its eigenvalues are 1; and *semisimple* if it is diagonalisable. These definitions extend to any affine algebraic group via the subgroup property.

Group $GL_1(k)$ is essentially k^*. Meanwhile the subgroup of upper unitriangular matrices in $GL_2(k)$ is isomorphic to $(k, +)$. A *torus* is an algebraic group isomorphic to $k^* \times k^* \times ... \times k^*$.

A *Borel subgroup* is a maximal connected solvable subgroup. There is a unique maximal closed connected solvable normal subgroup of any affine algebraic group G, denoted $R(G)$. There is also a unique maximal closed connected solvable normal subgroup of unipotent elements of G, denoted $R_u(G)$.

Connected reductive groups are examples of connected algebraic groups such that $R_u(G) = 1$. Examples include $GL_n(k)$. If B is a Borel subgroup and T a maximal torus contained in B then there is a unique Borel B^- such that $B \cap B^- = T$.

In the example $GL_n(k)$, the diagonal matrices are a maximal torus, and the upper triangular matrices are a Borel subgroup.

(**23.1.2**) Now fix G, and fix B and T as above. There are finitely many minimal closed subgroups of $R_u(B)$ normalised by T, and each is isomorphic to $(k, +)$. Conjugations of these subgroups by elements of T give automorphisms of $(k, +)$. With $c \in k^*$ the map $x \mapsto cx$ is an algebraic automorphism of $(k, +)$. All such automorphisms of $(k, +)$ take this form, so $Aut((k, +)) \cong k^*$.

Hence the action of T by conjugation on one of the subgroups gives an element of $\mathrm{Hom}(T, k^*)$. These elements are called the *positive roots*.

(**23.1.3**) Now let sl_n be the (simple) Lie algebra of traceless $n \times n$ matrices (see (23.3.1)). Set h as the (abelian) sub-Lie algebra of diagonal matrices; $n_\pm$ as the Lie algebras of strictly above/below diagonal matrices. Thus as k-vector space:

$$sl_n = h \oplus n_+ \oplus n_-$$

Define $b = h \oplus n_+$. Define $a_{ij} \in h^* = \mathrm{hom}(h, k)$ by $a_{ij}(H) = H_{ii} - H_{jj}$. Set $R_+ = \{a_{ij} \mid i < j\}$ and $R = \{a_{ij}\}$, the *set of roots*. The roots $a_i := a_{i,i+1}$ are *fundamental roots*.

Recall that E_{ij} is the ij-th elementary matrix. Define $H_{a_{ij}}$ as the diagonal matrix $E_{ii} - E_{jj}$. We have

$$a_{ij}(H_{a_{ij}}) = 2.$$

Note, for $H \in h$ and setting $X_{a_{ij}} = E_{ij}$ so that $\{X_{a_{ij}} \mid a_{ij} \in R_+\}$ is a basis of n_+, that

$$[H, E_{ij}] = a_{ij}(H)E_{ij}$$

$$[E_{ij}, E_{ji}] = H_{a_{ij}}.$$

(**23.1.4**) Now let V be an sl_n-module (so that the universal enveloping associative algebra Usl_n acts), and $x \in h^*$. Write V_x for the subspace of $v \in V$ such that $Hv = x(H)v$ for all $H \in h$. This v is called an eigenvector of h of *weight* x. For such a v we have $E_{ij}v \in V_{x+a_{ij}}$.

(**23.1.5**) Proposition. *V is the direct sum of its 'eigenspaces'.*

(**23.1.6**) The x's with $V_x \neq 0$ are the *weights of* V. An element $v \in V$ is *primitive* if it is an eigenvector and $E_{ij}v = 0$ for all $i < j$.

Note that any finite V has a primitive element.

If V is an sl_n-module and $v \in V$ primitive of weight x, then
(i) submodule Usl_nv is irreducible.
(ii) weights of Usl_nv are of form $x - \sum_{i=1}^{n-1} m_i a_i$ with $m_i \geq 0$.
(iii) a weight-x element in Usl_nv is a scalar multiple of v.

(**23.1.7**) Theorem. *If V is irreducible then it contains only one primitive element v up to scalars.* (The weight of v is the *highest weight* of V — since it is annihilated by any E_{ij} which might 'raise' its weight further.)

(**23.1.8**) Irreducible modules with the same highest weight are isomorphic.

(**23.1.9**) The preceding points mean that irreducible representations are indexed by highest weights, so the next question is: What are valid highest weights? We omit the details for now, but an interesting point is the following.

The set of highest weights is closed under addition.

23.2 Preliminaries

The Murnaghan convention for the Kronecker product of matrices is

$$\otimes : M_m(R) \times M_n(R) \to M_{mn}(R)$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... \\ a_{21}B & a_{22}B & ... \\ \vdots & & \end{pmatrix}$$

For fixed m, n there is a matrix P such that, for all A, B,

$$A \otimes B = P(B \otimes A)P^{-1}$$

We have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) \;=\; A_1 A_2 \otimes B_1 B_2$$

$$\det(A \otimes B) = \det(A)^n \det(B)^m$$

Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aa & ab \\ ac & ad \end{pmatrix} & \begin{pmatrix} ba & bb \\ bc & bd \end{pmatrix} \\ \begin{pmatrix} ca & cb \\ cc & cd \end{pmatrix} & \begin{pmatrix} da & db \\ dc & dd \end{pmatrix} \end{pmatrix}$$

Let us write e_1, e_2 as a basis for the first factor space and similarly for the second. Thus $e_1 \otimes e_1$, $e_1 \otimes e_2 + e_2 \otimes e_1$ and $e_2 \otimes e_2$ are vectors in the product space invariant under swapping the order of the factors. Since $A \otimes A$ is invariant under swapping the order, it will fix the subspace spanned by the invariant vectors. That is

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix} A \otimes A \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} aa & ab & bb & 0 \\ 2ac & ad+bc & 2bd & 0 \\ cc & cd & dd & 0 \\ 0 & 0 & 0 & ad-bc \end{pmatrix} \quad (23.1) \quad \boxed{\texttt{eq:sA}}$$

where

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{12}+e_{21} \\ e_{22} \\ e_{12}-e_{21} \end{pmatrix}$$

If A is $N \times N$ we can write $e_1, e_2, ..., e_N$ for the basis of the space V (say) that it acts on. Symmetrisation of $V \otimes V$ gives $e_{11}, e_{22}, ..., e_{NN}$, $e_{12}+e_{21}, e_{13}+e_{31}, ..., e_{N-1,N}+e_{N,N-1}$ — that is, $N + \frac{N(N-1)}{2}$ vectors altogether. (And a complementary collection of vectors antisymmetric under permutation.)

Let us define $S^{(n)}(A)$ as the 'symmetrised' part of $A^{\otimes n}$, i.e. the part acting on the permutation invariant vectors in the sense of the $n = 2$ case defined above (i.e. the 3x3 block in (23.1)). (Caveat: The details depend on the normalisation of the symmetrised vectors.)

23.3 Lie group

We start with an exposition along lines recalled from lectures by R.C.King (unpublished 1970s).

A continuous group is a group with elements $A(\alpha)$ parameterisable by a set $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ of real parameters (one arranges for $A(0) = 1$), such that group multiplication:

$$A(\alpha)A(\alpha') = A(f(\alpha, \alpha'))$$

and inverse are continuous functions. An example is the complex general linear group.

A continuous group is a Lie group if f satisfies some differentiability conditions, and

$$X_k = \frac{\partial A(\alpha)}{\partial \alpha_k}|_{\alpha=0}$$

is well-defined. Thus

$$A(\alpha) = 1 + \sum_{k=1}^{r} \alpha_k X_k + ...$$

de:liealg (**23.3.1**) The 'generator' matrices X_k lead to a representation of a *Lie algebra*. A Lie algebra is a vector space with a 'Lie bracket' bilinear antisymmetric binary operation satisfying the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

(And roughly speaking we can get back from the algebra to the group by exponentiating.)

Example: Any associative algebra defines a Lie algebra by $[a, b] = ab - ba$. Ado's Theorem says that every finite dimensional real or complex Lie algebra has a faithful representation by matrices using this Lie bracket.

In the Lie algebra our generators above may be extended to a basis. Consider the expansion of the Lie bracket on (a suitable such closure of the set of) the generators above, or any other basis,

$$[X_m, X_k] = \sum_r C^r_{mk} X_r$$

The coefficients are called structure constants. These depend on the basis. There is a *Chevalley basis* such that they are integers.

(**23.3.2**) EXAMPLE. Consider $SU(2)$. Unitarity restricts from 4 to 2 complex parameters, and the determinant constraint reduces this to 3 real parameters. Then for $a, b, c \in \mathbb{R}$ the matrix

$$D(a, b, c) = \begin{pmatrix} e^{-ia/2} & 0 \\ 0 & e^{ia/2} \end{pmatrix} \begin{pmatrix} \cos(b/2) & -\sin(b/2) \\ \sin(b/2) & \cos(b/2) \end{pmatrix} \begin{pmatrix} e^{-ic/2} & 0 \\ 0 & e^{ic/2} \end{pmatrix}$$

is a general element. Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Differentiating $D(a, b, c)$ we get two of these as generators. Then $D(a, b, c)$ is expressible as

$$D(a, b, c) \;=\; \exp(-i(a/2)\sigma_3) \exp(-i(b/2)\sigma_2) \; \exp(-i(c/2)\sigma_3)$$

Realising the Lie bracket as a commutator, the matrix generators σ_i obey

$$[X_i, X_j] = \epsilon_{ijk} X_k$$

where $\epsilon_{ijk} = 1$ for an even perm of 123; $= -1$ for an odd perm; and $= 0$ otherwise. Also

$$X_i^\dagger = X_i$$

We can look for (other) Hermitian irreducible matrix representations of the Lie algebra defined by these relations. It is useful here to define $X^2 = \sum_i X_i^2$ and $X_\pm = X_1 \pm i X_2$. We have $[X^2, X_i] = 0$. The elements X^2 and X_3 span a maximal commutative subalgebra (in our existing representation they span the subspace of diagonal matrices), so we can choose a basis for a representation in which these act diagonally. Pick an initial vector in the corresponding module. For some pair of scalars we have

$$X^2 v_0 = \lambda v_0 \qquad \textit{and} \qquad X_3 v_0 = m v_0$$

Thus

$$X_3(X_\pm v_0) = ([X_3, X_\pm] + X_\pm X_3) v_0 = \ (m \pm 1)(X_\pm v_0)$$

In other words, either $X_- v_0 = 0$ or it is independent of v_0 (and similarly for $X_+ v_0$).

For example, suppose $X_- v_0 = 0$ but not $X_+ v_0$. Then we can add $X_+ v_0$ to the basis and go again, applying $X_\pm$ to $X_+ v_0$. In this way we can keep getting new basis elements. In particular $X_3(X_+^j v_0) = (m+j)(X_+^j v_0)$. The other thing to note is that if $m = -k$ negative integer then our basis generation programme produces a decoupled pair of subspaces, with $(X_+^k v_0)$ and above decoupled from the lower basis elements. Thus here the lower elements form a finite-dimensional representation.

SOMETHING NOT QUITE RIGHT HERE YET!!! We did not impose Hermiticity. (Do we really want to?)

There is more on this in my notes [100].

23.3.1 Example: $SU(2)$ 'polynomial' representations

For any N the special unitary group is the subgroup of $GL(N, \mathbb{C})$ of special (det=1) unitary ($A^\dagger A = 1$) matrices. Thus $SU(2)$ is the set of complex matrices of form

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad \text{so} \qquad A^\dagger = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix},$$

with $aa^* + bb^* = 1$. This A acts on an arbitrary vector in $\mathbb{C}^2$ by $z \mapsto Az$. Consider

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^\dagger \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

This induces an action on functions $f(z_1, z_2) \mapsto f(z_1', z_2')$, and in particular on the space of polynomials in z_1, z_2 of fixed total degree $2n$, say (n integer or half-integer):

$$A.(\sum_{i=0}^{2n} \beta_i z_1^{2n-i} z_2^i) \ := \ \sum_{i=0}^{2n} \beta_i (z_1')^{2n-i} (z_2')^i$$

We can use this to construct a representation of $SU(2)$ of dimension $2n+1$. We define

$$v_{(\beta_0, \beta_1, \ldots, \beta_{2n})}(z_1, z_2) = v_\beta(z) = \sum_{i=0}^{2n} \beta_i z_1^{2n-i} z_2^i$$

so that

$$A.v_\beta(z) \; = \; v_\beta(Az) = v_\beta(z')$$

and define β' by

$$v_{\beta'}(z) = v_\beta(z')$$

For example case $n = 1$ is

$$A.v_{(\beta_0,\beta_1,\beta_2)}(z_1, z_2) = v_\beta(a^* z_1 - bz_2,\; b^* z_1 + az_2,)$$

$$= \beta_0(a^* z_1 - bz_2)^2 + \beta_1(a^* z_1 - bz_2)(b^* z_1 + az_2) + \beta_2(b^* z_1 + az_2)^2 \qquad (23.2) \quad \boxed{\texttt{eq:l=1-}}$$

$$= (\beta_0(a^*)^2 + \beta_1 a^* b^* + \beta_2(b^*)^2)z_1^2 \;\; + (-2\beta_0 a^* b + \beta_1(aa^* - bb^*) + \beta_2 b^* a)z_1 z_2 \;\; + (\beta_0 b^2 - \beta_1 ba + \beta_2 a^2)z_2^2 \qquad (23.3) \quad \boxed{\texttt{eq:l=1}}$$

We then define a map from $SU(2) \to GL(2n + 1, \mathbb{C})$ by

$$\beta' = R_n(A)\beta$$

We claim that this is a group representation.

For $n = 1/2$ this is just the defining representation. For $n = 1$ we have, from (23.3):

$$R_1(A) = \begin{pmatrix} (a^*)^2 & a^* b^* & (b^*)^2 \\ -2a^* b & aa^* - bb^* & 2ab^* \\ b^2 & -ab & a^2 \end{pmatrix}$$

For example, setting

$$g_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

in $SU(2)$ we have, for $t \in \mathbb{R}$,

$$R_1(g_t) = \begin{pmatrix} e^{-2it} & & \\ & 1 & \\ & & e^{2it} \end{pmatrix}$$

Indeed we claim

$$R_n(g_t) = \begin{pmatrix} e^{-2int} & & & \\ & e^{-2i(n-1)t} & & \\ & & \ddots & \\ & & & e^{2int} \end{pmatrix}$$

Note that the form of $R_n(g_t) \otimes R_1(g_t)$ is clear.

23.3.2 Lie algebra

Let $A(t)$ be any smooth path in $SU(2)$ such that $A(0) = 1$. For example, $A(t) = g_t$ above gives such a path. Then $\frac{dA(t)}{dt}$ makes sense (matrix entry by matrix entry) and can be evaluated at $t = 0$. The set of all these evaluations

$$T_1(SU(2)) = \{\frac{dA(t)}{dt}|_{t=0} \mid A(t) \text{ a smooth path; } A(0) = 1\}$$

is the *tangent space* at 1. This can be given the structure of a Lie algebra.

Chapter 24

Examples from classical representation theory

chex

This Chapter recalls some classical results that we shall need later. (It also strays in to areas which I thought were classical, but might be newer!) The Chapter is here for reference, and it is safe to skip it at first reading. Useful sources include Green [57], Fulton–Harris [**?**], Hamermesh [61], Goodman–Wallach [**?**], Jacobson [**?**], Murnaghan [**?**].

24.1 Preliminaries

Let N be a natural number, and $\underline{N} := \{1, 2, ..., N\}$.

Following Green [57, §2] we let K be an infinite field and $\Gamma = GL_N(K)$. We write K^Γ for the set of maps $f : \Gamma \to K$ (regarded as a commutative K-algebra by pointwise operations). For each pair $\mu, \nu \in \underline{N}$ let $c_{\mu\nu} \in K^\Gamma$ be the function which associates to each $g \in \Gamma$ its (μ, ν)-coefficient $g_{\mu\nu}$. Write A for the K-subalgebra of K^Γ generated by the functions $c_{\mu\nu}$ — the polynomial functions on Γ. This A can be regarded as the algebra of polynomials over K in n^2 indeterminates $c_{\mu\nu}$.

Write $A_K(n, r)$ for the subspace of A of elements expressible as polynomials which are homogeneous of degree r in the $c_{\mu\nu}$.

24.2 Invariant theory

Again we begin following Green [57, §2]. We write $I(N, r)$ for the set of all functions $i : \underline{r} \to \underline{N}$. We write $i = (i_1, ..., i_r)$ for the case $i(\mu) = i_\mu$. The symmetric group S_r acts on the right by $i\pi = (i_{\pi(1)}, ..., i_{\pi(r)})$.

We write $\Lambda(N, r)$ for the set of S_r-orbits in $I(N, r)$. The elements of $\Lambda(N, r)$ are the classical *weights* of GL_N of dimension r. A weight is specified by a vector $a = (a_1, ..., a_N) \in \mathbb{N}^N$ such that a_μ is the number of times that $i_k = \mu$ for some i in the orbit a. It will be clear that this specification is well-defined; and that a can be seen as a composition (an unordered partition) of r into (up to) N parts.

As we run over possible values of r we populate more of $\mathbb{N}^N$.

The group S_N acts on $I(N,r)$ on the left:

$$wi = (w(i_1), ..., w(i_r)).$$

We have $w(i\pi) = (wi)\pi$, so S_N acts on $\Lambda(N,r)$. Each S_N orbit of $\Lambda(N,r)$ contains a *dominant weight*. Denote by $\Lambda^+(N,r)$ the set of dominant weights.

de:SWdual1 (**24.2.1**) Let E be an N-dimensional K-space with basis $\{e_1, ..., e_N\}$, on which Γ acts naturally. Thus Γ acts diagonally on $E^{\otimes r}$. The space $E^{\otimes r}$ has K-basis

$$\{e_i = e_{i_1} \otimes ... \otimes e_{i_r} \ : \ i \in I(N,r)\}$$

and S_r acts on the right by $e_i\pi = e_{i\pi}$. This action commutes with the Γ action.

24.2.1 Schur functor, case GL_N

(**24.2.2**) By (24.2.1) we thus have a bimodule ${}_{K\Gamma}E^{\otimes r}{}_{KS_r}$ which gives us a functor

$$\begin{aligned} KS_r - mod &\rightarrow K\Gamma - mod \qquad (24.1)\\ M &\mapsto {}_{K\Gamma}E^{\otimes r}{}_{KS_r} \otimes_{KS_r} M \qquad (24.2)\end{aligned}$$

In particular for each partition of r and hence Young symmetriser $c_\lambda \in KS_r$, and left KS_r ideal $KS_r c_\lambda$, we may define a $K\Gamma$-module by

$$\mathbb{S}_\lambda E \ := \ {}_{K\Gamma}E^{\otimes r}{}_{KS_r} \otimes_{KS_r} KS_r c_\lambda$$

gl01 **Proposition 24.1.** *(see for example [50]) For $K = \mathbb{C}$ this $\mathbb{S}_\lambda E$ is either irreducible or zero. It is zero if $\lambda_{N+1} \neq 0$. If D_k, $k \in \mathbb{Z}$, is the representation*

$$D_k(g) = (\det(g))^k$$

then

$$\mathbb{S}_{(\lambda_1+k, ..., \lambda_N+k)}E = \mathbb{S}_\lambda E \otimes D_k$$

We extend this to define $\mathbb{S}_\lambda E$ for λ of depth N with some λ_i negative. This gives (see for example [50])

GLcomop **Proposition 24.2.** *Still over $\mathbb{C}$, the $\mathbb{S}_\lambda E$ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ are a complete set of irreducible Γ modules. The subset of these with $\lambda_N = 0$ are (by restriction) a complete set of irreducible SL_N modules.*

24.2.2 Now do $O(V)$, $Sp(V)$

1. $Sp(2n)$ (See for example [50, §17.3].)

Now let E be a complex space (necessarily even-dimensional) with a non-degenerate skew-symmetric bilinear form $\underline{\Omega} : E \times E \rightarrow \mathbb{C}$. Certain $g \in GL(E)$ will obey $\underline{\Omega}(gx, gy) = \underline{\Omega}(x, y)$ for all $x, y \in E$. Compositions of such elements also preserve the form. The subgroup of these elements is the symplectic group $Sp(\dim_{\mathbb{C}} E) = Sp(E) = Sp(E, \underline{\Omega})$ (we assume the form is fixed, even if not specified, most of the time).

Where no ambiguity arises we will streamline the notation for a basis of $E^n = E^{\otimes n}$ still further, and write simply $ijk...$ for $e_i \otimes e_j \otimes e_k \otimes ....$ For example $111 = e_1 \otimes e_1 \otimes e_1$.

For $i < j$ we define $\underline{\Omega}_{ij} : E^n \to E^{n-2}$ to map out the i-th and j-th factor. For example with $n = 3$

$$\underline{\Omega}_{13}(e_1 \otimes e_1 \otimes e_1) = \underline{\Omega}(e_1, e_1)e_1 = 0$$

and we can choose $\underline{\Omega}_{12}(12) = 1$ provided $\underline{\Omega}_{12}(12) = -1$ so

$$\underline{\Omega}_{12}(121 + 211) = e_1 - e_1 = 0$$

Classically one may write $E^{<n>_{\underline{\Omega}}}$, or just $E^{<n>}$, for the intersection of the kernels of all the $\underline{\Omega}_{ij}$s in E^n. For example $111 \in E^{<n>}$. One can check that $E^{<n>}$ is an $Sp(E)$ submodule of E^n (which is an $Sp(E)$-module by restricting the $GL(E)$ action). Accordingly we define another (possibly zero) $Sp(E)$ submodule of E^n by $\mathbb{S}'_\lambda E = E^{<n>} \cap \mathbb{S}_\lambda E$ (note that $\mathbb{S}_\lambda E$, defined above, is an $Sp(E)$-module again by restriction).

(**24.2.3**) Theorem. *For $Sp(E)$ with $E = \mathbb{C}^{2m}$, $\mathbb{S}'_\lambda E$ is nonzero iff $\lambda_{m+1} = 0$ (that is, $\lambda^t_1 \leq m$). Each nonzero $\mathbb{S}'_\lambda E$ is a distinct inequivalent simple module.*

include some 'duality' results a la Gavarini lemma 4.2...
... so that we can prepare for the stuff we are now using in the walk combinatorics section!

24.3 Weyl Character Formula

See e.g. Goodman and Wallach [**?**, §7.4].

Here G is a connected classical group, $\mathfrak{g}$ its Lie algebra, H a maximal torus and $\mathfrak{h}$ its Lie algebra. Here we can take $G \subset GL_n(\mathbb{C})$ a group of matrices, and H the diagonal matrices in G. Then every semisimple $g \in G$ is fhf^{-1} for some $f \in G$ and $h = \text{diag}(h_1, ..., h_n) \in H$. Set

$$\Delta(h) = \prod_{1 \leq i < j \leq n} (h_i - h_j)$$

(**24.3.1**) Exercise. Finish this!

24.3.1 Irreducible dimensions for $SL(V)$

In case $SL_n(\mathbb{C})$ then λ is an integer partition of at most $n-1$ rows (we take $\lambda_n = 0$), and

$$\dim V^\lambda \;=\; \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

24.3.2 Irreducible dimensions for $O(V)$, $Sp(V)$

WCF

The Weyl character formula (WCF) gives us dimensions for irreducible representations of the associated Lie algebras (see also El-Samra–King and King–El-Sharkaway [139, 88]). These irreps then give us the irreps for our Lie groups $O(V)$ and $Sp(V)$ by a bit of fiddling (depending on the case) which we will describe after.

We have

(1) Case $so_{2n+1}\mathbb{C}$ (leading to $O(2n+1)$):

Highest weights are n-tuples $\lambda = (\lambda_1, .., \lambda_n)$ with $\lambda \in \mathbb{Z}^n$ or $\lambda - (1,1,...,1)/2 \in \mathbb{Z}^n$ and $\lambda_1 \geq ... \geq \lambda_n \geq 0$; and

$$dim(M_\lambda) = \frac{\left(\prod_{i<j}(a_i - a_j)(a_i + a_j + 1)\right)\prod_i (2a_i + 1)}{\prod_{i=1}^n (2n - (2i-1))!} \tag{24.3}$$ eq:WCF-01

where $a_i = \lambda_i + n - i$ (this formulation can be found for example in Fulton–Harris [?, §24.2]).

Equivalently we have, writing x for $2n+1$,

$$dim(M_\lambda) = \left(\prod_i \prod_{j>i} \frac{(\lambda_i - \lambda_j - i + j)}{-i+j} \prod_i \prod_{j>i} \frac{(\lambda_i + \lambda_j + x - i - j)}{x - i - j}\right) \prod_i \frac{(2\lambda_i + x - 2i)}{x - i - j}$$

The products are all nominally up to n.

(**24.3.2**) Exercise. Evidently the integral λs can be regarded as Young diagrams; and each λ is a valid weight for infinitely many values of n. Consider what happens as n grows large for given λ.

One sees immediately that as soon as both i and j lie outside the range for which $\lambda_i > 0$ then the corresponding factors trivialise. Thus we can replace $\prod_i$ by $\prod_{i=1}^{\lambda'_1}$.

(**24.3.3**) Example. Case $\lambda = \square$:

$$dim(M_\square) = \left(\prod_{j>1} \frac{(j)}{-1+j} \prod_{j>1} \frac{(x-j)}{x-1-j}\right) \frac{(x)}{x-2}$$

$$= \left(\underbrace{\frac{2.3.4.....(n-1).n}{1.2.3.....(n-2).(n-1)}}_{\frac{n}{1}}\right)\left(\underbrace{\frac{x-2}{x-3}\frac{x-3}{x-4}\frac{x-4}{x-5}...\frac{x-(n-1)}{x-n}\frac{x-n}{n}}_{\frac{x-2}{n}}\right)\frac{x}{x-2} \quad = x$$

Note the almost complete cancellation of factors within the bracketed factors; the cancellation of (n) across the first two factors; and of $(x-2)$ across the second and third.

(**24.3.4**) Example. Case $\lambda = \square\square$:

$$dim(M_{\square\square}) = \left(\prod_{j>1} \frac{(1+j)}{-1+j} \prod_{j>1} \frac{(1+x-j)}{x-1-j}\right) \frac{(x+2)}{x-2}$$

$$= \left(\underbrace{\frac{3.4.5.....(n).(n+1)}{1.2.3.....(n-2).(n-1)}}_{\frac{n(n+1)}{2}}\right)\left(\underbrace{\frac{x-1}{x-3}\frac{x-2}{x-4}\frac{x-3}{x-5}...\frac{x-(n-2)}{x-n}\frac{x-(n-1)}{n}}_{\frac{(x-1)(x-2)}{(n+1)n}}\right)\frac{x+2}{x-2}$$

$$= \frac{(x+2)(x-1)}{2}$$

(**24.3.5**) EXAMPLE. Case $\lambda = \ydiagram{1,1}$:

$$dim(M_{\ydiagram{1,1}}) = \underbrace{\frac{1}{1}\frac{x-1}{x-3}}_{i=1,j=2} \left(\prod_{j>2}\frac{(j)}{-1+j}\prod_{j>2}\frac{(x-j)}{x-1-j}\right)\left(\prod_{j>2}\frac{(-1+j)}{-2+j}\prod_{j>2}\frac{(x-1-j)}{x-2-j}\right)\frac{(x)}{x-2}\frac{x-2}{x-4}$$

$$= \frac{x-1}{x-3}.\left(\frac{n}{2}.\frac{(x-3)}{n}\right).\left(\frac{(n-1)}{1}.\frac{x-4}{n-1}\right).\frac{(x)}{x-2}.\frac{x-2}{x-4} \quad = \frac{x(x-1)}{2}$$

(**24.3.6**) Each $SO(2n+1)$ simple gives rise to an $O(2n+1)$ simple directly, and another by tensoring with the determinant. (And the set of pairs together give all $O(2n+1)$ irreducibles.) We need to match these modules to their $O(2n+1)$ labels (at least in the integral cases). In the integral cases the weight λ, viewed as a partition, has $\lambda'_1 \leq n$ so $\lambda'_1 + \lambda'_2 \leq 2n < 2n+1$ by construction, and $\lambda'_2 < 2n+1-\lambda'_1$. It follows that there is a 'Weyl-associated' partition μ with $\mu'_1 = 2n+1-\lambda'_1 > n$ (and so *not* an so_{2n+1} highest weight) and all other columns the same. The original weight and the associated partition label the pair of irreducible representations of the same dimension for $O(2n+1)$ described above.

For example, for so_3 we have $n = 1$ so $\lambda = (\lambda_1)$ and

$$dim(M_\lambda) = \frac{2\lambda_1 + 1}{1}$$

As the integral weights are $\lambda = 0, (1), (2), ...$ these dimensions are $1, 3, 5, ...$; so the other 'associated' $O(3)$ irreducibles, with labels: $(1,1,1), (1,1), (1), ...$ (respectively), have dimensions $1, 3, 5, ...$ also. (The half-integral cases have dimensions $2, 4, 6, ...$, so that altogether the simples of so_3 coincide with those of sl_2 as required. However these simples do not appear in the ordinary tensor space.) Altogether for $O(3)$:

diagram	0		$\ydiagram{1}$		$\ydiagram{2}$		$\ydiagram{3}$	
associated diagram	$\ydiagram{1,1,1}$		$\ydiagram{1,1}$		$\ydiagram{2,1}$		$\ydiagram{3,1}$	
dimension	1	2	3	4	5	6	7	

<u>(2) Case $so_{2n}\mathbb{C}$ (leading to $O(2n)$):</u>

Highest weights are $(\lambda_1, .., \lambda_n)$ with $\lambda_1 \geq ... \geq |\lambda_n| \geq 0$ and

$$dim(M_\lambda) = \frac{2^{n-1}\left(\prod_{i<j}(a_i - a_j)(a_i + a_j)\right)}{(2n-2)!!}$$

where $a_i = \lambda_i + n - i$. If $\lambda_n = 0$ then λ is a partition and we may define the conjugate partition. The weight λ has $\lambda'_1 < n$ so $\lambda'_1 + \lambda'_2 < 2n$ by construction, and $\lambda'_2 < 2n - \lambda'_1$. It follows that there is a 'Weyl-associated' partition μ with $\mu'_1 = 2n - \lambda'_1 > n$ (and so *not* an so_{2n} highest weight) and all other columns the same. The original weight and the associated partition each label a distinct irreducible rep of the same dimension for $O(2n)$. If $\lambda_n \neq 0$ then the two so_{2n} irreps with $\lambda = (\lambda_1, ..., \pm\lambda_n)$ combine to make a single $O(2n)$ irrep. (And the set of pairs and the set of singletons together give all $O(2n)$ irreducibles.)

For example, for so_2 we have $n = 1$ so $\lambda = (\lambda_1)$ and

$$dim(M_\lambda) = \frac{1}{1}$$

As $\lambda = 0, (1), (2), ...$ these dimensions are $1, 1, 1, ...$; so the derived $O(2)$ irreps: $0, (1, 1), (1), (2), ...$ (respectively) have dimensions $1, 1, 2, 2, ...$

(3) Case sp_{2n} (see e.g. [50, §24.2])...

$$\dim L_{sp}(\lambda) = \prod_{i<j} \frac{\lambda_i - i - \lambda_j + j}{j - i} \prod_{i \leq j} \frac{\lambda_i - i + \lambda_j - j + 2n + 2}{-j - i + 2n + 2}$$

Examples: $\lambda = (a)$, $n = 1$: $\dim L_{sp}((a)) = 2(a - 1 + 1 + 1)/(-1 - 1 + 2 + 2) = a + 1$;
$\lambda = (a, b)$, $n = 2$:

$$\dim L_{sp}((a, b)) =$$

$$\frac{a - 1 - b + 2}{2 - 1} . \frac{a - 1 + a - 1 + 4 + 2}{-1 - 1 + 4 + 2} . \frac{a - 1 + b - 2 + 4 + 2}{-2 - 1 + 4 + 2} . \frac{b - 2 + b - 2 + 4 + 2}{-2 - 2 + 4 + 2}$$

$$= (a - b + 1). \frac{2a + 4}{4} . \frac{a + b + 3}{3} . \frac{2b + 2}{2}$$

so $\dim L_{sp}((1)) = 2.6/4.4/3.2/2 = 4$; $\dim L_{sp}((2)) = 2.5 = 10$; $\dim L_{sp}((1^2)) = 6/4.5/3.4/2 = 5$.

24.3.3 Hilbert space

A Hilbert space is a K-vector space ($K = \mathbb{C}$ or $\mathbb{R}$) H together with a map $H \times H \to K$ written $(x, y) \mapsto x.y$ such that
1) $x.x = 0$ if and only if $x = 0$;
2) $x.x \geq 0$ for all x;
3) $x.y$ linear in x;
Cauchy sequences have limits ...etc.

Examples:
For S any set, define $l^2(S)$ as the set of complex valued functions f of S such that $f(s) = 0$ for all but countably many s; and that the sum

$$\sum_s |f(s)|^2$$

is finite; with $f.g = \sum_s f(s)g(s)^*$.

Note that the integral lattice $\mathbb{Z}^f[S]$ in $l^2(S)$ consists of those elements of $\mathbb{Z}[S]$ of finite support.

Consider the triple $C = (\mathbb{S}, H(-, -), x)$ consisting of the class of sets; and for each pair of sets the set $H(S, T)$ of linear operators from $l^2(S)$ to $l^2(T)$; and composition of linear operators.

24.4 Matrices encoding invariant theory

For any (Lie) group G let $R(G)$ be the representation ring, generated as a free abelian group by the irrep classes, and with multiplication given by $[M].[N] = [M \otimes N]$. Write simply $\lambda \in \Lambda(G)$ for the class of irrep with label λ, and write a vector $m_G(M) = m(M)$ for the collection of coefficients of irreducibles in element $[M]$ arranged in some definite order (note that the notation $m(M)$ makes sense). Write e_λ for the vector $m(\lambda) = (0, 0, \ldots, 0, 1, 0, ...)$.

Thus for each $[N]$ there is an (infinite) integral matrix $\mathcal{L}_N(G)$ such that

$$m(N \otimes M) \;=\; \mathcal{L}_N(G)\, m(M)$$

(It is convenient to consider think of infinite integral matrix operators — homs in the category of Hilbert spaces $l^2(S)$ with orthonormal basis, in cases $S = \Lambda(G)$.)

For $G' \subset G$, a matrix $D^G_{G'}$ operating on $\mathbb{Z}^f[\Lambda(G)]$ to $\mathbb{Z}^f[\Lambda(G')]$, is defined by

$$m_{G'}(\mathrm{Res}^G_{G'} M) \;=\; D^G_{G'}\; m_G(M)$$

Note that $D^G_{G'}$ does not depend on M. In particular write D_V for the matrix operating on $\mathbb{Z}^f[\Lambda(GL(V))]$ to $\mathbb{Z}^f[\Lambda(O(V))]$ defined by

$$m_O(\mathrm{Res}^{GL(V)}_{O(V)} M) \;=\; D_V\; m_{GL}(M)$$

That is, $D_V = D^{GL(V)}_{O(V)}$.

The point of studying $\mathcal{L}_V(G)$ is that

cent-dim **Theorem 24.3.** *The multiplicities of simples in $V^{\otimes n}$ are given by the V-row of $(\mathcal{L}_V(G))^{n-1}$:*

$$m(V^{\otimes n}) = (\mathcal{L}_V(G))^{n-1}\, m(V)$$

Thus in char.0 the dimensions of the simple modules of the centraliser algebra $End_G(V^{\otimes n})$ are the same numbers.

24.4.1 Case $\mathcal{L}_V(GL(V))$

For $G = GL(V)$ we may order $\Lambda(G)$ as $0, (1), (2), (1^2), (3), (2, 1), (1^3), (4), (3, 1), (2^2), ...$ (truncated only to include partitions of at most $\dim V$ parts). In general then

$$\mathcal{L}_V(G)\; e_\lambda \;=\; \sum_{\mu \in I_{\dim V}(\lambda)} e_\mu$$

where $I_n(\lambda)$ is the set of diagrams with one more box than λ, but at most n parts. This is just the special case of the Littlewood-Richardson rule when the first module is V, corresponding to the Young diagram of a single box.

For $\dim V \geq 4$ this begins:

$\mathcal{L}_V(G) =$

	0	(1)	(2)	(1^2)	(3)	(21)	(1^3)	(4)	(31)	(2^2)	(21^2)	(1^4)	(5)	(41)
0	0	1												
(1)	0	0	1	1										
(2)	0	0	0	0	1	1								
(1^2)	0	0	0	0	0	1	1							
(3)	0	0	0	0	0	0	0	1	1					
(21)	0	0	0	0	0	0	0	0	1	1	1			
(1^3)	0	0	0	0	0	0	0	0	0	0	1	1		
(4)	0	0	0	0	0	0	0	0	0	0	0	0	1	1
(31)	0	0	0	0					$\ddots$					

Truncation on the matrix for smaller V is simply to omit the rows and columns not appropriate to $GL(V)$.

Example:
For $\dim V = 1$ we have

$\mathcal{L}_V(G) =$

	0	(1)	(2)	(3)	(4)	(5)	$\cdots$
0	0	1					
(1)	0	0	1				
(2)	0	0	0	1			
(3)	0	0	0	0	1		
(4)	0	0	0	0	0	1	
$\vdots$							$\ddots$

In this case V is the determinant representation, so $V \otimes V$ is just the square, and so on (N.B. in SL these all collapse to the trivial module).

24.4.2 Cases $\mathcal{L}_V(SL(2))$, $\mathcal{L}_V(O(3))$

By Proposition **??** we have

$$D^{GL(V)}_{SL(V)} e_\lambda = e_{\lambda - \lambda_N(1,1,..,1)}$$

where $N = \dim V$. Thus for $N = 2$

$D =$

	0	(1)	(2)	(1^2)	(3)	(21)	(4)	(31)	(2^2)	(5)	(41)	(32)
0	1			1					1			
(1)	0	1				1						1
(2)	0	0	1	0				1				
(3)	0	0	0	0	1	0					1	
(4)	0	0	0	0	0	0	1					
(5)										1		

and so on. Let us write $SL(\dim V)$ for $SL(V)$ for convenience. Still for $\dim V = 2$, we have

$$\mathcal{L}_V(SL(2)) = \begin{array}{c|ccccccc} & 0 & (1) & (2) & (3) & (4) & (5) & \cdots \\ \hline 0 & 0 & 1 \\ (1) & 1 & 0 & 1 \\ (2) & 0 & 1 & 0 & 1 \\ (3) & 0 & 0 & 1 & 0 & 1 \\ (4) & 0 & 0 & 0 & 1 & 0 & 1 \\ \vdots & & & & & & & \ddots \end{array}$$

while

$$\mathcal{L}_{(2)}(SL(2)) = \begin{array}{c|ccccccc} & 0 & (1) & (2) & (3) & (4) & (5) & \cdots \\ \hline 0 & 0 & 0 & 1 \\ (1) & 0 & 1 & 0 & 1 \\ (2) & 1 & 0 & 1 & 0 & 1 \\ (3) & 0 & 1 & 0 & 1 & 0 & 1 \\ (4) & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \vdots & & & & & & & \ddots \end{array}$$

This follows from the GL case above, or from the Clebsch-Gordan formula [?], or from the $SL(2)$ identity

$$\mathcal{L}_{(2)} - \mathcal{L}_0 \;=\; (\mathcal{L}_{(1)})^2$$

using $\mathcal{L}_0 = 1$.

Note that $SO(2) \cong GL(1)$ and $SO(3) \cong SL(2)$. We can thus work out the $\mathcal{L}$-matrices in these cases just by noting which representation label goes to which. (For example, the label (1) for $SO(3)$ corresponds to V with $\dim V = 3$; while for $SL(2)$ the label for this module is (2).) We can then work out the L-matrices in the cases $O(2)$ and $O(3)$ by fiddling.

In particular, refering to the $O(3)$ representation construction from $SO(3)$, and the corresponding table, in Section 24.3.2, and $\mathcal{L}_{(2)}(SL(2))$ above, we get

$$\mathcal{L}_{\Box}(O(3)) = \begin{array}{c|ccccccc} & 0 & (1^3) & (1) & (1^2) & (2) & (21) & \cdots \\ \hline 0 & 0 & 0 & 1 \\ (1^3) & 0 & 0 & 0 & 1 \\ (1) & 1 & 0 & 1 & 0 & 1 \\ (1^2) & 0 & 1 & 0 & 1 & 0 & 1 \\ (2) & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \vdots & & & & & & & \ddots \end{array} \tag{24.4}$$

L03

Proof: This follows directly from $\mathcal{L}_{(2)}(SL(2))$ on noting that the $O(3)$ module with label (l) coincides with the $SO(3)$ module whose $SL(2)$ label is $(2l)$; the $O(3)$ module with label (1^3) is the determinant representation; while the $O(3)$ module with label $(l, 1)$ is the tensor product of the determinant with the module with label (l). For example, in the obvious notation

$$(1) \otimes (1^3) = (1) \otimes \det \;=\; (1^2)$$

as encoded in the second row above. $\Box$

Let $\{e_i \mid i = 1, 2, 3, ...\}$ be the basis of $l^2 = l^2(\mathbb{N})$, and let L^+ be the operator on l^2 defined by $L^+ e_1 = 0$ and

$$L^+ e_i = e_{i-1} \qquad i > 1$$

and $L^- = (L^+)^t$. Then we have the formal matrix identity (i.e. ignoring basis labels)

$$\mathcal{L}_{\square}(O(3)) = 1_2 \otimes \left(L^- L^+ + L^+ + L^- \right) \tag{24.5}$$ $\boxed{\texttt{03f}}$

24.4.3 Cases

The matrix $\mathcal{L}_V(G)$ is apparently not known for arbitrary $G \subset GL(V)$. However we can do the following. (Indeed we can proceed in various ways. Later we will take the known forms for the corresponding simple Lie algebras as starting points. In particular we can treat $Sp(V)$ via $sp(V)$.)
1. Orthogonal group case.

A partition μ is *admissible* if $\mu'_1 + \mu'_2 \leq \dim V$. For $G = O(V)$ we may order $\Lambda(G)$ as $0, (1), (2), (1^2), (3), (2, 1), (1^3), (4), (3, 1), (2^2), ...$ truncated to exclude inadmissible partitions (note that for given V this is different from the GL truncation). We have (by the usual Littlewood restriction rules [94])

$$D_V = \left(\begin{array}{c|c|cc|ccc|c}
1 & 0 & & & & & & \\ \hline
0 & 1 & & & & & & \\ \hline
1 & 0 & 1 & 0 & & & & \\
0 & 0 & 0 & 1 & 0 & & & \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & \ddots
\end{array} \right)$$

(NB the Littlewood rules *per se* do not apply to arbitrary highest weights — see [**?**] for details.) That is (for sufficiently large $\dim V$ that λ is admissible)

$$D_V e_\lambda = e_\lambda + \sum_{\mu \in J(\lambda)} e_\mu$$

where $J(\lambda)$ is the set of diagrams obtained from λ by removing two boxes *not* in column array, together with diagrams obtained by iterating this removal (NB in general a given diagram can arise in more than one way here, and these multiplicities are respected).

This is Littlewood's $O(V)$ restriction rule. Note that the truncations depending on V are different in the two spaces, and that Littlewood only gives the rule when the $GL(V)$ weight coincides with an $O(V)$ weight (i.e. $\lambda'_1 + \lambda'_2 \leq \dim(V)$) but that the multiplicities have no further dependence on V. (See e.g. [**?**, Cor.4.7].) That is, giving the result for 'large' $\dim(V)$, where we do not need to worry about truncation in low rank, we can recover the result for any V simply by truncation, and without need for further modifications.

We want to work out $L_V(O(V))$. The idea is to convert our vector into a 'virtual' $GL(V)$ one, then use $L_V(GL(V))$, then convert back. For this we need a right inverse to D. Consider $\bar{D}$ acting on $\mathbb{Z}^f[\Lambda(O(V))]$ to $\mathbb{Z}^f[\Lambda(GL(V))]$ by

$$\bar{D} e_\lambda = e_\lambda - \sum_{\mu \in J_1(\lambda)} e_\mu$$

where $J_1(\lambda)$ is the set of diagrams obtained from λ by removing precisely two boxes *not* in column array. **Claim** This obeys $D\bar{D} = 1_{\mathbb{Z}^f[\Lambda(O(V))]}$.
Proof: Combining the definitions

$$D\bar{D}e_\lambda = e_\lambda + \sum_{\mu \in J(\lambda)} e_\mu - \sum_{\mu \in J_1(\lambda)} \sum_{\alpha \in J(\mu)} e_\alpha - \sum_{\mu \in J_1(\lambda)} e_\mu = e_\lambda$$

as required. (Here we have used

$$\sum_{\mu \in J(\lambda)} e_\mu = \sum_{\mu \in J_1(\lambda)} \left(e_\mu + \sum_{\alpha \in J(\mu)} e_\alpha \right)$$

which follows from the definition of $J(\lambda)$.) $\square$

Then

$$\mathcal{L}_V(O(V)) \;=\; D\,\mathcal{L}_V(GL(V))\,\bar{D}$$

OR SOMETHING!!!
which gives

$$\mathcal{L}_V(O(V)) = ...$$

2. Symplectic group case.

Recall that irreducibles of $Sp(E = \mathbb{C}^{2m})$ are labelled by partitions obeying $\lambda_1^t \leq m$. Simple modules $L_{sp}(\lambda)$ for $sp(E)$ are labelled similarly. According, for example, to Orellana–Ram [130] we have

$$E \otimes L_{sp}(\lambda) = \oplus_{\lambda^\pm} L_{sp}(\lambda^\pm)$$

where the sum is over partitions obtained by adding or removing a box to λ (while preserving the column length rule).

(**24.4.1**) Exercise. Check this against the WCF/other known data in low rank!!!

24.5 Combinatorial invariant theory for $O(N)$

Recall that $\mathcal{Y}$ is the Young graph. We write $\mathcal{Y}(-2n)$ for the truncation to Young diagrams λ with at most n rows (i.e. $\lambda_1' \leq n$); and $\mathcal{Y}(N)$ for the truncation to Young diagrams λ such that

$$\lambda_1' + \lambda_2' \leq N$$

24.5.1 Berele's and Proctor's results for $O(N)$, $Sp(2n)$

Here we summarize some relevant parts of Proctor [**?**] (which is in turn partly a review — see the references therein).

(**24.5.1**) An *N-semistandard tableau of shape λ* is a filling of the squares in λ with numbers from $\{1, 2, ..., N\}$ such that the letters strictly increase down each column and weakly increase across each row.

(**24.5.2**) A *$2n$-symplectic tableau of shape λ* is a $2n$-semistandard tableau such that the letters $2i - 1$ and $2i$ do not occur below the i-th row.
(Hence $2n$-symplectic tableaux cannot have more than n rows.)

(**24.5.3**) An *n-oscillating tableau of final shape λ and length k* is a sequence of $k+1$ shapes $0, \lambda(1), \lambda(2), ..., \lambda(k) = \lambda$, each of which has no more than n rows, such that each shape is obtained from the preceding one by adding or removing a box.
Let $g_\lambda(n, k)$ be the number of such tableaux.

(**24.5.4**) Remark. Note that n-oscillating tableaux of final shape λ and length k are in bijection with walks on the $2n$-symplectic truncation $\mathcal{Y}(-2n)$ of the Young graph of length k, from $\emptyset$ to λ.

We have [8, 133]

$$(2n)^k = \sum_\lambda g_\lambda(n, k) Sp^{\#}_{2n}(\lambda)$$

where the sum is over shapes λ of degree congruent to k mod.2; and $Sp^{\#}_{2n}(\lambda)$ is the number of $2n$-symplectic tableau of shape λ.

This corresponds to the decomposition of $(\mathbb{C}^{2n})^{\otimes k}$ as an $Sp(2n)$-module; with irreducible multiplicities $g_\lambda(n, k)$. Thus the $g_\lambda(n, k)$s are the dimensions of simple modules of the Brauer algebra $B_k(-2n)$ that appear in the action on tensor space.

(**24.5.5**) Shape λ is *N-orthogonal* if

$$\lambda'_1 + \lambda'_2 \leq N$$

An N-semistandard tableau that has $\leq q$ entries in the first two columns that are $\leq q$ satisfies the *q-th orthogonal condition.*

(**24.5.6**) An *N-orthogonal tableau* is an N-semistandard tableau that satisfies the q-th orthogonal condition for all $q \leq N$.
Let $O^{\#}_N(\lambda)$ be the number of N-orthogonal tableau of shape λ.

(**24.5.7**) An *oscillating N-orthogonal tableau of final shape λ and length k* is an N-oscillating tableau of final shape λ and length k such that each shape in the sequence is N-orthogonal.
Let $h_\lambda(N, k)$ be the number of such tableaux.

(**24.5.8**) Remark. Oscillating N-orthogonal tableau of final shape λ and length k are in bijection with walks on $\mathcal{Y}(N)$ of length k from $\emptyset$ to λ.

Proctor asserts:

$$N^k = \sum_\lambda h_\lambda(k, n) O^{\#}_N(\lambda)$$

where the sum is over all N-orthogonal shapes λ with $k - 2c$ squares, $c \geq 0$.

(**24.5.9**) Proposition. [133, Prop.8] *Write simply λ (an N-orthogonal shape) for the corresponding complex irreducible representation of $O(N)$. In particular $\square$ is the defining representation. Then*

$$\square \otimes \lambda = \bigoplus_\mu \mu$$

where is sum is over all N-orthogonal shapes adjacent to λ on the Young graph.

(**24.5.10**) Remark. This may be stated in terms of shapes adjacent to λ on $\mathcal{Y}(N)$.

We have an almost identical statement for the tensor product $\square \otimes \lambda$ in $Sp(2n)$, simply replacing $\mathcal{Y}(N)$ with $\mathcal{Y}(-2n)$.

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