Chapter 11

Basic representation theory of the symmetric group

ch:Sn

Recall that the symmetric group S_n is the group of permutations of a set of n objects. I.e. the group of self-isomorphisms and their compositions as functions. It is also a Coxeter group, as per §5.2 (with $M = M_{A_n}$).

Background references for this Chapter are discussed at the end. Particularly important contributions include Young [?], Schur [139] and James [73]. Books on the subject include Hamermesh [60], Boerner [12] and Robinson [36].

11.1 Introduction

11.1.1 Conventions

We take it here that the set permuted by S_n is $\underline{n} = \{1, 2, ..., n\}$ (as in §3.2 for example). Note that $\underline{n}$ is an ordered set, in the natural way. Let us further assume that n is single digit (or otherwise that each element of $\underline{n}$ somehow has a symbol that is connected). Then *'string notation'* for $w \in S_n$ is the string $w(1)w(2)...w(n)$, a string-perm of $12...n$. In particular $12...n$ is the notation for the identity element in S_n.

By convention then, for the composition $w' \circ w$, we have for example

$$213 \circ 132 = 231 \qquad (11.1)$$ eq:213x

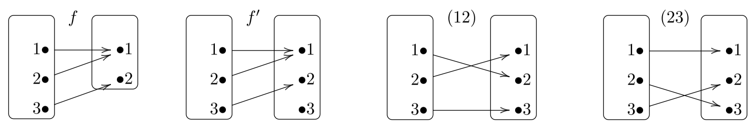
In particular here, $132(1) = 1$ because $w(1)w(2)w(3) = 132$ yields $w(1) = 1$; while $w'(1)w'(2)w'(3) = 213$ so $w'(1) = 2$, so $(w' \circ w)(1) = w'(w(1)) = 2$.

The group S_n is isomorphic to the group with the opposite multiplication, but it is not commutative, so the order in the convention matters 'internally'. Indeed

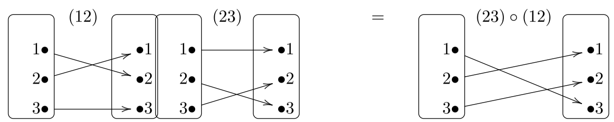
$$132 \circ 213 = 312 \qquad (11.2)$$ eq:312

here. Note that the convention with composition given by $f * g = g \circ f$ is also in fairly common use. Indeed in cases where S_n arises as a subalgebra of another structure the latter composition may be natural.

Here are some function diagrams in the spirit of §3.2:



The function $f \in \underline{2}^{\underline{3}}$. Its string notation is $f = f(1)f(2)f(3) = 112$. Meanwhile $f', (12), (23) \in \underline{3}^{\underline{3}}$, with the last two invertible and hence in S_3 — see below for the general form of the (12) notation. For composition here we have a concatenation in a suitable order:



This verifies $(23) \circ (12) = 312$ as in (11.2).

(**11.1.1**) If n is fixed, we write (ij) for the elementary transposition in the symmetric group S_n. That is

$$(12) = 21345...n \in S_n$$

and so on. Note that:

LEMMA. Group S_n is generated by $\{(i\, i{+}1) \mid i = 1, 2, ..., n-1\}$. □

(**11.1.2**) THEOREM. *The symmetric group S_n is isomorphic to the group with presentation*

$$\langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \rangle / \sim$$

where the relations are $\sigma_i^2 = 1$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

and $\sigma_i \sigma_j = \sigma_j \sigma_i$ *if* $|i - j| \neq 1$. *The isomorphism is given by* $\sigma_i \mapsto (i\, i{+}1)$.

(**11.1.3**) The *trivial representation* of S_n is given by $R_{(n)}((12)) = 1$; and the *alternating representation* by $R_{(1^n)}((12)) = -1$.

(**11.1.4**) We write $(i_1, i_2, ..., i_k) \in S_n$ for the cyclic permutation of the listed distinct elements of $\{1, 2, ..., n\}$. Note that $(i_1, i_2, ..., i_k) = (i_k, i_1, i_2, ..., i_{k-1})$. We declare a canonical expression for this element to be the one in which the lowest number appears first.

Any two such cyclic elements of S_n commute if they have no list element in common. Thus a set partition of $\{1, 2, ..., n\}$ and an ordering of the elements in each part defines an element w of S_n. Furthermore, any element $w \in S_n$ can be built this way.

The *cycle structure* of $w \in S_n$ is the integer partition associated to the set partition as above. For example the cycle structure of $(12)(34) \in S_7$ is $22111 = 2^2 1^3$.

Two elements of S_n are in the same class if and only if they have the same cycle structure. We have shown the following.

th:classn (**11.1.5**) THEOREM. *The classes of S_n may be indexed by the integer partitions of n.*

11.1.2 Preamble

(**11.1.6**) The study of the representation theory of S_n is built in large part, of course, on general machinery for the representation theory of finite groups (character theory, Maschke's Theorem, Brauer–modular systems and so on). The case specific aspects, however, involve various historically distinct approaches.

Theorem 11.1.5 can be seen as giving rise to one of the main 'combinatorial' approaches to S_n representation theory (perhaps largely attributable to Young). We have *Young diagrams* as pictures of integer partitions, and *Young tableaux*, which are various combinatorial embellishments. These involve Young subgroups, the Robinson-Schensted correspondence, and various iterative (n-varying) approaches.

A different approach is through the 'dual' study of the 'combinatorial' representation theory of the general linear groups, including 'Schur–Weyl duality'. (Yet another is Springer theory, which uses geometrical properties of general linear groups.)

One also has an approach (specifying the general machinery of character theory) involving symmetric polynomials. And there is a collection of approaches triggered by the so-called Jucys–Murphy elements of the group algebra. There are approaches which could be characterised as 'probablistic' (via the natural connection between combinatorics and probability), and others perhaps 'asymptotic'.

From a contemporary perspective some of these approaches manifest commonalities between them beyond the fact that they all speak to the symmetric group.

There are also contemporary approaches which, while not producing new results, shed interesting new light on the subject and its remaining open problems. An example here is the study of **Set** as a symmetric group module.

11.1.3 Integer partitions, Young diagrams and the Young lattice

In light of Th.11.1.5, integer partitions and their Young diagrams (see e.g. Fig.11.1) play a useful role in S_n representation theory. See §5.7 for some basic properties. The details we shall need are given in §11.4.

As we shall see, the symmetric group algebras are split semisimple over the rationals. The irreducible representations of S_n over the rationals (and hence over the containing complex field) are indexed by the integer partitions of n.

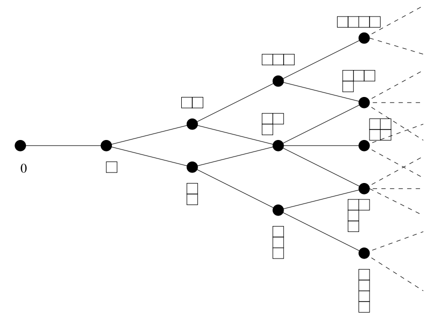
(**11.1.7**) Note that there is an inclusion of S_{n-1} in S_n given in string notation by

$$w \mapsto wn$$

- where wn is understood as the concatenation of the singleton string n onto the end of string w.

(**11.1.8**) The corresponding restriction rule for irreducible representations over the splitting field is that the representation indexed by integer partition λ restricts to the direct sum of irreducible representations indexed by integer partitions of $n-1$ that are sub-partitions of λ (i.e. their Young diagrams are sub-diagrams of the diagram for λ).

(**11.1.9**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice (as in Figure 5.8 - reproduced here in Fig.11.1).



Younggraph2 Figure 11.1: The start of the Young graph (covering DAG of the Young lattice, increasing from left to right).

11.1.4 Realisation of S_n as a reflection group

ss:Snrefl

Consider S_n acting on the standard ordered basis of $\mathbb{R}^n$ by permutation: $we_i = e_{w(i)}$. This lifts to an action on $\mathbb{R}^n$. In particular, setting $\langle e_i, e_j \rangle = \delta_{i,j}$, then perm (ij) is a reflection fixing the hyperplane $H_{ij} = \{x = (x_1, x_2, ..., x_i, ..., x_j = x_i, ..., x_n) \mid x \in \mathbb{R}^n\}$. See also §5.2.

Note that reflection in a *different* hyperplane does not fix H_{ij} in general, but does take it to a hyperplane. In fact the set H of hyperplanes of this form closes under the group generated by the corresponding reflections. (Note that these are all the reflections in the group, but they are not all the involutions in general. We also have (12)(34) for example.)

A single chamber of $\mathbb{R}^n \setminus H$ has open facets touching only a subset of H. For example there are chambers touching only the hyperplanes $H_{i\ i+1}$ (the chamber of strictly ascending sequences and the chamber of strictly descending sequences).

The chamber of ascending sequences a_+ contains vector $v_+ := (1, 2, ..., n)$. If we act with (ij) this is taken to the chamber containing $(1, 2, ..., j, ..., i, ..., n)$, and so on. In this way we may place the chambers of $\mathbb{R}^n \setminus H$ in bijective correspondence with the elements of S_n. That is, writing v for the chamber containing v, $(1, 2, ..., n) \overset{\beta}{\rightsquigarrow} 12...n = 1 \in S_n$. Then writing the reflection action of (ij) on the left we have $(ij)(1, 2, ..., n) \overset{\beta}{\rightsquigarrow} (ij)12...n = (ij)1$, and so on (note that we have to be careful with notation here).

(**11.1.10**) REMARK. Note that we have chosen a representative element of the chamber a_+. There is another action of S_n permuting the entries regarded as distinct symbols (instead of the positions of the entries).

(**11.1.11**) Note that the vectors $S_n v_+$ define a convex polytope P_n embedded in $\mathbb{R}^n$. Note that P_n is codimension-1, since the sum of entries in a vertex vector is fixed.

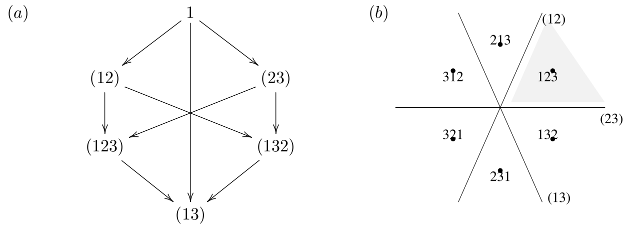


Figure 11.2: (a) Bruhat relation. (b) hyperplanes. fig:bruhat

The *permutahedron of order n* (see e.g. Ziegler [150]) is the polytope P_n.

The vertices of P_n (the faces of dimension zero) are precisely the set $S_n v_+$. The vertices and edges (faces of dimension one) of P_n define a graph (directed, by choice of a root vertex, such as v_+, away from which paths then flow, until converging on the image $(n, n-1, ..., 1)$ of the longest element of S_n), also denoted P_n. For example, P_3 is a hexagon; P_4 is a polyhedron with 4- and 6-sided faces.

(**11.1.12**) Let G be a group and $S \subset G$ a set of generators. The *Cayley graph* $\Gamma(G, S)$ of (G, S) is the digraph with vertex set G and an edge (a, b) whenever $b = as$ for some $s \in S$.

The *colour Cayley graph* is the Cayley graph together with edge labels from S — if $b = as$ then the edge label on (a, b) is $s = a^{-1}b$. This partitions the set of edges into parts labelled by S.

Lemma. Let $S = \{(i\ i{+}1) \mid i = 1, 2, ..., n-1\}$ be the set of Coxeter generators of S_n. Then

$$\Gamma(S_n, S) \cong P_n.$$

Proof. One needs to show that the faces of dimension one in P_n join vertices related by adjacent transpositions. See Ziegler [150]. □

Note that the edge label s on (a, as) is not in general the same as the elementary transposition w that takes a to $wa = as$. Note however that $w = asa^{-1}$ so that any w, s in this 'left-versus-right-action' relationship to each other are in the same class, and in particular here are both reflections. Although w is not generally a simple reflection. See (5.3.15).

(**11.1.13**) Considering $G(S_n, a_+ = (1, 2, ..., n))$ as defined in §5.2 we have $G(S_n, a_+ = (1, 2, ..., n)) \cong \Gamma(S_n, S)$.

The undirected version $G(S_n)$ of $G(S_n, a_+ = (1, 2, 3))$ as defined in §5.2 is the geometric dual graph to the chamber geometry in Fig.11.2 (b) seen as a complex (one considers the alcoves and their walls from the complex — these become the vertices and edges in the dual).

Note that the automorphism group of graph $G(S_n)$ is not S_n, but none-the-less acts transitively (i.e. fixes no proper subset of vertices), so that each single vertex is of equal 'standing'. (This corresponds to the freedom to choose a fundamental chamber.)

de:bruhat1 (**11.1.14**) The *Bruhat order* $(S_n, >)$ is a partial order defined as follows. We first define a relation $(S_n, \to)$ by $w \to w'$ if the chambers of w, w' are images either side of a hyperplane (not necessarily 'adjacent'), and w is on the same side of the hyperplane as $(1, 2, ..., n)$. The order $(S_n, >)$ is the transitive closure of the $\to$ relation (i.e. $w < w'$ if $w \to w'$ and so on).

Example: Fig.11.2 (a) digraph of the $\to$ relation for S_3; (b) hyperplanes and chambers in $\mathbb{R}^3$ viewed down the $(1, 1, 1)$ line.

(**11.1.15**) NB graph (a) in Fig.11.2 is not the Hasse graph of the Bruhat order — for this we should omit the long vertical line, even though it is direct in the $\to$ relation.

Note: The digraph $G(S_3, a_+ = (1, 2, 3))$ from §5.2 is a directed hexagon. Its transitive closure, the *cone order*, is not equal to the Bruhat order since the diagonal lines in Fig.11.2 (a) are not present in the cone order.

NB the Bruhat order is not the same as the *weak order*, the order generated by Coxeter reflections (reflections in walls 'touching' the fundamental chamber), since that order contains none of the vertical lines in (a) (the long one is not needed for Bruhat, but the shorter ones are).

11.2 Representations of S_n from the category Set

Recall the category **Set** from §6.1. For convenience define $\mathbf{Set}(m, n) = \mathbf{Set}(\underline{m}, \underline{n})$. Recall that the category composition equips $\mathbf{Set}(m, m)$ with the property of monoid; and $\mathbf{Set}(m, n)$ with the property of left $\mathbf{Set}(m, m)$-set; and right $\mathbf{Set}(n, n)$-set. For any commutative ring R these sets extend R-linearly to modules. Since $S_m \subset \mathbf{Set}(m, m)$ we can also build a left S_m-module by restriction (respectively a right S_n-module).

What does $\mathbf{Set}(3, 2)$ look like as a left $\mathbf{Set}(3, 3)$- or left S_3-module? In our notation the (ordered) basis is $\{111, 112, 121, 122, 211, 212, 221, 222\}$ and we have

$$R_{3,2}((12)) = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 0 && 1 &&& \\ &&& 0 && 1 && \\ && 1 && 0 &&& \\ &&& 1 && 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$$

We return to consider the general case in §11.2.3.

11.2.1 Connection with Schur's work and Schur functors

In this subsection we follow Green[57, §3,§6] closely. (Note that §6.2 of Green is also focal to §13.4.)

(**11.2.1**) Note that Green uses $I(n, r)$ for $\mathbf{Set}(r, n)$, and writes $i \in I(n, r)$, that is $i : \underline{r} \to \underline{n}$, as a vector or multi-index: $i = (i_1, ..., i_r)$.

(**11.2.2**) Let $G = GL_n(K)$ be the group of invertible matrices over an infinite field K. We regard $\mathbf{Set}(G, K)$ as a commutative K-algebra by $ab(g) = a(g)b(g)$; $(a + b)(g) = a(g) + b(g)$.

The identity element is given by $1(g) = 1_K$ for all g.

For each $s \in G$ we define $L_s \in \text{End}_K(\mathbf{Set}(G,K))$ by $L_s f(g) = f(sg)$ (and define $R_s f(g) = f(gs)$ similarly).

PROPOSITION. The map $R(s) = R_s$ is a representation of G (and L gives an antirepresentation). Thus $\mathbf{Set}(G,K)$ is a left KG-module:

$$sf = R_s f \qquad \text{(similarly } fs = L_s f)$$

The two actions commute.

(**11.2.3**) Let $c_{ij} : G \to K$ be given by $c_{ij}(g) = g_{ij}$. Write A for the K-subalgebra of $\mathbf{Set}(G,K)$ generated by all the c_{ij}s.

For each r we write $A(n,r)$ for the subspace of A of polynomials homogeneous of degree r in the c_{ij}s. For example $A(n,1) = K\{c_{11}, c_{12}, ..., c_{1n}, c_{21}, c_{22}, ..., c_{2n}, ..., c_{nn}\} = K\{c_{ij}\}_{ij}$ (a K-space of dimension n^2). Thus A has grading

$$A = \sum_r A(n,r)$$

as a K-algebra.

For $i,j \in \mathbf{Set}(r,n)$

$$c_{ij} \; := \; c_{i_1 j_1} ... c_{i_r j_r} \tag{11.3}$$

eq:codet1

Then $A(n,r)$ is spanned by these monomials. Note that the RHS of (11.3) does not determine i,j.

(**11.2.4**) Define

$$S_K(n,r) = \text{hom}_K(A(n,r), K)$$

This has basis $\{\xi_{ij} \mid i,j \in \mathbf{Set}(r,n)\}$ with ξ_{ij} given by

$$\xi_{ij}(c_{kl}) = \begin{cases} 1 & if \quad (i,j) \sim (k,l) \\ 0 & o/w \end{cases}$$

where $\sim$ means the orbit of the diagonal action of S_r.

Algebra $A(n,r)$ is a coalgebra, so the dual $S_K(n,r)$ is an associative algebra. We have (see Schur [139], Green [57, p.21], or Martin–Woodcock [119])

$$\xi_{ij}\xi_{kl} \;=\; \sum_{p,q} Z(i,j,k,l,p,q).1_K \; \xi_{pq} \tag{11.4}$$

eq:Zorder2

where the sum is over a transversal of $\sim$; and

$$Z(i,j,k,l,p,q) = |\{s \in \mathbf{Set}(r,n) \mid (i,j) \sim (p,s),\ (k,l) \sim (s,q)\}|$$

(**11.2.5**) For example $\xi_{ii}^2 = \xi_{ii}$ and $\xi_{ii}\xi_{jj} = 0$ if i,j not in the same orbit of the S_r-action (i.e. if $\xi_{ii} \neq \xi_{jj}$). Indeed

$$1_{S_K(n,r)} = \sum_i \xi_{ii}$$

where the sum is over the distinct elements.

(**11.2.6**) Note from (11.4) that the $\mathbb{Z}$-submodule $S_{\mathbb{Z}}(n,r)$ of $S_{\mathbb{Q}}(n,r)$ generated by the ξ_{ij} is multiplicatively closed. That is, it is a $\mathbb{Z}$-order in $S_{\mathbb{Q}}(n,r)$. For any field K there is an isomorphism of K-algebras $S_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} K \cong S_K(n,r)$. In this sense, for fixed n,r, the 'scheme' or family of algebras $S_K(n,r)$ is 'defined over $\mathbb{Z}$'.

(**11.2.7**) Let $\Lambda(n,r)$ be the set of S_r-orbits in $\mathbf{Set}(r,n)$ (the set of 'weights'). For example, for $r \leq n$ there exist functions $i \in \mathbf{Set}(r,n)$ of form $i = (s(1), s(2), ..., s(r))$, where $s \in S_r$. The weight w of any such i is

$$w = (1,1,...,1,0,0,...,0) \in \mathbb{Z}^n$$

(r nonzero entries).

Consider the commuting S_n action on $\mathbf{Set}(r,n)$. Each S_n-orbit contains one dominant weight. Write $\Lambda^+(n,r)$ for the set of dominant weights.

(**11.2.8**) If $i \in \mathbf{Set}(r,n)$ belongs to $a \in \Lambda(n,r)$ we may write ξ_a for ξ_{ii}.

(**11.2.9**) PROPOSITION. There is an isomorphism of K-algebras

$$\xi_w S_K(n,r) \xi_w \;\cong\; KS_r$$

which takes $\xi_{us,u}$ to s for all $s \in S_r$.

(**11.2.10**) This allows us to construct a 'Schur' functor relating the representation theory of the Schur algebra $S_K(n,r)$ (and hence part of the representation theory of the general linear group) to the symmetric group S_r:

$$F : S_K(n,r) - \text{mod} \;\to\; KS_r - \text{mod} \tag{11.5}$$

eq:schur functor

where $FM = \xi_w M$.

(**11.2.11**) (TO CONTINUE we should summarize [57, §3.2].)

(**11.2.12**) A closely related idea is that, CLAIM:

$$S_K(n,r) \cong \text{End}_{KS_r}((K^n)^{\otimes r})$$

(**11.2.13**) Schur–Weyl duality. Let $V = k\{e_1, e_2, ..., e_N\}$ for some field k. For physics $k = \mathbb{C}$. Now see Fig.11.3.

11.2.2 Idempotents and other elements in $\mathbb{Z}S_n$

(**11.2.14**) For $x \subset \underline{n}$ we write S_x for the subgroup of S_n in which only the elements in x may be permuted nontrivially. If $p = \{p_1, p_2, ...\}$ is a set partition of $\underline{n}$ then each S_{p_i} is a subgroup; these subgroups commute pairwise; and we write S_p for the Young subgroup

$$S_p = S_{p_1} S_{p_2} ...$$

(Note that the given part names in p suggest an ordering of parts, but this is not intrinsic to a set partition in general. In our case it is easy to give an ordering rule - for example 'children first' - but note that in any case S_p will not depend on this order.)

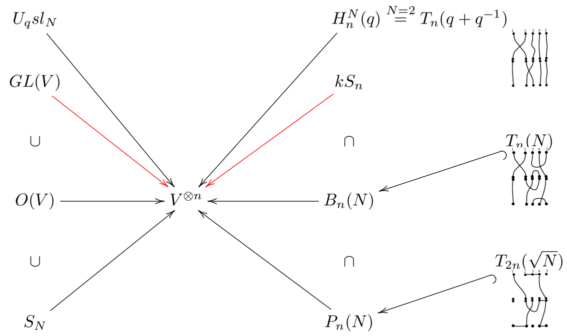


Figure 11.3: Schur–Weyl duality schematic. For N-state Potts on n-site wide lattice. fig:SW01

(**11.2.15**) A *composition* of n (into m parts) is a sequence of elements of $\mathbb{N}_0$ (of length m) that sums to n. We may associate a composition to each ordered set partition of $\underline{n}$ (or indeed of any finite set), $p = \{p_1, p_2, ...\}$, by

$$\lambda_i = |p_i|$$

It will be evident that

$$S_p \cong \times_i S_{\lambda_i}$$

For each integer partition $\lambda \vdash n$ we associate a set partition by

$$p(\lambda) = \{\{1, 2, .., \lambda_1\}, \{\lambda_1 + 1, .., \lambda_1 + \lambda_2\}, ...\}$$

We write $S_\lambda = S_{p(\lambda)}$. When the factor groups are simply arranged side-by-side in this way we write $\otimes_i w_i$ for the image of $(w_1, w_2, ...) \in \times_i S_{\lambda_i}$ in S_n.

(**11.2.16**) Define elements of $\mathbb{Z}S_n$ by

$$e'_{(n)} = \sum_{w \in S_n} R_{(n)}(w)\, w$$

$$e'_{(1^n)} = \sum_{w \in S_n} R_{(1^n)}(w)\, w$$

Note that

$$e'_{(n)} e'_{(1^m)} = 0 \qquad \text{for } m, n > 1 \tag{11.6}$$

eq:orth1

Using the side-by-side notation define

$$e'_\lambda = \otimes_i e'_{(\lambda_i)}$$

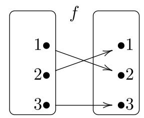


Figure 11.4: Mapping diagram of $(12) \in S_3$. fig:map diag

$$f'_\lambda = \otimes_i e'_{(1^{\lambda'_i})}$$

where λ' is the transpose partition to λ.

(**11.2.17**) Define a map C_L from S_n to set partitions of $\underline{n}$ as follows. Draw the mapping diagram of $w \in S_n$ (as illustrated in Figure 11.4), and put i, j $(i < j)$ in the same part of $C_L(w)$ if the lines from i, j cross, i.e. if $i < j$ and $w(i) > w(j)$. Define $C_R(w) = C_L(w^{-1})$.

Example: $C_L((12)) = \{\{1, 2\}, \{3\}, ...\}$.

(**11.2.18**) For any set partition $p \in P_{\underline{n}}$ we call a $w \in S_n$ left p-noncrossing if $i \sim^p j$ implies $i \not\sim^{C_L(w)} j$.

The idea here is that if we look at any of the parts of p regarded as a subset of $\underline{n}$, we will find that the corresponding strings (as labelled on the left) do not cross each other in w.

For example, if p is the partition into singletons then every $w \in S_n$ is p-noncrossing. If $p = \{\underline{n}\}$ then only $1 \in S_n$ is p-noncrossing.

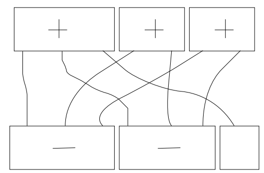
Define right p-noncrossing similarly. Write $S_n(p, q)$ for the set of left p-noncrossing right q-noncrossing elements of S_n. Write $S'_n(p, q)$ for the subset of elements such that no two strings in the same part at p are in the same part at q.

pr:pig1 (**11.2.19**) Proposition. *There is a $w_\lambda \in S_n$ such that $e'_\lambda w_\lambda f'_\lambda \neq 0$ and*

$$e'_\lambda w f'_\lambda \;=\; \epsilon_\lambda(w)\; e'_\lambda w_\lambda f'_\lambda$$

with $\epsilon_\lambda(w) \in \{0, \pm 1\}$, for all $w \in S_n$.

Proof. A conceptual/diagramatic proof is effective here. Draw a box on the left for each factor in e'_λ and one on the right for each factor in f'_λ. For example (rotating so that boxes 'on the left' are on the top, just to save space) the case $\lambda = (3, 2, 2)$ is:



By (11.6) if w connects any box on the left to any box on the right by more than one line, then $e'_\lambda w f'_\lambda = 0$. Also, if two lines coming out of any box cross this crossing can be removed (at cost a factor -1 if it is a box on the right). Since λ'_i is the number of boxes on the left with at least i lines, one sees that there is precisely one w producing a non-vanishing product and with no removable crossings. This is w_λ. One readily confirms that $e'_\lambda w_\lambda f'_\lambda = w_\lambda + ...$ is non-vanishing. □

(**11.2.20**) Note from the proof that there is more than one choice for w_λ above in general, but there is a unique choice of least length.

(**11.2.21**) Set $E_\lambda = e'_\lambda w_\lambda f'_\lambda$. Evidently $E_\lambda x E_\lambda = m_x E_\lambda$ for all $x \in kS_n$ for some scalar m_x. Thus the left ideal $I = kS_n e'_\lambda w_\lambda f'_\lambda$ obeys $E_\lambda I \subseteq kE_\lambda$.

(**11.2.22**) Suppose we work over a field k of char.0. Then $I^2 \neq 0$, so $E_\lambda g E_\lambda \neq 0$ for some $g \in S_n$. Thus there exists a g for which $E_\lambda g E_\lambda g = m_g E_\lambda g \neq 0$, so that $E_\lambda g$ is an unnormalised idempotent.

(**11.2.23**) Suppose J is a subideal of I. Then either $E_\lambda J = kE_\lambda$ or $E_\lambda J = 0$. In the first case $J = kS_n E_\lambda J = I$; in the second $J^2 = \{0\}$ so if $k \supseteq \mathbb{Q}$ then kS_n is semisimple and $J = 0$. Thus:

(**11.2.24**) Lemma. *If $k \supseteq \mathbb{Q}$ then left ideal $I = kS_n E_\lambda$ is minimal.* □

(**11.2.25**) Set $s_\lambda = e'_\lambda (w_\lambda f'_\lambda w_\lambda^{-1})$. We claim this is the unnormalised idempotent in the char.0 case.

In this case $s^2 = m_w s \neq 0$. Write simply $s = ef$ for a moment. Then $s^2 = efef \neq 0$, so $efe \neq 0$. Note that efe is fixed under the standard involutive antiautomorphism map $g \mapsto g^{-1}$. We claim this efe is then a fixed (unnormalised) primitive idempotent conjugate to s.

11.2.3 Young modules

ss:youngm1

Recall from §6.2 and 6.3 that for k a ring, $k\mathbf{Set}^f$ is the k-linear category over category $\mathbf{Set}^f$, and $kC_{\mathbb{N}}$ is a corresponding skeleton. We may write $\hom(m,n)$ for $\mathbf{Set}(m,n)$, a basis for $k\mathbf{Set}^f(m,n)$.

As noted in §6.3, S_n is the group of isomorphisms in $\hom(n,n)$, so $k\hom(n,m)$ is a left kS_n right kS_m bimodule by restriction. Recall that the basis $\hom(n,m)$ of functions $f : \underline{n} \to \underline{m}$ may be written as words $f(1)f(2)...f(n)$ in $\underline{m}$. Then the action of S_n is to permute the n entries in this list; while the action of S_m is to permute the set of symbols. For example

$$(23)1244 = 1424$$

$$1244(23) = 1344$$

(**11.2.26**) We may associate a composition λ of n to each function f in $\hom(n,m)$ by

$$\lambda_i = \#\{j \mid f(j) = i\}$$

For example $\lambda(12444) = (1,1,0,3)$.
(Note that the notation $12444 \in \hom(n,m)$ tells us that $n = 5$, but only that $m > 3$. In the above we have assumed $12444 \in \hom(5,4)$. If we regard $12444 \in \hom(5,6)$ then we have $\lambda(12444) = (1,1,0,3,0,0)$.)

For $\lambda \in \mathbb{N}_0^m$, we write $\hom(n,\lambda)$ for the subset of $\hom(n,m)$ of functions of fixed λ. For example

$$\hom(4,(3,1)) = \{1112, 1121, 1211, 2111\}$$

(For the moment we leave this simply as an abuse of notation, as far as the categorical context is concerned.)

Note that the left action of S_n on $\hom(n,m)$ fixes this composition of n, so we may decompose $k\hom(n,m)$ as a direct sum of left modules indexed by compositions:

$$k\hom(n,m) \cong \bigoplus_{\lambda \in \mathbb{N}_0^m} k\hom(n,\lambda)$$

Example:

$$k\hom(3,2) \cong k\{111\} \oplus k\{112,121,211\} \oplus k\{122,212,221\} \oplus k\{222\}$$

Any two such left submodules are isomorphic if their compositions are related by a reordering of the terms in the sequence λ (since the invertible right action by S_m achieves all such reorderings). We will take integer partitions to be representative elements of the orbits of compositions under this action.

(**11.2.27**) We now consider the integral, or ring independent, decomposition of the regular left $\hom(n,n)$-module $k\hom(n,n)$. We are interested in this as a left kS_n-module by restriction.

We have seen that formally ignoring the difference in codomain gives an inclusion of sets $\hom(n,m) \hookrightarrow \hom(n,m{+}1)$. This gives an example of an injection $k\hom(n,m) \hookrightarrow k\hom(n,m{+}1)$, and indeed this injection is split. The sections contain sums of the $k\hom(n,\lambda)$'s with precicely $m+1$ parts.

It follows that we can find all the simple left modules for kS_n by looking in the $k\hom(n,\lambda)$'s with $\lambda \vdash n$.
(From now on we shall mean $\lambda \vdash n$ by λ, unless otherwise stated.)
(Indeed the claim follows directly on noting that $k\hom(n,(1^n))$ is isomorphic to kS_n as a left module. But we shall make use of the others too.)

(**11.2.28**) In particular if k is a field, all simple left kS_n-modules will appear as composition factors for this collection of modules — $\{k\hom(n,\lambda) \mid \lambda \vdash n\}$ — which we shall call the *Young modules* of kS_n. However it will be evident that these modules are not themselves simple in general. For example

$$e'_{(n)} k\hom(n,\lambda) \neq \{0\}$$

for any λ, so $e'_{(n)} k\hom(n,\lambda)$ is a proper submodule of $k\hom(n,\lambda)$ for any $\lambda \neq (n)$.

On the other hand

$$e'_{(1^n)} k\hom(n,\lambda) = \{0\} \qquad \text{for any } \lambda \neq (1^n).$$

This gives us a clue as to how to extract useful submodules from the Young modules more systematically.

pr:yyoung1

(**11.2.29**) Proposition. *(I) Let $\lambda \vdash n$. The left kS_n module $kS_n e'_\lambda$ is k-free with basis the elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor $e'_{(\lambda_i)}$ in e'_λ. (II) There is a bijection between this basis and $\hom(n,\lambda)$ obtained by modifying w to an element of $\hom(n,\lambda'_1)$ by making each of these subsets of λ_i noncrossing lines meet at a point i on the target side. (III) This bijection extends k-linearly to an isomorphism of left modules:*

$$kS_n e'_\lambda \cong k\hom(n,\lambda)$$

Proof. (I) A spanning set for the LHS is elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor in e'_λ. One can check that we'_λ contains the group element w with coefficient 1, and no other such group element, so the set is k-free and a basis. (II,III) The map described is evidently a set bijection and hence an isomorphism of free k-modules, but it also commutes with the S_n action (indeed the computation of this action is essentially the same computation on each side). □

ss:specht1

11.2.4 Specht modules

See for examples James' Lecture Notes [73]. Here we give a quick summary; with more details in the next section.

specht is ideal

(**11.2.30**) Comparing Prop. (11.2.29) with Prop. (11.2.19) it follows that $f'_\lambda k \hom(n,\lambda)$ is a rank-1 k-module.

QUESTION/CAVEAT: How do we know that the non-vanishing claim in Prop. (11.2.19) does not fail in finite characteristic?

(**11.2.31**) The kS_n-modules of form

$$\mathcal{S}(\lambda) \;:=\; S_n f'_\lambda k \hom(n,\lambda) \quad \cong S_n f'_\lambda k S_n e'_\lambda$$

are called *Specht* modules after [143].

These are free modules of finite rank over $\mathbb{Z}$, and hence the rank is not affected by base change to k. (For this reason the dependence of $\mathcal{S}(\lambda)$ on k is often left implicit. However some properties do depend on k.)

(**11.2.32**) Proposition. [James] For field k of characteristic $p \neq 2$ this $\mathcal{S}(\lambda)$ is an indecomposable kS_n submodule of $k\hom(n,\lambda)$ (cf. (9.1.4)).

However for $p = 2$ the Specht module with $\lambda = (5,1,1)$ is decomposable (and Murphy shows that there are infinitely many others such).

(**11.2.33**) An integer partition λ is *p-regular* if no part is repeated p or more times.

(**11.2.34**) Let k be a field of characteristic $p > 0$. James has shown that if λ is p-regular then $\mathcal{S}(\lambda)$ has simple head over k, and that

$$\{L^k(\lambda) = \text{head}\,^k\mathcal{S}(\lambda) \;\mid\; \lambda \;\; p\text{-regular}\}$$

is a complete set of kS_n-modules.

(**11.2.35**) Basis / restriction rules — see below.

The notation $\mathcal{S}(\lambda)^\perp$ for the next Theorem is explained in the following section (where the module $S^\lambda \cong \mathcal{S}(\lambda)$).

(**11.2.36**) Let $\triangleleft$ denote the dominance order. Note that dictionary order is a total order refining the dominance order.

For k be a field of char. p, let

$$D^p_n = [\mathcal{S}(\lambda) : L^k(\mu)]_{\lambda,\mu}$$

be the S_n Specht decomposition matrix over k; where for the rows p-regular partitions and then other partitions (and for the columns, p-regular partitions) are written out in the dictionary order.

(**11.2.37**) Theorem. *[James78 12.2] Fix k a field of char. p, and any n. Then $L^k(\lambda) = \mathcal{S}(\lambda)/(\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp)$ if λ is p-regular; and $[\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp \;: L^k(\mu)] \neq 0$ implies $\mu \triangleright \lambda$ for all λ; and $[\mathcal{S}(\lambda) : L^k(\mu)] \neq 0$ implies $\mu \triangleright \lambda$ for λ non-p-regular.*

That is, D^p_n is lower unitriangular.

pa:specht filt

(**11.2.38**) In e.g. Hemmer [**?**] (see also Green [57, §6.3]) it is asserted that the Schur functor from (11.5) takes Weyl and coWeyl modules to Specht and dual Specht modules respectively; and takes projective modules to Young permutation modules.

The Schur functor is exact, and one then has (again from Hemmer) that *Young permutation modules have Specht and dual-Specht filtrations.* In particular the regular module has such filtrations. Distinct such filtrations do not necessarily have the same multiplicities, unless the characteristic is $p > 3$.

Note that filtration of the regular module does not necessarily imply that an indecomposable projective module has a filtration, since a Specht module may not be indecomposable. But this is only an issue for $p = 2$.

11.3 Characteristic p, Nakayama and the James abacus

See for example James–Kerber [77].

(**11.3.1**) A *rim-hook* of a Young diagram d is a skew-subdiagram of d whose dual graph is a chain. Define *the rim* of d as the maximal rim-hook.

Note that a rim-hook is not necessarily a hook

For $p \in \mathbb{N}$, a *p-hook* of diagram d is a rim-hook of length p. We define a partial order on Young diagrams by $d' \overset{p}{<} d$ if $d' \subset d$ and the skew is a p-hook. A *p-core* of d is a minimal element in any chain containing d.

FACT: All p-cores of d coincide. (This follows from (11.3.5) below.)

th:Nakayama

(**11.3.2**) Theorem. *[Nakayama conjecture] Let k be a field of prime characteristic p. Then Specht module $\mathcal{S}(\lambda)$ lies in the same block as $\mathcal{S}(\lambda')$ iff their diagrams have the same p-core.*

Proof. See e.g. James–Kerber [77, p.245].

(**11.3.3**) Remark. Note that we have not shown that the block of $\mathcal{S}(\lambda)$ is well-defined if $p = 2$.

(**11.3.4**) Abacus: see James–Kerber [77, p.77-78]. A q-abacus is an abacus with q (vertical) runners, with the upper frame fixed and the lower frame very far away. The abacus may be considered to be filled with equal sized beads, almost all of which are 'empty' beads. The bead positions are numbered from 0 in reading order (i.e. left to right then top to bottom). A *bead configuration* records the position of the non-empty beads.

To each bead configuration there corresponds a Young diagram as follows. Associate to each non-empty bead the number of empty beads encountered in reading to that point. This gives a non-decreasing sequence from $\mathbb{N}_0$. Thus writing the sequence in reverse order and ignoring any 0s we get a non-increasing sequence in $\mathbb{N}$, and hence a Young diagram.

A useful fact is the following.

de:pcores2

(**11.3.5**) CLAIM: Suppose that removing a rim q-hook from d gives d'. Then replacing any bead configuration for d with a bead configuration in which one bead has been moved one space up, gives a bead configuration for d'.

(**11.3.6**) Example. (Omitting the vertical runners etc for reasons of laziness)

$$\begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \end{pmatrix} \begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \end{pmatrix}$$

The first abacus gives 00011233334 and hence $43^4 21^2$. The second gives 00011112234 and hence $432^2 1^4$. The difference is a rim 5-hook:

$$\begin{pmatrix} x & x & x & x \\ x & x & x & \\ x & x & x' & \\ x & x & x' & \\ x & x' & x' & \\ x & x' & & \\ x & & & \\ x & & & \end{pmatrix}$$

11.4 James–Murphy theory

JamesMurphy

This section is a summary of standard symmetric group results, cast in a form following [77, Ch.7]. See also §10.1.7. The objective is to give a definite (if not canonical) construction for symmetric group Specht module Gram matrices.

We use various notations for S_n elements. If $f \in S_n$ then $(f(1), f(2), ..., f(n))$ is a permutation. On the other hand in *cycle* notation, if $S = \{i, j, ..., k\} \subseteq \underline{n}$ then by $(ij..k) \in S_n$ we mean $f(i) = j$,..., $f(k) = i$ and $f(l) = l$ for $l \notin S$. Thus

$$(13).(12) = (123)$$

is an example of group multiplication.
(Then again, the common *diagram algebra* notation effectively composes permutations backwards, i.e. as in the opposite group. This is not a major issue, since the groups are isomorphic.)

(**11.4.1**) A *tableau* is an ordering of the boxes in a Young diagram, usually given by writing the counting numbers in the boxes. A β-tableau is a tableau for diagram β. If $\beta \vdash n$ then $\pi \in S_n$ acts on tableau in the obvious way:

$$(12)\begin{smallmatrix}1&2\\3\end{smallmatrix} = \begin{smallmatrix}2&1\\3\end{smallmatrix}$$

We have

$$(13)(12)\begin{smallmatrix}1&2\\3\end{smallmatrix} = (13)\begin{smallmatrix}2&1\\3\end{smallmatrix} = \begin{smallmatrix}2&3\\1\end{smallmatrix} = (123)\begin{smallmatrix}1&2\\3\end{smallmatrix}$$

so this is a 'left' action, as written.

(**11.4.2**) A β-tabloid is an equivalence class of tableaux under permutations within rows. (Abusing notation somewhat) One writes $\{t\}$ for the equivalence class of t. Thus for example

$$\{\begin{smallmatrix}1&2\\3\end{smallmatrix}\} = \{\begin{smallmatrix}1&2\\3\end{smallmatrix}, \begin{smallmatrix}2&1\\3\end{smallmatrix}\}$$

Note that $\pi\{t\} = \{\pi t\}$ is well-defined.

There is a sequence notation for tabloids, in which one writes $s(t)_i = j$ if the number i appears in row j in a tableau $u \in \{t\}$. (Note that this does not depend on the choice of class representative u.) Thus for example

$$s(t) = 112$$

for the tabloid above.

(**11.4.3**) Let F be an arbitrary field. Define S_n module

$$M^\beta = F\{\{t\}\}_{t \in \beta\text{-tableau}}$$

Note that $M^\beta = FS_n\{t\}$ for any suitable t. Examples: $M^{(2,1)}$ has basis

$$\{\{\begin{smallmatrix}1&2\\3\end{smallmatrix}\}, \{\begin{smallmatrix}1&3\\2\end{smallmatrix}\}, \{\begin{smallmatrix}2&3\\1\end{smallmatrix}\}\}$$

$M^{(1^3)}$ has basis

$$\left\{\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\}, \{\begin{smallmatrix}1\\3\\2\end{smallmatrix}\}, \{\begin{smallmatrix}2\\1\\3\end{smallmatrix}\}, \{\begin{smallmatrix}2\\3\\1\end{smallmatrix}\}, \{\begin{smallmatrix}3\\1\\2\end{smallmatrix}\}, \{\begin{smallmatrix}3\\2\\1\end{smallmatrix}\}\right\}$$

Note that the different treatment of rows and colums is a source of non-canonicalness. This is unavoidable.

Let $V(t) \in \mathbb{Z}S_n$ be the unnormalised column antisymmetriser associated to tableau t. Examples:

$$V(\begin{smallmatrix}1&2&5\\4&3\end{smallmatrix}) = (1-(14))(1-(23))$$

and

$$V(\begin{smallmatrix}1\\2\\3\end{smallmatrix}) = 1 - (12) - (23) - (13) + (123) + (321)$$

Note that for all $\pi \in S_n$

$$\pi V(t) = V(\pi t)\pi.$$

Then

$$e_t := V(t)\{t\}$$

is a β-polytabloid in M^β.
Examples:

$$e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} = (1-(13))\{\begin{smallmatrix}1&2\\3\end{smallmatrix}\} = \{\begin{smallmatrix}1&2\\3\end{smallmatrix}\} - \{\begin{smallmatrix}2&3\\1\end{smallmatrix}\}$$

$$e_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} = (1-(12))\{\begin{smallmatrix}1&3\\2\end{smallmatrix}\} = \{\begin{smallmatrix}1&3\\2\end{smallmatrix}\} - \{\begin{smallmatrix}2&3\\1\end{smallmatrix}\}$$

$$e_{\begin{smallmatrix}2&3\\1\end{smallmatrix}} = (1-(12))\{\begin{smallmatrix}2&3\\1\end{smallmatrix}\} = \{\begin{smallmatrix}2&3\\1\end{smallmatrix}\} - \{\begin{smallmatrix}1&3\\2\end{smallmatrix}\}$$

de:S_n specht

(**11.4.4**) The Specht module S^β is the subspace of M^β spanned by β-polytabloids:

$$\pi e_t = \pi V(t)\{t\} = V(\pi t)\pi\{t\} = V(\pi t)\{\pi t\} = e_{\pi t}.$$

Cf. §11.2.4.

11.4.1 Gram matrices

de:S_n Gram

(**11.4.5**) Define a bilinear form (as in §10.1.5) on the FS_N-module M^β by

$$\Phi(\{t\},\{t'\}) = \begin{cases} 1 & \{t\} = \{t'\} \\ 0 & o/w \end{cases}$$

This form is a contravariant form with respect to the $g \mapsto g^{-1}$ antiautomorphism (since $g\{t\} = \{t'\}$ iff $g^{-1}\{t'\} = \{t\}$ — see e.g. §10.1.7).

(**11.4.6**) Define $Gram(\beta)$ as the matrix with entries $\Phi(e_t, e_{t'})$, with t, t' varying over standard β-tabloids in the lexicographic total order.
Examples: $Gram((1^3)) = (6)$; $Gram((3)) = (1)$;

$$Gram((2,1)) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(**11.4.7**) Define $S^{\beta\perp}$ as the submodule of M^β spanned by elements orthogonal to β-polytabloids.

oonee

Theorem 11.1. *Let F be any field. Suppose $A \hookrightarrow M^\beta$ is an inclusion of FS_n-modules. Then either $S^\beta \hookrightarrow A$ or $A \hookrightarrow S^{\beta\perp}$.*
Proof: Suppose $a \in A$, and t a β-tabloid. Then it is easy to see that $V(t)a \in Fe_t$. Thus if $V(t)a \neq 0$ then $e_t \in A$, so $S^\beta = FS_n e_t \hookrightarrow A$. If $V(t)a = 0$ then

$$0 = \Phi(V(t)a, \{t\}) = \Phi(a, V(t)\{t\}) = \Phi(a, e)$$

so $A \hookrightarrow S^{\beta\perp}$. □

Theorem 11.2. *Let F be any field. Then $S^\beta / S^\beta \cap S^{\beta\perp}$ is either an absolutely irreducible FS_n-module (i.e. remains irreducible under any field extension) or zero.*
Proof: If A is a submodule of S^β (and hence of M^β) then by Theorem 11.1 it is either S^β or else in $S^\beta \cap S^{\beta\perp}$. Thus $S^\beta / S^\beta \cap S^{\beta\perp}$ has no submodule, so it is irreducible. Absolute irreducibility follows from a consideration of the rank of the Gram matrix over the prime subfield. □

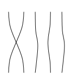
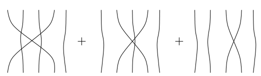
11.4.2 Murphy elements

For $(i,j) \in S_n$ the pair permutation define Murphy elements of S_n:

$$\mathcal{M}_m = \sum_{i=1}^{m-1} (i,m)$$

ss:Murphy1

Example: In $\mathbb{Z}S_5$ we have

$\mathcal{M}_2 =$  , $\mathcal{M}_4 =$ 

Since $(ij)(im)(ij) = (jm)$ we have $(ij)\mathcal{M}_m(ij) = \mathcal{M}_m$ whenever $i,j < m$ or $i,j > m$. Thus

$$[\mathcal{M}_m, S_{m-1}] = 0$$

and so for all m, m':

$$[\mathcal{M}_m, \mathcal{M}_{m'}] = 0$$

That is, the Murphy elements of $\mathbb{Z}S_n$ form a commutative subalgebra. Also

$$\sigma_m \mathcal{M}_m \sigma_m \;=\; \mathcal{M}_{m+1} - \sigma_m \qquad \text{so} \qquad \sigma_m \mathcal{M}_{m+1} \sigma_m \;=\; \mathcal{M}_m + \sigma_m$$

so

$$\sigma_m(\mathcal{M}_m + \mathcal{M}_{m+1})\sigma_m \;=\; \mathcal{M}_m + \mathcal{M}_{m+1}$$

pa:Murphy1SNF

From this one can show that symmetric polynomials in $\mathcal{M}_2, ..., \mathcal{M}_n$ are central in $\mathbb{Z}S_n$.
Murphy [126] (see also, e.g. Green–Diaconis [38]) computed the representation matrices for these elements in Young's seminormal form (see §11.5) of each Specht module $\mathcal{S}_\lambda$:

$$\rho_\lambda(\mathcal{M}_i) = \; diag(c_1(i), c_2(i), ...)$$

where $c_l(i)$ is the content of the box containing i in the l-th standard Young tableau of shape λ (in some chosen total order of tableau, which will not be important to us).
Since $\sum_i \mathcal{M}_i$ is central it acts like a scalar, so we only need the first diagonal entry to determine this scalar. This is, then, the sum of the contents of all the boxes in the 'first' (or indeed any) standard tableau of shape λ. That is

$$\rho_\lambda(\sum_i \mathcal{M}_i) = \; (\sum_{b \in \lambda} c(b)) I$$

where the sum is simply over the contents of boxes of λ; and I is the identity matrix.
For (a rather trivial) example, $\rho_{(2)}(\sum_i \mathcal{M}_i) = \rho_{(2)}(\mathcal{M}_2) = 1$ while $\rho_{(1^2)}(\sum_i \mathcal{M}_i) = -1$. We deduce that these representations are in different blocks unless $1 = -1$ in the ground field k. (Cf. the Nakayama conjecture (11.3.2).)
We return to these elements in §19.1.1.

11.5 Young forms for S_n irreducible representations

ss:SNF

Young obtained explicit constructions of the irreducible representations of S_n over $\mathbb{C}$ satisfying various specific properties as matrices (of importance in physical applications, for example). See e.g. Boerner [12].

(**11.5.1**) Young's Orthogonal form, normal form and SNF all require the notion of hook length.

11.5.1 Hooks, diamond pairs and the Young Forms for S_n

ss:diamond1

de:diamondpair

(**11.5.2**) Fix a digraph G that is a multiplicity-free Bratteli diagram for a tower of algebras starting from the ground field (hence a single root vertex in G). It follows immediately that the set of walks from the root to $\lambda \in G$ is a basis for the corresponding simple module.
Following Martin[103, §7.4], a pair of partial walks of length 2 on G is called a *diamond pair* if they start at the same label and finish at the same label. That is, supposing we start in layer $m-2$ (some m), we may write

$$S \;=\; (s(m-2),\; s(m-1),\; s(m)),$$

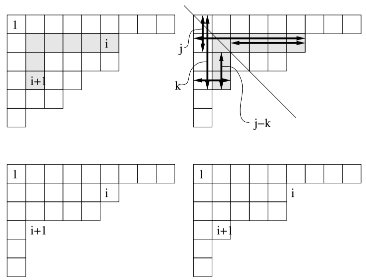


Figure 11.5: Schematic for (a) a tableau t and a hook length; and, (b,c,d) associated diagrams. fig:hookcontent1

and $T = (s(m-2), t(m-1), s(m))$. We say that an algebra element u acts *locally* on the basis of walks if it mixes between two walks only if they differ at a single vertex. That is, they agree except for forming a single diamond pair. (We say such a pair of full walks form a diamond pair at the vertex in question.)

(**11.5.3**) The Bratteli diagram for the tower of symmetric group algebras over $\mathbb{C}$ is the directed Young graph. Walks on this are encoded by standard tableaux.

The matrix for a representation of σ_i in Young's SNF, or OF, (see e.g. Boerner [12]) mixes between two tableaux only if they differ, *as walks*, at a single point — i.e. in two steps. (Thus each tableau is mixed by σ_i with at most one other.)

(**11.5.4**) The mixing is determined by the *tableau hook distance* between i and $i+1$ in the first tableau t say (or equivalently the second, u), denoted $h_t(i,i+1)$. See Fig.11.5.

The *tableau hook length* between i and $i+1$ (shaded in Fig.) is

$$h_t(i,i+1) = (\nu_j - j) - (\nu_k - k) \;=\; c_t(i) - c_t(i+1)$$

where $c_t(i)$ means the content of the box determined by the position of symbol i in tableau t.

(**11.5.5**) The *box hook length* $h(i,j) = h_\lambda(i,j)$ for a box (i,j) in a partition λ is the length of the hook in λ with axis box (i,j).

Tableau hook length is not to be confused with this.

(**11.5.6**) Recall (see e.g. Hamermesh [60], or Yamanouchi [147]) that the S_n orthogonal action on a standard tableau t as above is then:

$$\sigma_i t \;=\; \frac{1}{h_t(i,i+1)} t \;+\; \sqrt{\frac{h_t(i,i+1)^2 - 1}{h_t(i,i+1)^2}} \sigma_i(t) \qquad (11.7)$$ ortho1

where $\sigma_i(t)$ denotes the (not necessarily standard) image of t under swapping i and $i+1$.

Note that if $\sigma_i(t)$ nonstandard then $h_t(i,i+1) \in \{1,-1\}$ so nonstandard tableau can be ignored.

11.5.2 Asides on geometry

(**11.5.7**) Remark. It is useful to think about Young forms geometrically. To this end, one can extract a certain foursome of Young diagrams from the t,u tableaux mixing setup as follows:
(1) the diagram containing $1, 2., , , .i-1$ in t;
(2_t) the diagram containing $1, 2., , , .i$ in t;
(2_u) the diagram containing $1, 2., , , .i$ in u;
(3) the diagram containing $1, 2., , , .i+1$ in t.

The figure on the top left in Figure 11.5 illustrates a tableau t, of shape λ say. The figure on the top right is the diagram $\nu \subset \lambda$ containing $1, 2., , , .i+1$. The diagrams in the second row in the figure are cases 2_t and 2_u respectively.

(**11.5.8**) Chat: Consider whether 2_t and 2_u are taken in to each other by reflection in a suitable affine wall? What does this mean?
(1) We must embed the diagrams in a Euclidean space;
(2) We must associate a reflection group to this space.

There are at least two ways of embedding diagrams in $\mathbb{R}^{\mathbb{N}}$, since we can embed diagrams or their transposes, by regarding either as an integral vector in the obvious way. It is not obvious how to *combine* these! Reflection usually involves row permutation by convention, since the vector associated to rows seems marginally more natural than that associated to columns. Thus any row perm take a partition to a composition at worst. Column perms do not preserve row compositions in general, so they do not play together well with row actions. Perhaps the structure preserved by both is that subsets of the boxes in the 'matrix' quadrant are take into each other.

Note however that *some* column operations on *some* diagrams *do* preserve compositions, partitions even,... Consider our example above.

Also in the S_n case we are focussing on certain kinds of diagram 'progressions' — forward walks on the Young graph. If we do not require this, then maybe more freedom arises... ...indeeed we can try to relate to the direct product rep construction, or variants thereof (semidirect products say).

11.6 Outer product and related representations of S_n

ss:outer1

See e.g. Robinson [36], Hamermesh [60], Hoefsmit [65], Martin–Woodcock–Levy [121]. There are several aspects to this topic. Later we look at the classical construction (modules induced from representations of Young and other special subgroups), but there is also a geometric aspect related to affine Hecke algebras and also to the invariant theory, and the theory of highest weight and Verma modules, of sl_n. (This leads to connections with Yang–Baxter equations and elsewhere.)

11.6.1 Multipartitions and their tableaux

ss:multipart

We start with some combinatorial preliminaries.

Recall from §5.7 that Λ is the set of integer partitions. For $d \in \mathbb{N}$ define the set of *d-component multipartitions*

$$\Gamma^d = \Lambda^{\times d}$$

and Γ^d_n as the subset of multipartitions of total degree n (we take this notation from Martin–Woodcock–Levy [121]).

We write

$$\mu = (\mu^1, \mu^2, ..., \mu^d) \; \in \; \Gamma_n^d$$

The *shape* of μ is the composition of n whose i-th component is $|\mu^i|$.

(**11.6.1**) The natural inclusion of $\Gamma^d \hookrightarrow \Gamma^{d+1}$ is given by appending an empty component. We define Γ^{fin} as the inverse limit of these inclusions.

(**11.6.2**) We define a digraph $\mathcal{Y}_*$ on vertex set Γ^{fin} such that there is an edge from μ to ν if, regarded as multi(-Young)diagrams, they differ by the addition of a single box.

Note that this digraph is rooted, simple, loop-free and non-tree. However there are infinitely many edges out of every vertex, since there are infinitely many component partitions in the multitude (most of them the empty partition) to which a box may be added.

(**11.6.3**) A *tableau* of shape $\mu \in \Gamma_n^d$ is an arrangement of the symbols $1, 2, ..., n$ in the n boxes of μ regarded as a multi-Young diagram. As a *set* we may regard μ as a certain set of n boxes. Thus the set of tableaux is $\mathbf{Set}^{\sim}(\mu, \underline{n})$ (here $\sim$ means to take the subset of bijective maps).

(**11.6.4**) A tableau of shape μ is *standard* if each component tableau μ^i is standard. We write T^μ for the set of standard tableau of shape μ. Assuming a given total order on this set, we write

$$T^\mu = \{T_1^\mu, T_2^\mu, ..., T_t^\mu\}$$

A tableau in T^μ determines a walk on the digraph $\mathcal{Y}_*$ from $\emptyset$ to μ.

11.6.2 Actions of S_n on tableaux

de:nat act

(**11.6.5**) We define a 'natural' (or 'regular' or 'Cayley') action of S_n on tableaux of shape $\mu \vdash n$ so that $\sigma_i(T_p^\mu)$ (and indeed $\sigma_i(T)$ for any tableau $T \in \mathbf{Set}^{\sim}(\mu, \underline{n})$) is obtained by interchanging symbols i and $i+1$ in T_p^μ.

Note that this action does not necessarily take standard to standard, depending on μ (NB, in the example $\mu = ((1), (1), ..., (1))$ all tableaux are standard). Over all (not necessarily standard) tableau, however, this action evidently makes $\mathbf{Set}^{\sim}(\mu, \underline{n})$ a basis for the regular module for any μ.

(**11.6.6**) Note that this can be seen as a Cayley representation (i.e. a (left) regular permutation representation). Consider the reflection group action $\sigma_i \mapsto (i\; i{+}1)$ of S_n on $\mathbb{R}^n$, with reflection hyperplanes H_{ij} (as in §**??**), associated chambers of $\mathbb{R}^n \setminus \cup_{ij} H_{ij}$, 'dominant' chamber containing $(1, 2, ..., n)$, and so on. Recall S_n acts simply transitively on the set of chambers (which may be indexed by the perms of $(1, 2, ..., n)$). We associate $1 \in S_n$ to $(1, 2, ..., n)$ and $w \in S_n$ to $w(1, 2, ..., n)$. (The 'left' action.) Taking $T_1^\mu = (1, 2, ..., n)$ (in the obvious notation for tableau of this shape) we get the required correspondence. Note that the reflection action of S_n also closes on the 'integral box' $\underline{n}^{\times n} \subset \mathbb{R}^n$ containing all our chamber representative points and some 'singular' points. Here the action is isomorphic (indeed identical) to the 'tensor space' action as for example on $\mathbf{Set}(3, 3)$ in §**??**.

We want to describe various other actions of S_n on $\mathbf{Set}^{\sim}(\mu, \underline{n})$ to $\mathbb{C}\mathbf{Set}^{\sim}(\mu, \underline{n})$, some of which will make certain submodules *manifest*.

(**11.6.7**) For the moment suppose that $\mu = ((1), (1), ..., (1))$ (that is, an n-tuple of single boxes), and let $p(i)$ denote the position in the tuple in which i appears in T_p^μ. Let x be an n-tuple of

complex numbers. Define $\sigma_i T_p^\mu$ by

$$\sigma_i \begin{pmatrix} T_p^\mu \\ \sigma_i(T_p^\mu) \end{pmatrix} = \frac{1}{h} \begin{pmatrix} -1 & -(h-1) \\ -(h+1) & 1 \end{pmatrix} \begin{pmatrix} T_p^\mu \\ \sigma_i(T_p^\mu) \end{pmatrix} \qquad (11.8)$$

eq:YBx1

where $h = x_{p(i)} - x_{p(i+1)}$.

In this way σ_i acts by a linear transformation on the vector space with tableaux basis, and hence as a particular matrix on the tableau basis. Let us write R^x for this map from Coxeter generators to matrices (for x indeterminate, and otherwise when x such that this map is well-defined per se).

(**11.6.8**) Note that x may be chosen so that all h are large magnitude (complex). In the large magnitude limit, then action σ_i- coincides with the regular (as in Cayley, as in simply transitive) representation $\sigma_i(-)$.

pa:rep x

(**11.6.9**) We claim that R^x extends to a representation. To verify this we need to show:
(1) $\sigma_i\sigma_i - 1 \overset{R^x}{=} 0$:

This is clear since every matrix $R(\sigma_i)$ falls into 2x2 blocks as above, each of which is traceless and has det$= -1$.
(2) $\sigma_i\sigma_{i+1}\sigma_i - \sigma_{i+1}\sigma_i\sigma_{i+1} \overset{R^x}{=} 0$:

The action of $\{\sigma_i, \sigma_{i+1}\}$ evidently falls into blocks of tableaux of size six. The identity can therefore be checked by looking at σ_i and σ_{i+1} on a typical block of tableaux. Here $h = x_{p(i)} - x_{p(i+1)}$, $h_2 = x_{p(i)} - x_{p(i+2)}$ and $h_1 = x_{p(i+1)} - x_{p(i+2)}$:

$$\sigma_i \begin{pmatrix} T_p^\mu \\ \sigma_i(T_p^\mu) \\ \sigma_{i+1}(T_p^\mu) \\ \sigma_i(\sigma_{i+1}(T_p^\mu)) \\ \sigma_{i+1}(\sigma_i(T_p^\mu)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T_p^\mu))) \end{pmatrix} = \begin{pmatrix} \frac{-1}{h} & \frac{1-h}{h} & 0 & & & \\ \frac{-h-1}{h} & \frac{1}{h} & 0 & & & \\ 0 & 0 & \frac{-1}{h_2} & \frac{1-h_2}{h_2} & 0 & 0 \\ 0 & 0 & \frac{-h_2-1}{h_2} & \frac{1}{h_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{h_1} & \frac{1-h_1}{h_1} \\ 0 & 0 & 0 & 0 & \frac{-h_1-1}{h_1} & \frac{1}{h_1} \end{pmatrix} \begin{pmatrix} T_p^\mu \\ \sigma_i(T_p^\mu) \\ \sigma_{i+1}(T_p^\mu) \\ \sigma_i(\sigma_{i+1}(T_p^\mu)) \\ \sigma_{i+1}(\sigma_i(T_p^\mu)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T_p^\mu))) \end{pmatrix}$$

On the same part of the basis we have:

$$R_{part}(\sigma_{i+1}) = \begin{pmatrix} \frac{-1}{h_1} & 0 & \frac{1-h_1}{h_1} & 0 & & \\ 0 & \frac{-1}{h_2} & 0 & 0 & \frac{1-h_2}{h_2} & 0 \\ \frac{-h_1-1}{h_1} & 0 & \frac{1}{h_1} & 0 & & \\ 0 & 0 & 0 & \frac{-1}{h} & 0 & \frac{1-h}{h} \\ 0 & \frac{-h_2-1}{h_2} & 0 & 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & 0 & \frac{-h-1}{h} & 0 & \frac{1}{h} \end{pmatrix}$$

One can then verify the relation by brute force[1].

(3) distant commutation...

(**11.6.10**) It follows from (11.6.9) that R^x is a representation of S_n over $\mathbb{C}$ for almost every $x \in \mathbb{C}^n$. Thus there are a continuum of such representations. Thus these representations are all isomorphic.

[1]For example, *Maxima* [**?**] the open source algebraic computation package does this. See my Maxima file "defns.mac".

We next investigate special cases of x (and their relationship to other choices of μ) that illuminate important features of this class of representations.

(**11.6.11**) Consider now the case $x = (3, 2, 1)$. We have $h = h_1 = 1$ and $h_2 = 2$.
Note that the tableau $T_p^\mu = (1, 2, 3) = 123$ (in the obvious notation) spans a submodule.

(**11.6.12**) If we set $h = x_1 - x_2 = -1$ and $h_2 = x_1 - x_3 = 1$ then $h_1 = x_2 - x_3 = 2$.

11.6.3 Generalised hook lengths and geometry

(**11.6.13**) Hook length: ...

(**11.6.14**) We define $h_{ij}(T_p^\mu)$ as the hook length between the boxes containing i and j in T_p^μ, with $h_{ij}(T_p^\mu) = \infty$ if i, j in different parts. We define $h_{ij}^0(T_p^\mu)$ as the hook length between the boxes containing i and j in T_p^μ, with all Young diagrams overlaying so that the $(1,1)$-box is in the same position.

For x a d-tuple of complex numbers we define

$$h_{ij}^x(T_p^\mu) \;:=\; h_{ij}^0 + x_{\#i} - x_{\#j}$$

where $\#i$ is the position in the tuple corresponding to the Young diagram containing i in T_p^μ.

11.6.4 Connections to Lie theory and Yang–Baxter

To Do!

11.7 Outer products continued — classical cases

(**11.7.1**) Let G, G' be groups and $R_G, R_{G'}$ representations. Then the Kronecker product $R_G \otimes R_{G'}$ is a representation of $G \otimes G'$.

11.7.1 Outer products over Young subgroups

(**11.7.2**) If $\lambda \vdash n$ then

$$S_\lambda := \otimes_i S_{\lambda_i}$$

is a subgroup of S_n (each factor acts nontrivially on a corresponding subset of $\underline{n}$).

Let $\mu \in \Gamma_n^d$ of shape λ. Fix a representation R_ν for each Specht module $\Delta(\nu)$. Then we have a representation $\otimes_i R_{\mu^i}$ of S_λ. Fix a commutative ring K and write M'_μ for the corresponding left KS_λ-module. We define a left KS_n-module by

$$M_\mu \;:=\; KS_n \otimes_{KS_\lambda} M'_\mu$$

This M_μ is called an *outer product* representation.

Our next task is to construct an explicit basis and action for each M_μ.

11.7.2 Outer products over wreath subgroups

Whenever a partition occurs more than once in an outer product representation there is an automorphism acting permuting the identical factors. We can use this to decompose the representation.

11.7.3 The Leduc–Ram–Wenzl representations

As kS_n-modules the Leduc–Ram modules are particular examples of mixtures of the previous two types of modules. (They are of interest via their extensions to Brauer algebra representations, but we shall start by considering them as S_n representations.)

For each n and m such that $n = 2m + k$, and $\lambda \vdash k$ we have the subgroup $S_2 \wr S_m \subset S_{2m} \subset S_n$.

11.8 Finite group generalities

11.8.1 Characters

Let G be a finite group and let $\{\rho_i\}_{i=1,2,...,r}$ be a (complete) set of irreducible representations over $\mathbb{C}$, with dimensions $d_1, d_2, ..., d_r$. Then for each pair i, j and $d_i \times d_j$ matrix C define

$$P_C = \sum_{g \in G} \rho_i(g) C \rho_j(g^{-1})$$

If $P_C \neq 0$ then it intertwines ρ_i and ρ_j,

$$\rho_i(g) P_C = P_C \rho_j(g)$$

Thus by Schur's Lemma $P_C = 0$ unless $i = j$, in which case it is a (possibly zero) scalar multiple of the identity matrix.

In the former case, choosing any pair h, h' and puting $C = e_{hh'}$ (the elementary matrix) we have

$$\sum_g (\rho_i(g))_{kh} (\rho_j(g^{-1}))_{h'k'} = 0$$

for any k, k'. Puting $k = h$ and $k' = h'$ and summing over k and k' we get

$$\sum_g \sum_k \sum_{k'} (\rho_i(g))_{kk} (\rho_j(g^{-1}))_{k'k'} = 0$$

That is, with character $\chi_i(g) = \sum_k (\rho_i(g))_{kk}$,

$$\sum_g \chi_i(g) \chi_j(g^{-1}) = 0$$

Similarly

$$\sum_g \chi_i(g) \chi_i(g^{-1}) = \; |G|$$

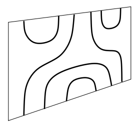
Puting $k = k'$ and summing we get

$$\sum_g \sum_k (\rho_i(g))_{kh} (\rho_j(g^{-1})_{h'k} = 0$$

Chapter 12

The Temperley–Lieb algebra and related problems

Ch:TL



Fix a commutative ring k, and choose an element $\delta \in k$. Then for each $n \in \mathbb{N}_0$ the Temperley–Lieb algebra $T_n(\delta)$ is a certain finite rank k-algebra. There is in particular a notable basis consisting of certain 'diagrams' (the subset of non-crossing diagrams among the 'Brauer diagrams' [**?**] – diagrams representing set partitions of $2n$ vertices into pairs, see later). We already had a first encounter with them in Chapter 2.

Temperley–Lieb algebras appear in several significantly different settings. These settings lead to various distinct applications, and also inform Temperley–Lieb representation theory in different ways. In §12.3 we look at the Hecke algebra context — where Temperley–Lieb algebras are defined by a presentation. In §12.11 we look at the 'diagram algebra' context. In ...

One objective is to describe the structures of these algebras over the complex field (hopefully complementing the method of [103, §7.3]; and see also our §2.1 *et seq*, and in particular §2.6). It is convenient to do this by bringing together features from different realisations. For example, it is relatively easy to determine the dimension of the diagram algebra; and to construct a surjective map from the Hecke quotient to the diagram algebra. ... We continue with representation theory in Ch.13.

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