Relations

Fundamental Idea The basic idea of a relation is some rule for associating objects of one set with objects of another

Example 1 If we associate the people in a room with the numbers from 0 to 25 by asking "how old are you?", we have set up a relation. This produces a set of pairs:
{(Samina,19),(Richard,19),(Maureen,20), ...} .
Clearly not everyone in the room will necessarily be associated with a number (especially if I'm in the room — you would need much larger numbers). Conversely, not every number will necessarily be used in the association.

Example 2 A real number x is related to a real number y by the equation $y = x^2$ Again this produces a set of pairs, or if you plot a graph, a set of points, each defined by a pair of numbers (x, y) such that $y = x^2$

Example 3 Given the set $A = \{0, 1, 2, 3, 4, 5, 6\}$ we form the partition

$$A_0 = \{0, 1\} \quad A_1 = \{2, 3, 4\} \quad A_2 = \{5\} \quad A_3 = \{6\}$$

We define a relation **on** A by saying that one number is related to the other if they are both in the same partition set. ie x is related to y if x and y are in A_i for some i. Again we form pairs, namely $\{(0, 0), (0, 1), (1, 1), (2, 2), (2, 3), \ldots\}$

Definition A **relation** from a set A to a set B is any subset of $A \times B$, where $A \times B$ is the set of all pairs (a, b) with $a \in A$ and $b \in B$.

Denoting this subset by ρ, if $(a, b) \in \rho$ then we say that a is related to b and write $a\rho b$.
An alternative notation for ρ is $\sim$

Definition If $A = B$ then we say the relation is **on** the set A

In example 3 the relation is **on** A and is a subset of $A \times A$ containing only 15 of the possible 49 pairs. These are given by:
$\rho = \{(0, 0), (0, 1), (1, 1), (1, 0), (2, 2), (3, 3), (4, 4),$
$(2, 3), (3, 2), (2, 4)(4, 2), (3, 2), (4, 3), (5, 5), (6, 6)\}$

In example 2 the relation is on $\mathbf{R}$ and is defined by $\rho = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = x^2\}$

Relations and Functions A relation is a more general idea than a function. ie a function is an example of a relation but not all relations are functions. Recall that a function relates **each** of the elements x of one set to a unique element y of another set. In example 2, for each element $x \in \mathbf{R}$ there exists a unique element y, also in $\mathbf{R}$, given by $y = x^2$.

Example 1 and Example 3 are not functions - why?

Example 4 Consider the following relation **on R**

$$\rho = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x = y^2\}$$

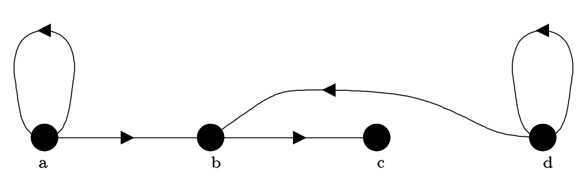
This example fails to be a function on two counts:

i) clearly it is not possible to use all values of x, indeed x must be greater or equal to zero.

ii) for each x there is not a unique y. If $x = 4$ then y could equal $+2$ or -2.

Relations on a Set

We now focus our attention to relations defined **on** a set. Such relations are closely linked with sorting and ordering. Consider the following graph consisting of four points labelled a,b,c and d.



In this example the set $A = \{a, b, c, d\}$ and we say that two elements,x and y, are related if there is a directed path from x to y. Thus the relation ρ is defined by the following subset of $A \times A$
$\rho = \{(a, a), (a, b), (b, c), (d, b), (d, d)\}$.

In this example we make the following observations:

- although $a\rho a$ and $d\rho d$ it is not true that $c\rho c$ and $b\rho b$ since the pairs (c, c) and (b, b) are not in the list. ie not all elements are related to themselves.

- Since the routes are only oneway even though $a\rho b$ it does not follow that $b\rho a$ ie (a, b) is in the list but (b, a) is not.

- we also note that $d\rho b$ and $b\rho c$, however it is not true that $d\rho c$

In Example 3 we note that, in contrast to the above, the following properties do hold:

- **Reflexive**: $x\rho x$ for all x in A.
  This is a simple consequence of each element being in the same subset as itself.
  The list contains
  $(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ and $(6, 6)$

- **Symmetric**: **If** $x\rho y$ **then** $y\rho x$
  ie if x is related to y then y is related to x This is a simple consequence of if x is in the same subset as y then clearly y is in the same subset as x.

- **Transitive**: **If** $x\rho y$ and $y\rho z$ then $x\rho z$
  This is not so easy to see and is often the most difficult to prove. In this example it is simply saying that it x and y are in the same subset and y and z are in the same subset then x and z must be in the same subset.

Definition A relation **on** a set A that is reflexive, symmetric and transitive is said to be an **Equivalence** relation on A.

Example 5 The relation in Example 3 was an equivalence relation generated by partitioning the set. We now consider the reverse of this problem and see how an equivalence relation generates a partition of the set.

Let $A = \{0, 1, 2, 3, 4, \ldots\}$ and define the relation $\sim$ on A by:

$$x \sim y \text{ if } x - y \text{ is divisible by } 3$$

ie there exists an integer k such that $x - y = 3k$

In terms of subsets:

$$\sim = \{(x, y) \in A \times A : x - y \text{ divisible by } 3\}$$

We now prove that $\sim$ is reflexive, symmetric and transitive.

R. $x \sim x$ for all x since $x - x = 3k$ with $k = 0$

S. if $x \sim y$ then there exists integer k such that $x - y = 3k$
This implies that $y - x = 3(-k)$
As $-k$ is also an integer we have that $y \sim x$

T. if $x \sim y$ and $y \sim z$ then there exists integers k_1 and k_2 such that

$$x - y = 3k_1 \text{ and } y - z = 3k_2$$

Adding gives $x - z = 3(k_1 + k_2)$
As $k_1 + k_2$ is an integer, k (say) we have
$x - z = 3k$ hence $x \sim z$.

Consider now the sets of elements of A related to 0, 1 and 2, denote these as $[|0|]$, $[|1|]$ and $[|2|]$ respectively Thus

$$[|0|] = \{0, 3, 6, 9 \ldots\}$$
$$[|1|] = \{1, 4, 7, 10, \ldots\}$$
$$[|2|] = \{2, 5, 8, 11, \ldots\}$$

These sets form a partition of the set A. Thus the equivalence relation has split the set A into partitions.

Theorem A partition $\{A_i\}$ of a set A defines a natural equivalence relation $\sim$ defined by:
$x \sim y$ if and only if there exists i such that $x \in A_i$ and $y \in A_i$.

Conversely, given an equivalence relation $\sim$ on A the set A is naturally partitioned. The sets forming the partition are called the **equivalence classes** of $\sim$

In Example 5 the equivalence classes are the three sets $[|0|], [|1|]$ and $[|2|]$.

Example 6 We now consider an example where the equivalence classes turn out to be concentric circles centre the origin in the Argand plane.

Define the relation $\sim$ on the set of complex number $\mathbf{C}$ by
$z_1 \sim z_2$ if there exists a real θ such that

$$z_1 = (\cos\theta + i\sin\theta)z_2$$

Show that $\sim$ is an equivalence relation on $\mathbf{C}$.

R. with $\theta = 0$ we see that $z \sim z$ for all z

S. if $z_1 \sim z_2$ then $z_1 = (\cos\theta + i\sin\theta)z_2$
Multiplying by $\cos\theta - i\sin\theta$ gives

$$z_1(\cos\theta - i\sin\theta) = z_2$$
$$z_2 = (\cos(-\theta) + i\sin(-\theta))z_1$$

Thus $z_2 \sim z_1$

T. If $z_1 \sim z_2$ and $z_2 \sim z_3$ then there exists real θ_1 and θ_2 such that

$$z_1 = (\cos\theta_1 + i\sin\theta_1)z_2$$
$$z_2 = (\cos\theta_2 + i\sin\theta_2)z_3$$

Combining these we can obtain:

$$z_1 = (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))z_3$$

Thus $z_1 \sim z_3$.

From the definition of $\sim$ we can see that if $z_1 \sim z_2$ then $|z_1| = |z_2|$. Thus z_1 and z_2 lie on the same circle centre the origin. Indeed all the elements of $\mathbf{C}$ that are related to z_1 lie on the same circle as z_1, thus the equivalence classes are these circles.