Calculus

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Chapter 1

Introduction

We assume that you have attended a first year calculus course. This would have covered topics such as Limits, Continuity and Differentiability for functions of one variable. Just in case you are feeling rusty, here is a brief review.

1.1 Review of Differentiation

We will need to recall the notions of Function, Limit, and Differentiation.

1.1.1 Functions

There are many kinds of function, and a formal mathematical definition can be given which encompasses them all. You may know it. For our purposes it will be sufficient to start by thinking about *real functions of real variables.*

A real function is something which takes as input a real number, and gives as output another real number.

There are all sorts of special cases. None of the special cases we will need are any cause for alarm!

Example 1.1.

$$f(x) = x^2$$

Example 1.2.

$$f(x) = \sin(x)$$

1.1.2 Limits

Some functions $f(x)$ *break down* at certain values of x.

Example 1.3. Consider

$$f(x) = x^{-1}$$

at $x = 0$!

Other functions have similar looking problems at certain values of x, but can usefully be studied by using the notion of limit.

A function $f(x)$ has a limit l at $x = c$ (c some given constant), if we can make the value of $f(x) - l$ arbitrarily small, by choosing x suitably close to c.

We write

$$\lim_{x \to c} f(x) = l$$

if such a limit exists.

The function $f(x) = x^{-1}$ has no limit at $x = 0$. The closer we get to $x = 0$, the bigger it gets, without bound! We say $f(x)$ is *divergent* at $x = 0$.

However,

Example 1.4. Consider

$$f(x) = x^{-1} \sin(x)$$

at $x = 0$. Again there is a problem, caused by the x^{-1} part. We can see that the $\sin(x)$ part, which gets small close to $x = 0$, *might* restrain this divergence, but we can't tell just by staring at it! In fact, by carefully calculating the value of this function for values of x close to (but not at) 0 we can see that

$$\lim_{x \to 0} f(x) = 1$$

1.1.3 Differentiation

Not every function can be differentiated. If a function can be differentiated (at x), the differential is given by

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Note that this is exactly what we mean by the *tangent* to the curve of the graph of $f(x)$ at x.

Example 1.5. For

$$f(x) = x^2$$

we get

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh}{h} + \lim_{h\to 0} \frac{h^2}{h} = 2x$$

(I hope this is what you were expecting!)

1.2 Other things we expect you to know!

de Moivre's theorem etc.:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

so, for example,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \tag{1.1}$$

1.3 Other support material

Check out the course homepage.
Check out the vast amount of related material on the web.
Check out past exam papers and solutions.
Check out books called 'CALCULUS' (or similar); such as the SCHAUM OUTLINE SERIES book on this topic, and other books mentioned on the course homepage.
Check out 'Mathematical Methods for Science Students' by G Stephenson (Longman). [1]
Check out the DERIVE mathematical software package.

[1] Some of the examples in the present draft of this work are taken from my own undergraduate Calculus notes (from a course given by the legendary Dr Ian Ketley). This means that they are inspired by, or directly taken from, examples and exercises to be found in Stephenson's excellent book. As a result, the present version of this work is possibly suitable *only* for its present educational purpose, and *not* for publication or any other form of redistribution. Buy Stephenson's book.

Chapter 2

Functions of more than one variable

2.1 Introduction

If we collate the data from enough actuarial tables (say), we can construct a function which, when we input the age of a person in years, returns the probability that they will have a serious illness within the next twelve months.

This is, potentially, a very useful function to help assess risk in health insurance.

Of course it is a very crude model of the health profile of any given candidate for insurance. There are a number of other factors which directly affect the probability of illness, besides age. For example, the number of cigarettes smoked each day.

Based on this and countless other examples it will be obvious that functions which model aspects of real life are likely to need to be models with more than one input parameter. That is, functions of more than one variable.

Example 2.1.

$$f(x, y) = x^2 - 2xy$$

2.2 Partial Differentation

The graph of a function $f(x,y)$ has two base axes (x,y) and a value axis f. This makes it difficult to draw on the page, but the concept is just the same as for graphing $f(x)$. Such functions can have maxima, and minima, and it is very useful to be able to locate these features.

Finding the maxima and minima of ordinary funtions is one of the jobs which differentiation helps us enormously with, as we know. Is there some version of differentiation we can use for $f(x,y)$?

There are *two* partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Note that in each case we pick a variable to differentiate with respect to, and then differentiate treating the other variable as if it were a constant.

Example 2.2. We can't differentiate $f(x,y) = x^2 - 2xy$ in the ordinary way, but if we fix $y = 3$ (say), then $f(x,3) = x^2 - 6x$, which we can differentiate. We can do it again for any other 'fixed' value of y. All these differentiations can be summarized by the *partial* differentiation with respect to x:

$$f_x = 2x - 2y.$$

Similarly

$$f_y = -2x.$$

Note that the differentiated function can still be a function of one or more variables!

Example 2.3. Yes, that's right

$$\frac{\partial x^2}{\partial y} = 0.$$

Here we are holding x constant, so we are differentiating a constant!

Exercise 2.4.

$$f(x,y) = \sin^2(x)\cos(x) + \frac{x}{y^2}$$

Determine f_x and f_y.

2.2.1 Higher derivatives

Any function may, in principle, be differentiated. The output of the operation of differentiating a function is another function, so we may differentiate multiple times. The same idea works for partial differentiation.

The notion is straightfoward! What we have to grasp is really just notation.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = f_{yx} == \frac{\partial^2 f}{\partial xy} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

(In this last one we have been a bit glib. Think about it!)

These are the second derivatives. Higher derivatives work similarly.

Example 2.5. For

$$f(x,y) = x^2 - 2xy$$

then

$$f_{xx} = 2$$

$$f_{yy} = 0$$

$$f_{xy} = -2$$

$$f_{xxxx} = 0$$

2.3 Chain Rule

Suppose

$$z = f(x, y)$$

given, but

$$x = x(r, s)$$

$$y = y(r, s)$$

How do we compute $\frac{\partial z}{\partial r}$? (Compare this with the similar problem for the one–variable case.)

Answer:

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

(and similarly for s).

Exercise 2.6. Verify that this reduces to the usual one–variable case.

Example 2.7.

$$z = x^3 + y^3$$

where

$$x = 2r + s$$

$$y = 3r - 2s$$

Find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$ [1]

2.4 Maxima and Minima

We can find the maxima of $f(x)$ by differentiation, but the underlying definition is more straightforward. The point $x = c$ is a (local) maximum of

[1] Partial answer:

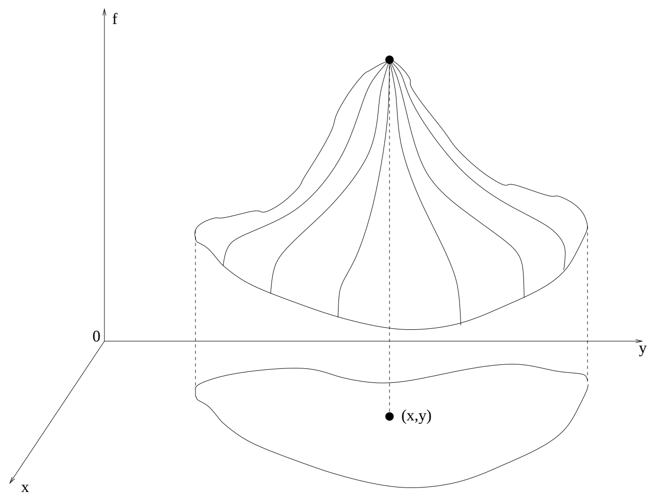
$$\frac{\partial z}{\partial r} = 6(2r + s)^2 + 9(3r - 2s)^2$$

$f(x)$ if the value of $f(x)$ is lower than $f(c)$ for every point in the immediate neighbourhood of $x = c$.

This definition generalises in a simple way to $f(x, y)$.

A point $(x, y) = (c, d)$ is a maximum of $f(x, y)$ if the value of $f(x, y)$ is lower than $f(c, d)$ for every point in the immediate neighbourhood of (c, d).

Here is a picture of a simple mountain. The maximum is the highest point.



The coordinate system in this picture is not intrinsic to the mountain. It is something we put on so that we can give the location of the maximum, and other features, in a convenient format. Once we have a coordinate system we can, in principle, give the height of the mountain at each point of the (x, y) base plane.

This height function will thus be a function of two variables.

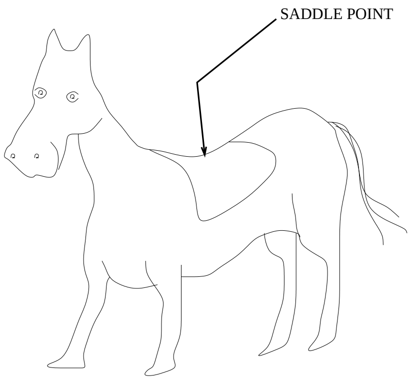
Given this function, $f(x, y)$ say, how do we find the maximum (minimum)?

Once again consider $f(x, y)$ as a function of one variable by holding y constant. If $f(x, y)$ is a maximum at, say, $(x, y) = (x_c, y_c)$ (x_c, y_c given constants), it is also a maximum of the function $f(x, y_c)$.

Thus a *necessary* condition for an extremum is

$$f_x = f_y = 0$$

However, NB, this is not a sufficient condition. It includes saddle points...



...so, define

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

Suppose $(x, y) = (a, b)$ satisfies the first order conditions.

If $D(a, b) > 0$ then

$$f_{xx}(a, b) < 0$$

implies a MAXIMUM.

$$f_{xx}(a, b) > 0$$

implies a MINIMUM.

If $D(a, b) < 0$ then SADDLE POINT.

Example 2.8.

$$f(x, y) = -x^2 - y^3 + 12y^2$$

[2]

Example 2.9.

$$f(x, y) = x^4 + 4x^2y^2 - 2(x^2 - y^2)$$

[2] Answer: (0, 0) SADDLE; (0, 8) MAX.

[3]

2.4.1 Several variables

Suppose $z = f(x_1, x_2, .., x_n)$. (Note the variable names.)

This f maps each point $\underline{a} \in \mathbb{R}^n$ to a real number. (We also write $\underline{x} = (x_1, x_2, .., x_n)$.)

Necessary condition for extremum (n variable case)

$$f_{x_1} = f_{x_2} = \cdots = f_{x_n} = 0 \tag{2.1}$$

Sufficient condition:

Form HESSIAN MATRIX

$$\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots \\ f_{x_2x_1} & f_{x_2x_2} & \cdots \\ & & \ddots \end{pmatrix}$$

(Also let H_m denote the top left $m \times m$ submatrix.)

Now suppose $\underline{a} = (a_1, a_2, .., a_n)$ satisfies (2.1). Then if evaluated at $\underline{a}$ we have

$$|H_1| < 0 \qquad |H_2| > 0 \qquad |H_3| < 0 \qquad \text{etc.}$$

implies MAXIMUM; while

$$|H_1| > 0 \qquad |H_2| > 0 \qquad |H_3| > 0 \qquad \text{etc.}$$

implies MINIMUM.

We will provide some examples shortly.

---

[3]Answer: $(0, 0)$ SADDLE; $(1, 0)$ MIN.; $(-1, 0)$ MIN. (Now try to sketch the function, or get a computer to do it for you!

Note that the DERIVE computer package is excellent for this kind of thing.)

2.5 Lagrange Multipliers

Problem:

Maximize $f(x_1, x_2)$ subject to a constraint on the allowable values of x_1, x_2, let us say of the form $g(x_1, x_2) = k$, where g is a given function and k a constant.

(Typically an analyst might be asked to determine the production parameters for a company so as to maximize profit.

Suppose the company makes TVs and RADIOs, and the profit function depends on the number of each made.

Naively we maximize profit by making infinitely many of each, but of course there will be practical limitations on the total number of devices which can be produced in the real world factory.

We want to maximize subject to these realistic constraints.)

Solution:

Form

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[g(x_1, x_2) - k]$$

(λ is called the LAGRANGE MULTIPLIER.)

Now find *un*constrained maximum of L:

Necessary:

$$L_{x_1} = L_{x_2} = L_\lambda = 0$$

(Note that the last condition enforces the original constraint!)

Example 2.10.

$$f(x_1, x_2) = 25 - x_1^2 - x_2^2$$

subject to

$$2x_1 + x_2 = 4$$

so

$$L = 25 - x_1^2 - x_2^2 - \lambda[2x_1 + x_2 - 4]$$

$$L_{x_1} = -2x_1 - 2\lambda$$

$$L_{x_2} = -2x_2 - \lambda$$

$$L_\lambda = -2x_1 - x_2 + 4 = 0$$

The first order conditions are thus three linear (homogeneous) equations in three unknowns. Linear algebra (!) gives

$$x_1 = 1.6 \qquad x_2 = 0.8 \qquad \lambda = -1.6$$

Now, is this a MAX/MIN/??

Form BORDERED HESSIAN

$$H_B = \begin{pmatrix} 0 & g_{x_1} & g_{x_2} \\ g_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ g_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{pmatrix}$$

then for (a_1, a_2, l) satisfying the necessary conditions,

$$|H_B| > 0 \qquad \text{implies MAX;}$$

$$|H_B| < 0 \qquad \text{implies MIN.}$$

For our example above we have

$$|H_B| = \left| \begin{pmatrix} 0 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \right| = 10 > 0$$

so MAX.

Example 2.11.

$$f(x, y) = x^2 + y^2$$

subject to

$$x^3 + y^3 - 6xy = 0$$

Get

$$L = x^2 + y^2 - \lambda[x^3 + y^3 - 6xy]$$

Necessary conditions:

$$L_x = 2x - 3x^2\lambda + 6y\lambda = 0 \tag{2.2}$$

$$L_y = 2y - 3y^2\lambda + 6x\lambda = 0 \tag{2.3}$$

$$L_\lambda = -(x^3 + y^3 - 6xy) = 0$$

Together (2.2) and (2.3) imply

$$\lambda = \frac{2x}{3x^2 - 6y} = \frac{2y}{3y^2 - 6x}$$

so

$$x^2y - xy^2 + 2x^2 - 2y^2 = 0$$

so

$$(x - y)(2x + 2y + xy) = 0$$

Evidently $x = y$ is a solution to this. Substitute back into the constraint:

$$2x^3 - 6x^2 = 0$$

with solutions $x = 0, 3$.

So, we have $(0, 0)$ (a MINIMUM) and $(3, 3)$ (a local MAXIMUM).

(Sketch the curve implied by the constraint to confirm this.)

Example 2.12. In a suitable coordinate system, the Chicago Beltway lies on the curve

$$3(x^2 + y^2) + 4xy = 2$$

You are tendering for the job of building a connecting road from the Beltway to Soldier Field sports stadium (at coordinates (0,0)).

It costs your company $\$5 \times 10^7$ per kilometre to build road. What is the minimum tender you could bid without making a loss?

[4]

[4] Answer: $(\frac{\pm 1}{\sqrt{5}}, \frac{\pm 1}{\sqrt{5}})$ min.; $(\pm 1, \mp 1)$ max., so $\$\sqrt{\frac{2}{5}}.5.10^7$.

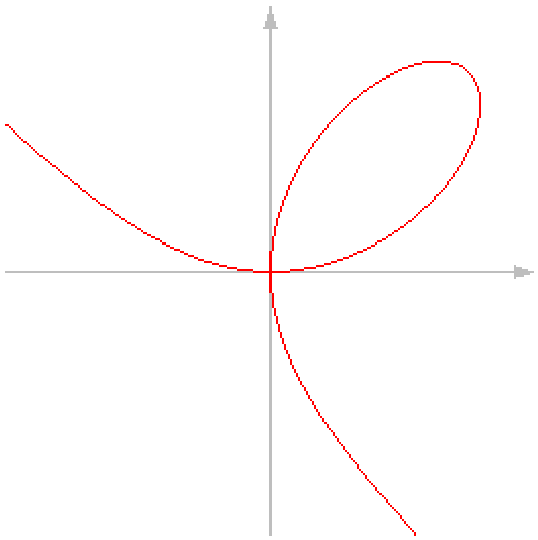


Figure 2.1: Plot of $x^3 + y^3 - 6xy = 0$ in the interval $-4 < x, y < 4$.

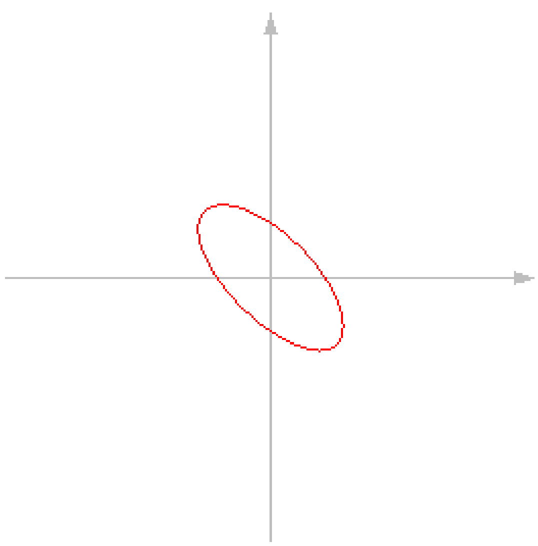


Figure 2.2: Plot of $3x^2 + 3y^2 + 4xy = 2$ in the interval $-4 < x, y < 4$.

The first question here is WHAT IS THE FUNCTION TO OPTIMIZE? It is the distance l (say) from the point (x, y) to the stadium, i.e.

$$l^2 = f(x, y) = x^2 + y^2$$

(Note that we have taken the square. This makes for simpler calculations, and of course the square has its minimum at the same moment as l itself.)

WHAT IS THE CONSTRAINT?

$$g(x, y) = 3(x^2 + y^2) + 4xy = 2$$

Thus

$$2x + \lambda(6x + 4y) = 0$$

$$2y + \lambda(6y + 4x) = 0$$

Together these give

$$4\lambda(y^2 - x^2) = 0$$

so $y = \pm x$.

In case $y = x$ we have $10x^2 - 2 = 0$, so $x = \frac{\pm 1}{\sqrt{5}}$.

In case $y = -x$ we have $2x^2 - 2 = 0$, so $x = \pm 1$.

It will be *evident* that the first of these is the MINIMUM! As an exercise, check this using the Hessian method.

2.6 Taylor series

Exercise 2.13. Recall that the *Taylor series* for a function of one variable x about the point $x = a$ (a a constant) may be written in the form

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \ldots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \ldots$$

where $f'(a)$ denotes the derivative of f evaluated at a, and $f^{(n)}$ the n^{th} derivative.

Suppose that $x = a$ is in fact an extremal point of f. Indeed, for the sake of definiteness let us say that it is a maximum. Then of course $f(x) < f(a)$ for values of x in the neighbourhood of $x = a$. That is to say,

$$f(x) - f(a) < 0 \tag{2.4}$$

for such values of x.

Now say we know that $x = a$ is an extremal point (i.e. we know that $f'(a) = 0$), but we do not know the nature of the extremum. Considering the Taylor series we have

$$f(x) - f(a) = \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots \tag{2.5}$$

For points close to $x = a$ then $(x - a)$ is small, so $(x - a)^2$ is bigger than $(x-a)^3$ and so on. Now note that the sign of $f(x) - f(a)$ is thus determined by the sign of $f''(a)$ for points close to $x = a$. Comparing equation (2.4) with equation (2.5), and noting that $(x - a)^2$ cannot be negative, this is one way to understand why the second order condition $f''(a) < 0$ means MAXIMUM.

An exactly analogous argument shows that $f''(a) > 0$ means MINIMUM.

Now, suppose that $f''(a) = f'(a) = 0$.

(a) Give the new necessary condition for f to be an extremal point, carefully explaining the reasons for your answer.

(b) Give the new sufficient condition for f to be MAXIMUM, carefully explaining the reasons for your answer.

(Marks will be given for the explanation, as well as the answer.)

Exercise 2.14. Let a, b be constants, and let $h = x - a$ and $k = y - b$. Let $\mathcal{D}$ denote the *operator* $h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$. That is to say

$$\mathcal{D}f(x, y) = hf_x + kf_y.$$

Then the Taylor series for $f(x, y)$ about $(x, y) = (a, b)$ may be written

$$f(x, y) = f(a, b) + \mathcal{D}f(a, b) + \frac{1}{2!}\mathcal{D}^2 f(a, b) + \dots$$

where we take $\mathcal{D}f(a, b)$ to mean $\mathcal{D}f$ evaluated at (a, b), and so on.

(a) Show that the usual necessary condition for an extremal point of $f(x, y)$ at (a, b) implies $\mathcal{D}f(a, b) = 0$.

(b) Write out explicitly what is meant by the shorthand $\mathcal{D}^2 f(a, b)$.

(c) Hence derive the usual full second order condition for a maximum at (a, b).

Exercise 2.15. OPTIONAL: By considering the above two exercises together, think about how to proceed to investigate the nature of extremal points $f(a, b)$, when the quatities evaluated in the second order test (as defined in section 2.4 of the notes) turn out to vanish.

Another exercise relevant for this section is Exercise 6.1 below.

2.7 Further Notes and Exercises

Exercise 2.16. The trajectory through space of a 150 x 10^6 kg chunk of cosmic debris lies in a plane through the Earth. Suitably coordinatizing this plane, with the centre of the Earth at the point $(0, 0)$, the trajectory is given by the curve

$$x^2 + 8xy + 7y^2 - 225 = 0$$

(units are thousands of kilometres). What is the closest distance which the object comes to the Earth?
(HINT: Use the method of Lagrange multipliers. What is the function to be minimized? What is the constraint equation? Give a sketch of the trajectory if it helps you to envisage the problem and/or check the reasonableness of your answer.)

OPTIONAL: Is it safe for your company to insure against damage resulting from the passage of this object?

Exercise 2.17. Using the method of Lagrange's multipliers, find the shortest distance from the point $P(0, 0, 1)$ on the z-axis, to the curve $y^3+x^3+y^2+x^2 = 0$ which lies in the (x, y)-plane.

Exercise 2.18.

1. Find all the stationary points (maxima, minima and saddle points) of the function
$$g(x,y) = x^2y^2.$$

2. Show that the function
$$f(x,y) = -xy\, e^{-\frac{x^2+y^2}{2}}$$
has a stationary point at $x = 0$, $y = 0$.

   Find all the other stationary points of $f(x,y)$.

   Determine the nature of the stationary point at $(0,0)$.

The next few exercises really are just that, i.e. relatively routine optimisation problems designed to build up your mathematical 'muscles'. They can all be solved simply by other means, but the point here is to practice Lagrange multipliers!

Exercise 2.19. Find all the stationary points of the function
$$g(x,y) = x^2 + y^2$$
subject to
$$x^2 + 2y^2 = 1$$
using the method of Lagrange Multipliers. (Check your answer by sketching the curve defined by the constraint; and separately by elementary substitution!)

Exercise 2.20. Find all the stationary points of the function
$$g(x,y) = xy$$
subject to
$$x + y - 6 = 0$$

using the method of Lagrange Multipliers. (Check your answer by sketching the curve defined by the constraint; and separately by elementary substitution!)

Exercise 2.21. Find all the stationary points of the function

$$g(x, y) = x^2 + y^2$$

subject to

$$x + y - 1 = 0$$

using the method of Lagrange Multipliers. (Check your answer by sketching the curve defined by the constraint; and separately by elementary substitution!)

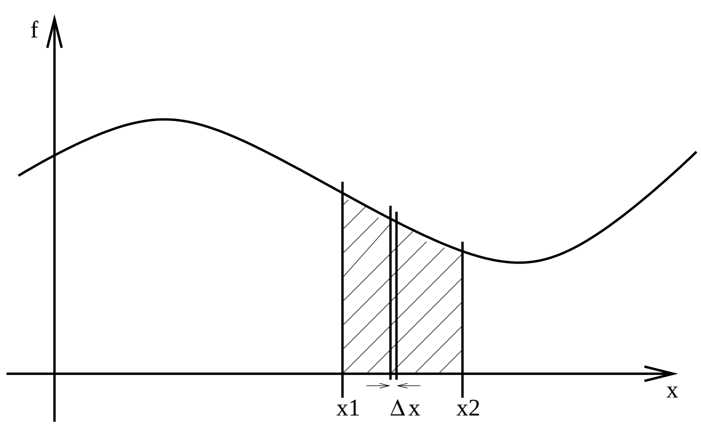
Chapter 3

Multiple integration

We will start by reviewing ordinary integration. Then we will look at the meaning and uses of integration, when functions of more than one variable are involved.

3.1 Ordinary integration revisited

Recall:

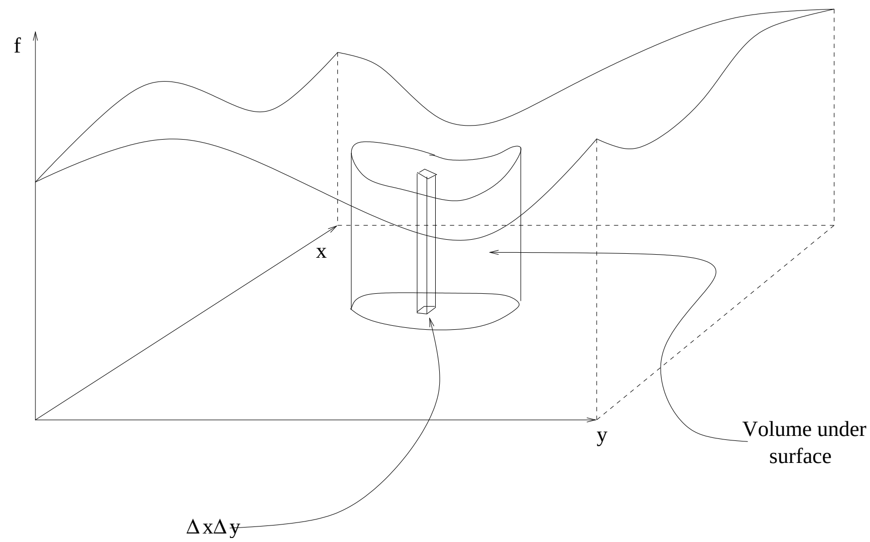


$$I = \int_{x_1}^{x_2} f(x)dx = \text{Area under curve.}$$

$$\cong \sum_i f(x_i)\Delta x$$

3.2 Multiple integration

Now:



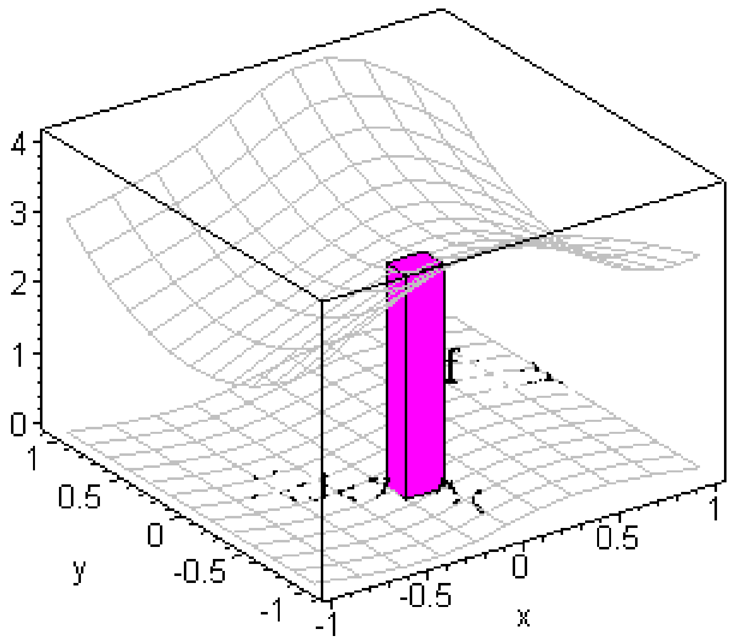
Here the generalisation of the usual *limits of integration* (which define an interval of the real line — a region of the real line) is an 'interval' of the base plane — a region of the base plane.

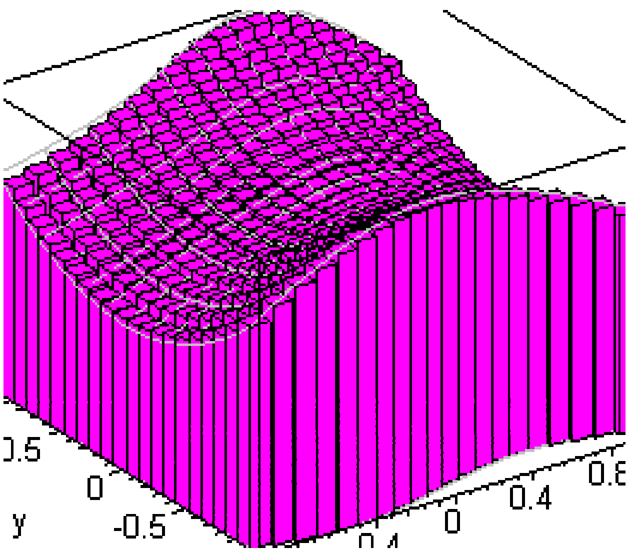
Such a thing is more complicated to specify precisely, but the concept is not so complex.

Let us call the region of the base plane which we are integrating over R. The volume under the surphace is

$$I = \int_R f(x,y)dR = \int\int f(x,y)dxdy$$

$$\cong \sum_i f(x_i, y_i) \underbrace{\Delta x \Delta y}_{\Delta R}$$

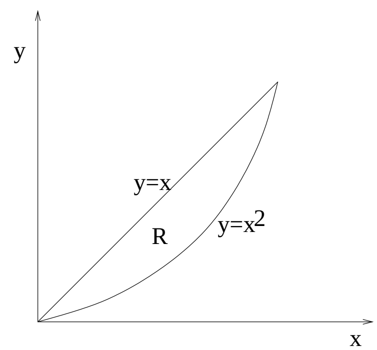

4
3
2
1
0
1
0.5
0
y
-0.5
-1
-1
-0.5
0
0.5
1
x


0
y
-0.5
0
0.4

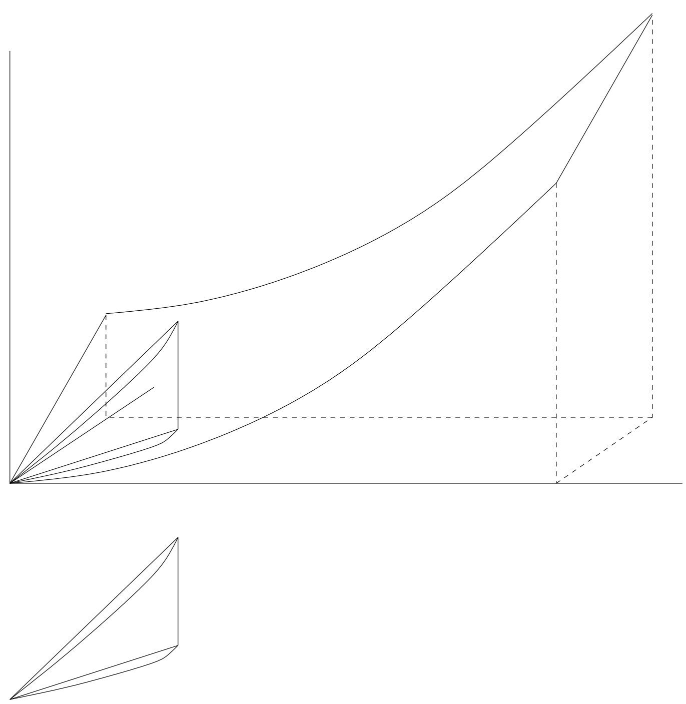
Example 3.1. Integrate to find the volume under the surface

$$f(x, y) = 2x^2 + y$$

with base given by the region



A sketch of the surface, the base region *and* the resultant volume looks like this:



(Here at the bottom we have extracted a little sketch of the volume on its own, as a guide to the eye.)

The integral is

$$I = \iint_R (2x^2 + y)dxdy$$

Basic idea:

$$I = \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=x} (2x^2 + y)dy \right] dx$$

In the part in square brackets we regard x as constant (!) — then we *can do the integral* (but the answer will depend on x):

$$I = \int_{x=0}^{x=1} F(x)dx$$

where

$$F = \int_{y=x^2}^{y=x} (2x^2 + y)dy = [2x^2y + \frac{y^2}{2}]_{y=x^2}^{y=x}$$

$$= 2x^3 + \frac{x^2}{2} - 2x^4 - \frac{x^4}{2}$$

so

$$I = \int_0^1 (2x^3 + \frac{x^2}{2} - \frac{5x^4}{2})dx$$

$$= \left[\frac{2x^4}{4} + \frac{x^3}{6} - \frac{5}{2}\frac{x^5}{5} \right]_0^1 = \frac{1}{2} + \frac{1}{6} - \frac{1}{2} = \frac{1}{6}$$

Exercise 3.2. Compute the same integral, but performing the $\int dx$ first. (Hint: the main problem is to correctly give the limits.)

Example 3.3. Consider the cylinder (tube) in 3d given by

$$x^2 + y^2 = 2ax \qquad (a \text{ a constant})$$

Consider further that we cut through this tube in 2 planes: $z = mx$ and $z = nx$ (m, n constants).

Find the volume of the resultant wedge.

Idea: Volume = Volume1 - Volume2 where

$$\mathrm{Vol}_1 = \int\int_R mx\; dxdy$$

(volume under $f = mx$)

$$= 2m \int_{x=0}^{x=2a} \left[\int_{y=0}^{y=\sqrt{2ax-x^2}} x \, dy \right] dx$$

$$= 2m \int_0^{2a} x\sqrt{2ax - x^2} dx = 2m \int_0^{2a} x\sqrt{a^2 - (a-x)^2} dx$$

(set $u = a - x$)

$$= 2m \int_{-a}^{a} (a-u)\sqrt{a^2 - (u)^2} du$$

(set $u = a\sin(\theta)$)

$$= 2m \int_{-\pi/2}^{\pi/2} a(1 - \sin\theta)(a\cos\theta)^2 d\theta$$

$$= 2ma^3 \int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2} d\theta = \pi m a^3$$

(Here we use $a^2 - u^2 = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$ and $\frac{du}{d\theta} = a\cos\theta$; and the double angle formula (1.1).)

Similarly,

$$\mathrm{Vol}_2 = \int\int_R nx \, dxdy = n\pi a^3$$

so the volume of the wedge is $\pi(m-n)a^3$.

(NB, this is very easy to check!)

3.3 Further Notes and Exercises

Exercise 3.4. Verify

$$\int_0^1 \int_{x^2}^{x} 2dydx = \frac{1}{3}$$

(yes! the argument of the integral is 2). Explain exactly what geometrical quantity this integral is computing.

Exercise 3.5. Verify

$$\int_1^2 \int_y^{3y} \frac{(x+y)}{9} dx dy = \frac{14}{9}$$

Explain exactly what geometrical quantity this integral is computing.

Exercise 3.6. Verify

$$\int_{-1}^2 \int_{2(x+1)(x-1)}^{x(x+1)} \frac{x}{2} dy dx = \frac{9}{8}$$

Explain exactly what geometrical quantity this integral is computing.

Exercise 3.7. A cylindrical well is bored vertically down into the New Mexico desert, with interior sectional radius 3m. The well wall is made of a relatively thin layer of concrete.

A cylindrical tunnel is bored horizontally under the desert, again with interior sectional radius 3m, by a separate Engineering company, but following a similar construction method. Unfortunately the central axes of the two tubes coincide at one point under the desert! What is the volume of the region common to these two tubes.

(HINT: Set up a coordinate system with the y and z axes being the axes of the two tubes. Then they can meet at $(0, 0, 0)$ say. You could draw a picture of this situation. Now construct a suitable double integral by considering the equations describing the shapes of the two tubes.)

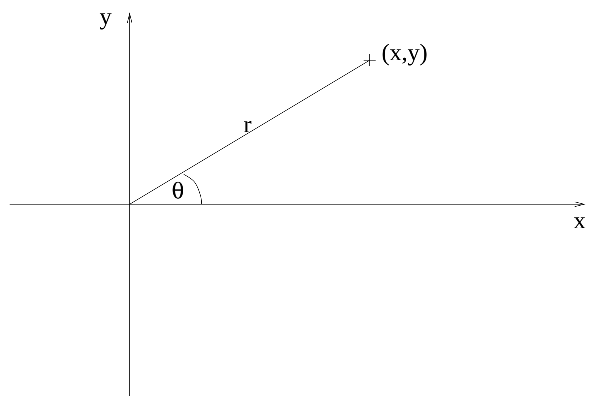
2003 COURSEWORK 1: Do exercises 2.13, 2.14, 2.16, 2.17, and 3.7.

You must read the separate sheet of advice and rules for coursework submission. The rules are binding on any submission.

3.4 Polar coordinates

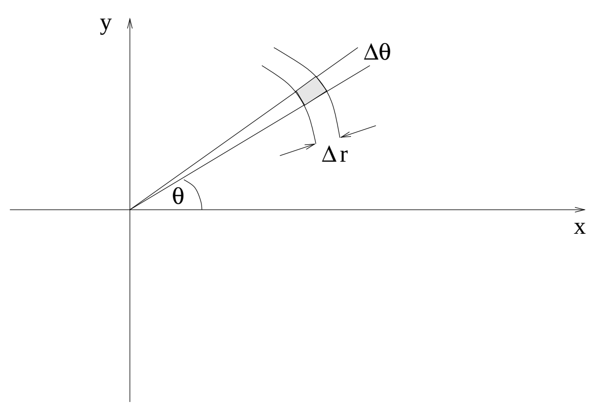
In ordinary integration, we are familiar with the method of change of variables (we just used it several times above!). There is a multiple integral generalisation which can be just as useful.

We begin with an example. Recall:



That is, $x = r\cos\theta$, $y = r\sin\theta$.

The approximate area of the shaded region in



is $r\Delta r\Delta\theta$.

Thus we can make a change of variables in a double integral (corresponding to moving from the Cartesian to the polar coordinate system) as follows:

$$I = \iint_R f(x,y)dxdy = \iint_R f(r\cos\theta, r\sin\theta)rdrd\theta$$

(Note the 'extra' r!)

To see the point of this we need an example.

Example 3.8.

$$I = \iint_R \{1 - \sqrt{x^2+y^2}\}dxdy$$

where the region R is the disk bounded by the unit circle.

$$I = \int_0^{2\pi}\int_0^1 (1-r)rdrd\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_0^1 d\theta = 2\pi\frac{1}{6} = \frac{\pi}{3}.$$

Example 3.9. The integral

$$I = \int_0^\infty \exp(-x^2)dx$$

is a very important (Gaussian) and difficult single integral. (Sketch the curve of the argument, and recall the role of this function in, for example Statistics, to see why it is so important.)

$$I^2 = I.I = \left(\int_0^\infty \exp(-x^2)dx\right)\left(\int_0^\infty \exp(-y^2)dy\right)$$

$$= \int_0^\infty\int_0^\infty \exp(-x^2-y^2)dxdy = \int\int_R \exp(-x^2-y^2)dxdy$$

$$= \int\int_R \exp(-r^2)rdrd\theta = \int_0^{\pi/2}\left[\int_0^\infty \exp(-r^2)rdr\right]d\theta$$

(We are doing an improper integral here. The answer we will get is correct, but to be more careful we could have considered

$$I = \lim_{a\to\infty}\int_0^a e^{-x^2}dx$$

for which the change of variables is slightly harder to implement.) So

$$I^2 = \int_0^{\pi/2} \left[-\frac{1}{2} \exp(-r^2) \right]_0^\infty d\theta = \frac{\pi}{4}.$$

So $I = \frac{\sqrt{\pi}}{2}$.

3.5 General change of variables

Just as

$$x = r\cos\theta = x(r, \theta)$$

$$y = y(r, \theta) = r\sin\theta$$

we could have $x = x(u, v)$, $y = y(u, v)$. Then

$$\int\int_R f(x, y) dx dy = \int\int_R f(x(u, v), y(u, v)) \, |\mathcal{D}| \, du dv$$

where $\mathcal{D}$ is the JACOBIAN:

$$\mathcal{D} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

This generalises the single change of variables

$$\int f(x) dx = \int f(x(u)) \frac{dx}{du} du.$$

Example 3.10. $x = r\cos\theta$, $y = r\sin\theta$

$$|\mathcal{D}| = r\cos^2\theta + r\sin^2\theta = r$$

as we argued explicitly earlier!

Exercise 3.11. Compute

$$I = \int\int_R \sqrt{1 - (x^2 + y^2)} dx dy$$

where R is the interior of the unit disk. Interpret your answer geometrically.

$$I = \int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\; r dr d\theta$$

Substituting $r = \sin\phi$ we get

$$I = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\phi\ \sin\phi\ \cos\phi\ d\phi d\theta.$$

Now

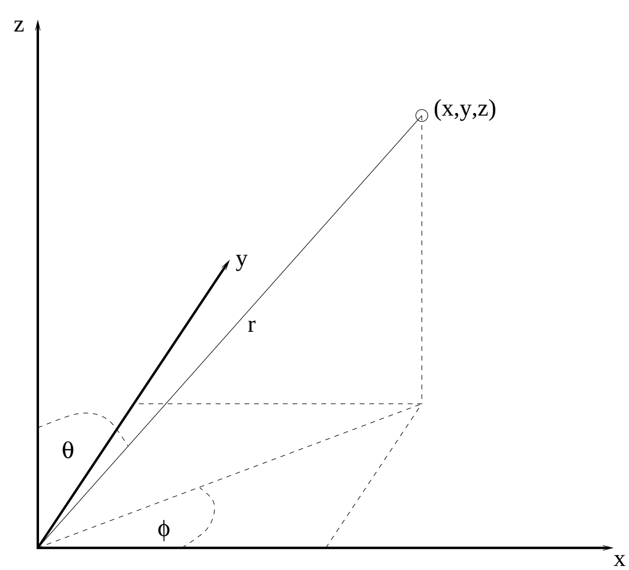
$$\cos\phi\ \sin\phi\ \cos\phi = \sin\phi - \sin^3\phi = \frac{1}{4}(\sin 3\phi + \sin\phi)$$

(hint: compute $\sin^3\phi = (\frac{e^{i\phi}-e^{-i\phi}}{2i})^3$). So

$$I = \int_0^{2\pi} \left[\frac{1}{4}(\frac{-\cos 3\phi}{3} - \cos\phi)\right]_0^{\frac{\pi}{2}} d\theta = \frac{2\pi}{3}.$$

3.6 Spherical polar coordinates

Spherical polar coordinates are the generalisation of polar coordinates to three dimensions. We have $(x, y, z) \to (r, \theta, \phi)$ as follows.



That is

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

so the Jacobian is

$$\mathcal{D} = \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}\right| = \left|\begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & \dots \\ \sin\phi\sin\theta & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}\right|$$
$$= r^2\sin\theta.$$

Example 3.12. A solid object S has density $\delta(x, y, z)$. Then it has mass

$$m(S) = \int\int\int_S \delta(x, y, z)\ dxdydz. \tag{3.1}$$

Consider a hemispherical object with density

$$\delta = 17\sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2}\ Kg\ m^{-3},$$

where (x_c, y_c, z_c) is the centre of the corresponding *full* sphere. What is its mass?

$$m(S) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a 17r(r^2\sin\theta)drd\theta d\phi$$
$$= 17\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\sin\theta\ d\theta d\phi = \frac{17a^4}{4}\int_0^{2\pi}[-\cos\theta]_0^{\frac{\pi}{2}}\ d\phi = \frac{17\pi a^4}{2}.$$

The CENTRE OF MASS of S is $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{\int\int\int_S\ x\ \delta(x, y, z)\ dxdydz}{m(S)}$$
$$\overline{y} = \frac{\int\int\int_S\ y\ \delta(x, y, z)\ dxdydz}{m(S)}$$

$$\overline{z} = \frac{\int\int\int_S \ z \ \delta(x,y,z) \ dxdydz}{m(S)}$$

Exercise 3.13. Compute the CoM of our hemisphere. I.e.

$$\overline{x} = \frac{\int\int\int_S \ r \ \cos\phi \ \sin\theta \ (17r)r^2 \sin\theta \ drd\theta d\phi}{m(S)} = 0 = \overline{y}$$

$$\overline{z} = \frac{\int\int\int_S \ r \ \cos\theta \ (17r)r^2 \sin\theta \ drd\theta d\phi}{m(S)}$$

$$= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \ r \ \cos\theta \ (17r)r^2 \sin\theta \ drd\theta d\phi}{m(S)}$$

$$= \frac{2}{\pi a^4}\frac{a^5}{5}2\pi \int_0^{\frac{\pi}{2}} \cos\theta \ \sin\theta \ d\theta = \frac{4}{5}a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{2a}{5}$$

Thus the CoM is $(0, 0, \frac{2a}{5})$.

Is this realistic??!!

3.7 Examples and Exercises

3.7.1 Actuarial Science, Quantity and Physics

This section is not examinable. It is here to provide an intuitive and applicable context for problems in integration.

Recall the following facts from elementary Newtonian mechanics. [1] If you push an object along a line through its centre of mass you will accelerate it. For a given force applied F, the acceleration a is inversely proportional to the mass m of the object:

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = \frac{dM}{dt}$$

[1] See any standard Physics or Engineering text, for example. (One such is W. Wilson's *Theoretical Physics* (Methuen).)

where t is time, v is velocity, x is distance and $M = mv$ is *momentum*, all in appropriate units.

If you accelerate the object over some distance, raising its velocity, you have put *energy* into it. This movement energy is called kinetic energy. The total kinetic energy an object has (relative to some coordinate system in which velocity is measured) is given by

$$E = \frac{1}{2}mv^2 = \frac{1}{2}Mv$$

in appropriate units.

Now suppose that our object and forces applied to it 'live' in two dimensions rather than one. There is an equation of motion for each direction. Suppose for definiteness that the force F acts along the line between the object and the centre of coordinates (i.e. this is how we choose our coordinate frame — which we are free to do); and let (x_1, x_2) be the coordinates of our object. We can resolve the force into x_1 and x_2 components by elementary trigonometry:

$$m\frac{d^2x_i}{dt^2} = F\frac{x_i}{\sqrt{x_1^2 + x_2^2}} \qquad (i = 1, 2)$$

Considering the combination of these equations $x_2(i = 1) - x_1(i = 2)$ we have

$$\frac{d(\Omega = m(x_1x_2' - x_2x_1'))}{dt} = 0$$

that is, Ω is constant in this situation. This Ω is called the *angular momentum* of the object. It is the angular equivalent of the ordinary momentum mv. As we have seen, it is not changed by a radial force. It is changed by a tangential force.

If our object is moving with velocity v at an angle θ to the radial line from the centre to the object, then its velocity component in the tangential direction is $v\sin(\theta)$ (elementary trigonometry) and its *angular velocity* is the analogue of ordinary velocity defined by

$$\omega = \frac{v\sin(\theta)}{r}$$

where r is the distance from the origin. The angular momentum is proportional to the mass times the angular velocity:

$$\Omega = mr^2\omega$$

This is the same as for ordinary momentum (except for the factor r^2 which allows for the fact that, for a given angular velocity, the further away from the origin we are the greater the 'linear' velocity).

Now consider an object in the xy plane rotating about a fixed axis (for simplicity let us make it the z-axis). If ϕ measures the angle made by the line from the object to the origin and the x-axis, then

$$v = r\frac{d\phi}{dt}$$

so the kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2$$

Now suppose we have a rigid object made up of many particles (m_1, m_2, ..., say), again rotating about the fixed axis. Since it is rigid, all the particles have the same angular velocity. The total kinetic energy, the sum of individual energies computed above, is thus

$$T = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\left(\frac{d\phi}{dt}\right)^2$$

The quantity $(\sum_i m_i r_i^2)$ is the *moment of inertia* of the composite object with respect to the given axis. It plays the same role in rotating motion as the *mass* does in the expression $(1/2)mv^2$ for the kinetic energy in linear motion.

By considering a solid object as the limit of a collection of tiny individual chunks we can reduce the computation of its moment of inertia to an integral. Suppose that $\rho(x, y, z)$ gives the density per unit volume, and that the object occupies a volume of shape V. Since the distance from the z-axis is $r^2 = x^2 + y^2$ we have

$$I_z = \int\int\int_V \rho(x, y, z)(x^2 + y^2)\ dx\ dy\ dz$$

where I_z denotes that this is the moment with respect to the z-axis. (This generalises in a natural way to rotation about an arbitrary axis, but this need not concern us here.)

3.7.2 Integration related examples

A straight line in the xy plane partitions the plane into two parts ($y = 0$ partitions the plane into a region above the line and a region below the line, for example). A curved line without an endpoint (i.e. either closed or infinite) partitions the plane similarly.

Sets of curves will further partition the plane into subregions. For example, three straight lines, none of which are parallel, will partition the plane into a number of regions. Most of these regions are infinite in extent ('unbounded'), but one is finite — a triangle (unless all three lines intersect at the same point!). In such a case we say the finite region is *bounded* by the set of lines. Generalising from this: even if the set of lines is curved, provided that precisely one part of the partition of the plane they describe is finite, we call this the region bounded by the curves.

Exercise 3.14. Sketch the region of the xy plane bounded by each of the following sets of curves:

$x^2 + y^2 = 16$;

$x = y$, $x = 2$, $y = 3$;

$x = 1$, $x = 2$, $y = 3$, $y = 6$;

$y = \sin(x)$ $(0 \leq x \leq \pi/3)$, $x = \pi/3$, $y = 0$.

Express the area of the region as a double integral (or multiple thereof) in each case. Evaluate this double integral.

Exercise 3.15. Sketch the volume bounded by $x = 0$, $y = 0$, $z = 6$, $z = x + y$. Compute this volume by means of a double integral. (Ans=36)

Exercise 3.16. Sketch and Evaluate:
$\int_0^a \int_0^{a-x} 1\; dy\; dx$;
$\int_0^1 \int_x^{x^{1/2}} xy^2\; dy\; dx$;
$\int_0^a \int_0^x (x^2+y^2) dy\; dx$;
$\int_{-\pi/2}^{\pi/2} \int_0^{2a\cos(\theta)} r^2 \cos(\theta)\; dr\; d\theta$.
(Ans=$a^2/2; 1/35; a^4/3; \pi a^3$)

Exercise 3.17. A new design of motorised food slicer blade (or helocopter rotor, take your pick!) has shape given by the region R of the x_1x_2-plane bounded by the curves $x_1^2 = x_2$, $x_2 = 1$, $x_1 = 2$. It has uniform thickness and density in this region (and has unit mass per unit area). Sketch the shape of the slicer blade. Work out the moment of inertia with respect to the x_3-axis:

$$\int\int_R (x_1^2 + x_2^2)\; dx_1\; dx_2$$

(Ans=1006/105)

Exercise 3.18. Formulate and compute a double integral which determines the volume of a sphere.
Formulate and compute a triple integral which determines the volume of a sphere.

Exercise 3.19. Recall that the *region of integration* of the double integral

$$I_1 = \int_0^1 \left(\int_0^1 3\; dy \right) dx$$

is the square in the (x, y) plane with corners (0,0),(1,0),(1,1),(0,1). Sketch this region. What is the value of I_1? (Hint: you do not need to do an integral!)

Exercise 3.20. Sketch the region of integration in the double integral

$$I = \int_0^1 \left(\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy \right) dx.$$

Re–express I with the x–integral as the inner (first) integral. By thus changing the order of integration, evaluate I. (Hint: you will need to make at least one change of variables.)

Exercise 3.21. Show that in polar coordinates the equation of the circle $(x-1)^2 + y^2 = 1$ takes the form

$$r = 2\cos\theta,$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Hence, by using the cylindrical coordinate system (r, θ, z) or otherwise, find the volume of the solid enclosed in the vertical cylinder $(x-1)^2 + y^2 = 1$ bounded below by the plane $z = 0$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

Chapter 4

Differential Equations

Examples:

$$\frac{df(x)}{dx} = 3 \tag{4.1}$$

$$\frac{d^2 f(x)}{dx^2} = 6x \tag{4.2}$$

$$\frac{d^2 f(x)}{dx^2} + 7\frac{df(x)}{dx} = (f(x))^2 + 7\log_e(x) \tag{4.3}$$

Solution: Find a function $f(x)$ such that the Differential Equation (DE) is a true identity. E.g.

(4.1) $f(x) = 3x + 7$

(4.2) $f(x) = x^3 + \pi x - 4$

(4.3) ?!

A DE is *linear* if it can be expressed in the form

$$P(D)\ f(x) \ = \ q(x)$$

where D is the "differential operator" $\frac{d}{dx}$ (which is only meaningful when acting on a $f(x)$ on its right!) and $P(D)$ is a polynomial; and $q(x)$ is a given function.

Example 4.1.

$$\frac{d^2 f}{dx^2} + 7x\frac{df}{dx} - 2f = \sin x$$

$$= (D^2 + 7xD - 2)f = \sin x$$

If the coefficients in P are constants it is a CONSTANT COEFFICIENT LINEAR DE. The order of P is the ORDER of the DE. For example,

$$(D^3 - 7D + 3)f(x) = \tan x$$

is a third order linear constant coefficient DE.

Exercise 4.2. Find a 2nd order linear DE for which $f(x) = e^{3x}$ is a solution.

In general, solving DEs is HARD. So, in general, we use *guesswork* and *standard recipes* to find solutions. Here we will look at many such recipes.

4.1 Solutions I

1. If

$$\frac{df}{dx} = q(x)$$

$$f = \int \frac{df}{dx} dx = \int q(x) dx + c$$

(note the constant of integration c).

2. If

$$\frac{df}{dx} = 7f$$

$$\int \frac{df}{f} = \int 7dx$$

"separation of variables", giving

$$\log_e f = 7x + c$$

so $f = e^c e^{7x} = Ce^{7x}$ where C is a constant.

DEs may have multiple possible solutions in general (or none!). For each $\frac{d}{dx}$ we can (think of) *undoing* it with an $\int -dx$. Each such integration has an integration constant, so a DE of order n has n undetermined constants in its *general* solution. That is, n constants which can be determined freely to obtain a specific solution.

DEs often appear like this:
Solve

$$\frac{d^2f}{dx^2} + 3\frac{df}{dx} = 2x$$

subject to $f(0) = 0$, $f'(0) = 7$. These $x = 0$ conditions are called INITIAL CONDITIONS. Using them (if there are enough) we can *fix* the constants of integration.

4.1.1 Homogeneous Linear DEs

A linear DE of form

$$P(D)f(x) = 0 \tag{4.4}$$

is called HOMOGENEOUS.

Proposition 4.3. *Suppose* $f_1(x)$, $f_2(x)$ *are specific given solutions to (4.4). (This means that* $P(D)f_1(x) = 0$ *and* $P(D)f_2(x) = 0$ *are* true *identities.) Then* $A_1f_1(x) + A_2f_2(x)$ *is also a solution for any constants* A_1, A_2.

This is the miracle of LINEARITY.

Proof:

$$P(D)(A_1f_1(x) + A_2f_2(x)) = A_1P(D)f_1(x) + A_2P(D)f_2(x) = 0 + 0 = 0.$$

Done.

Example 4.4. $(D - 3)(D - 4)f(x) = 0$

Trial form of solution:

$$f(x) = e^{\lambda x}$$

Now,

$$De^{\lambda x} = \frac{de^{\lambda x}}{dx} = \lambda e^{\lambda x}$$

so substitution gives

$$(\lambda - 3)(\lambda - 4)e^{\lambda x} = 0.$$

This is true for all x only when $\lambda = 3$ or 4, so $f_1(x) = e^{3x}$ and $f_2(x) = e^{4x}$ are both solutions.

Exercise 4.5. So is $7e^{3x} + 21\pi e^{4x}$. Confirm.

More generally, suppose

$$\prod_{i=1}^{n}(D-\lambda_i)f(x)=0$$

where the product is the factored form of $P(D)$. Then

$$f(x)=\sum_i A_i e^{\lambda_i x}$$

is the *general* solution.
(Note

Proposition 4.6. *Every complex polynomial may be factorised with linear factors over the complex numbers.*

so we can ALWAYS solve constant coefficient $P(D)f=0$ in this way.)

Example 4.7.

$$\frac{d^2f}{dx^2}=-9f$$

$$(D^2+9)f=0=(D+3i)(D-3i)$$

so

$$f=A_1e^{3ix}+A_2e^{-3ix}.$$

Note as usual that

$$\frac{e^{3ix}+e^{-3ix}}{2}=\cos 3x$$
$$\frac{e^{3ix}-e^{-3ix}}{2i}=\sin 3x$$

so

$$f=C\cos 3x+D\sin 3x$$

is an equally good way of writing the answer.

Note here that 2nd order DE, implies 2 undetermined constants in the general solution.

Overall, solving an nth order linear homogeneous constant coefficient DE requires *only* to find the roots of the associated $P(D)$.

The polynomial equation $P(m) = 0$ is sometimes called the *auxilliary equation* (the dummy variable name m here has just been chosen to avoid confusion with commonly used variable names, already possibly occuring in the problem).

4.1.2 Inhomogeneous DEs

$$P(D)f(x) = q(x) \neq 0 \tag{4.5}$$

where $q(x)$ is given.

We break the problem of finding the general solution into 2 parts:
1. find any single solution by guesswork (!) (called the particular solution f_P);
2. find the general solution to $P(D)f(x) = 0$ (called the *associated* homogeneous DE) as above. (NB, we again refer to the corresponding polynomial equation as the auxilliary equation.)

The general solution to the associated homogeneous DE is called the Complementary Function f_{CF}.

Proposition 4.8. *The general solution to (4.5) is the sum of these.*

Proof: Exercise.

Example 4.9.

$$\frac{df}{dx} - 3f = 7x^2 + 3x$$

Looking at the form of the RHS, we guess that $f(x)$ a polynomial might be a solution. Try it! ...

More Clues:
If the RHS looks like ae^{bx} then try

$$f_P = Ax^k e^{bx}$$

where k is such that the auxilliary equation has $m = b$ as a root of multiplicity k.
If the RHS looks like $a\sin(bx)$ then try

$$f_P = x^k(A\sin(bx) + B\cos(bx))$$

where k is such that the auxilliary equation has $m^2 + b^2$ as a real quadratic factor of multiplicity k.

Example 4.10.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$$

The auxilliary equation is $m^2 - 3m + 2 = 0$, so $m = 1, 2$, so try a solution of form

$$Y(x) = Axe^{2x}$$

Substituting in gives

$$A(x4e^{2x} + 2.2e^{2x} - 3x2e^{2x} - 3e^{2x} + 2xe^{2x}) = e^{2x}$$

so that $A = 1$. Thus the general solution is

$$y = Ce^x + De^{2x} + xe^{2x}$$

where C, D are arbitrary constants.

Example 4.11.

$$\frac{d^2y}{dx^2} + 4y = 3\sin 2x$$

$$m^2 + 4 = 0$$

so try

$$Y(x) = x(A\sin 2x + B\cos 2x).$$

Substitute into the DE to get

$$Y = -\frac{3}{4}x\cos 2x$$

then

$$y = Ce^{2ix} + De^{-2ix} - \frac{3}{4}x\cos 2x = E\cos 2x + F\sin 2x - \frac{3}{4}x\cos 2x.$$

Exercise 4.12.

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 6x + \sin x$$

[1]

4.2 Wronskians, Variation etc.

Consider

$$a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=f(x) \tag{4.6}$$

($f(x)$ given), with Complementary Function (CF)

$$y=A_1y_1+A_2y_2$$

(A_1, A_2 constants). Now try to find

$$Y=v_1(x)y_1+v_2(x)y_2$$

a solution of (4.6), given that

$$v_1'y_1+v_2'y_2=0 \tag{4.7}$$

(so that

$$\frac{dY}{dx}=v_1y_1'+v_2y_2'$$

$$\frac{d^2Y}{dx^2}=v_1y_1''+v_2y_2''+v_1'y_1'+v_2'y_2'$$

etc.). Substituting into (4.6):

$$v_1'y_1'+v_2'y_2'=\frac{f(x)}{a_0}$$

so from (4.7) we have

$$v_1'=\frac{-y_2f}{a_0(y_1y_2'-y_1'y_2)}$$

[1] Answer for Particular solution:

$$Y(x)=3(x+\frac{1}{2})+\frac{1}{10}(2\sin x+\cos x).$$

($y_1y_2' - y_1'y_2$ is called the WRONSKIAN)

$$v_2' = \frac{y_1 f}{a_0(y_1y_2' - y_1'y_2)}$$

so

$$v_1 = -\frac{1}{a_0}\int \frac{y_2 f}{y_1y_2' - y_1'y_2}dx$$

and v_2 similarly. Thus the general solution to (4.6) is

$$y = (A_1 + v_1)y_1 + (A_2 + v_2)y_2.$$

Example 4.13.

$$\frac{d^2y}{dx^2} - y = e^x$$

CF: $y_1 = e^x$, $y_2 = e^{-x}$

$$Y(x) = v_1e^x + v_2e^{-x}$$

where

$$v_1 = -\int \frac{e^{-x}e^x}{-e^xe^{-x} - e^xe^{-x}}dx = x/2$$

$$v_2 = \ldots = -\frac{-e^{2x}}{4}.$$

Thus the general solution is

$$y = (A_1 + \frac{x}{2})e^x + (A_2 - \frac{e^{2x}}{4})e^{-x}$$

4.3 General linear second order equations

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$$

with p, q, f given, CANNOT always be solved!

Sometimes it can be ... E.g. by substitution, or by knowing one solution already (somehow!).

Example 4.14. Suppose $y = v(x)$ IS a solution of

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

A second solution can be found by putting

$$y = u(x)v(x)$$

and solving for $u(x)$. This gives

$$\frac{1}{w}\frac{dw}{dx} + \frac{2}{v}\frac{dv}{dx} + \frac{1}{x} = 0$$

($w = \frac{du}{dx}$), so

$$\frac{du}{dx} = \frac{A}{xv^2}$$

(by integration, with A the integration constant) and

$$u(x) = A\int \frac{dx}{xv^2} + B$$

so

$$y = Av\int \frac{dx}{xv^2} + Bv.$$

In particular, consider

Example 4.15.

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0$$

One solution, note, is $y_1 = x$. So, try $y_2 = xu(x)$. Substituting:

$$\frac{d^2u}{dx^2} - \frac{du}{dx} = 0$$

so $u = Ae^x + B$ and

$$y = x(Ae^x + B)$$

is the general solution.

4.4 Series solution methods

Example 4.16. To solve

$$(1-x^2)\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 3y = 0 \tag{4.8}$$

guess a solution of the form

$$y = y(0) + xy^{(1)}(0) + \frac{x^2}{2!}y^{(2)}(0) + \ldots + \frac{x^r}{r!}y^{(r)}(0) + \ldots$$

where $y^{(r)}(0) = \frac{d^r y}{dy^r}\Big|_{x=0}$ a *constant* (as before).

This is the MacLaurin Expansion (ME) of an arbitrary differentiable function $y(x)$. Repeatedly differentiating (4.8) gives

$$(1-x^2)y^{(n+2)} - x(2n+5)y^{(n+1)} - (n+1)(n+3)y^{(n)} = 0$$

(exercise!). Substituting our ME in:

$$y^{(n+2)}(0) = (n+1)(n+3)y^{(n)}(0)$$

(using $x = 0$).

Hence

$$y^{(2)}(0) = 1 \times 3 \times y(0)$$

$$y^{(3)}(0) = 2 \times 4 \times y^{(1)}(0)$$

$$y^{(4)}(0) = 3 \times 5 \times y^{(2)}(0) = 1 \times 3^2 \times 5 \times y(0)$$

etc., giving

$$y = y(0)\left(1 + \frac{1.3}{2!}x^2 + \frac{1.3^2.5}{4!}x^4 + ...\right)$$
$$+y^{(1)}(0)\left(x + \frac{2.4}{3!}x^3 + \frac{2.4^2.6}{5!}x^5 + ...\right)$$

Note as usual 2 constants undetermined. Fixing these with 2 initial conditions we would have a good (!) approximation to y *close to* $x = 0$.

To obtain an approximate solution which is good close to $x = a$ use the TAYLOR EXPANSION.

Example 4.17. Solve

$$x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0 \tag{4.9}$$

near $x = 0$.

Differentiate n times:

$$xy^{(n+2)} + (1+n+x)y^{(n+1)} + (n+2)y^{(n)} = 0$$

taking $x = 0$:

$$y^{(n+1)}(0) = -\frac{n+2}{n+1}y^{(n)}(0).$$

Thus the coefficients $y^{(r)}(0)$ in the ME are:

$$y^{(1)}(0) = -2y(0)$$

$$y^{(2)}(0) = -\frac{3}{2}y^{(1)}(0) = 3y(0)$$

$$y^{(3)}(0) = -\frac{4}{3}y^{(2)}(0) = -4y(0)$$

$$y^{(4)}(0) = -\frac{5}{4}y^{(3)}(0) = 5y(0)$$

so

$$y = y(0)\left(1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \frac{5}{4!}x^4 + \ldots + (-1)^r\frac{r+1}{r!}x^r + \ldots\right)$$

($y(0)$ arbitrary).

NB, only one arbitrary constant, so this is not the general solution. Following our previous notes we may call this solution y_1 and try $y_2 = u(x)y_1$ in (4.9).

Exercise 4.18. Solve for $u(x)$.

Answer:

$$u = c\int \frac{e^{-x}}{xy_1^2}dx.$$

Chapter 5

Laplace Transforms

$$\mathcal{L}_p[f(x)] = \int_0^\infty e^{-px} f(x)\, dx$$

is the Laplace transform of $f(x)$ $(x > 0)$.
IT IS A LINEAR OPERATOR:

$$\mathcal{L}_p[kf(x)] = k\mathcal{L}_p[f(x)]$$

$$\mathcal{L}_p[f(x) + g(x)] = \mathcal{L}_p[f(x)] + \mathcal{L}_p[g(x)]$$

Simple ones:

$$\mathcal{L}_p[e^{ax}] = \int_0^\infty e^{-px} e^{ax}\, dx = \frac{1}{p-a}$$

(provided $p > a$).

$$\mathcal{L}_p[x^n] = \int_0^\infty e^{-px} x^n\, dx = ?$$

Recall the mnemonic for integration by parts:

$$duv = udv + vdu$$

or

$$\int udv = uv - \int vdu.$$

In our case we have $u = x^n$ and $dv = e^{-px}$ so we get

$$\mathcal{L}_p[x^n] = \left[\frac{x^n e^{-px}}{-p}\right]_0^\infty - \int_0^\infty \frac{e^{-px}}{-p} n x^{n-1} dx = \frac{n!}{p^{n+1}}.$$

Exercise 5.1. Use deMoivre's theorem etc. to compute $\mathcal{L}_p[\sin ax]$, $\mathcal{L}_p[\cos ax]$, $\mathcal{L}_p[\sinh ax]$.

"Shift theorem":

$$\begin{aligned}\mathcal{L}_p[e^{-ax} f(x)] &= \int_0^\infty e^{-px} e^{-ax} f(x)\, dx \\ &= \int_0^\infty e^{-(p+a)x} f(x)\, dx = \mathcal{L}_{p+a}[f(x)]\end{aligned}$$

Thus

$$\mathcal{L}_p[e^{-ax}\sin(bx)] = \frac{b}{(p+a)^2 + b^2}$$

$(p > -a)$.

5.1 Inverse operator

$$\mathcal{L}_x^{-1}[\mathcal{L}_p[f(x)]] = f(x)$$

Again $\mathcal{L}_x^{-1}$ is linear.

E.g. Since $\mathcal{L}_p[1] = \frac{1}{p}$

$$\mathcal{L}_x^{-1}[\frac{1}{p}] = 1$$

and $\mathcal{L}_x^{-1}[\frac{k}{p}] = k$.

Example 5.2. $\mathcal{L}_x^{-1}\left[\frac{1}{(p+a)(p+b)}\right]$ — use PARTIAL FRACTIONS.

$$\frac{1}{(p+a)(p+b)} = \frac{A}{(p+a)} + \frac{B}{p+b}$$

implies $B = -A = \frac{1}{a-b}$ $(a \neq b)$, so

$$\mathcal{L}_x^{-1}\left[\frac{1}{(p+a)(p+b)}\right] = \frac{1}{b-a}\left(e^{-ax} - e^{-bx}\right).$$

Exercise 5.3. Use partial fractions to determine

$$\mathcal{L}_x^{-1}\left[\frac{p}{(p^2+a^2)(p^2+b^2)}\right]$$

and

$$\mathcal{L}_x^{-1}\left[\frac{1}{p^2(p^2+a^2)}\right]$$

[1]

Now, what is the point of all this?!

5.2 Transforms of differentials

$$\mathcal{L}_p[\frac{dy}{dx}] = \int_0^\infty e^{-px}\frac{dy}{dx}dx = \left[ye^{-px}\right]_0^\infty + p\int_0^\infty e^{-px}y\,dx$$
$$= -y(0) + p\mathcal{L}_p[y]$$

(assume $e^{-px}y \to 0$ as $x \to \infty$).

Similarly,

$$\mathcal{L}_p[\frac{d^2y}{dx^2}] = \quad (exercise)$$

$$\mathcal{L}_p[\frac{d^ny}{dx^n}] = p^n\mathcal{L}_p[y] - p^{n-1}y(0) - p^{n-2}y^{(1)}(0) - \ldots - py^{(n-2)}(0) - y^{(n-1)}(0).$$

5.3 Solution of Ordinary Differential Equations

Solve

$$(D+2)y = \cos x$$

[1] Answers:

$$\frac{1}{b^2-a^2}\left(\cos ax - \cos bx\right)$$
$$\frac{x}{a^2} - \frac{1}{a^3}\sin ax.$$

given $y(0) = 1$.

Apply the Laplace transform to both sides:

$$p\mathcal{L}_p[y] - y(0) + 2\mathcal{L}_p[y] = \frac{p}{p^2+1}$$

so

$$\mathcal{L}_p[y] = \frac{p}{(p+2)(p^2+1)} + \frac{y(0)}{p+2}.$$

Now

$$\frac{p}{(p+2)(p^2+1)} = \frac{A}{p+2} + \frac{Bp+C}{p^2+1}$$

implies $A = -B = -2/5$, $C = 1/5$. Thus

$$\mathcal{L}_p[y] = \frac{1}{5}\frac{1}{p^2+1} + \frac{2}{5}\frac{p}{p^2+1} + \frac{3}{5}\frac{1}{p+2}.$$

Now invert the Laplace transform:

$$y(x) = \frac{1}{5}\sin x + \frac{2}{5}\cos x + \frac{3}{5}e^{-2x}.$$

Example 5.4. Solve $(D^2 + a^2)y = \sin bx$.

Laplace:

$$p^2\mathcal{L}_p[y] - py(0) - y^{(1)}(0) + a^2\mathcal{L}_p[y] = \frac{b}{p^2+b^2}$$

so

$$\mathcal{L}_p[y] = \frac{b}{(p^2+a^2)(p^2+b^2)} + \frac{y(0)}{(p^2+a^2)} + \frac{y^{(1)}(0)}{(p^2+a^2)}$$

Partial fractions:

$$\mathcal{L}_p[y] = \frac{b}{b^2-a^2}\left(\frac{1}{(p^2+a^2)} - \frac{1}{(p^2+b^2)}\right)$$
$$+\frac{y(0)p}{(p^2+a^2)} + \frac{y^{(1)}(0)}{a}\frac{a}{(p^2+a^2)}$$

Invert Laplace:

$$y(x) = \frac{b\sin ax}{a(b^2-a^2)} - \frac{\sin bx}{(b^2-a^2)} + y(0)\cos ax + \frac{y^{(1)}(0)}{a}\sin ax.$$

Example 5.5.

$$(D^2 + 2)y(t) - x(t) = 0$$
$$(D^2 + 2)x(t) - y(t) = 0$$

Solve simultaneously! for x and y, subject to $x(0) = 2$, $y(0) = 0$, $x^{(1)}(0) = 0$, $y^{(1)}(0) = 0$.

First Laplace transform everything:

$$p^2\mathcal{L}_p[y] - py(0) - y^{(1)}(0) + 2\mathcal{L}_p[y] - \mathcal{L}_p[x] = 0$$
$$p^2\mathcal{L}_p[x] - px(0) - x^{(1)}(0) + 2\mathcal{L}_p[x] - \mathcal{L}_p[y] = 0$$

Rearranging:

$$(p^2 + 2)\mathcal{L}_p[y] - \mathcal{L}_p[x] = 0$$
$$(p^2 + 2)\mathcal{L}_p[x] - \mathcal{L}_p[y] = 2p$$

Eliminate, say, $\mathcal{L}_p[y]$:

$$\mathcal{L}_p[x] = \frac{2p(p^2 + 2)}{(p^2 + 1)(p^2 + 3)}$$

Apply partial fractions method:

$$x(t) = \cos t + \cos\sqrt{3}t$$

Now eliminate $\mathcal{L}_p[x]$ instead:

$$y(t) = \cos t - \cos\sqrt{3}t.$$

Note that it is relatively easy to check answers by substitution. So, always check! CHECK NOW!

Exercise 5.6. Verify

$$\mathcal{L}_p[a + bx] = \frac{ap + b}{p^2}$$
$$\mathcal{L}_p[\frac{1}{\sqrt{x}}] = \sqrt{\frac{\pi}{p}}.$$

Exercise 5.7. Solve

$$(D^2 + 4D + 8)y = \cos 2x$$

given $y(0) = 2$, $\frac{dy}{dx}\big|_{x=0} = 1$, using Laplace transforms. (And check your answer!)

Chapter 6

More exercises

Exercise 6.1. This question begins with some revision of Taylor series for functions of one variable, before going on to investigate the integral of a function of two variables.

(a) Show that every Taylor series is a Maclaurin series for a certain related function.

(b) For $g(x)$ a differentiable function, let $g_i(x)$ denote the Maclaurin series for g up to the term in x^i.

Recall that the Taylor series for $\cos(x)$ about $x = 0$ (the Maclaurin series) begins $1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$. Thus we have the approximation

$$\cos(.1) \cong 1 - \frac{.01}{2} + \frac{.0001}{24} + ... = 1 - .005 + .000004 + ... \cong .9950$$

(angles in radians) correct in all four significant figures. Note that each term in the series contributes in a lower decimal place than the one before, which makes it clear that the series is convergent.

Using our notation above, if $g = \cos(x)$ then $g_2 = 1 - \frac{x^2}{2}$. Sketch the functions g and g_2 on the same graph, in the region $-.1 < x < .1$. Make a separate sketch showing the functions g and g_2 on the same graph, in the region $-1 < x < 1$. Comment on the relationships between g and g_2 illustrated by your sketches.

(c) Let $f(x,y) = \cos(x)\cos(y)$. Let f_{22} denote the Taylor series about $(0,0)$ for f up to terms quadratic in x and y (i.e. neglecting cubic, biquadratic, and higher order terms). Determine f_{22} and hence compute

$$I_1 = \int\int_R f_{22} dx dy$$

where R is the square region given by $-.1 < x, y < .1$. Compute

$$I_2 = \int\int_R f dx dy$$

directly for the same region R. Compare and comment on your answers.

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