COURSEWORK AND FURTHER EXERCISES I

Attempt all of the following exercises. Solutions to exercises 1.1, 1.3, 1.4 are to be submitted as coursework.

Exercise 1.1.

(a) As noted in lectures, the ideas of domain, limit, and derivative of a function of one real variable all have their counterparts for functions of two variables. Recall that the domain of the function $f(x) = \frac{1}{1-x}$ must exclude the point $x = 1$. We may write the domain as domain$(f) = \mathbb{R} \setminus \{1\}$ in set notation.

The domain of a function of two real variables is, then, some subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. What is the domain of $\frac{1}{1-\exp(xy)}$?

Given $\psi(x, y) = 2x^3y + e^{xy^2} + 14$, what is the domain of ψ? Determine the partial derivatives ψ_x, ψ_{xx} and ψ_y.

(b) Find all the stationary points (maxima, minima and saddle points) of the function

$$g(x, y) = x^2y^2 + 2x^2 + y^2.$$

(c) Find all the stationary points (maxima, minima and saddle points) of the function

$$h(x, y) = 4x^3 + 4y^3 - 3x - 12y + k$$

where $k = 3$.

(d) Show that the function

$$f(x, y) = xy\, e^{-\frac{x^2+y^2}{2}}$$

has a stationary point at $x = 0$, $y = 0$.

Find all the other stationary points of $f(x, y)$. Determine the nature of the stationary point at $(0, 0)$.

Exercise 1.2. Use Taylor's theorem to expand the function $f(x, y) = x^3y^3$ up to second order in the displacements h, k from the point $(1, 1)$.

Verify your result directly by setting $x = 1 + h$ and $y = 1 + k$ in the function $f(x, y)$.

Exercise 1.3.

(a) The *region of integration* of the double integral

$$I_1 = \int_0^{l_x} \left(\int_0^{l_y} l_z\, dy \right) dx$$

is the rectangle in the (x, y) plane with corners $(0, 0), (l_x, 0), (l_x, l_y), (0, l_y)$. Sketch this region in case $l_x = 2$, $l_y = 3$, $l_z = 8$. What is the value of I_1? (Hint: you do not need to do an integral!)

Note that the equation $x = 0$ defines a *plane* in three dimensions. Sketch the volume bounded by $x = 0$, $y = 0$, $z = 10$ and $z = x + y$. Compute the volume by means of a double integral. (You must show the working for your double integral, even if you can work out the answer by direct geometrical means.)

(b) Sketch the region of integration in the double integral

$$I = \int_0^1 \left(\int_{\sqrt{x}}^1 \pi \sin\left(\frac{y^3 + 1}{2}\right) dy \right) dx.$$

Re–express I with the x–integral as the inner (first) integral. By thus changing the order of integration, evaluate I. (Hint: you will need to make at least one change of variables.)

(c) Show that in polar coordinates the equation of the circle $(x - 1)^2 + y^2 = 1$ takes the form $r = 2\cos\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Hence, by using the *cylindrical* coordinate system (r, θ, z) or otherwise, find the volume of the solid enclosed in the vertical cylinder defined by the circular section $(x - 1)^2 + y^2 = 1$, bounded below by the plane $z = -1$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

Exercise 1.4.

(a) Common sense shows that the point of closest approach of the curve $2y^3 + 2x^3 + y^2 + x^2 = 0$ lying in the (x, y)-plane to the point $(x, y, z) = (0, 0, 1)$ on the z-axis is the point $(0, 0, 0)$. Show explicitly that the method of Lagrange's multipliers confirms this fact.

(b) Find the shortest distance from the point $P = (0, 0, 2)$ to the curve $x^2 + 8xy + 7y^2 = 45$ in the (x, y)-plane.

Exercise 1.5. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

where A, B are arbitrary constants.

Find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2.$$

Hence determine the general solution of this inhomogeneous equation.