Calculus COURSEWORK 2

1. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0,$$

where A, B are arbitrary constants.

Find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x^2$$

and of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 1$$

Hence determine the general solution of each of these inhomogeneous equations. Deduce the general solution to

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x^2 + 1.$$

2. (a) *Carefully* write down the definition of the Laplace transform $\mathcal{L}[f(x)]$ of the function $f(x)$.

   (b) Illustrate the meaning of this definition by working out the transform of the funtion $f(x) = \cos(x)$, showing all working.

   (c) Using your definition of $\mathcal{L}[f(x)]$ above show that if a, b are constants and f, g functions then

$$\mathcal{L}[af(x) + bg(x)] = a\mathcal{L}[f(x)] + b\mathcal{L}[g(x)]$$

(i.e., the Laplace transform is LINEAR).
(HINT: you may use the well known identity

$$\int (a\ f(x) + b\ g(x))dx \ = \ a\int f(x)dx \ + \ b\int g(x)dx$$

i.e., linearity of integration.)

PTO

3. Solve the following *by the Laplace transform method.*
(Here we use the shorthand

$$D = \frac{d}{dx}$$

as usual.
You may use your notes, or other sources, to obtain the Laplace transforms and inverse Laplace transforms of any functions you need. However, be sure to clearly quote in your coursework submission any such result which you use, *before* you use it! For example, you might write:
I will use the standard Laplace transform $\mathcal{L}[1] = \frac{1}{p}$.)

(a) Solve

$$(D + 2)y = 0$$

for the function $y(x)$, given that $y(0) = 12$.

(b) Solve

$$(D + 4)y = 7$$

for the function $y(x)$, given that $y(0) = 1$.

(c) Solve

$$(D^2 + 4D + 8)y = 2\cos(2x)$$

for the function $y(x)$, given that $y(0) = 4$ and $\frac{dy}{dx} = 1$ at $x = 0$.
(HINT: $p^2 + 4p + 8 = (p + 2)^2 + 2^2$.)

4. Solve *simultaneously* for functions $y(x)$ and $z(x)$:

$$(D + 1)y + Dz = 0$$

$$(D - 1)y + 2Dz = 3e^{-x}$$

where $y(0) = 3/2$ and $z(0) = 0$.

5. The *Havanga Tower* in the City of London (headquarters building of the *Allied HTN Banking Group*) has the shape of a section of a quarter of a parabolic cylinder. To be precise, in a suitable coordinate system, its floor lies in the plane $z = 0$, indeed in the positive quadrant of this (x, y)-plane. Its roof/curved-wall shape is given by $z = 4 - x^2$, and its other three walls lie in the planes $x = 0$, $y = 0$ and $y = 3$. Sketch this shape.

Your insurance company insures such buildings by volume. Write down a triple integral whose value gives the volume of this building, in appropriate units. Evaluate your integral.