MATH 3225 Topology Final Assessment Solutions

NB this is a 'take home' exam. Some questions are relatively open, and a broad range of nice correct answers are possible and can achieve marks. Indeed quite generally marking schemes in this setting are intended as indicative rather than totally rigid.

1. (a) We saw in the course that the natural projection

$$p : X \longrightarrow X/\sim$$

defined by $p(x) = [x]$ is continuous and we trivially have $p(X) = X/\sim$. We also saw that continuous images of compact, connected, path-connected topological spaces are compact, connected, path-connected respectively. Thus these three invariants are passed from X to $X/\sim$.

The remaining two invariants are not passed from X to $X/\sim$.

A counterexample in the notes shows that it is possible that X is Hausdorff and $X/\sim$ is not Hausdorff.

Consider another example from the notes: the cylinder. We know that the square $X = [0,1] \times [0,1]$ is simply connected since (for example) $[0,1]$ is clearly simply connected, and the product of two simply connected spaces is simply connected.

However by exercise sheet 5 question 7 (or other good argument), we know that

$$\pi_1(X/\sim) \cong \mathbb{Z} \ncong \pi_1(X).$$

[6 marks]

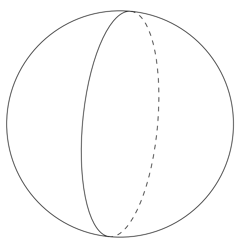
(b) i. We proved in exercise sheet 4 question 10 that

$$D^2/\partial D^2 \cong S^2$$

where $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. Let $\tilde{f} : [0,1] \longrightarrow D^2$ be defined by

$$\tilde{f}(s) = (-1 + 2s, 0).$$

This is continuous since both of its components are continuous as maps into $\mathbb{R}$. Letting $f(s) := p \circ \tilde{f}(s)$, we have immediately that $f : [0,1] \longrightarrow S^2$ is continuous (composition of continuous functions) and $f(0) = f(1) = [(1,0)]$ making f a loop. The image of f is a great circle (equator) passing through the north pole of S^2.



There are many possibilities for the second loop. For example

$$\tilde{f}_1(s) = (0, -1 + 2s).$$

[3 marks]

ii. We cannot here use either of our previous loops anyway since they pass through the excluded point. Any two chords not through (0,0) will do.

[2 marks]

iii. Any two chords avoiding the two excluded points will do.
(Reserve the last mark for some observation about homotopy.)

[3 marks]

(c) Our definition of T^2 is, schematically, the unit square (let's label the edges ABCD) with parallel orientation arrows on the parallel edges — meaning that B and D are identified pointwise according to the arrows; and A,C similarly.
One way to proceed is to build something homeomorphic to the square from two disconnected squares (quotiented together pointwise along an edge, say), and then use the fact that the quotient of a quotient is a quotient. In other words we finish with a T^2-building quotient analogous to the defining construction but on the new 'square'.
(Other well-explained constructions are acceptable.)

[7 marks]

(d) It is fine here to follow the lecture notes fairly closely at first (Theorem 11.15 in ver.2.10 of the notes). The key extra thing (for full marks) is to discuss the completeness property of $\mathbb{R}$ and how it is used.

[4 marks]

2. (a) For $n \geq 3$ consider $\mathbb{R}^n$ equipped with the usual topology and let

$$L = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, x_2) = (0, 0)\}.$$

Prove that $\mathbb{R}^n \setminus L$ is path connected and $\pi_1(\mathbb{R}^n \setminus L) \cong \mathbb{Z}$.

[7 marks]

(b) Consider $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ and let

$$S^{n-2} := \{x \in S^n : (x_1, x_2) = (0, 0)\}.$$

Prove that $S^n \setminus S^{n-2}$ is always path connected and $\pi_1(S^n \setminus S^{n-2}) \cong \mathbb{Z}$.

[8 marks]

(c) Find an open subset $U \subset \mathbb{R}^3$ so that U is path connected, but not simply connected and $\pi_1(U) \not\cong \mathbb{Z}$. You should justify that your set U has the desired properties, and write down what $\pi_1(U)$ is. [10 marks]

Solution:

(a) **(The proof of this question is potentially quite long, but it is a translation of case $n = 3$ which was set as an exercise, and the students have a sketch solution.)**

Recall that $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^2$,

$$\pi(x_1, x_2, \ldots, x_n) = (x_1, x_2)$$

is continuous and by definition we have $L = \pi^{-1}(\{(0, 0)\})$. We also have that $\mathbb{R}^n \setminus L = \{x \in \mathbb{R}^n : (x_1, x_2) \neq (0, 0)\}$.

We first show that $\mathbb{R}^n \setminus L$ is path connected. Picking $a, b \in \mathbb{R}^n \setminus L$ we have that $\pi(a), \pi(b) \neq (0, 0)$ and, trivially $(1 - t)(a_1, a_2) + t\pi(a) = (a_1, a_2) \neq (0, 0)$ for all $t \in [0, 1]$ (similarly for b). Thus we may define

$$\alpha(t) = (1 - t)a + t(\pi(a), 0, \ldots, 0)$$

which is a path from a to $(\pi(a), 0 \ldots, 0)$ in $\mathbb{R}^n \setminus L$. Let

$$\beta : [0, 1] \longrightarrow \mathbb{R}^n \setminus L$$

be the equivalent path from b to $(\pi(b), 0 \ldots, 0)$.

[2 marks]

We saw in the lectures that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is path connected, thus there is a path $\tilde{\gamma}$ from $\pi(a)$ to $\pi(b)$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Defining $\gamma(t) = (\tilde{\gamma}(t), 0, \ldots, 0)$ we see that γ is a path from $(\pi(a), 0, \ldots, 0)$ to $(\pi(b), 0 \ldots, 0)$ in $\mathbb{R}^2 \setminus \{(0, 0)\} \times \{(0, 0, \ldots 0)\}$. Thus $\alpha * \gamma * \overline{\beta}$ is our path from a to b.

[2 marks]

This allows us to define a function

$$H : \mathbb{R}^n \setminus L \times [0, 1] \longrightarrow \mathbb{R}^n \setminus L$$

by $H(x, t) = (1 - t)x + t\pi(x)$. H is continuous since all of its components are continuous. Furthermore, $H(x, 0) = x$ for all $x \in \mathbb{R}^n \setminus L$, $H(y, t) = y$ for all $y \in \mathbb{R}^2 \setminus \{(0, 0)\} \times \{(0, 0, \ldots 0)\}$ and $t \in [0, 1]$ and finally, $H(x, 1) \in$

$\mathbb{R}^2 \setminus \{(0,0)\} \times \{(0,0,\dots 0)\}$ for all x. Therefore H is a strong deformation retraction from $\mathbb{R}^n \setminus L$ to $\mathbb{R}^2 \setminus \{(0,0)\} \times \{(0,0,\dots 0)\}$.

[1 mark]

It remains to check that $\mathbb{R}^2 \setminus \{(0,0)\} \cong \mathbb{R}^2 \setminus \{(0,0)\} \times \{(0,0,\dots 0)\}$. So we have

$$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\} \times \{(0,0,\dots 0)\}) \cong \mathbb{Z}$$

and we are done by invoking a Proposition in the notes. (Explicit proof step also acceptable.)

[1 mark]

The function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ given by

$$f(x_1, x_2) = (x_1, x_2, 0, \dots, 0)$$

is continuous and injective with $im(f) = \mathbb{R}^2 \times \{(0,0,\dots 0)\}$. If we restrict π to $\pi| : \mathbb{R}^2 \times \{(0,0,\dots 0)\} \longrightarrow \mathbb{R}^2$ we see that $\pi|$ is continuous and furthermore $f \circ \pi| = Id_{\mathbb{R}^2 \times \{(0,0,\dots 0)\}}$ and $\pi| \circ f = Id_{\mathbb{R}^2}$ so in fact f is a homeomorphism between $\mathbb{R}^2$ and $\mathbb{R}^2 \times \{(0,0,\dots 0)\}$.

[1 mark]

Alternatively: One could prove that $\mathbb{R}^n \setminus L \cong \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}^{n-2}$ and use other results from the exercise sheets/course to obtain the result.

(b) The stereographic projection $f : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$ is defined in the notes by

$$f(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots x_n)$$

and it is a a homeomorphism.

[3 marks]

A Proposition in the notes now tells us that if $Z \subset S^n \setminus \{N\}$ then $f : Z \longrightarrow f(Z) \subset \mathbb{R}^n$ is a homeomorphism.

[1 mark]

Notice that setting $Z = S^n \setminus S^{n-2}$ we have $Z \subset S^n \setminus \{N\}$ and $Z = \{x \in S^n : (x_1, x_2) \neq (0,0)\}$.

[1 mark]

Therefore

$$f(S^n \setminus S^{n-2}) = \{x \in \mathbb{R}^n : (x_1, x_2) \neq (0,0)\} = \mathbb{R}^n \setminus L.$$

[2 marks]

So we are done by the first part of this question, and the fact that path connectedness and the fundamental group are topological invariants.

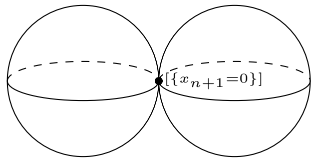
[1 mark]

(c) There are lots of options here. For instance we could let U be $\mathbb{R}^3$ with the coordinate axes removed. There is a strong deformation retraction from U to a 2-sphere with six points removed, which is homeomorphic to $\mathbb{R}^2$ with five points removed. This space then deformation-retracts to a bouquet of five circles whose fundamental group is a free group on five generators - this is not Abelian so $\pi_1(U) \ncong \mathbb{Z}$.

Similarly if we remove n disjoint lines from $\mathbb{R}^3$ we have a path-connected subset whose fundamental group is the free group on n generators: it's easiest to prove this when we take n lines $\{L_i\}_{i=1}^n$ which are all contained in some plane and are mutually parallel. We can then deformation-retract this space to $\mathbb{R}^2$ with n-points removed.

3. (a) X^n is a pair of spheres touching at a single point so it is not a topological manifold.

[2 marks]



When $n = 1$ this is a bouquet of two circles giving $\pi_1(X^1) \cong \langle a, b; \cdot \rangle$ directly from example...in the notes.

[1 mark]

When $n \geq 2$ we will apply Van Kampen's theorem with U and V as suggested below.

[1 mark]

Notice that $U \cap V$ deformation retracts to a point $[\{x_{n+1} = 0\}]$, and furthermore that U (resp. V) deformation retracts to a sphere S^n where $n \geq 2$. [1 mark]

We have seen in the lectures that $\pi_1(S^n)$ is trivial when $n \geq 2$, thus Van Kampen's theorem immediately tells us that $\pi_1(X^n) \cong \{0\}$ the trivial group when $n \geq 2$. [2 marks]

(b) Below you can find a picture of X followed by choices of U and V respectively according to Van Kampen's Theorem.

[2 marks]

From the picture of X we see that it is not a topological manifold.

[1 mark]

Notice that $U \cap V$ deformation retracts to a point $[\Gamma]$, and furthermore that U (resp. V) deformation retracts to a torus.

[1 mark]

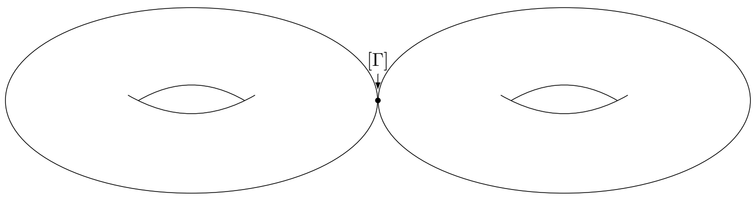
Thus from what we have proved in the lectures: $\pi_1(U, [\Gamma]) \cong \langle a, b; aba^{-1}b^{-1} \rangle$ and $\pi_1(V, [\Gamma]) \cong \langle c, d; cdc^{-1}d^{-1} \rangle$ and $\pi_1(U \cap V, [\Gamma]) \cong \langle \cdot ; \cdot \rangle$.

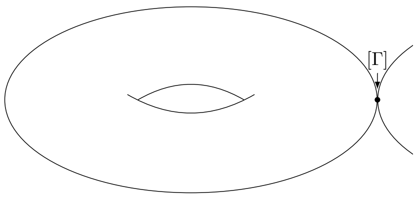
[2 marks]

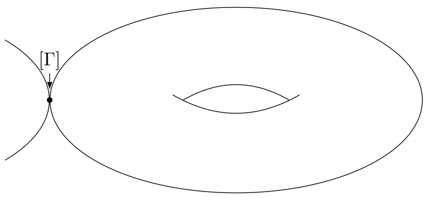
Van Kampen's Theorem tells us immediately that

$$\pi_1(X) \cong \langle a, b, c, d; aba^{-1}b^{-1}, cdc^{-1}d^{-1} \rangle.$$

[2 marks]







(c) If we take the basepoint to be the touching point then our two paths can be each a trip around one of the loops and back to base.

Call the two loop-paths a (around circle A, say) and b; and the group elements $[a]$ and $[b]$ respectively. It is clear that the path compositions ab and ba are well-defined and not equal, so we need to consider whether there is a path of paths between them. But observe that there is no way to drag even the path around A away from A completely, so every path in $[ab]$ starts with a bit that may visit both A and B, but only 'irreversibly' loops around A — and necessarily does so. So $[ab] \neq [ba]$, so $[a][b] \neq [b][a]$.

[10 marks]

4. (a) There are many acceptable ways to do this. For example recall that finite topologies are Alexandroff, so every point has a smallest neighbourhood $n(x)$. Writing these in the order $(n(x), n(y))$ — and writing just x for singleton $\{x\}$ — we have: $T_Y \rightsquigarrow \{(Y,Y),(x,Y),(Y,y),(x,y)\}$. The answer must check that each claimed element is or gives a topology (by construction in this format); and that the list is complete. Here the latter follows since it is the complete subset of the power set of the power set satisfying the first axiom.

[4 marks]

(b) Here again we may refer to the power set of the power set, which is a superset, and finite.

[2 marks]

(c) If X has only one element then the power set of X contains two elements: $\emptyset$ and $X = \{x\}$. Since a topology on X is a collection of subsets of X, τ, which satisfies, in particular, that $X, \emptyset \in \tau$, then the only possible topology is given by the discrete (equivalently indiscrete) topology in this case.

[2 marks]

(d) Throughout the course we have only seen two topologies that can be put on arbitrary sets. The discrete topology and the indiscrete topology.

[1 mark]

The discrete topology on X, that we will denote by τ_1, is the collection of all subsets of X. In other words it is the power set of X. In this case *all* subsets of (X, τ_1) are both open and closed. [2 marks]

The indiscrete topology on X, that we denote by τ_2, only contains X and $\emptyset$. So the *only* open sets in (X, τ_2) are X and $\emptyset$. These are also the only closed subsets in X. [2 marks]

$(X, \tau_1) \cong (X, \tau_2)$ if and only if there exists a homeomorphism $f : (X, \tau_1) \longrightarrow (X, \tau_2)$. We saw in the course that a homeomorphism is a continuous bijection whose inverse is also continuous; and we concluded that f thus induces a bijection between the open sets in the domain, to the open sets in the target. Thus $(X, \tau_1) \cong (X, \tau_2)$ implies that $|\tau_1| = |\tau_2|$.

[2 marks]

When X has at least two elements then certainly $|\tau_1| > 2$. Since $X, \emptyset, \{x\} \in \tau_1$ for any $x \in X$ and $\{x\} \neq X$ by assumption. However, $|\tau_2| = 2$ for any X. So we must have that $(X, \tau_1) \ncong (X, \tau_2)$.

[2 marks]

There are many alternatives to the above argument, e.g.: There can be no continuous bijection $g : (X, \tau_2) \longrightarrow (X, \tau_1)$ when X has at least two elements: since $|g^{-1}(\{x\})| = 1$ so $g^{-1}(\{x\})$ can never be open in τ_2.

(e) There are various acceptable methods here. For example the following, which is not specific to the given Z but works generally:
We can use the same $n(x)$ notation as before. If $|\tau| = 3$ then the list of $n()$'s can (indeed must) have repeated entries, but has exactly two types of entry — one of which is Z itself. The other is $Y \subsetneq Z$, say. (Since then the topology is $\{\emptyset, Y, Z\}$.)
Note that the entry Y is $n(a)$ for each $a \in Y$; and otherwise $n(a) = Z$. Since $Y \neq \emptyset$, there are $2^{|Z|} - 2$ choices. So this is the number of topologies. Two of these topologies are homeomorphic if an action of the symmetric group on Z takes one in to the other, i.e. their Y sets have the same size, and not otherwise. So there are $|Z| - 1$ homeomorphism classes.

If $|\tau| = 4$ then we must choose 2 proper subsets of Z. The first can be chosen freely per se. Since we are working up to homeomorphism it is enough to choose the order of this set. We must choose the second such that the set created is closed under union and intersection. Thus either they are related as $Y \subset Y'$; or they give a partition of Z. In the first case, working up to homeomorphism, again it is enough to choose the order. Thus we have a contribution of $\frac{n(n-1)}{2}$ classes, where $n = |Z| - 1$. In the partition case the second set is determined. To avoid double-counting we can simply restrict the first set order to be in the range $1, ..., |Z|/2$. These constructions are disjoint, so the numbers are simply added to give the total number of homeomorphism classes.

[8 marks]