**Topology** 

**School of Mathematics** **Semester Two 202021**

**Academic integrity:**

- This is an Open Book assessment. You may refer to lecture notes, books, and websites to help you complete the work (as long as you do not disobey the instructions below).
- The work that you submit must be your own work and must represent your own understanding.
- You must not ask others to help you to complete your assessments.
- You must not discuss this assessment with other students.
- You must attach a completed Academic Integrity form at the end of your solutions before you submit them.

**Instructions:**

- There are 5 pages to this assessment.
- There will be **48 hours** to complete this assessment, and an additional **2 hours** to upload your solutions. Late submissions will not be accepted.
- If you believe the assessment contains a mistake or is not clearly written, send an email to the module leader. The module leader will normally respond within four hours of receiving a query, but will not respond outside of normal working hours (09.00-17.00 UK time).
- If you have trouble submitting your work, write to `maths.taught.students@leeds.ac.uk`.
- If a module leader issues a correction this will be posted on Minerva and you will also be notified by email.
- Answer all questions.
- You must show all your calculations. You must write clearly, in sentences.
- There is no page limit, but we expect most students' handwritten solutions to be between $4$ and $12$ pages long.

**1.** Let $X$ be a topological space. Let $\sim$ be an equivalence relation on $X$. Consider the quotient space $X/_\sim$ endowed with the quotient topology.

(a) Consider the following topological invariants: the separated (Hausdorff) property, connectedness, path-connectedness, compactness, the fundamental group.

For each of these invariants decide whether for all $X$ and $\sim$ on $X$, the space $X$ having the property implies that the space $X/_\sim$ has the property (i.e., in the last case, has the same fundamental group). Justify your answer in each case.

(b) Consider the following examples of quotient spaces:
(Here for $X$ a set and $A$ a subset, we use the notation $X/A$ for the relation with equivalence classes singletons for $x \in X \setminus A$ and the other class is $A$.)
(i) $D^2/\partial D^2$ where

$$D^2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$$

with the usual topology, and $\partial D^2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ ;
(ii) $(D^2 \setminus \{(0,0)\})/\partial D^2$
(iii) $(D^2 \setminus \{(0,0), (\frac{1}{10}, \frac{1}{10})\})/\partial D^2$

In each case write down two explicit continuous functions $f : [0,1] \longrightarrow X/_\sim$ with $f(0) = f(1)$. (Thus ending up with six *distinct* functions in total.) Describe your 'loops' in words and pictures, briefly justifying that your examples are indeed loops.

(c) Consider the torus $T^2$ (the surface of the donut) with the usual topology. Find a disconnected subspace $X \subset \mathbb{R}^2$, and an equivalence relation on $X$ so that $X/_\sim \cong T^2$.

(d) Consider $\mathbb{R}$ with the usual topology. Carefully prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval.
(In lectures we gave an indicative proof, leaning on facts about the real line. You can look at this. But take care to explain the nature of any such facts that you use.)

**2.** (a) For $n \geq 3$ consider $\mathbb{R}^n$ equipped with the usual topology and let

$$L = \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, x_2) = (0, 0)\}.$$

Prove that $\mathbb{R}^n \setminus L$ is path connected and that the fundamental group is given by $\pi_1(\mathbb{R}^n \setminus L) \cong \mathbb{Z}$.

(b) Consider $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ and let

$$S^{n-2} := \{x \in S^n : (x_1, x_2) = (0, 0)\}.$$

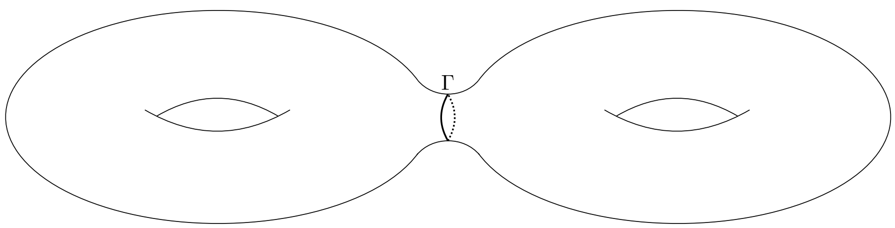
Prove that $S^n \setminus S^{n-2}$ is always path connected and that $\pi_1(S^n \setminus S^{n-2}) \cong \mathbb{Z}$.

(c) Find an open subset $U \subset \mathbb{R}^3$ (equipped with the usual topology) so that $U$ is path connected, but not simply connected and $\pi_1(U) \ncong \mathbb{Z}$. You should justify that your set $U$ has the desired properties, and write down what $\pi_1(U)$ is.

**3.** Here $S^n = \{\underline{x} \in \mathbb{R}^{n+1} : |\underline{x}|^2 = 1\}$. As a topological space it will have the usual topology.

(a) For $n \geq 1$ consider the relation $\sim$ on $S^n$ defined by
$\underline{x} \sim \underline{y} \iff x_{n+1} = y_{n+1} = 0$ or $\underline{x} = \underline{y}$.
For each $n$ say if $S^n/\sim$ is a manifold. For each $n$ describe a pair $U, V$ of path-connected open subsets of $S^n/\sim$ such that $U \cup V = S^n/\sim$ and $U \cap V$ is non-empty and path-connected, suitable for an application of the Seifert-Van Kampen Theorem. Hence give the isomorphism class of the fundamental group $\pi_1(S^n/\sim)$. (You should get a different answer if $n = 1$ or $n \geq 2$. You may use results about $\pi_1(S^n)$ without proof.)

(b) Let $\tilde{X}$ be the orientable surface of genus two depicted below. Let set $\Gamma \subset \tilde{X}$ be as shown ($\Gamma$ is the image of a loop in $\tilde{X}$).




Consider $\sim$ on $\tilde{X}$ defined by $x \sim y \iff x, y \in \Gamma$ or $x = y$. Let $X = \tilde{X}/\sim$. Is $X$ a manifold? Determine $\pi_1(X)$.

(c) Consider again $S^1/\sim$ from (a) above. By first picking a base point, carefully give two paths that are representative of two elements that generate the fundamental group $\pi_1(S^1/\sim)$. Explain whether these generators commute or not.

**4.** For $S$ a set, let $T_S$ denote the set of all topological spaces whose underlying set is $S$.

(a) Let set $Y = \{x, y\}$. Write out the set $T_Y$ explicitly. Prove that the set you wrote out is $T_Y$.

(b) Prove that if $S$ is a finite set then so is $T_S$.

(c) If $X = \{x\}$ is a set with just one element, is there more than one topology we can put on $X$? Explain your answer.

(d) On any set $S$ with $|S| > 1$, show that there are two topologies $\tau_1, \tau_2$ with $(S, \tau_1)$ not homeomorphic to $(S, \tau_2)$.

(e) Let $Z$ be a finite set. State and prove a proposition giving the number of topological spaces $(Z, \tau) \in T_Z$ such that the set $\tau$ of open subsets has order $|\tau| = 3$; and also giving the number of homeomorphism classes of such spaces in $T_Z$.
Then state and prove a proposition giving the number of homeomorphism classes of topological spaces $(Z, \tau) \in T_Z$ such that the set $\tau$ has order $|\tau| = 4$.
(Partial answers, clearly annotated as such, will still win at least partial marks here.)

Before you submit your solutions remember to attach a completed Academic Integrity form. We recommend `www.ilovepdf.com` for signing and merging pdf documents.