Topology

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Based partly on some lovely notes by Ben Sharp,
Josh Cork, Derek Harland and others

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Contents

Chapter 1

Introduction

"Continuity" is a very useful notion in human thought. But what does it mean, exactly? A topology is the minimal extra structure with which we must equip a set (such as physical space) so that the idea of "continuity" makes sense.

The following two sections are introductory descriptions of topology, from different perspectives. (The two sections are similar — almost the same. What does it mean to say that they are almost the same? That itself is a question in topology.)

1.1 Introduction: a generalist's viewpoint

Are you like me? If you are, then you wake up on Wednesday morning feeling that you are pretty much the same person you were when you went to bed on Tuesday night. This notion of same-ness is part of our sense of self. While sleep may have interupted the continuity of consciousness, there is a deeper feeling of continuity of the self that survives sleep. The subject of 'Topology' is about studying and using the general notion of continuity. It underpins our sense of self, but it is also very useful in countless other ways, as we will see.

A set X becomes a topological space if it has 'a topology'. A topology is a collection of subsets of X that must include X and $\emptyset$ and be closed under finite intersections and all unions (see Definition 6.1). To have a reasonable notion of continuity for functions f on a space X to a space Y, we simply require the existence of a topology on each.

(This remarkable simplicity seems just to be a piece of good luck. It is worth admitting that. Just so that we can grasp the intellectual starting point for this branch of study. But because of this good luck,...)
...We can study many profound and useful organisational tools for life at quite a good level of mathematical rigour.

Having said that, our definition allows for many different topologies on a set. And each one can give rise to a different version of continuity. Depending on the application, some are more useful than others, as we will see.

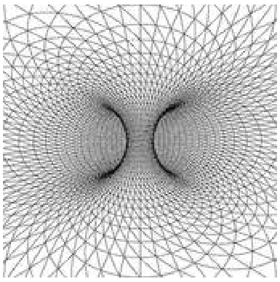
We will determine rigorous and abstract properties of topological spaces which will help us distinguish between topological spaces. Specifically, we will introduce the organisational notions of connectedness, compactness and 'shape'.

When it comes down to detail, different people may have different notions of sameness (think of the different possible notions associated to "she bought the same newspaper every day"). A Topologist's notion of sameness equates any two spaces if (and only if) one can be continuously mapped to the other, bijectively, and such that the inverse map is also continuous. We will explain all these terms.

Why is this useful? (Apart from in understanding one's sense of self.)
Well, its a big scary world. Some tiger-sized things will eat you. Some won't. We need a reliable and practical way to detect the dangerous ones. (Choosing the tiger example in a country with no wild tigers is to choose a lighthearted example, but hopefully the transferability of the point will be clear.) If a given tiger eats you (or someone) then we can mark that tiger down as dangerous. But do you want to give other tigers the benefit of the doubt, or lump them together, just to be safe? Topology is the maths behind the how and what of this 'lumping together'.

Tigers have quite complicated shapes, so let's start with something simpler. Topology sees equivalence between the boundary of a cube, and the boundary of a solid ball (a sphere) — there is a continuous bijection from one to the other whose inverse is also continuous. (An example of such a map?: Imagine putting the cube inside the sphere, and picture light rays radiating from a point inside the cube!) Similarly, the surface of a ring donut is equivalent to the surface of a coffee mug, from the perspective of 'holdability' topology (both have a 'handle').

Can one continuously, and bijectively map the surface of a solid ball to that of a donut with a map whose inverse is continuous? We can rigorously prove that the answer is "NO!"



One can sometimes think of topological manifolds (special kinds of topological spaces) as being made out of some flexible material (flesh, fabric or rubber, say) so that a lot of bending, stretching and shrinking is allowed without changing the underlying topological structure.

Another place where we use topology heavily (again usually subconsciously) is in reading and writing. And indeed in communication generally. The notion that the words BUBBBLEGUM and NARNIA can be found in this picture:

involves a lot of 'experiments' with topological spaces and continuous functions. There are only finitely many words in a dictionary (certainly only finitely many that we need to know), but very many more different ways of writing each letter and each word.

One of our assumptions in these notes is that you can read them! (Passing from formal reading to understanding is another story, but we have to — indeed can — assume that you can read... or you would not have got this far.) If you can read then you already understand and are familiar with a lot of complex concepts and operations. Shortly it will be helpful to us to draw some of these operations (such as arranging objects into ordered sequences) to more explicit attention.

1.1.1 Overview

There are several useful kinds of relationships explained by topology. Useful because they explain how we classify unmanageably large collections of things into useful smaller groupings. One kind is a natural extension of the idea of cardinality (organising sets by size) to sets that are spaces: 'homeomorphism'. A second kind combines topology with everybody's favourite example of a mathematical structure — the real line, and uses it to relate different images of one space in another space: 'homotopy'. Finally, classification is all very well in principle, but how do we do it in practice? How do we tell which class each element belongs to? What can be useful here is a (computable) function on the set being classified that takes all elements of the same class to the same image point — an 'invariant'.

After studying foundational definitions in section 2-6; we investigate homeomorphism in section 8-9; invariants in section 10; homotopy in section 11.

The tools and ideas of topology are used wherever organisation and classification are useful, and hence across all realms of thought. In the course we will prove some powerful results in different fields. E.g. the fundamental theorem of algebra, and a 'ham sandwich' theorem (which proves that any sandwich made from bread, butter, and ham can always be sliced (with a single cut) into two parts, so that each part consists of equal quantities of the three separate ingredients). We can also prove that at each moment in time, there exist antipodal points on the surface of the earth which have the same temperature *and* pressure (assuming that temperature and pressure vary continuously in space).

In the realm of risk assessment we mentioned before — the risk posed by various tiger-sized things — what we need is a classification scheme for such things: a way of quickly identifying them into a grouping of established risk level. For tiger-like things in particular we tend to do this by looking at the surface (assuming the appropriate size of course). The exact shape is not helpful because tigers articulate their bodies when they run, but can we classify things like tigers, say, according to some common properties of their surfaces? In fact we can classify all surfaces up to the kind of articulations and movements that tigers can do. (In practice most hunted animals do their risk assessment classifications essentially subconsciously, rather than with maths research. But it is empowering, and transferable, to know how this works.)

We can also see topology as a bridge between the major sub-disciplines of modern mathematics, sometimes called algebra, analysis, geometry and logic. As we go through, we will see many examples of this bridge in action. Indeed the way it connects the disciplines shines a light on the nature of the disciplines themselves, and the reasons for their existence!

Thanks. I thank Ben for letting me have his lovely notes to use as a starting point (and note that Ben in turn thanked Derek and Josh). I also thank Paula, Joao and Fiona for many useful conversations.

1.2 Introduction: a pure maths viewpoint

(I have borrowed this beautiful short essay directly from Ben's notes for comparison.)

At its heart, topology is concerned with spaces upon which it is possible to discuss/define continuous functions (a topological space), and is geared towards rigorously classifying all such spaces. A space X is a topological space if it has a topology. A topology is a collection of special subsets of X that must satisfy three requirements (see Definition 6.1) - these special subsets are usually called **open subsets**. Remarkably, in order to have a reasonable notion of continuity for functions f on a space X to a space Y, we only require the existence of a topology on X and Y.

We will determine rigorous and abstract properties of topological spaces - topological invariants - which will help us distinguish between topological spaces. Specifically, we will introduce notions of connectedness (what does it mean for a topological space to be connected?), compactness (perhaps the most important concept to pure mathematicians) and 'shape' (or more precisely, homotopy). Topologists equate any two spaces if one can be continuously mapped to the other, bijectively, and whose inverse is also continuous[1]. If no such map exists then the spaces are topologically different.

A topologist sees no difference between the boundary of a cube, and the boundary of a football (a sphere); there is a continuous bijection from one to the other whose inverse is also continuous; what does this map look like?[2] Similarly, the surface of a donut is no different to the surface of a coffee mug, from the perspective of topology (can you imagine why?).

Question: can one continuously, and bijectively map the surface of a football to that of a donut with a map whose inverse is continuous? We'll be able to rigorously prove that the answer is "NO!" by the end of the course. Your intuition should tell you that this would be impossible without tearing one or the other surface.

You can sometimes think of topological spaces (more precisely topological manifolds) as being made out of rubber, so that any bending, stretching or shrinking is allowed without changing the underlying topological structure. However the rubber is so strong that an 'infinite amount' of stretching may survive this process[3], but the following are **not** allowed: tearing of the rubber; folding so hard that two regions become merged; or squeezing so hard that you 'lose dimensions'.

The abstract tools/ideas of topology are used heavily across all subfields of mathematics. We will not have time to go into the more algebraic side of things (via homology and cohomology), however we will introduce homotopy groups and use these to distinguish between different topological spaces. By the end of the course we will also be able to prove some powerful results in different fields: e.g. the fundamental theorem of algebra, and the ham sandwich theorem[4]. One more thing we'll be able to prove by the end of the course: at any moment in time, there exist antipodal points on the surface of the earth which have the same temperature *and* pressure[5]. To give you an idea of the power of topology, see if you can prove this before reading the notes...

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[1] such a map is called a *homeomorphism*

[2] Put the cube inside the sphere and think of light rays emanating from a point inside!

[3] e.g. the continuous function $f : (0, 1) \to (1, \infty)$, $f(x) = \frac{1}{x}$ continuously stretches out a bounded interval to an unbounded one: you can check that the inverse exists and is also continuous

[4] which proves that any sandwich made from bread, butter, and ham can always be sliced (with a single cut) into two parts, so that each part consists of equal quantities of the three separate ingredients

[5] We are making the assumption that temperature and pressure vary continuously in space here

Chapter 2

Preliminaries

We assume familiarity with set theory ideas from earlier, but we will review some of them here in Chapter 2 (and see also appendix Section A).

2.1 Some reminders on sets

We assume here that you are reasonably happy with the idea of a collection of "objects". This is a bit vague and potentially troublesome. But it *is* very useful, and we have to start somewhere. We will use the term 'set' for a collection of objects.

Suppose that we (you and I) both have in mind a set. Let's call it S. To say that we both have it is to say that we agree on what the "elements" are — the objects that are collected in S. Thus if we both have in mind an object x (say), we can agree if the statement 'x is in S' (written $x \in S$) is true or false (if false then we write $x \notin S$).

What might constitute a good "object"? In practice this is anything that we can agree is a good object. Just to get things started with a minimum of trouble, we can say that a set itself can be an object. Let us also say that there is one formal set, call it $\emptyset$, that does not contain any objects — thus postponing the general issue of what an object is by avoiding it. Thus the statement '$x \in \emptyset$' is false for every object x.

Putting these two ideas together, we have another set: the set containing only the set $\emptyset$.

If we have given a name to an object, like $\emptyset$, or X perhaps, then we can write the set containing only that object as $\{X\}$. The only concrete example of this that we have so far is $\{\emptyset\}$. For this at least we can say $\emptyset \in \{\emptyset\}$ and $x \notin \{\emptyset\}$ for all other objects x.

We say that two sets are equal if they contain the same elements; and otherwise they are unequal. Thus $\emptyset \neq \{\emptyset\}$.

Suppose that x and y and z represent objects, somehow agreed between us. One way of writing that x and y and z are in S (that $x, y, z \in S$) is $S = \{x, y, z, ...\}$. Another way is $S = \{y, x, z, ...\}$. If x, y, z are the only elements in S then we can write $S = \{x, y, z\}$. The extension of this notation to more (or fewer) elements can be guessed. (For the moment the question of precisely what the objects x, y, z here are remains mysterious.)

And then, using this notation, another set with un-mysterious objects is $\{\emptyset, \{\emptyset\}\}$. Notice that this is not equal as a set either to $\emptyset$ or to $\{\emptyset\}$. And notice that we can 'iterate' this construction: the set containing all the sets we have so far as elements is a new set; and now we can make another new set by adding this new set as a new element.

With such unappealing constructions of new sets, and hence new objects, we can at least delay the discussion of more interesting (but maybe not clearly defined) objects. We do now have many objects available — just by iterating the construction of adding a new set to a set of sets.

One more device before we really get started. Suppose that a and b represent objects (not even necessarily distinct). An *ordered pair*, denoted (a, b), is a set $\{\{a\}, \{a, b\}\}$.
(Caveat: this notation (a, b) can be used in other contexts as well, to represent other things. So, to be safe, if we do mean it to denote an ordered pair then we will say so explicitly.)

Because of the way we write (and talk; and think) it sometimes looks like there is order in expressions like $\{a, b\}$ already. But note from above that $\{a, b\} = \{b, a\}$ so there is not. However note that $(a, b) \neq (b, a)$ (unless $a = b$) — this is a good exercise to prove.

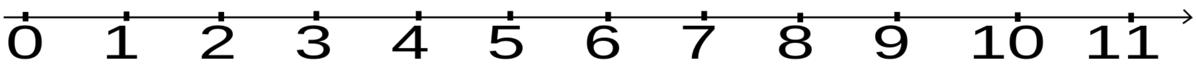
Some further reading:
Beginning Finite Mathematics (Schaum's Outline Series), S Lipschutz et al.
Discrete Mathematics, J K Truss.
Sets, Logic and Categories, P J Cameron (Springer).
Algebra Volume 1, P M Cohn.

2.2 Elementary set theory notations and constructions

Notation: Let
$$\underline{n} \;:=\; \{1, 2, .., n\}$$
Similarly here $\underline{n}' \;:=\; \{1', 2', .., n'\}$, $\underline{n}'' \;:=\; \{1'', 2'', .., n''\}$ and so on.

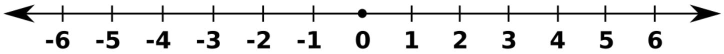
This familiar, unchallenging-seeming notation represents a huge store of knowledge and power derived from set theory: categories, cardinality and computation! The number 1 represents the class of sets in bijection with a certain 'cardinal', $\{\emptyset\}$. And so on. Thus

$$\mathbb{N}_0 : ...$$



Disjoint union yields an algebraic structure on $\mathbb{N}_0 = \{0, 1, 2, ...\}$. There is also an order structure. We order $i > j$ if there exist $a \in i$ and $b \in j$ such that $a \supset b$. The idea of lost kittens gives us negative numbers:

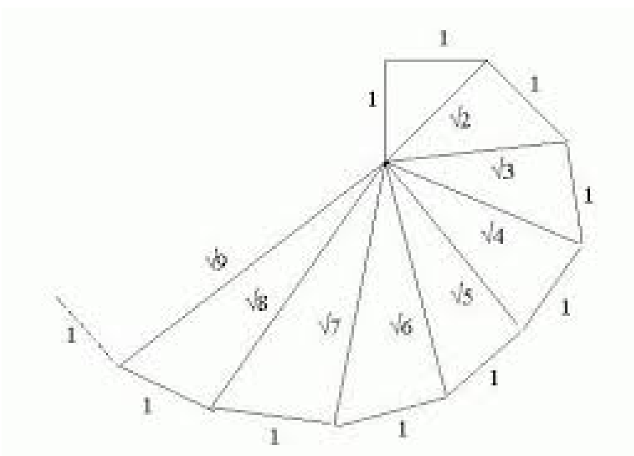
$$\mathbb{Z} : .....$$



One milk shared 3 ways gives us the idea for rational numbers.

$$\mathbb{Q} : ......$$

And we can go on. For example to real numbers. See later.



Some Real Numbers

(**2.2.1**) Think about the Newton–Raphson method for finding roots of, say, $f(x) = x^2 - 2$. Our first guess can be $r_0 = 1$, say. Then the next guess is $r_1 = r_0 - \frac{f(r_0)}{f'(r_0)}$ and so on. Thus every approximation is rational. But the approximations get better and better (this is not obvious, but true). And of course $\sqrt{2}$ is not rational.

2.2.1 Power sets

(**2.2.2**) For S a set, let $\mathtt{P}(S)$ denote the *power set*, the set of subsets of S.

(We may consider $\mathtt{P}(S)$ to be partially ordered by inclusion. As such it has the structure of a lattice — see 3.6.9 below.)

Let $\mathtt{P}_n(S) \subset \mathtt{P}(S)$ be the subset of elements of order n.

(**2.2.3**) Example: The power set $\mathtt{P}(\mathbb{N})$ is a very interesting set! We can place it in correspondence with the set of all possible assignments of an element from $\{0, 1\}$ to $\mathbb{N}$. Let b be such an assignment (so $b(7) = 0$ or 1, and so on). Then

$$n(b) = \sum_{i \in \mathbb{N}} \frac{b(i)}{2^i}$$

For example if a is the assignment $a(i) = 1$ for all i then

$$n(a) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

2.2.2 Relations

(**2.2.4**) For S, T sets, let $U_{S,T}$ denote the set of relations on S to T. That is,

$$U_{S,T} = \mathsf{P}(S \times T).$$

Let $U_S = U_{S,S}$. Even in this case we may consider the left-hand 'input' set to be distinct from the right-hand 'output' set — elements are distinguished by their position in the ordered pair.

A relation on S to T is *'simple'* if no element of the left-hand set S appears more than once.

(**2.2.5**) A relation on S to T is a *'function'* if every element of the left-hand set S appears once.

Notation: Given a relation $\rho \subseteq A \times B$ we may write $x\rho y$ as shorthand for $(x, y) \in \rho$.
Having established ρ used as above on a set S to itself, we may write (S, ρ) for the 'relational structure'.
Example 2.1. (a) Let S be a non-empty set, and $s \in S$. Then $\rho_s = \{(s, s)\}$ is a relation on S to itself.
(b) A relation on set S to itself is given by $\rho_{id} = \{(a, b) \mid a, b \in S,\ a = b\}$.
(c) Can you give another (infinite) relation on the set $\mathbb{N}$? How about on $\mathbb{R}$? (We have assumed that we know these sets as sets, but if you want to use any structure on them you must first introduce it.)

Discuss: The axiom of choice.

Example 2.2. Let S be a set, and $\mathsf{P}(S)$ the power set. Then $(\mathsf{P}(S), \subseteq)$ gives a relation on $\mathsf{P}(S)$.

Given a relation $\rho \subset A \times B$ we may write

$$x\rho- \ := \ \{b \in B \mid x\rho b\},$$

the subset of B such that $x\rho b$ for $b \in B$.

(**2.2.6**) **Definition**. If $x\rho-$ contains a single element for every $x \in A$ then ρ is a *function*.

The *opposite* of a relation $\rho \subseteq A \times B$ is

$$\rho^\circ \ := \ \{(y, x) \mid (x, y) \in \rho\}$$

which is a relation $\rho^\circ \subseteq B \times A$.

A relation $\sim$ on a set X is:

a) **reflexive:** if for all $a \in X$, $a \sim a$ (i.e. $(a, a) \in\sim$).
b) **symmetric:** if for all $a, b \in X$, if $a \sim b$, then $b \sim a$.
b) **antisymmetric:** if for all $a \neq b \in X$, if $a \sim b$, then $b \not\sim a$ (i.e. $(b, a) \notin\sim$).
c) **transitive:** if for all $a, b, c \in X$ if $a \sim b$ and $b \sim c$ then $a \sim c$.

An **equivalence relation** on a set X is a relation on X that is reflexive, symmetric and transitive.

An **partial order** on a set X is a relation $\sim$ on X that is reflexive, antisymmetric and transitive — and then $(X, \sim)$ is called a *poset*.

Notation: We write $\mathsf{E}(S)$ for the set of equivalence relations on a set S.

(**2.2.7**) **Definition**. Given an equivalence relation $\sim$ on set S, a transversal of $\sim$ is a subset of S containing exactly one element of each equivalence class $[s]$.

2.2.3 Functions

(**2.2.8**) For $(a,b) \in S \times T$, set $\pi_1(a,b) = a$. For $\rho \in U_{S,T}$ let $\mathrm{dom}(\rho) := \pi_1(\rho)$. Let $T^S = \hom(S,T) \subset U_{S,T}$ be the subset of simple relations with $\mathrm{dom}(\rho) = S$, or *functions.*

For example

$$\underline{2}^{\underline{2}} = \{\{(1,1),(2,1)\},\{(1,1),(2,2)\},\{(1,2),(2,1)\},\{(1,2),(2,2)\}\} \tag{2.1}$$

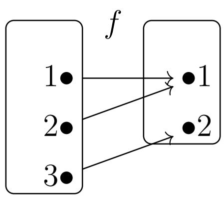
(**2.2.9**) For I a set and S_i a non-empty set for each $i \in I$, a choice function is a function that takes an element $i \in I$ as input and returns an element from S_i. (The axiom of choice says that such a function exists for any such family of sets.) Then $\prod_{i \in I} S_i$ is the set of all such functions.

2.2.4 Composition of functions

(**2.2.10**) It will be useful to have in mind the *mapping diagram* realisation of finite functions such as in (2.1). For example

$$f = \{(1,1),(2,1),(3,2)\} \in \underline{2}^{\underline{3}}$$

is



(**2.2.11**) If T,S finite it will be clear that any total order on each of T and S puts T^S in bijection with $\underline{|T|}^{\underline{|S|}}$. We may represent the elements of T^S as S-ordered lists of elements from T. Thus

$$\underline{2}^{\underline{2}} = \{11,12,21,22\}, \qquad \underline{2}^{\underline{3}} = \{111,112,121,122,211,212,221,222\}$$

(for example $22(1) = 2$, since the first entry in 22 is the image of 1).

(**2.2.12**) A *composition* of n is a finite sequence λ in $\mathbb{N}_0$ that sums to n. We write $\lambda \vDash n$.

We define the *shape* of an element f of $\underline{m}^{\underline{n}}$ as the composition of n given by

$$\lambda(f)_i = |f^{-1}(i)|$$

Example: for $111432525 \in \underline{6}^{\underline{9}}$ we have $\lambda(111432525) = (3,2,1,1,2,0)$.

If $\lambda \vDash n$ we write $|\lambda| = n$.

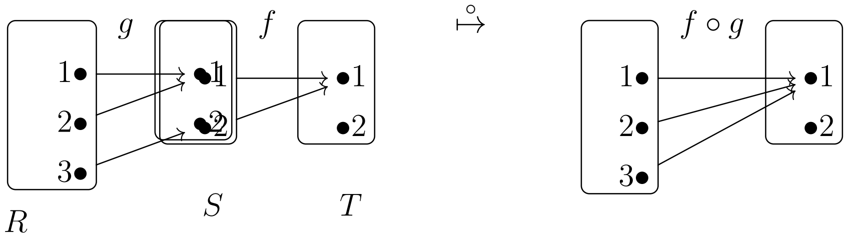
(**2.2.13**) Composition of functions defines a map

$$\hom(S,T) \times \hom(R,S) \;\to\; \hom(R,T) \tag{2.2}$$

$$(f,g) \mapsto f \circ g \tag{2.3}$$

where as usual $(f \circ g)(x) = f(g(x))$. For example $11 \circ 22 = 11$ (since $11(22(1)) = 11(2) = 1$; and so on).

The *mapping diagram realisation* of composition is to first juxtapose the two functions so that the two instances of the set S coincide, then define a direct path from R to T for each path of length 2 so formed:



(2.4)

(**2.2.14**) If the image $f(S)$ of a map $f : S \to T$ is of finite order we shall say that f has order $|f(S)|$ (otherwise it has infinite order).

For $R \xrightarrow{f} S \xrightarrow{g} T$ we have the *bottleneck principle*

$$|(g \circ f)(R)| \;\leq\; \min(|g(S)|, |f(R)|)$$

To see this note that evidently $g(S) \supseteq g \circ f(R)$, from which the first inequality follows; meanwhile clearly $|f^{-1}(R)| = |f(R)|$ for any $f \in \hom(R, -)$, leading to the second inequality.

(**2.2.15**) Proposition. *(i) For S a set, $S^S = \hom(S,S)$ is a monoid under composition of functions.*
(ii) For each $d \in \mathbb{N}$ then set $\hom^d(S,S) \;:=\; \{f \in S^S \mid |f(S)| < d\}$ is an ideal (hence a sub-semigroup) of S^S.

Proof. (i) Hint: $(f \circ (g \circ h))(x) = f(g(h(x))) = ((f \circ g) \circ h)(x)$
Exercise: explain this argument in terms of mapping diagrams.
(ii) Consider $g \circ f$, say. Evidently $g(S) \supseteq g \circ f(S)$. Thus $\hom^d(S,S) \circ f \subset \hom^d(S,S)$ for all f. Meanwhile $f(s) = f(t)$ implies $g \circ f(s) = g \circ f(t)$ so the partition $p = f^{-1}(S)$ of S implied by f cannot be refined in passing to the partition implied by $g \circ f$. Of course $|f^{-1}(S)| = |f(S)|$ for any f. Thus $g \circ \hom^d(S,S) \subset \hom^d(S,S)$ for all g. □

2.3 Setting the scene

2.3.1 'Intuitive' continuity

How can we tell if a function $f : A \to B$ is continuous?

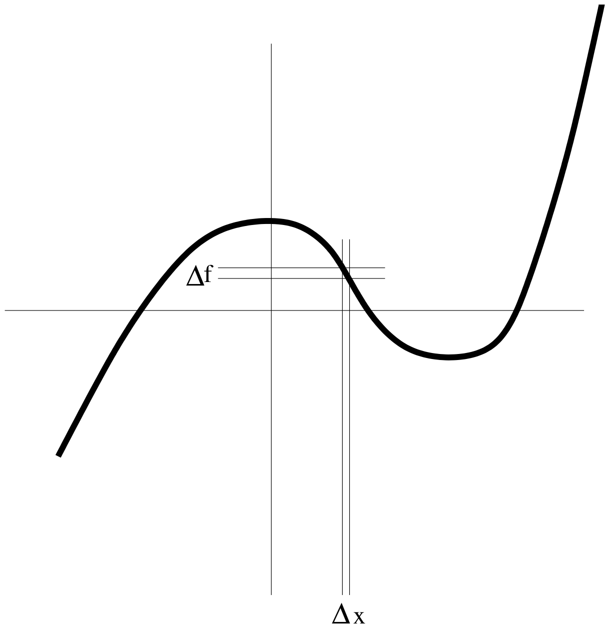
"Compare $f(x)$ with $f(x+\Delta x)$. As Δx gets small, $\Delta f = f(x+\Delta x) - f(x)$ gets small."

Several aspects of this heuristic will benefit from closer inspection if we want to use on $f : A \to B$ generally, say.

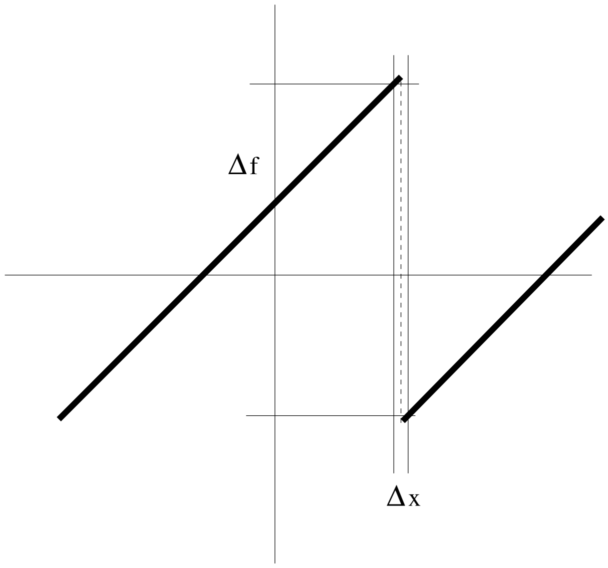
1. $x \in A$, and maybe $\Delta x \in A$ too, but what is $x + \Delta x$?
2. What is 'gets small' in A?
3. What is $f(x+\Delta x) - f(x)$ in B?
4. What is 'gets small' in B?

The familiar setting for examples is $f : \mathbb{R} \to \mathbb{R}$. Here all the questions have answers, and we can picture specific examples.

Continuous:



Not continuous:



We can also interpret Δx as giving a small 'ball' around x (in this case a 1d ball) $B_\delta(x)$ (small if δ is small), and ask about the size of ball that we would need in the codomain to contain $f(B_\delta(x))$. For continuity we are saying that it should be possible to make this containing ball 'correspondingly' small.

For some reason (habit) analysts always call the measure of smallness on the codomain side ϵ, along with δ on the domain side.

More formally we say that
Function $f : \mathbb{R} \to \mathbb{R}$ is 'continuous at $x \in \mathbb{R}$' iff for all $\epsilon > 0$ there exists $\delta > 0$ such that
(I) if $y \in \mathbb{R}$ is such that $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

(II) $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. (using $d(x, y) = |x - y|$)

(III) $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

(IV) $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$.
At the last re-write here we have used one of the set theory identities, applying f^{-1} to both sides.

(**2.3.1**) **Definition**. A function $f : (X, d) \to (Y, d')$ between metric spaces is **metric-continuous** at $x \in X$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$B_\delta(x) \subseteq f^{-1}(B'_\epsilon(f(x))).$$

(here B' flags that we are using the metric d' — later we'll just write B).

We say that $f : X \to Y$ is metric-continuous if and only if it is continuous for all $x \in X$.

(**2.3.2**) N.B.: This is the usual definition of continuity when X, Y are the vector spaces $\mathbb{R}^n, \mathbb{R}^m$ equipped with the standard metrics.

Theorem 2.3. *A function $f : (X, d) \to (Y, d')$ is metric-continuous if and only if for all d'-open $U \subset Y$, $f^{-1}(U) \subset X$ is d-open.*

Proof. Suppose that for all open $U \subset Y$, $f^{-1}(U) \subset X$ is open. Then for all $\epsilon > 0$ we know that $f^{-1}(B_\epsilon(f(x))) \subseteq X$ is open, since $B_\epsilon(f(x))$ is open in Y. By definition (since $x \in f^{-1}(B_\epsilon(f(x))))$ this implies there exists $\delta > 0$ so that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$, thus f is metric-continuous.

For the converse suppose that f is metric continuous. Let $U \subseteq Y$ be open so we want to prove that $f^{-1}(U)$ is open. Pick $x \in f^{-1}(U)$ giving $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ so that $B_\epsilon(f(x)) \subseteq U$ and since f is metric continuous, there exists $\delta > 0$ so that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))) \subseteq f^{-1}(U)$ so we are done (remember that, since X is a metric space, $f^{-1}(U)$ is open if and only if, for all $x \in f^{-1}(U)$ there exists $\delta > 0$ so that $B_\delta(x) \subseteq f^{-1}(U)$). $\square$

2.3.2 And why is it useful?...

... And why is it useful? Physics, Biology, Chemistry, Computing, Sociology, etc. ... See later for more on this.

Chapter 3

Basic tools for topology: first pass

Next we give a relatively brief overview of topics to be addressed in these notes. Then more details (examples and so on) will be given later.

Some people say that maths organises thought by introducing and codifying structures along three main lines. In this view, the lines are:
∘ ordered structures (less than)
∘ algebraic structures (times)
∘ topological structures (near).

In this view, a 'structure' is a pair consisting of a set S and a collection of sets built on S. For example a *group* is an algebraic structure and is a set together with a closed binary operation. Taken together with our Introduction above, these remarks set us up in two ways. Firstly they indicate how topology fits into organisational thought, and secondly they tell us that we can characterise it as a 'structure'. Now read on.

3.1 Overview

See e.g. Armstrong [Arm79], Mendelson [Men62], Hartshorne [Har77].

(**3.1.1**) A *sigma-algebra* over a set S is a subset Σ of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under countable unions, and complementation in S.

Any subset S' of $\mathtt{P}(S)$ defines a sigma-algebra — the smallest sigma-algebra generated by S'. For example $\{\{1\}\} \subset \mathtt{P}(\{1,2,3\})$ generates $\Sigma = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$.

(**3.1.2**) A *topological space* is a set S together with a subset T of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under unions and finite intersections.

The set T is called a *topology* on S. The elements of T are called the *open sets* of this topology. A set is *closed* if it is the complement in S of an open set.

(**3.1.3**) A function between topological spaces is *continuous* if the inverse image of every open set is open.

(**3.1.4**) Two spaces are *homeomorphic* if there is a bijection between them, continuous in both directions.

(**3.1.5**) We write $\mathsf{Top}((X,\tau),(X',\tau'))$ (or just $\mathsf{Top}(X,X')$) for the set of continuous functions from topological space (X,τ) to topological space (X',τ').

Note that each $\mathsf{Top}(X,X') \subseteq hom(X,X')$. In this sense composition of functions restricts to a composition $\mathsf{Top}(Y,Z) \times \mathsf{Top}(X,Y) \to hom(X,Z)$.
Prop. The image is contained in $\mathsf{Top}(X,Z)$, so composition of functions gives a composition

$$\mathsf{Top}(Y,Z) \times \mathsf{Top}(X,Y) \to \mathsf{Top}(X,Z)$$

(**3.1.6**) Example. /Proposition. Consider the set $\mathbb{R}^n$ together with the set of subsets 'generated' by unions and finite intersections of the set of open balls. This is a topological space. (We will prove this in §3.1.2.)

In particular this makes $M_n(\mathbb{R})$ a topological space — as a topological space it is $\mathbb{R}^{n^2}$. The subgroup $GL_n(\mathbb{R})$ of invertible matrices may be considered as a topological space by restriction. Note that $GL_n(\mathbb{R})$ is open and not closed (its complement is not open) in $M_n(\mathbb{R})$, but it is open and closed in the restricted topology.

3.1.1 Subspace topology

(**3.1.7**) Given a topological space (S,T), the restriction of T to $S' \subset S$

$$T' = \{t \cap S' | t \in T\}$$

is a topology on S', called the *subspace topology.*

A subset S' of a topological space (S,T) is *irreducible* if $S' = S_1 \cup S_2$ with S_1 closed implies S_2 not closed.

3.1.2 Set theory aspects

(**3.1.8**) A set β of sets is
finite-intersection-closed if $A,B \in \beta$ implies $A \cap B \in \beta$.
intersection-closed if closed under all intersections.
finite-union-closed if $A,B \in \beta$ implies $A \cup B \in \beta$.
union-closed if closed under all unions.
weak-intersection-closed (WIC) if for $A,B \in \beta$ then $x \in A \cap B$ implies $\exists C \ni x$, $C \subseteq A \cap B$.

(**3.1.9**) The union-closure of a set of sets β is the set of all unions. Denote it by $C_\cup \beta$. (See also (6.1).)

(**3.1.10**) The 'universe' of a set of sets β is the union of all the sets. Denote it by U_β.

(**3.1.11**) The union-closure of a set of sets β is a topology iff β is WIC. If so, then $(\mathsf{U}_\beta, C_\cup\beta)$ is a topological space.

(**3.1.12**) Exercise: prove it.

(**3.1.13**) A WIC set of sets β is called a **basis** for a topology on U_β. The topology is that 'generated' by β.

(**3.1.14**) The metric topology on a metric space (X, d) (see e.g. §**??**) is generated by the set $\{B_r(x) \mid x \in X, r > 0\}$ of open balls.

Exercise: Prove it!

(**3.1.15**) A topological space is 'second countable' if it can be generated by a countable basis.

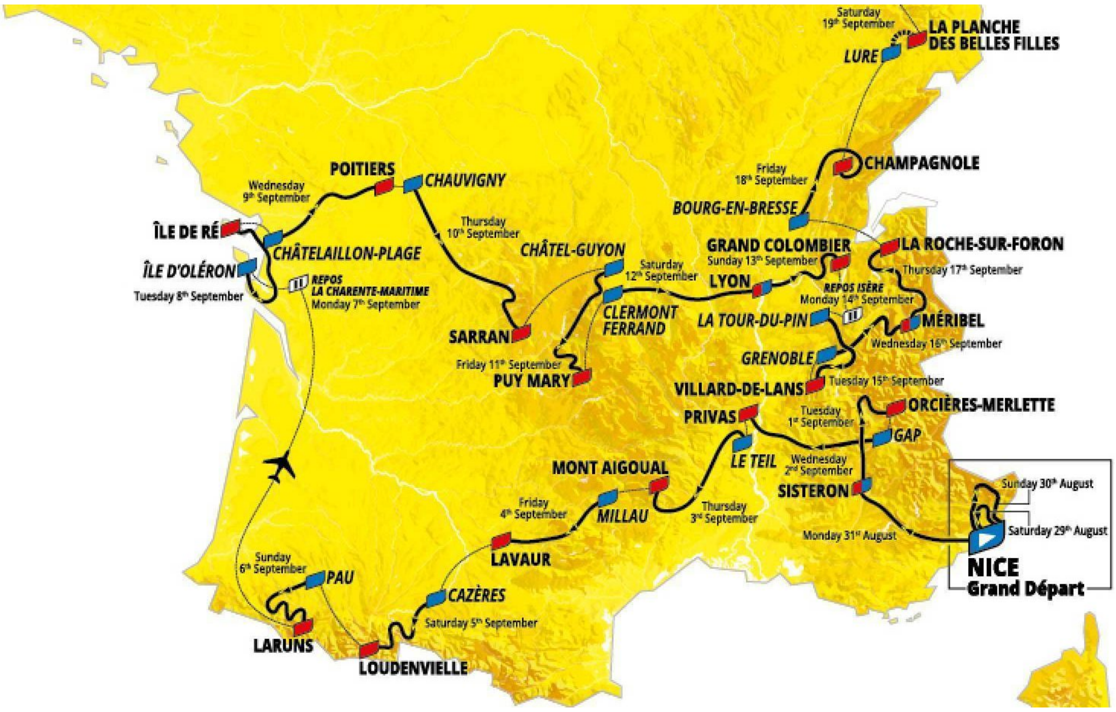
(**3.1.16**) Example: $\mathbb{R}^n$ (with Euclidean metric topology) is second countable. The set of all balls of rational radius centered at the points in $\mathbb{Q}^n$ is a basis.

Exercise: Prove it!

3.2 Basic 'low-dimensional' topology

3.2.1 Preliminaries for paths and pictures

Let us introduce some notation for journeys and chaining journey stages together. Such chains of journeys are common. But they contain a lot of topological ideas.



For example, informally we could partition the set of all journeys according to where they start. And end. Let us introduce some general notation for this.

(**3.2.1**) A function $G \xrightarrow{s} X \in hom(G, X)$ induces a partition

$$G = \cup_{x \in X} G(x)$$

where

$$G(x) \;=\; s^{-1}(x) \;=\; \{g \in G \mid s(g) = x\}$$

A pair of functions $G \underset{t}{\overset{s}{\rightrightarrows}} X$ induces

$$G = \cup_{x,y \in X} G(x, y)$$

similarly.

(**3.2.2**) A *magmoid* is a pair $G \underset{t}{\overset{s}{\rightrightarrows}} X$ with, for each triple $x, y, z \in X$, a function

$$* : G(x,y) \times G(y,z) \to G(x,z)$$

(**3.2.3**) A *category* is a magmoid with associative composition, so that each $(G(x,x),*)$ is a semigroup; and an identity element in each $(G(x,x),*)$, so that it is actually a monoid.

(**3.2.4**) A *groupoid* is a category in which each $g \in G$ has an inverse.

(**3.2.5**) Recall that given a normal subgroup of a group G we may form the quotient group.

More generally, an equivalence relation on the underlying set of an algebraic structure is called a *congruence* if the class $[a * b] = [a' * b']$ whenever $a' \in [a]$ and $b' \in [b]$.

3.2.2 Paths and journeys

(**3.2.6**) Here $\mathbb{I}^n$ means $[0,1]^n$ with the usual metric topology.

(**3.2.7**) Let (X, τ) be a topological space. An element $\alpha \in \mathsf{Top}(\mathbb{I}, (X,\tau))$ is called a path in (X, τ).

An element in $\cup_l \mathsf{Top}([0,l], (X,\tau))$ is called a journey in (X,τ).

(**3.2.8**) We can partition the set of elements $\alpha \in \mathsf{Top}(\mathbb{I}, (X,\tau))$ according to $\alpha(0)$ and $\alpha(1)$. Let

$$\mathsf{P}_{(X,\tau)}(x,y) = \{\alpha \in \mathsf{Top}(\mathbb{I}, (X,\tau)) | \alpha(0) = x,\ \alpha(1) = y\}$$

$$\mathsf{P}'_{(X,\tau)}(x,y) = \cup_l \{\alpha \in \mathsf{Top}([0,l], (X,\tau)) | \alpha(0) = x,\ \alpha(l) = y\}$$

Note that we can stop a journey part way, and thus obtain a restricted function that is in fact a shorter journey (since continuity of the original journey ensures continuity). Indeed any interval of the domain yields a journey.

Note that there is a natural function

$$* : \mathsf{P}'_{(X,\tau)}(x,y) \times \mathsf{P}'_{(X,\tau)}(y,z) \to \mathsf{P}'_{(X,\tau)}(x,z)$$

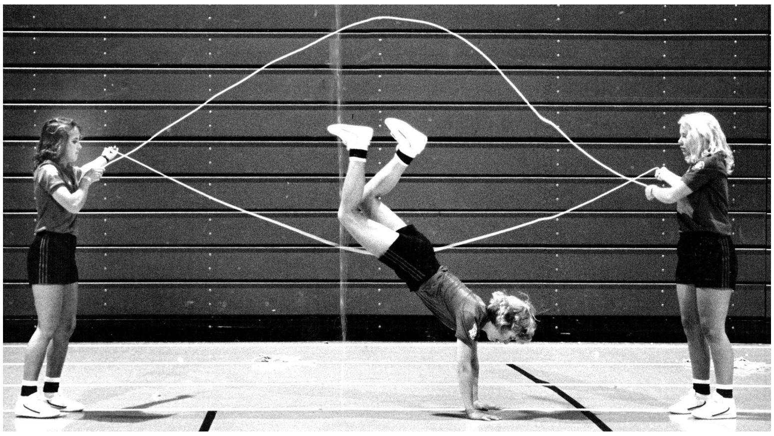
by doing one journey then the other. (It is an exercise to prove continuity.)

Note that this composition is associative and (allowing duration $l = 0$) unital. Thus $\mathsf{P}'_{(X,\tau)}$ is a category.

Note that $\mathsf{P}'_{(X,\tau)}$ is not a groupoid, and $\mathsf{P}'_{(X,\tau)}(x,x)$ is not a group, because we don't have inverses. There are 'reverse' trips, but composition with the reverse is a longer trip.

(**3.2.9**) To understand why $\alpha \in \mathsf{P}_{(X,\tau)}(x,y)$ is a 'path in X from x to y' it is perhaps helpful to have a picture. This raises the question of the relationship between pictures and spaces, and between pictures and $\mathbb{R}$ and $\mathbb{I}$ in particular.

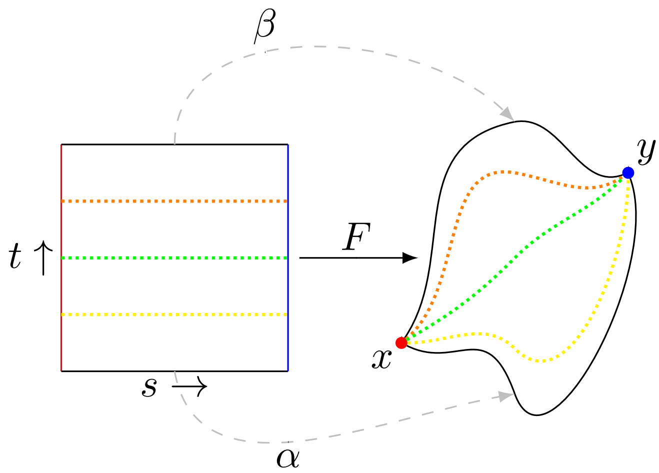
A picture is a representation of something (possible an abstract thing; possibly a 'real world' thing) in the real world — albeit squashed onto a sheet of paper or other physical surface. Constructions in mathematics are not in the business of 'being' real physical things. But they do sometimes attempt to model physical things.



(**3.2.10**) **Definition**. L et (X,τ) be a topological space. Let $x,y \in X$ and let $\alpha,\beta \in \mathsf{P}_{(X,\tau)}(x,y)$ be paths from x to y. Then α is **path homotopic** to β (written $\alpha \simeq \beta$) if there exists $F \in \mathsf{Top}(\mathbb{I}^2,(X,\tau))$ such that

$$\begin{aligned} F(s,0) &= \alpha(s) \quad \forall s \in [0,1] \\ F(s,1) &= \beta(s) \quad \forall s \in [0,1] \\ F(0,t) &= x \quad \forall t \in [0,1] \\ F(1,t) &= y \quad \forall t \in [0,1]. \end{aligned}$$

Such a function F is a **path homotopy** from α to β.



(**3.2.11**) **Lemma.** The relation on $\mathsf{P}_{(X,\tau)}(x,y)$ given by path homotopy is an equivalence relation.
Proof. See later.

(**3.2.12**) **Definition**. L et (X,τ) be a topological space. Let $x,y,z \in X$. Let $\alpha \in \mathsf{P}_{(X,\tau)}(x,y)$,

$\beta \in \mathsf{P}_{(X,\tau)}(y, z)$. The **meld** of α and β is the function $\alpha * \beta : [0,1] \to X$ defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \beta(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Note that $\alpha * \beta \in \mathsf{P}_{(X,\tau)}(x, z)$, since at $s = 1/2$ we have $\alpha(0) = y = \beta(0)$. (We will give a more careful proof later, using a 'glue lemma' **??**.)
The **reverse** of α is the function $\overline{\alpha} : [0,1] \to X$ defined by

$$\overline{\alpha}(s) = \alpha(1-s).$$

Note that $\overline{\alpha} \in \mathsf{P}_{(X,\tau)}(y, x)$, since the change of argument is continuous. (Again see later for a careful proof.)
For any point $x \in X$, the **constant path at** x is the function $e_x : [0,1] \to X$ defined by

$$e_x(s) = x \quad \forall s \in [0,1].$$

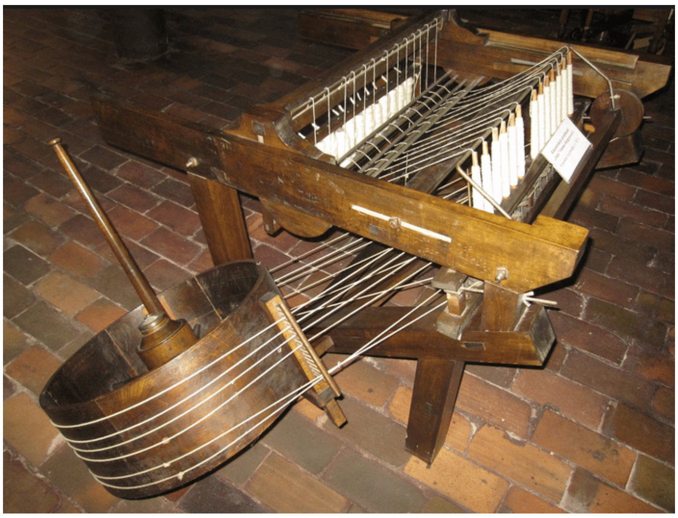
Note that $e_x \in \mathsf{P}_{(X,\tau)}(x, x)$.
Exercise: check that $\alpha * \beta$, $\overline{\alpha}$ and e_x are paths.

(**3.2.13**) **Lemma.** Path homotopy yields a congruence on our journey category $\mathsf{P}'_{(X,\tau)}$, making a groupoid — called the fundamental groupoid of (X, τ).
Proof. See later.

(**3.2.14**) Why?! ... (More interesting than the how is the why.) $\mathsf{P}'_{(X,\tau)}$ is very big. And really contains more information than we can handle or want. Every fine details of every journey. No-one needs this much information. Most tiny variations in the route are not interesting to us. Just as we do not care that a tiger has a tuft of fur slightly out of place while it is eating us.

Path homotopy gives a way of brushing over the high level of detail.

So the big question is: what information does it keep?



3.2.3 Algebraic aspects

This section can be skipped on first reading.

(**3.2.15**) Let k be a field. A polynomial $p \in k[x_1, ..., x_r]$ determines a map from k^r to k by evaluation. For $P = \{p_i\}_i \subset k[x_1, ..., x_r]$ define

$$Z(P) = Z(\{p_i\}_i) = \{x \in k^r \ : \ p_i(x) = 0 \ \forall \ i\}$$

An *affine algebraic set* is any such set, in case k algebraically closed. An *affine variety* is any such set, that cannot be written as the union of two proper such subsets. (See for example, Hartshorne [Har77, I.1].)

(**3.2.16**) EXAMPLE. $Z(x_1x_2 - 1) = Z(\{p(x_1, x_2) = x_1x_2 - 1\})$ is a variety in k^2. Its points $(x_1, x_2) = (\alpha, \beta)$ may be given by a free choice of α (say) from $k^\times$, with β then determined. (Note that this latter characterisation looks like an open subset of k (specifically the complement of $Z(x)$), but the original formulation makes it clear that it is closed in k^2.)

(**3.2.17**) The set of affine varieties in k^r satisfy the axioms for closed sets in a topology. This is called the *Zariski topology*. The Zariski topology on an affine variety is simply the corresponding subspace topology.

The set $I(P) \in k[x_1, ..., x_r]$ of all functions vanishing on $Z(P)$ is the ideal in $k[x_1, ..., x_r]$ generated by P. We call

$$k_P = k[x_1, ..., x_r]/I(P)$$

the *coordinate ring* of $Z(P)$.

(**3.2.18**) Let Z be an affine variety in k^r and $f : Z \to k$. We say f is *regular* at $z \in Z$ if there is an open set containing z, and $p_1, p_2 \in k[x_1, ..., x_r]$, such that f agrees with p_1/p_2 on this set.

(**3.2.19**) A morphism of varieties is a Zariski continuous map $f : Z \to Z'$ such that if V is open in Z' and $g : V \to k$ is regular then $g \circ f : f^{-1}(V) \to k$ is regular.

(**3.2.20**) Given affine varieties X, Y then $X \times Y$ may be made in to an algebraic variety in the obvious way.

(**3.2.21**) An *algebraic group* G is a group that is an affine variety such that inversion is a morphism of algebraic varieties; and multiplication is a morphism of algebraic varieties from $G \times G$ to G.

3.3 More new from old: Quotients

(**3.3.1**) **Lemma.** Let (X, τ) be a topological space and Y a set. We have a function

$$\tau_{\_} : hom(X, Y) \to \mathtt{P}(Y)$$

given for $f : X \to Y$ a function by

$$\tau_f = \{S \subseteq Y \mid f^{-1}(S) \in \tau\}$$

This τ_f is a topology on Y.
Proof. (T0,1): note $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$.
(T2): Suppose $U, V \in \tau_f$. Then

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

and the RHS is in τ because it is a topology.
(T3): Suppose $\{S_\lambda\}_{\lambda\in\Lambda}$ a collection in τ_f. Then

$$f^{-1}(\bigcup_{\lambda\in\Lambda} S_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(S_\lambda)$$

The RHS is a union in τ by construction, and hence in τ as required. $\square$

(**3.3.2**) Given an equivalence relation $\sim$ on a set X, recall that $X/\sim$ denotes the set of classes.

(**3.3.3**) Let (X,τ) be a topological space. Let $\sim$ be an equivalence relation on X and let $f_\sim : X \to X/\sim$ be the map $x \mapsto [x]$. The topology $\tau_{f_\sim}$ is the 'quotient topology' on $X/\sim$. The space $(X/\sim, \tau_{f_\sim})$ is the 'quotient space'.

3.4 Homeomorphism and 'shapes'

(**3.4.1**) Let $(X,\tau) \in$ Top. Let $U, V \in \tau \setminus \{\emptyset\}$. If $\{U, V\}$ a partition of X then (X,τ) is 'disconnected'. If there is no such U, V then (X,τ) is 'connected'.

(**3.4.2**) Path-connected: ...

(**3.4.3**) Compact: ...

(**3.4.4**) Linearity: Note that $[0, 1]$ is a topological space, but not a vector space. ...

3.5 Set partitions

(**3.5.1**) Let $\mathsf{E}_S \subset U_S$ denote the set of equivalence relations (reflexive, symmetric, transitive/RST relations) on set S. Let P_S denote the set of partitions of S. Note the natural bijection

$$\mathsf{E}_S \underset{\kappa}{\overset{\epsilon}{\leftrightarrows}} \mathsf{P}_S.$$

For $\rho \in U_S$ let $\bar\rho \in U_S$ be the smallest transitive relation containing ρ. The relation $\bar\rho$ is called the *transitive closure* of ρ.

(**3.5.2**) Let a, b be RS relations on any two finite sets. Then $a \cup b$ is an RS relation on the union. Let $ab := \overline{a \cup b}$ be the transitive closure of $a \cup b$.

Note that $\overline{a \cup b}$ is an equivalence relation on the union of the two finite sets. Note that

$$\overline{\bar{a} \cup b} \;=\; \overline{a \cup b} \tag{3.1}$$

If a, b partitions then $\epsilon(a), \epsilon(b)$ are RS (indeed RST), and we will understand by ab the partition given by $ab = \kappa(\epsilon(a)\epsilon(b))$.

(**3.5.3**) Proposition. *For a, b, c RS relations*

$$a(bc) = (ab)c$$

Proof.

$$(ab)c \;=\; \overline{\overline{(a\cup b)} \cup c} \overset{(3.1)}{=} \overline{(a\cup b)\cup c} \;=\; \overline{a\cup b\cup c} = \overline{a \cup \overline{b\cup c}} \;=\; a(bc)$$

$\square$

(**3.5.4**) Let $\mathsf{P}_{n,m} = \mathsf{P}_{\underline{n}\cup\underline{m'}}$; and $\mathsf{P}_n = \mathsf{P}_{n,n}$. Let $\mathsf{E}_{n,m} = \mathsf{E}_{\underline{n}\cup\underline{m'}}$ similarly. For $a \in \mathsf{P}_{n,m}$ let a' be the partition of $\underline{n'} \cup \underline{m''}$ obtained by adding a prime to each object in every part.

For $a \in \mathsf{P}_{l,m}$, $b \in \mathsf{P}_{m,n}$ partitions (and hence $\epsilon(a)$, $\epsilon(b)$ equivalence relations) note that $\epsilon(a)\epsilon(b')$ is an equivalence relation on $\underline{l} \cup \underline{m'} \cup \underline{n''}$. Restricting to $\underline{l} \cup \underline{n''}$ this equivalence relation gives again a partition, call it $r(ab')$ (indeed if a, b are pair-partitions then so is $r(ab')$).

For $x \in \underline{l} \cup \underline{n''}$ let $u(x) \in \underline{l} \cup \underline{n'}$ be the image under the action of replacing double primes with single.

We may define a map

$$\circ : \mathsf{P}_{l,m} \times \mathsf{P}_{m,n} \to \mathsf{P}_{l,n}$$

by

$$a \circ b \;=\; u(r(ab')) \in \mathsf{P}_{l,n}$$

— the image under the obvious application of the u map.

(**3.5.5**) Proposition. *For each $n \in \mathbb{N}$ the map $\circ : (a, b) \mapsto u(r(ab'))$ defines an associative unital product on P_n, making it a monoid, with identity $1_n = \{\{1, 1'\}, \{2, 2'\}, ..., \{n, n'\}\}$.*

Proof. To show associativity note that ab' encodes $a \circ b$ directly, except that it is encoded via the unprimed and double-primed 'vertices'. Thus $(ab')c''$ encodes $(a \circ b) \circ c$ via the unprimed and triple-primed vertices. Meanwhile $b'c''$ encodes $b \circ c$ via the primed and triple-primed vertices; thus $a(b'c'')$ encodes $a \circ (b \circ c)$ via the unprimed and triple-primed vertices. But by Prop.3.5.3 we have $a(b'c'') = (ab')c''$.

To show unital with identity 1_n: Exercise. ■

(**3.5.6**) A convenient pictorial realisation of such a set partition p, i.e. a realisation as a picture in the plane, is as follows. (See also **??**.)

Firstly, a digraph G (as in 3.6.13) on vertex set V determines a relation on V in the obvious way. In particular a graph determines a symmetric relation. Hence a graph G determines an equivalence relation on V (take the RT closure); and hence also an equivalence relation on (or partition of) any subset of V. Thus it is enough to realise a suitable graph G of p as a picture.

To depict such a G one draws a set of points for the vertices V, and specifies an injective map from the underlying set of p to V; and then draws a 'regular' collection of 'edges'. Here a picture edge is a piecewise smooth line between two vertices. A collection is *regular* if two lines never meet at points where they do not have distinct tangents. The collection consists of one picture edge for each vertex pair that are associated by an edge in G. ('Incidental' vertices in the picture are those *not* associated to the underlying set.)

Note that two elements from the underlying set are in the same part in p if there is a path between their vertices.

(**3.5.7**) For a partition in P_n one may arrange the underlying-set vertices naturally as two parallel rows of vertices (if there are incidental vertices these are drawn between the two rows). In this realisation the product $\circ$ may be computed, schematically, by concatenating the two pictures so as to identify certain vertices in pairs between two rows — one row from each picture (thus forming a 'middle' row).

(**3.5.8**) Let $J_S \subset \mathsf{P}_S$ be the set of pair-partitions of S. Let

$$J_{n,m} = J_{\underline{n} \cup \underline{m'}} \subset \mathsf{P}_{n,m}$$

Set $J_n = J_{n,n}$.

(**3.5.9**) Proposition. *The compostion $\circ$ restricts to make J_n a monoid.*

Proof. Exercise. ■

(**3.5.10**) A partition is *non-crossing* if there is such a pictorial realisation having the property that all lines are drawn in the interior of the interval defined by the two rows, and no two lines cross. Let T_n denote the subset of non-crossing pair partitions.

One easily checks that the product above restricts to make T_n a monoid. This is sometimes called the n-th *Temperley–Lieb monoid*.

(**3.5.11**) One could similarly imagine drawing a realisation of a partition on a cylinder. This leads us to a notion of cylinder-non-crossing pair partitions. There are several further subsets of the set of partitions that are characterised in terms of geometrical embeddings.
Exercise: Find some more submonoids of P_n.

3.5.1 Exercises on closed binary operations

(**3.5.12**) A closed binary operation on a (finite) set S (of degree n) may be given by a multiplication table — an element of $S^{S\times S}$. There are $|S^{S\times S}| = n^{(n^2)}$ of these. Note that an ordering of S induces an ordering on the set of closed binary operations (read order the entries in the multiplication table and dictionary order the ordered lists).

Define a natural notion of isomorphism of closed binary operations on S, and determine the number of isomorphism classes for $n = 2$. Is commutativity a class property? If so, how many of these classes are commutative?

Which of the following are semigroups/ monoids / groups?:

$$\begin{array}{c|cc} & a & b \\ \hline a & aa & ab \\ b & ba & bb \end{array} \quad : \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & b \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & b \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & a \end{array} \quad \begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a \end{array}$$

(Hint: S,M,X ($b(ab) = b$),S,G,X ($(aa)b = a$).)

Explain the following statement: "For $n = 3$, most binary operations are not associative." (Hint: 113 of them are associative.)

3.6 Partial orders, lattices and graphs

3.6.1 Posets

General references on posets and lattices include Birkhoff [Bir48], and Burris and Sankappanavar [BS81, §1].

(**3.6.1**) A *relation* on a set S is a subset of $S \times S$ as in (2.2.2). Thus the intersection of any set of relations on S is certainly a relation. Indeed the intersection of any set of transitive relations is transitive.

The *transitive closure* of a relation ρ on S is the intersection of all transitive relations containing ρ. (This transitive relation is non-empty since $S \times S$ is a transitive relation.)

(**3.6.2**) A *poset* is a reflexive, antisymmetric, transitive relation.

An acylic (no cyclic chains) relation ρ on S defines a partial order, by taking the transitive reflexive closure $TR(\rho)$. Note that every relation in the interval $[\rho, TR(\rho)]$ (with respect to the inclusion partial order) is acyclic.

We may consider the set of all relations having the same transitive reflexive closure. If the closure is a poset then all the relations 'above' it are acyclic. A minimal such relation is a *transitive reduction* (of any element of this set).

If S is finite then there is a unique transitive reduction of an acylic relation. Otherwise there may be no (or one, or multiple) transitive reductions.

If there is a unique transitive reduction of an acyclic relation we call this the *covering relation*.

(**3.6.3**) If we use the notation $(S, \geq)$ for a poset then we may write $a > b$ for $a \geq b$ and $a \neq b$. In this case the relation $(S, >)$ induces the same poset. Further we may write $(S, \leq)$ for the opposite relation, which is another poset.

(**3.6.4**) Let $(S, \geq)$ be a poset, and $s, t \in S$. We say s *covers* t if $s > t$ and there does not exist $s > u > t$.

(**3.6.5**) The notion of cover/covering relation leads to the notion of *Hasse diagram*, as for example in [Bir48, BS81].

(**3.6.6**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

3.6.2 Lattice meet and join

(**3.6.7**) By convention if we declare a poset $(S, \leq)$ then $a \leq b$ can be read as *a is less than or equal to b* (although the opposite relation is a perfectly good poset, and we could in principle have associated the relation symbol $\leq$ to that).

(**3.6.8**) Let $(S, \geq)$ be a poset. A lower bound of subset $T \subset S$ is an element $b \in S$ such that $t \geq b$ for all $t \in T$. A greatest lower bound of $T \subset S$ is a $b \in S$ such that b is a lower bound and for each lower bound c we have $b \geq c$.
Upper bounds and least upper bounds reverse these definitions in the obvious way.

(**3.6.9**) With the above convention, a poset S is a *join semilattice* if every pair $s, t \in S$ has a least upper bound (join) in S.
A poset is a *lattice* if both it and its opposite are join semilattices.

(**3.6.10**) Example. The power set $\mathtt{P}(S)$ of a finite set with the inclusion order is a lattice. An upper bound of $s, t \in \mathtt{P}(S)$ is any set containing sets s, t; and the least upper bound is the union. That is

$$s \vee t \;=\; s \cup t.$$

(**3.6.11**) A lattice is *modular* if

$$S \wedge (T \vee U) \;=\; (S \wedge T) \vee U \qquad \Rightarrow \qquad S \geq U.$$

(**3.6.12**) For P a lattice, the interval

$$[a, b] := \; \{c \in P \mid a \leq c \leq b\}$$

is sometimes called a *quotient* (see e.g. Faith [**?**]).

3.6.3 Digraphs and graphs

(**3.6.13**) A *digraph* is a triple (V, E, f) where V, E are (finite) sets and f a function $f : E \to V \times V$.

(**3.6.14**) Given a digraph $G = (V, E, f)$, we say V is the set of 'vertices' and E the set of 'edges'. An edge $e \in E$ with $f(e) = (a, a)$ is a 'loop'. If $f(e) = (a, b)$ then e is an edge 'on' (a, b) or from a to b.

An *edge colouring* of a digraph is a map from E to a set of 'colours', and hence a partition of E into same-coloured subsets.

(**3.6.15**) A digraph can be represented by a picture with a labeled node for each vertex and a directed labeled arc for each edge. Examples:

$$G_1 = \begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, "y"'] & b \\ c \arrow[u, "z"] \arrow[ur, "t"'] & \end{tikzcd} \qquad\qquad G_2 = \begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, "y"'] & b \arrow[loop above, "u"] \arrow[dl, bend left, "s"] \\ c \arrow[u, "z"] \arrow[ur, "t"] & \end{tikzcd} \tag{3.2}$$

(**3.6.16**) A *simple digraph* is a digraph (V, E, f) in which f is an inclusion.

This amounts to saying that we can use a subset of $V \times V$ as the edge set. Thus we do not need labels on edges in a picture. That is, a simple digraph is just a relation on V.

(**3.6.17**) Given a digraph, if there is a proper path (along directed edges) from a to b then the *'distance'* from a to b is the minimum number of edges in such a path. (Note that this is not a true distance function. The distance from b to a may be different, for example.)

A digraph is *acyclic* if there is no proper path (along directed edges) from a to a for any $a \in V$.

(**3.6.18**) A digraph is *rooted* with root $r \in V$ if there is a vertex $r \in V$ such that every vertex is reachable by a directed path from r.

Note that if a digraph is acyclic then it has at most one root.

(**3.6.19**) Two simple digraphs (V, E, f) and (V', E', f') are *isomorphic* if there is a bijection $\psi : V \to V'$ such that (a, b) is in $f(E)$ iff $(\psi(a), \psi(b))$ is in $f'(E')$.

(**3.6.20**) We will say that two (not necessarily simple) digraphs are isomorphic if there is a bijection $\psi : V \to V'$ such that $f^{-1}(a, b)$ has the same order as $f'^{-1}(\psi(a), \psi(b))$ for all a, b (thus each of these pairs of sets could be placed in explicit bijection, but such a set of bijections is not necessarily given).

That is, two digraphs are isomorphic if their pictures can be 'morphed' into each other, using ψ, but ignoring the edge labels.

(**3.6.21**) Remark. We do not require the *finite* set condition for digraphs here. In practice our digraphs are either finite or inverse limits of sequences of finite graphs. This means in particular that there are only finitely many edges associated to any given pair of vertices, i.e. $f^{-1}(v, w)$ is always finite.

(**3.6.22**) Let G be a digraph with a countable vertex set. The *adjacency matrix* $M^G = A(G)$ is a vertex indexed square array such that entry M^G_{ij} is the number of edges from i to j in G.

Example from (3.2) above:

$$A(G_1) = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \hline \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

(**3.6.23**) The opposite graph of a digraph has the same V and E but $f^{op}(e) = f(e)^{op}$ (i.e. if $f(e) = (a, b)$ then $f^{op}(e) = (b, a)$).

(**3.6.24**) A *graph* is a digraph that is isomorphic to its opposite.

(**3.6.25**) We say a graph is *connected* if for any pair of vertices there is a finite chain of edges connecting them.

(**3.6.26**) Example. Let G be a group and S a set of elements. The *Cayley graph* $\Gamma(G, S)$ is the digraph with vertex set G and an edge s_a on (a, b) whenever $b = as$ for some $s \in S$.

(**3.6.27**) Notes:

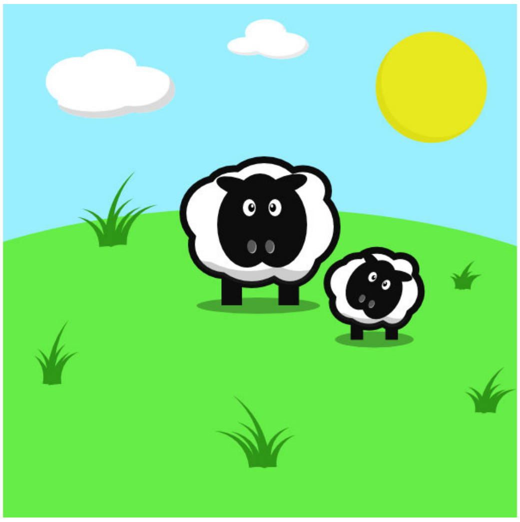
1. $s = a^{-1}b$ so there is at most one edge on (a, b), i.e. $\Gamma(G, S)$ is simple.
2. If $s \in S$ is an involution then edges involving s are effectively undirected. Some workers define S to include inverses (write this as $S = S^{-1}$), so again $\Gamma(G, S)$ is undirected.
3. We consider that S excludes the identity, so $\Gamma(G, S)$ is loop-free.
4. Some workers require that S generates G. Then $\Gamma(G, S)$ is connected. If $S = G$ then $\Gamma(G, S)$ is the complete graph.
5. If all generators are involutions (or $S = S^{-1}$) then the graph is effectively undirected by construction. However one can sometimes 'direct' such a $\Gamma(G, S)$, by using a *length function*... The root vertex is the identity 1, and there is a well-defined distance (really minimum distance, since there are undirected adjacent pairs) from 1 to any vertex g, denoted $l(g)$. If there is no edge between vertices of equal distance then we can 'direct' edges away from the root.

(**3.6.28**) Example. For $\Gamma(S_n, S)$ where S is the set of adjacent pair permutations, one can show that $l(gs) \neq l(g)$. See §**??**.

Chapter 4

The real field and geometry

For examples and motivation topology makes heavy use of properties of the real number field. These are familiar to us — the familiar properties of real arithmetic. But they are amazing and important. So let's review them a little. It is safe to skip this Chapter on first reading.



Our starting point here is the observation that the real line is a magical thing — meaning that it is amazing and familiar but hard to fully understand.

Here is a picture (of just a bit of it):

It is already quite amazing that I didn't need to tell you *which* bit that was. This is because, from one point of view, one chunk of the real line is much like another. If I mark a point anywhere on it, then that partitions the line into three parts: left of the point; the point; right of the point.

Notice that is not a property held by just any old set. If I have a random set and a pick an element, no further structual implications arise.

We often do mark a point on the real line. We call the first such marked point 0. We then have an additive structure on the line, with respect to which 0 is the identity. We can do 'slide-rule' addition, meaning we can compute $a + b$ by taking another copy of the line and sliding until the second copy of 0 is at a then looking at the position of the second copy of b. (Caveat: To do the slide we had to place the line in a bigger universe in which the second copy could also live! This was a physical-world operation rather than a maths one. To do it in maths-world we have to be a bit more patient.)

Notice that there was not actually anything special about the point 0 as far as the line was concerned. We chose it and made it special.