sub to Comm. Algebra

ON THE CENTRALISER ALGEBRAS OF QUANTUM LINEAR GROUPS

P. P. Martin and B. W. Westbury

26 June 1992

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Introduction

In this article we study a family of finite-dimensional associative algebras. These are usually referred to as the Hecke algebras of the general linear groups since they are a deformation of the group rings of the symmetric groups (see (James and Dipper [**1986**])). However they also arise in the study of knot theory and integrable two-dimensional lattice models, (Jones [**1987**]). In these applications they appear as centraliser algebras for the standard representation of the quantum enveloping algebra of the general linear group. In this article we consider certain quotients which arise from this alternative point of view.

In this article we have used a presentation of these algebras which is equivalent to, but not the same as, the deformation of the Coxeter presentation of the symmetric groups. We have also made this article self-contained, even though this has meant giving proofs of some results which are well-known.

Let V be a free module of rank m. Then the Hecke algebra of type A_{n-1}, H_n, acts on $\otimes^n V$. Define $H_n(m)$ to be the image of H_n in $\mathrm{End}(\otimes^n V)$. Assuming that the Laurent polynomial $[m]!$ (defined in 1.2) is invertible there is an idempotent $F_m \in H_n$ such that

$$H_n / \langle F_m \rangle \cong H_n(m) / \langle F_m \rangle \cong H_n(m-1).$$

Then the main result in this paper is that, for each n and m, there is an algebra isomorphism

$$F_m H_n(m) F_m \cong H_{n-m}(m).$$

This result can be seen as a dual to a result proved by Donkin and stated in (Erdmann [**1991**, §1.4]). The q-Schur algebra, $S_q(m,n)$, is also a subalgebra of $\mathrm{End}(\otimes^n V)$. In fact each of the algebras $H_n(m)$ and $S_q(m,n)$ is the centraliser of the other although we do not prove this. There is an idempotent $E \in S_q(m,n)$ such that $ES_q(m,n)E$ is isomorphic to $S_q(m-1,n)$ and there is an algebra isomorphism

$$S_q(m,n) / \langle E \rangle \cong S_q(m, n-m).$$

The main application of this result is to show that if the polynomial $[m]!$ is non-zero then, for each n, the algebra $H_n(m)$ is quasi-hereditary in the sense of (Scott [**1987**]).

As a second application of this result we give a new proof of a special case of the main theorem in (James [**1981**]). Let λ and μ be partitions of n with precisely m parts and let $\bar{\lambda}$ and $\bar{\mu}$ be the partitions of $n-m$ obtained by removing the first columns. Then

$$D(\lambda, \mu) = D(\bar{\lambda}, \bar{\mu})$$

as entries in the two decomposition matrices. Our result is a special case of this result since we assume that $[m]!$ is invertible. On the other hand our proof is elementary and does not use the q-Schur algebra. Also our method gives a refinement involving a formal parameter (and applies to the Hecke algebras).

The algebras $H_n(m)$ occur in two-dimensional solvable lattice models as the algebras generated by single bond transfer matrices (for a recent review see (Deguchi and Martin [**1992**])). In this situation n depends on the size of the finite rectangular lattice and in ice-type models, for example, the parameter m is the number of states on each edge. One of the consequences of the occurrence of the Hecke algebras in solvable models is the implication of a certain profound connection between each representation of $H_n(m)$ and a corresponding representation of $H_{n+m}(m)$. At the level of physics this mapping embodies (the realisation in solvable models of) a fundamental axiom of statistical mechanics, that of stability of expectation values of extensive observables under changes in system size small compared to n. The restriction to extensive observables (a standard example of which is mass density) is simply because intensive observables (such as mass) obviously always depend on the size of the system. In the algebraic formulation of statistical mechanics certain amongst the extensive observables of a particular model may be put in a correspondence with the indecomposable summands of the representation of $H_n(m)$ in that model. This suggests that there is a connection between representations of $H_n(m)$ for different values of n, namely between those representations corresponding to the 'same'extensive observable. The results of this paper can be considered as a precise formulation of this connection without further reference to physics.

A version of the results in this article presented from the point of view of solvable lattice models has appeared in (Martin [**1991**]).

1. HECKE ALGEBRAS

There are several conventions and notations in the literature for the parameters that will be used. The parameters we will use are δ and q. These are related by the formula:

$$\delta = q + q^{-1}$$

Let x be any indeterminate. Then the sequence $P_n(x)$ is a sequence of polynomials in x with integer coefficients where the degree of $P_n(x)$ is $n-1$ for $n \geqslant 1$. This sequence is defined by the initial conditions $P_0(x) = 0$ and $P_1(x) = 1$ together with the recurrence relation

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$$

These are the Chebychev polynomials of the second kind and their main properties can be deduced from the following two equivalent formulae.

$$P_n(q+q^{-1}) = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad P_n(2\cos\theta) = \frac{\sin n\theta}{\sin\theta} \qquad \text{for } n > 0.$$

The equivalence between these is given by putting $q = e^{i\theta}$.

Lemma 1.1. The polynomials P_n satisfy the following identity:

$$P_{n+1} = -\sum_{j=1}^{n+1} P_{j-2}\delta^{n-j+1}$$

Proof. By induction on n, assuming the result for $n-1$ and $n-2$. The basis for the induction is

$$\begin{aligned} P_1 &= -P_{-1} \\ P_2 &= -P_{-1}\delta - P_0 \end{aligned}$$

The inductive step is the following calculation:

$$\begin{aligned} P_{n+1} &= \delta P_n - P_{n-1} \\ &= -\delta\sum_{j=1}^{n} P_{j-2}\delta^{n-j} + \sum_{j=1}^{n-1} P_{j-2}\delta^{n-j-1} \\ &= -\delta\sum_{j=2}^{n+1} P_{j-3}\delta^{n-j+1} + \sum_{j=3}^{n+1} P_{j-4}\delta^{n-j+1} \\ &= -\sum_{j=3}^{n+1} (\delta P_{j-3} - P_{j-4})\delta^{n-j+1} - P_{-1}\delta^n \\ &= -\sum_{j=1}^{n+1} P_{j-2}\delta^{n-j+1} \end{aligned}$$

Notation 1.2. The polynomial $[n]!$ is given by

$$[n]! = P_1(\delta)P_2(\delta)\dots P_n(\delta).$$

Definition 1.3. For each n, the algebra H_n is generated by elements 1, $u_1, \ldots, u_{n-1}$ subject to the relations:

$$u_i^2 = \delta u_i$$
$$u_i u_{i+1} u_i - u_i = u_{i+1} u_i u_{i+1} - u_{i+1}$$
$$u_i u_j = u_j u_i \text{ if } |i-j| > 1$$

Definition 1.4. The Hecke algebra is generated by elements 1, $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations are

$$(\sigma_i + 1)(\sigma_i - q^2) = 0$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1$$

The inverse isomorphisms between the algebras defined by these two presentations are defined by:

$$u_i \mapsto q^{-1}(\sigma_i + 1) \qquad \sigma_i \mapsto q u_i - 1$$

This shows that for $q = 1$ and $\delta = 2$ the algebra H_n is the group algebra of the symmetric group on n letters.

Remark 1.5. For each n, there is an involution of H_n given by

$$u_i \mapsto \delta - u_i$$

Lemma 1.6. *There is a natural isomorphism of H_{n-1}-bimodules*

$$H_n \cong H_{n-1} \oplus H_{n-1} u_{n-1} H_{n-1}.$$

Proof. The proof is by induction on n. Clearly

$$H_{n+1} \cong H_n \oplus H_n u_n H_n + H_n u_n H_n u_n H_n + \cdots.$$

Hence it is sufficient to show that

$$H_n u_n H_n u_n H_n \to H_n u_n H_n \oplus H_n.$$

Using the inductive hypothesis, the observation that u_n commutes with H_{n-1} and the defining relations we have

$$\begin{aligned} H_n u_n H_n u_n H_n &\cong H_n u_n (H_{n-1} \oplus H_{n-1} u_{n-1} H_{n-1}) u_n H_n \\ &\cong H_n u_n u_n H_n \oplus H_n u_n u_{n-1} u_n H_n \\ &\to H_n u_n H_n \oplus H_n \end{aligned}$$

Proposition 1.7. *For $0 \leqslant p \leqslant n-1$, there is a natural isomorphism of H_{n-p}-bimodules*

$$H_n \cong \bigoplus H_{n-p}(u_{i_1}u_{i_1-1}\dots u_{j_1})H_{n-p}(u_{i_2}u_{i_2-1}\dots u_{j_2})H_{n-p}\dots$$
$$\dots H_{n-p}(u_{i_k}u_{i_k-1}\dots u_{j_k})H_{n-p}$$

where the sum is over all sequences $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ which satisfy the two conditions

$$n-p \leqslant i_1 < i_2 < \cdots i_k \leqslant n-1$$
$$n-p \leqslant j_l \leqslant i_l \text{ for } 1 \leqslant l \leqslant k$$

Example 1.8. The longest expression in this sum is given by $k = p$,

$$(i_1, i_2, \dots, i_p) = (n-p, n-p+1, \dots, n-1)$$
$$(j_1, j_2, \dots, j_p) = (n-p, n-p, \dots, n-p)$$

and is given explicitly by

$$H_{n-p}u_{n-p}H_{n-p}u_{n-p+1}u_{n-p}H_{n-p}\dots H_{n-p}(u_{n-1}u_{n-2}\dots u_{n-p})H_{n-p}.$$

Proof. By induction on p, using Lemma 1.6 for the inductive step.

Remark 1.9. This result, for $p = n-1$, gives a convenient basis of H_n consisting of reduced words. The basis consists of all words of the form

$$(u_{i_1}u_{i_1-1}\dots u_{j_1})(u_{i_2}u_{i_2-1}\dots u_{j_2})\dots(u_{i_k}u_{i_k-1}\dots u_{j_k})$$

where the sequences $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ satisfy the two conditions

$$1 \leqslant i_1 < i_2 < \cdots i_k \leqslant n-1$$
$$1 \leqslant j_l \leqslant i_l \text{ for } 1 \leqslant l \leqslant k$$

Corollary 1.10. *For each n and p, with $0 < p < n$, there is an isomorphism*

$$H_n \cong \left(\bigoplus_{j=n-p+1}^{n-1} H_j u_j u_{j-1}\dots u_{n-p}H_{n-p}\right) \bigoplus H_{n-p}$$

2. Ideals

The algebra H_n always has two irreducible one dimensional representations defined by

$$u_i \mapsto 0 \qquad \text{and} \qquad u_i \mapsto \delta \qquad \text{for } i \leqslant n-1.$$

Assuming that these representations are projective there are central idempotents E_n and F_n in H_n which satisfy

$$u_i E_n = 0 \qquad \text{and} \qquad u_i F_n = \delta F_n \qquad \text{for } i \leqslant n-1.$$

Definition 2.1. For each n, X_n is defined as an element of the algebra

$$H_n \underset{\mathbb{Z}[\delta]}{\otimes} \mathbb{Z}(\delta).$$

The sequence X_n is defined inductively by $X_1 = 1$ and

$$X_{n+1} = \frac{1}{[n]!} X_n \left(P_{n+1} - P_n u_n\right) X_n$$

The following Proposition uniquely determines the elements X_n and shows that if $P_m(\delta)$ is invertible for $1 \leqslant m \leqslant n$ then

$$E_n = \frac{X_n}{[n]!}.$$

Proposition 2.2. *The elements X_n satisfy*

(1) $X_n X_n = [n]! X_n$
(2) $u_i X_n = 0$ for $1 \leqslant i \leqslant n-1$
(3) $X_n u_i = 0$ *for* $1 \leqslant i \leqslant n-1$
(4) X_n *is an element of* H_n

This result is given in (Jones [**1983**]) and (Wenzl [**1988**]).

Proof. The proof is by induction on n. The basis for the induction is that the Theorem holds for $n = 1$. The inductive hypothesis is that the Theorem holds for X_n and that X_n also satisfies

$$X_n u_n X_n u_n = P_{n+1}[n-1]! X_n u_n$$
$$u_n X_n u_n X_n = P_{n+1}[n-1]! u_n X_n.$$

The inductive step follows from the following calculations. In each of these calculations, the previous calculations, as well as the inductive hypothesis are assumed. Note that u_{n+1} commutes with X_n.

$$\begin{aligned} u_n X_{n+1} &= \frac{1}{[n]!} u_n X_n \left(P_{n+1} - P_n u_n\right) X_n \\ &= u_n X_n \left(P_{n+1} - P_{n+1}\right) = 0 \end{aligned}$$

Similarly $X_{n+1}u_n = 0$.

$$\begin{aligned} X_{n+1}X_{n+1} &= X_{n+1}\left(P_{n+1} - P_n u_n\right) X_n \\ &= [n+1]!X_{n+1} \end{aligned}$$

$$\begin{aligned} X_{n+1}u_{n+1}X_{n+1}u_{n+1} &= X_{n+1}u_{n+1}\left(P_{n+1} - P_n u_n\right) X_n u_{n+1} \\ &= \delta P_{n+1}X_{n+1}u_{n+1}X_n - P_n X_{n+1}u_{n+1}u_n u_{n+1}X_n \\ &= P_{n+2}[n]!X_{n+1}u_{n+1} \end{aligned}$$

Similarly $u_{n+1}X_{n+1}u_{n+1}X_{n+1} = P_{n+2}[n]!u_{n+1}X_{n+1}$.

Definition 2.3. The sequence of elements $Y_n \in H_n$ is defined by $Y_1 = 1$ and

$$Y_{n+1} = \frac{1}{[n]!} Y_n \left(-P_{n-1} + P_n u_n\right) Y_n$$

Example 2.4. The sequence Y_n starts

$$\begin{aligned} Y_1 &= 1 \\ Y_2 &= u_1 \\ Y_3 &= u_1 u_2 u_1 - u_1 \end{aligned}$$

Corollary 2.5. *For each n, the element $Y_n \in H_n$ satisfies*

(1) $Y_n Y_n = [n]!Y_n$
(2) $u_i Y_n = \delta Y_n$ for $1 \leqslant i \leqslant n-1$
(3) $Y_n u_i = \delta Y_n$ *for* $1 \leqslant i \leqslant n-1$

Proof. For each n, apply the involution of Remark 2.5 to the element $X_n \in H_n$.

Lemma 2.6. *The sequence Y_n also satisfies*

$$Y_{n+1} = -Y_n \left(\sum_{j=1}^{n} P_{j-2}(u_n u_{n-1} \dots u_j) + P_{n-1} \right).$$