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ON THE CENTRALISER ALGEBRAS OF QUANTUM LINEAR GROUPS

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Introduction

In this article we study a family of finite-dimensional associative algebras. These are usually referred to as the Hecke algebras of the general linear groups since they are a deformation of the group rings of the symmetric groups (see (James and Dipper [**1986**])). However they also arise in the study of knot theory and integrable two-dimensional lattice models, (Jones [**1987**]). In these applications they appear as centraliser algebras for the standard representation of the quantum enveloping algebra of the general linear group. In this article we consider certain quotients which arise from this alternative point of view.

In this article we have used a presentation of these algebras which is equivalent to, but not the same as, the deformation of the Coxeter presentation of the symmetric groups. We have also made this article self-contained, even though this has meant giving proofs of some results which are well-known.

Let V be a free module of rank m. Then the Hecke algebra of type A_{n-1}, H_n, acts on $\otimes^n V$. Define $H_n(m)$ to be the image of H_n in $\mathrm{End}(\otimes^n V)$. Assuming that the Laurent polynomial $[m]!$ (defined in 1.2) is invertible there is an idempotent $F_m \in H_n$ such that

$$H_n/\langle F_m\rangle \cong H_n(m)/\langle F_m\rangle \cong H_n(m-1).$$

Then the main result in this paper is that, for each n and m, there is an algebra isomorphism

$$F_m H_n(m) F_m \cong H_{n-m}(m).$$

This result can be seen as a dual to a result proved by Donkin and stated in (Erdmann [**1991**, §1.4]). The q-Schur algebra, $S_q(m,n)$, is also a subalgebra of $\mathrm{End}(\otimes^n V)$. In fact each of the algebras $H_n(m)$ and $S_q(m,n)$ is the centraliser of the other although we do not prove this. There is an idempotent $E \in S_q(m,n)$ such that $ES_q(m,n)E$ is isomorphic to $S_q(m-1,n)$ and there is an algebra isomorphism

$$S_q(m,n)/\langle E\rangle \cong S_q(m,n-m).$$

The main application of this result is to show that if the polynomial $[m]!$ is non-zero then, for each n, the algebra $H_n(m)$ is quasi-hereditary in the sense of (Scott [**1987**]).

As a second application of this result we give a new proof of a special case of the main theorem in (James [**1981**]). Let λ and μ be partitions of n with precisely m parts and let $\bar{\lambda}$ and $\bar{\mu}$ be the partitions of $n-m$ obtained by removing the first columns. Then

$$D(\lambda,\mu) = D(\bar{\lambda},\bar{\mu})$$

as entries in the two decomposition matrices. Our result is a special case of this result since we assume that $[m]!$ is invertible. On the other hand our proof is elementary and does not use the q-Schur algebra. Also our method gives a refinement involving a formal parameter (and applies to the Hecke algebras).

The algebras $H_n(m)$ occur in two-dimensional solvable lattice models as the algebras generated by single bond transfer matrices (for a recent review see (Deguchi and Martin [**1992**])). In this situation n depends on the size of the finite rectangular lattice and in ice-type models, for example, the parameter m is the number of states on each edge. One of the consequences of the occurrence of the Hecke algebras in solvable models is the implication of a certain profound connection between each representation of $H_n(m)$ and a corresponding representation of $H_{n+m}(m)$. At the level of physics this mapping embodies (the realisation in solvable models of) a fundamental axiom of statistical mechanics, that of stability of expectation values of extensive observables under changes in system size small compared to n. The restriction to extensive observables (a standard example of which is mass density) is simply because intensive observables (such as mass) obviously always depend on the size of the system. In the algebraic formulation of statistical mechanics certain amongst the extensive observables of a particular model may be put in a correspondence with the indecomposable summands of the representation of $H_n(m)$ in that model. This suggests that there is a connection between representations of $H_n(m)$ for different values of n, namely between those representations corresponding to the 'same'extensive observable. The results of this paper can be considered as a precise formulation of this connection without further reference to physics.

A version of the results in this article presented from the point of view of solvable lattice models has appeared in (Martin [**1991**]).

1. Hecke algebras

There are several conventions and notations in the literature for the parameters that will be used. The parameters we will use are δ and q. These are related by the formula:

$$\delta = q + q^{-1}$$

Let x be any indeterminate. Then the sequence $P_n(x)$ is a sequence of polynomials in x with integer coefficients where the degree of $P_n(x)$ is $n-1$ for $n \geqslant 1$. This sequence is defined by the initial conditions $P_0(x) = 0$ and $P_1(x) = 1$ together with the recurrence relation

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$$

These are the Chebychev polynomials of the second kind and their main properties can be deduced from the following two equivalent formulae.

$$P_n(q+q^{-1}) = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad P_n(2\cos\theta) = \frac{\sin n\theta}{\sin\theta} \qquad \text{for } n > 0.$$

The equivalence between these is given by putting $q = e^{i\theta}$.

Lemma 1.1. The polynomials P_n satisfy the following identity:

$$P_{n+1} = -\sum_{j=1}^{n+1} P_{j-2}\delta^{n-j+1}$$

Proof. By induction on n, assuming the result for $n-1$ and $n-2$. The basis for the induction is

$$\begin{aligned} P_1 &= -P_{-1} \\ P_2 &= -P_{-1}\delta - P_0 \end{aligned}$$

The inductive step is the following calculation:

$$\begin{aligned} P_{n+1} &= \delta P_n - P_{n-1} \\ &= -\delta\sum_{j=1}^{n} P_{j-2}\delta^{n-j} + \sum_{j=1}^{n-1} P_{j-2}\delta^{n-j-1} \\ &= -\delta\sum_{j=2}^{n+1} P_{j-3}\delta^{n-j+1} + \sum_{j=3}^{n+1} P_{j-4}\delta^{n-j+1} \\ &= -\sum_{j=3}^{n+1} (\delta P_{j-3} - P_{j-4})\delta^{n-j+1} - P_{-1}\delta^n \\ &= -\sum_{j=1}^{n+1} P_{j-2}\delta^{n-j+1} \end{aligned}$$

Notation 1.2. The polynomial $[n]!$ is given by

$$[n]! = P_1(\delta)P_2(\delta)\dots P_n(\delta).$$

Definition 1.3. For each n, the algebra H_n is generated by elements 1, u_1, ..., u_{n-1} subject to the relations:

$$u_i^2 = \delta u_i$$
$$u_i u_{i+1} u_i - u_i = u_{i+1} u_i u_{i+1} - u_{i+1}$$
$$u_i u_j = u_j u_i \text{ if } |i-j| > 1$$

Definition 1.4. The Hecke algebra is generated by elements 1, σ_1, ..., σ_{n-1} and defining relations are

$$(\sigma_i + 1)(\sigma_i - q^2) = 0$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1$$

The inverse isomorphisms between the algebras defined by these two presentations are defined by:

$$u_i \mapsto q^{-1}(\sigma_i + 1) \qquad \sigma_i \mapsto q u_i - 1$$

This shows that for $q = 1$ and $\delta = 2$ the algebra H_n is the group algebra of the symmetric group on n letters.

Remark 1.5. For each n, there is an involution of H_n given by

$$u_i \mapsto \delta - u_i$$

Lemma 1.6. *There is a natural isomorphism of H_{n-1}-bimodules*

$$H_n \cong H_{n-1} \oplus H_{n-1} u_{n-1} H_{n-1}.$$

Proof. The proof is by induction on n. Clearly

$$H_{n+1} \cong H_n \oplus H_n u_n H_n + H_n u_n H_n u_n H_n + \cdots .$$

Hence it is sufficient to show that

$$H_n u_n H_n u_n H_n \to H_n u_n H_n \oplus H_n.$$

Using the inductive hypothesis, the observation that u_n commutes with H_{n-1} and the defining relations we have

$$\begin{aligned} H_n u_n H_n u_n H_n &\cong H_n u_n (H_{n-1} \oplus H_{n-1} u_{n-1} H_{n-1}) u_n H_n \\ &\cong H_n u_n u_n H_n \oplus H_n u_n u_{n-1} u_n H_n \\ &\to H_n u_n H_n \oplus H_n \end{aligned}$$

Proposition 1.7. *For $0 \leqslant p \leqslant n-1$, there is a natural isomorphism of H_{n-p}-bimodules*

$$H_n \cong \bigoplus H_{n-p}(u_{i_1}u_{i_1-1}\dots u_{j_1})H_{n-p}(u_{i_2}u_{i_2-1}\dots u_{j_2})H_{n-p}\dots \\ \dots H_{n-p}(u_{i_k}u_{i_k-1}\dots u_{j_k})H_{n-p}$$

where the sum is over all sequences $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ which satisfy the two conditions

$$n-p \leqslant i_1 < i_2 < \cdots i_k \leqslant n-1$$
$$n-p \leqslant j_l \leqslant i_l \text{ for } 1 \leqslant l \leqslant k$$

Example 1.8. The longest expression in this sum is given by $k = p$,

$$(i_1, i_2, \dots, i_p) = (n-p, n-p+1, \dots, n-1)$$
$$(j_1, j_2, \dots, j_p) = (n-p, n-p, \dots, n-p)$$

and is given explicitly by

$$H_{n-p}u_{n-p}H_{n-p}u_{n-p+1}u_{n-p}H_{n-p}\dots H_{n-p}(u_{n-1}u_{n-2}\dots u_{n-p})H_{n-p}.$$

Proof. By induction on p, using Lemma 1.6 for the inductive step.

Remark 1.9. This result, for $p = n-1$, gives a convenient basis of H_n consisting of reduced words. The basis consists of all words of the form

$$(u_{i_1}u_{i_1-1}\dots u_{j_1})(u_{i_2}u_{i_2-1}\dots u_{j_2})\dots(u_{i_k}u_{i_k-1}\dots u_{j_k})$$

where the sequences $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ satisfy the two conditions

$$1 \leqslant i_1 < i_2 < \cdots i_k \leqslant n-1$$
$$1 \leqslant j_l \leqslant i_l \text{ for } 1 \leqslant l \leqslant k$$

Corollary 1.10. *For each n and p, with $0 < p < n$, there is an isomorphism*

$$H_n \cong \left(\bigoplus_{j=n-p+1}^{n-1} H_j u_j u_{j-1}\dots u_{n-p}H_{n-p} \right) \bigoplus H_{n-p}$$

2. Ideals

The algebra H_n always has two irreducible one dimensional representations defined by

$$u_i \mapsto 0 \qquad \text{and} \qquad u_i \mapsto \delta \qquad \text{for } i \leqslant n-1.$$

Assuming that these representations are projective there are central idempotents E_n and F_n in H_n which satisfy

$$u_i E_n = 0 \qquad \text{and} \qquad u_i F_n = \delta F_n \qquad \text{for } i \leqslant n-1.$$

Definition 2.1. For each n, X_n is defined as an element of the algebra

$$H_n \underset{\mathbb{Z}[\delta]}{\otimes} \mathbb{Z}(\delta).$$

The sequence X_n is defined inductively by $X_1 = 1$ and

$$X_{n+1} = \frac{1}{[n]!} X_n \left(P_{n+1} - P_n u_n\right) X_n$$

The following Proposition uniquely determines the elements X_n and shows that if $P_m(\delta)$ is invertible for $1 \leqslant m \leqslant n$ then

$$E_n = \frac{X_n}{[n]!}.$$

Proposition 2.2. *The elements X_n satisfy*

(1) $X_n X_n = [n]! X_n$
(2) $u_i X_n = 0$ *for* $1 \leqslant i \leqslant n-1$
(3) $X_n u_i = 0$ *for* $1 \leqslant i \leqslant n-1$
(4) X_n *is an element of* H_n

This result is given in (Jones **[1983]**) and (Wenzl **[1988]**).

Proof. The proof is by induction on n. The basis for the induction is that the Theorem holds for $n = 1$. The inductive hypothesis is that the Theorem holds for X_n and that X_n also satisfies

$$X_n u_n X_n u_n = P_{n+1}[n-1]! X_n u_n$$
$$u_n X_n u_n X_n = P_{n+1}[n-1]! u_n X_n.$$

The inductive step follows from the following calculations. In each of these calculations, the previous calculations, as well as the inductive hypothesis are assumed. Note that u_{n+1} commutes with X_n.

$$\begin{aligned} u_n X_{n+1} &= \frac{1}{[n]!} u_n X_n \left(P_{n+1} - P_n u_n\right) X_n \\ &= u_n X_n \left(P_{n+1} - P_{n+1}\right) = 0 \end{aligned}$$

Similarly $X_{n+1}u_n = 0$.

$$\begin{aligned} X_{n+1}X_{n+1} &= X_{n+1}\left(P_{n+1} - P_n u_n\right) X_n \\ &= [n+1]! X_{n+1} \end{aligned}$$

$$\begin{aligned} X_{n+1}u_{n+1}X_{n+1}u_{n+1} &= X_{n+1}u_{n+1}\left(P_{n+1} - P_n u_n\right) X_n u_{n+1} \\ &= \delta P_{n+1} X_{n+1} u_{n+1} X_n - P_n X_{n+1} u_{n+1} u_n u_{n+1} X_n \\ &= P_{n+2}[n]! X_{n+1} u_{n+1} \end{aligned}$$

Similarly $u_{n+1}X_{n+1}u_{n+1}X_{n+1} = P_{n+2}[n]! u_{n+1}X_{n+1}$.

Definition 2.3. The sequence of elements $Y_n \in H_n$ is defined by $Y_1 = 1$ and

$$Y_{n+1} = \frac{1}{[n]!} Y_n \left(-P_{n-1} + P_n u_n\right) Y_n$$

Example 2.4. The sequence Y_n starts

$$\begin{aligned} Y_1 &= 1 \\ Y_2 &= u_1 \\ Y_3 &= u_1 u_2 u_1 - u_1 \end{aligned}$$

Corollary 2.5. *For each n, the element $Y_n \in H_n$ satisfies*

(1) $Y_n Y_n = [n]! Y_n$
(2) $u_i Y_n = \delta Y_n$ for $1 \leqslant i \leqslant n-1$
(3) $Y_n u_i = \delta Y_n$ *for* $1 \leqslant i \leqslant n-1$

Proof. For each n, apply the involution of Remark 2.5 to the element $X_n \in H_n$.

Lemma 2.6. *The sequence Y_n also satisfies*

$$Y_{n+1} = -Y_n \left(\sum_{j=1}^{n} P_{j-2}(u_n u_{n-1} \dots u_j) + P_{n-1} \right).$$

Proof. By induction on n, assume the result for $n-1$.

$$\begin{aligned}
Y_{n+1} &= \frac{1}{[n]!}Y_n(-P_n + P_{n+1}u_n)Y_n \\
&= -\frac{1}{[n]!}Y_n(-P_n + P_{n+1}u_n)Y_{n-1}\left(\sum_{j=1}^{n-1} P_{j-2}(u_{n-1}u_{n-2}\dots u_j) + P_{n-2}\right) \\
&= -\frac{1}{P_n}Y_n(-P_n + P_{n+1}u_n)\left(\sum_{j=1}^{n} P_{j-2}(u_{n-1}u_{n-2}\dots u_j) + P_{n-1}\right) \\
&= -\frac{P_n}{P_{n+1}}Y_n\left(\sum_{j=1}^{n+1} P_{j-2}\delta^{n-j+1}\right) \\
&\quad + Y_n u_{n+1}\left(\sum_{j=1}^{n} P_{j-2}(u_n u_{n-1}\dots u_j) + P_{n-1}\right) \\
&= Y_n \sum_{j=1}^{n+1} P_{j-2}(u_{n+1}u_{n-1}\dots u_j) + Y_n P_n
\end{aligned}$$

Corollary 2.7. *The image of Y_n in any specialisation is non-zero.*

Proof. The coefficient of the basis element

$$\prod_{k=1}^{n-1}(u_k u_{k-1}\dots u_1)$$

is one.

Remark 2.8. This also shows that if $[n]! = 0$ then H_n is not semi-simple because Y_n spans a one-dimensional nilpotent two-sided ideal.

Definition 2.9. If $1 \leqslant m < n$ then $H_n(m)$ is the quotient of H_n by the two-sided ideal generated by Y_{m+1}.

For example, $Y_3 = u_1u_2u_1 - u_1$, so the sequence of algebras $H_n(2)$ is the sequence of Temperley-Lieb algebras. These are studied in (Temperley and Lieb [**1971**]) and (Jones [**1983**]).

Since Y_{m+1} is in the two-sided ideal generated by Y_m this defines, for each n, a decreasing sequence of two-sided ideals of H_n.

Notation 2.10. For each p and q, with $q \geqslant p$, let $Y_{<p,q>}$ be the image of Y_{q-p+1} under the homomorphism from H_{q-p+1} to H_q determined by $u_i \mapsto u_{i+p-1}$. This notation has been chosen so that $Y_{<p,q>}$ is a linear combination of words in the generators $u_p, \dots, u_q$. In particular, $Y_{<p,q>}$ commutes with u_i if $i \leqslant p-2$ or if $i \geqslant q+2$.

Proposition 2.11. *For each n and p, there is a morphism of H_{n-p}-bimodules*

$$Y_{<n-p+1,n-1>}H_nY_{<n-p+1,n-1>} \to H_nY_{<n-p,n-1>}H_n \oplus H_{n-p}.$$

Furthermore the composite morphism

$$Y_{<n-p+1,n-1>}H_nY_{<n-p+1,n-1>} \to H_{n-p}$$

is surjective.

Remark 2.12. Note that putting $p = 1$ in this Proposition gives Lemma 2.6.

Proof. Let $(i_1, i_2, \dots, i_k)$ and $(j_1, j_2, \dots, j_k)$ be any two sequences which satisfy the two conditions

$$1 \leqslant i_1 < i_2 < \cdots i_k \leqslant n-1$$
$$1 \leqslant j_l \leqslant i_l \text{ for } 1 \leqslant l \leqslant k$$

Using Proposition 2.7, and the observation that

$$H_{n-p}u_{i_1}u_{i_1-1}\dots u_{j_1}H_{n-p}u_{i_2}\dots u_{j_2}H_{n-p}\dots H_{n-p}u_{i_{k-1}}\dots u_{j_{k-1}}H_{n-p} \subset H_{i_k}$$

it is sufficient to show that there is a morphism

$$Y_{<n-p+1,n-1>}H_{i_k}u_{i_k}\dots u_{j_k}H_{n-p}Y_{<n-p+1,n-1>} \to H_nY_{<n-p,n-1>}H_n \oplus H_{n-p}.$$

If $j_k > n-p$ then

$$(u_{i_k}u_{i_k-1}\dots u_{j_k})H_{n-p}Y_{<n-p+1,n-1>} \cong H_{n-p}Y_{<n-p+1,n-1>}.$$

Hence there is no loss of generality in assuming that $j_k = n-p$. This shows that it is sufficient to show that, for $n-p \leqslant i \leqslant n$, there is a morphism

$$Y_{<n-p+1,n-1>}H_{i-1}u_{i-1}\dots u_{n-p}Y_{<n-p+1,n-1>}H_{n-p} \to H_nY_{<n-p,n-1>}H_n \oplus H_{n-p}.$$

Now by Lemma 2.6 we have

$$Y_{<n-p+1,n-1>} = \prod_{m=p-1}^{1}\left(\sum_{j=1}^{m} P_{j-2}u_{n-j}u_{n-j-1}\dots u_{n-m} + P_{m-1}\right).$$

Define $L(i)_{<n-p+1,n-1>}$ by

$$L(i)_{<n-p+1,n-1>} = \prod_{m=p-1}^{n-i}\left(\sum_{j=1}^{m} P_{j-2}u_{n-j}u_{n-j-1}\dots u_{n-m} + P_{m-1}\right).$$

Then the above expression for $Y_{<n-p+1,n-1>}$ can be written as

$$Y_{<n-p+1,n-1>} = L(i)_{<n-p+1,n-1>} Y_{<i+1,n-1>}$$

The second factor, $Y_{<i+1,n-1>}$, commutes with $H_{i-1}u_{i-1}u_{i-2}\dots u_{n-p}$. So, making this substitution, it is sufficient to show that, for $n-p \leqslant i \leqslant n$, there is a morphism

$$L(i)_{<n-p+1,n-1>} H_{i-1}u_{i-1}\dots u_{n-p} Y_{<n-p+1,n-1>} H_{n-p} \\ \to H_n Y_{<n-p,n-1>} H_n \oplus H_{n-p}.$$

This is proved by induction on i. The basis of the induction is $i = n-p$ which is an application of Lemma 2.6. The inductive step is the following calculation. From the definition of $L(i+1)_{<n-p+1,n-1>}$ we have

$$L(i+1)_{<n-p+1,n-1>} H_i \\ = L(i)_{<n-p+1,n-1>} H_i \left(\sum_{j=1}^{n-i-1} P_{j-2} u_{n-j} u_{n-j-1} \dots u_{i+1} + P_{n-i-2} \right).$$

Now it is straightforward that:

$$\left(\sum_{j=1}^{n-i-1} P_{j-2} u_{n-j} u_{n-j-1} \dots u_{i+1} + P_{n-i-2} \right) u_i u_{i-1} \dots u_{n-p} \\ = \sum_{j=1}^{n-i} P_{j-2} u_{n-j} u_{n-j-1} \dots u_{n-p} \\ = \sum_{j=1}^{p} P_{j-2} u_{n-j} u_{n-j-1} \dots u_{n-p} + P_{p-1} \\ - \sum_{j=n-i+1}^{p} P_{j-2} u_{n-j} u_{n-j-1} \dots u_{n-p} - P_{p-1}$$

Note that the second term in this last expression is an element of H_i. These two formulae then show that there is a map

$$L(i+1)_{<n-p+1,n-1>} H_i u_i u_{i-1} \dots u_{n-p} Y_{<n-p+1,n-1>} H_{n-p} \\ \to L(i)_{<n-p+1,n-1>} H_i Y_{<n-p,n-1>} H_{n-p} \\ \oplus L(i)_{<n-p+1,n-1>} H_i Y_{<n-p+1,n-1>} H_{n-p}$$

There is obviously a map

$$L(i)_{<n-p+1,n-1>} H_i Y_{<n-p,n-1>} H_{n-p} \to H_n Y_{<n-p,n-1>} H_n$$

By Corollary 2.10 there is an isomorphism

$$H_i \cong \left(\bigoplus_{j=n-p+1}^{i-1} H_j u_j u_{j-1} \dots u_{n-p} H_{n-p} \right) \bigoplus H_{n-p}.$$

Now substitute this into $L(i)_{<n-p+1,n-1>}H_iY_{<n-p+1,n-1>}H_{n-p}$. This shows that there is a map

$$L(i)_{<n-p+1,n-1>}H_iY_{<n-p+1,n-1>}H_{n-p}$$
$$\to L(i)_{<n-p+1,n-1>}\left(\bigoplus_{j=n-p+1}^{i-1} H_ju_ju_{j-1}\dots u_{n-p}\oplus H_{n-p}\right)Y_{<n-p+1,n-1>}H_{n-p}$$

By the inductive hypothesis, the range maps to $H_nY_{<n-p,n-1>}H_n\oplus H_{n-p}$. This completes the inductive step and the proof of the Proposition.

The result that the composite morphism

$$Y_{<n-p+1,n-1>}H_nY_{<n-p+1,n-1>}\to H_{n-p}$$

is surjective follows from the observation that there is a splitting map

$$H_{n-p}\to Y_{<n-p+1,n-1>}H_nY_{<n-p+1,n-1>}.$$

Corollary 2.14. *If $[m]!$ is invertible then there is an algebra isomorphism*

$$\left(\frac{Y_m}{[m]!}\right)H_n(m)\left(\frac{Y_m}{[m]!}\right)\cong H_{n-m}(m).$$

By construction, the quotient of $H_n(m)$ by the two-sided ideal generated by Y_m is $H_n(m-1)$. In other words we have a short exact sequence of algebras

$$0\to H_n(m)Y_mH_n(m)\to H_n(m)\to H_n(m-1)\to 0.$$

This gives the following well-known sufficient condition for $H_n(m)$ to be semisimple. If the image of Y_n in $H_n(m)$ is non-zero then, by Remark 2.8, this condition is also necessary. Other proofs have been given in (James and Dipper [**1986**]) and (Wenzl [**1988**]).

Theorem 2.15. *For all n, if $[n]!$ is invertible then, for all m, the algebra $H_n(m)$ is a direct sum of matrix algebras.*

Proof. The proof is by induction on n with an inner induction on $n-m$. The basis for the induction on n is the case $n=1$. The basis for the inner induction on $n-m$ is the case $n=m$.

For the inductive step consider the exact sequence of algebras

$$0\to H_n(m)Y_{m-1}H_n(m)\to H_n(m)\to H_n(m-1)\to 0.$$

Then $H_n(m)Y_{m-1}H_n(m)$ and $H_n(m-1)$ are direct sums of matrix algebras by the inductive hypotheses.

The assumption that $[n]!$ is invertible implies that there are idempotents E_k, for $1\leqslant k\leqslant n-1$, defined before Proposition 2.2. This shows that the short exact sequence splits.

3. APPLICATIONS

In this section we assume that $[r] \neq 0$ for $r \leqslant m$ so that $Y_m/[m]!$ is defined and is an idempotent. In the previous section it was shown that there is an isomorphism of unital algebras

$$\left(\frac{Y_m}{[m]!}\right) H_n(m) \left(\frac{Y_m}{[m]!}\right) \cong H_{n-m}(m).$$

In this section we will apply the following general theory described in (Green [**1980**, §6.2]) and (Cline, Parshall and Scott [**1988**]) to this situation.

Let A be an algebra and $e \in A$ a non-zero idempotent. Denote the (unital) centraliser algebra $eAe \cong \mathrm{End}_A(eA)$ by B. Then there is a functor j^* from left A-modules to left B-modules defined on modules by

$$j^*M = eM$$

and on morphisms by restriction.

Lemma 3.1. *The functor j^* is a full embedding and has a right adjoint and a left adjoint.*

Proof. The functor j^* is exact. The functor j_* defined by

$$j_*N = \mathrm{Hom}_B(A, N) \cong \mathrm{Hom}_A(Ae, N)$$

is right adjoint to j^* and is a right inverse to j^* since

$$\mathrm{Hom}_A(j_*j^*M_1, M_2) \cong \mathrm{Hom}_B(j^*M_1, j^*M_2) = \mathrm{Hom}_B(eM_1, eM_2).$$

The functor $j_!$ defined by

$$j_!N = N \underset{eAe}{\otimes} A \cong N \underset{eAe}{\otimes} eA$$

is left adjoint to j^*.

The homomorphism $H_n(m) \to H_n(m-1)$ induces a restriction functor i_* from left A-modules to left B-modules. Since the homomorphism is a surjection this is a full embedding. The functor i_* has a left adjoint i^* given by induction and a right adjoint $i^!$ given by coinduction.

A finite dimensional algebra, A, is said to be quasi-hereditary if the category of left A-modules is a highest weight category in the sense of (Cline, Parshall and Scott [**1988**]). There is a different but equivalent definition in (Scott [**1987**]).

Theorem 3.2. *If $[m]! \neq 0$ then $H_n(m)$ is quasi-hereditary for all n.*

Proof. The six functors j^*, j_*, $j_!$, i_*, i^* and $i^!$ pass to functors on the derived categories and satisfy the conditions for a recollement given in (Cline, Parshall and Scott [**1988**]).

The theorem now follows by induction on m with an inner induction on n. The basis of the induction is the result that, since $[m]! \neq 0$, $H_n(m)$ is semi-simple for $n \leqslant m$. The inductive step follows from the recollement using the converse to (Cline, Parshall and Scott [**1988**, Theorem 3.9.]).

The following two results are deduced from Lemma 3.1 in (Green [**1980**, §6.2]).

Lemma 3.3. *Let $\{S_\lambda | \lambda \in \Lambda\}$ be a complete set of inequivalent irreducible A-modules. Then*

$$\{eS_\lambda | \lambda \in \Lambda \quad \text{and} \quad eS_\lambda \neq 0\}$$

is a complete set of inequivalent irreducible B-modules.

Lemma 3.4. *If $(\, , \,)$ is a symmetric bilinear form on V such that $(eV, (1-e)V) = 0$, and if $(\, , \,)^e$ denotes the restriction of this form to eV, then*

$$\mathrm{rad}(\, , \,)^e \cong e.\mathrm{rad}(\, , \,).$$

Definition 3.5. A partition of n into m parts is an ordered sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$$
$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$$

Note that every partition of n into m parts is also a partition of n into $m + 1$ parts.

Notation 3.6. The partition conjugate to λ is λ'. This means that λ'_i is the number of parts of λ which are at least i.

Definition 3.7. The set of partitions of n has a partial order, called the dominance order, defined by

$$\lambda \trianglerighteq \mu \quad \text{if and only if} \quad \sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i \quad \text{for all } j.$$

Definition 3.8. For each partition of n into m parts, λ, define elements X_λ and Y_λ in $H_n(m)$ by

$$X_\lambda = X_{<1,\lambda_1-1>} X_{<\lambda_1+1,\lambda_1+\lambda_2-1>} \cdots X_{<n-\lambda_m+1,n-1>}$$
$$Y_\lambda = Y_{<1,\lambda'_1-1>} Y_{<\lambda'_1+1,\lambda'_1+\lambda'_2-1>} \cdots Y_{<n-\lambda'_m+1,n-1>}$$

Notation 3.9. Let λ be a partition with exactly m non-zero parts. Then define the partition $\bar{\lambda}$ by

$$\bar{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_m - 1)$$

In terms of the diagram, $\bar{\lambda}$ is obtained from λ by removing the first column.

The irreducible representations are labelled by partitions, but, in general, not every partition corresponds to an irreducible representation.

Notation 3.10. The irreducible representation corresponding to λ is denoted by D_λ. The projective cover of D_λ is denoted by P_λ.

Definition 3.11. For each partition of n into m parts define a right $H_n(m)$-module M_λ, called a permutation module, by

$$M_\lambda = \frac{Y_\lambda}{[m]!} H_n(m)$$

One of the main properties of the permutation modules is the following Proposition. This Proposition is given in (James and Dipper [**1986**]).

Proposition 3.12. *Let λ and μ be partitions of n. Then*

(1) *if D_λ occurs as a composition factor of M_μ then $\lambda \trianglerighteq \mu$*
(2) *D_λ occurs as a composition factor of M_λ with multiplicity one.*

Taking e to be $\frac{Y_m}{[m]!}$ and applying Lemma 3.3 gives an inclusion of the set of isomorphism classes of $H_{n-m}(m)$-modules in the set of isomorphism classes of $H_n(m)$-modules. The following proposition identifies this map.

Proposition 3.13. *For each partition, λ,*

(1) $M_{\bar{\lambda}} \mapsto M_\lambda$
(2) $D_{\bar{\lambda}} \mapsto D_\lambda$
(3) $P_{\bar{\lambda}} \mapsto P_\lambda$

Proof. Since $Y_{<n+1,n+m-1>}$ commutes with Y_λ, there is an isomorphism

$$Y_\lambda H_n(m) \underset{H_n(m)}{\otimes} Y_{<n+1,n+m-1>} H_{n+m}(m) \cong Y_\lambda Y_{<n+1,n+m-1>} H_{n+m}(m)$$

There is also an isomorphism

$$Y_\lambda Y_{<n+1,n+m-1>} H_{n+m}(m) \cong M_{(\lambda_1+1,\lambda_2+1,\dots,\lambda_m+1)}$$

The statement for D_λ follows by induction on the dominance order using Proposition 3.12.

This implies the statement for P_λ since this is the projective cover of D_λ.

For each λ there is a symmetric bilinear form on M_λ, D_λ, and P_λ.

Definition 3.14. For each n and m, the Cartan matrix, C_t, of $H_n(m)$ is a square matrix. If λ and μ are partitions of n into m parts and there are irreducible representations D_λ and D_μ then $C_t(\lambda, \mu)$ is the polynomial in a formal variable t defined by

$$C_t(\lambda,\mu) = \sum_{j=0}^{\infty} \dim \operatorname{Hom}(D_\lambda, \operatorname{rad}^j P_\mu / \operatorname{rad}^{j+1} P_\mu) t^j.$$

Setting $t = 1$ gives the usual Cartan matrix. Note also that these polynomials depend only on the partitions λ and μ and are independent of m.

Theorem 3.15. *Let λ and μ be partitions with the same number of non-zero parts. Then*

$$C_t(\lambda,\mu) = C_t(\bar{\lambda},\bar{\mu})$$

Proof. This result follows from Proposition 3.13 and Lemma 3.4.

Example 3.16. The Cartan matrices of the Temperley-Lieb algebras, $H_n(2)$, over a field of characteristic zero are given in (Westbury [**1990**]) so, if λ and μ are partitions of n, then this gives $C(\lambda,\mu)$ over a field of characteristic zero provided $\lambda_i = \mu_i$ for $i > 2$.

There is a similar result for the decomposition matrices. For each λ let S_λ be the Specht module constructed in (James and Dipper [**1986**]). These modules are defined over the ring $\mathbf{Z}[q,q^{-1}]$ and are a complete set of inequivalent irreducible modules if the algebra $H_n(m)$ is semi-simple. For each λ, the module S_λ also has a symmetric bilinear form satisfying the hypothesis of Lemma 3.3.

Definition 3.17. For each n and m, the decomposition matrix, D_t, of $H_n(m)$ is defined as follows. If λ and μ are partitions of n into m parts and there is an irreducible representation D_λ then $D_t(\lambda,\mu)$ is the polynomial in a formal variable t defined by

$$D_t(\lambda,\mu) = \sum_{j=0}^{\infty} \dim \operatorname{Hom}(D_\lambda, \operatorname{rad}^j S_\mu / \operatorname{rad}^{j+1} S_\mu) t^j.$$

Setting $t = 1$ gives the usual decomposition matrix and these polynomials also depend only on the partitions λ and μ.

Theorem 3.18. *Let λ and μ be partitions with the same number of non-zero parts. Then*

$$D_t(\lambda,\mu) = D_t(\bar{\lambda},\bar{\mu})$$

Proof. This result follows from the fact that

$$j^* S_\lambda \cong S_{\bar{\lambda}}$$

and Lemma 3.3.

We conjecture that $D^t D = C$ for all t. This conjecture implies that Theorem 3.12 follows from Theorem 3.15. However we have only verified this conjecture for the Temperley-Lieb algebras, $H_n(2)$, over fields of characteristic zero.

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