A Template for Quantum Spin Chain Spectra

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Abstract We consider a series of N-state $L(\geq N)$ site quantum spin chains, characterised by the ordered partition of N into 2 parts, $N = P + M$. These (P/M) chains are invariant under an action of $U_q SU(P/M)$, and are built from a representation of the Hecke algebra $H_{L-1}(q)$. We establish that the intersection of the spectra of a (P/M) and (P'/M') chain of fixed length L is the spectrum of the $(min(P,P')/min(M,M'))$ chain of that length. We establish that the spectrum of the (P/M) chain breaks into blocks corresponding to irreducible representations of $H_{L-1}(q)$ (or equivalently irreducible representations of $U_q SU(P/M)$) characterised by Young diagrams with no rectangular subdiagrams of dimension $(P+1) \times (M+1)$ (height $\times$ width resp.). We give the corresponding quotient relations for the Hecke algebra. We discuss several implications of these results.

1 Introduction

There has been much interest recently on the one hand in computing the spectra for quantum spin chains and on the other hand in determining the structure of $H_n(q)$ and associated quantum groups at q a root of unity. In this paper we introduce technology for progress in both these areas!

Let us begin with the physics side. We consider the (P/M) quantum chains defined by the Hamiltonians with $L = n + 1$ sites:

$$H = \sum_{j=1}^{n} U_j^{(P/M)} \tag{1}$$

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where

$$U_j^{(P/M)} = \frac{q+q^{-1}}{2} - \left\{ \sum_{a\neq b} E_j^{ab} E_{j+1}^{ba} + \frac{q+q^{-1}}{2} \sum_{a=1}^{P+M} \epsilon_a E_j^{aa} E_{j+1}^{aa} + \right.$$

$$\left. + \frac{q-q^{-1}}{2} \sum_{a\neq b} sign(a-b) E_j^{aa} E_{j+1}^{bb} \right\} \qquad (2)$$

and, with $N = P + M$, the $N \times N$ matrices E^{ab} have elements

$$(E^{ab})_{cd} = \delta_c^a \delta_d^b \qquad (a, b, c, d = 1, 2, ..., N)$$

and finally

$$\epsilon_1 = \epsilon_2 = .. = \epsilon_P = -\epsilon_{P+1} = -\epsilon_{P+2} = .. = -\epsilon_{P+M} = 1.$$

These Hamiltonians were introduced for $q = 1$ by Sutherland[1] and generalised for $q \neq 1$ by Schultz[2]. If we consider the case $P + M = 2$, from equation 2 we get

$$U_j^{(P/M)} = \frac{q+q^{-1}}{2} - \left\{ s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+ + \frac{q+q^{-1}}{2} \left[\frac{(1+\epsilon_2)}{2} (s_j^z s_{j+1}^z + 1/4) \right.\right.$$

$$\left.\left. + \frac{(1-\epsilon_2)}{2} (s_j^z + s_{j+1}^z) \right] + \frac{q-q^{-1}}{2} (s_{j+1}^z - s_j^z) \right\} \qquad (3)$$

where s^+, s^- and s^z are $SU(2)$ spin 1/2 matrices. If we take $\epsilon_2 = 1$ (i.e. the (2/0) chain) we get the well known spin 1/2 Heisenberg chain which has a massless phase for $-2 \leq (x = q + q^{-1}) \leq 2$ and a massive phase for $|x| > 2$. The (1/1) chain has also been extensively studied[3]. It has a floating phase for $-2 < x < 2$, a Pokrovsky-Talapov[4] phase transition for $|x| = 2$ and a frozen phase for $|x| > 2$. This model might describe the coverage dependence on fugacity for xenon adsorbed on copper[5].

Recently it was found[6] that the (2/1) chain is relevant for the understanding of Anderson's t-J model[7]. This model might in turn be relevant for desribing high T_c superconductivity. The very different properties of the chains (2/0) and (1/1), as well as the existence of other possible physical applications, motivated us to try to understand the properties of all the chains (P/M). The burgeoning interest in these models amongst the solid state community, and there relevance, from a very different point of view, to q-representation theory, motivated us to try to present the results in two parts, encompasing both traditional physical and mathematical perspectives.

A simple inspection of equation 2 shows that for $q = 1$ the chains are $SU(P/M)$ invariant (for the definition and properties of the super algebras

$SU(P/M)$ see ref.s 8). It was also shown by Deguchi and Akutsu[9,10] and by Ge, Liu and Hue[11] that for $q \neq 1$ the chains (P/M) are invariant under (an action of) the so called quantum algebra $U_q SU(P/M)$. This is true specifically for the open chains defined by equation 1. If one considers the chains with periodic boundary conditions (hence losing the $U_q SU(P/M)$ symmetry) one can use the Bethe ansatz, and in this way de Vega and Lopes[12] have obtained some properties of the spectra in the continuum limit. In this paper we will be concerned only with the open chains. The spectrum for real x is real here, even for q on the unit circle. This is not the case in general for periodic boundary conditions.

For open chains the Bethe ansatz equations were derived only for the (2/0) case (see ref.13). The (1/1) chain can be written in terms of free fermions through a Jordan-Wigner transformation. We hope that our appraoch to the problem will help to develop the proper Bethe ansatz equations for the general case.

Let us consider the problem of diagonalising a given chain (P/M) with L sites (for the moment we assume q generic). We can first use the $U_q SU(P/M)$ invariance to bring the Hamiltonian to a block diagonal form. Each block is then labelled by an irreducible representation (λ) of the quantum algebra. Each block will give a matrix of some dimension $D(\lambda)$ which has to be diagonalised, and this block will appear $d(\lambda)$ times, where $d(\lambda)$ is the dimension of the irreducible representation (λ).

Naively one would expect that the same procedure must be repeated for each chain (P/M) separately. This is not the case! For a given number of sites L, different chains have some of the $D(\lambda) \times D(\lambda)$ blocks in common, the *degeneracy* $d(\lambda)$ generally being different for the various chains. Thus if we have already met a given block in one chain, and have diagonalised it, we can use this information for many other chains. Moreover it turns out that we can write the $D(\lambda) \times D(\lambda)$ matrices directly without needing to go through the standard procedure of using Clebsch-Gordan machinery.

This enormous simplification of the problem is based on the observation of Deguchi and Akutsu[10] that the $U_j^{(P/M)}$'s defined by equation 2 satisfy the relations for the generators of the (unital associative) Hecke algebra H_n

$$U_i U_i = x U_i \tag{4}$$

$$U_i U_{i\pm1} U_i - U_i = U_{i\pm1} U_i U_{i\pm1} - U_{i\pm1} \tag{5}$$

$$U_i U_{i+j} = U_{i+j} U_i \qquad (j \neq 1) \tag{6}$$

(the dimension of H_n is $(n+1)!$ independent of x). Moreover, for real q and for all the chains (P/M) with L sites for which we have non-zero P and M, the lowest and largest eigenvalues are known. They are zero and $(L-1)x$ respectively (an outline proof is given in section 3.4).

The paper is organised as follows. In section 2 we first describe the content of irreducible representations (λ) of $U_q SU(P/M)$ for a (P/M) chain with L

sites, using super-Young tableaux. To each super-Young tableau (Y) there corresponds an irreducible representation of the quantum algebra, of dimension $d_{P/M}(Y)$. As indicated, this depends on the chain (P/M). The same irreducible representation (Y) will appear $D(Y)$ times in the representation of the quantum algebra which commutes with our Hamiltonian. This multiplicity is, in fact, the dimension of the irreducible representation of the symmetric group S_L described by the same tableau Y. Note that a given tableau Y thus appears in two 'different' ways. Its second use (here associated to the symmetric group) is more generally related to the Hecke algebra H_n. It turns out that if for a given chain (P/M) a certain Young tableau appears (super or not) then the block does not depend on (P/M). The reader should be able to understand the block structure of the (P/M) chains after finishing this section.

In section 3 we write down a set of representations R_{PM} of the Hecke algebra H_n which we identify with those in equation 2. The new definition exhibits a block diagonal structure (in a different way to that above) which establishes contact with the q-deformed version of some standard representation theory. Using this we obtain the precise irreducible content of the Hamiltonian representation.

In section 4 we show how to construct the irreducibles directly from the Hecke algebra (i.e. independent of the (P/M) chain) and discuss the diagonalisation of the Hamiltonian within these irreducibles.

In section 6 we draw conclusions and discuss a number of open questions raised by this work.

In appendix A we consider the $(1/1)$ chain, and show the connection between the fermionic representation given by equation 3 and the irreducible representations given by Young tableaux. Appendix B is a very pedestrian presentation of the $(2/1)$ chain with 4 sites and $q = 1$. We present the usual approach of the representation theory of the superalgebra $SU(2/1)$, make the connection with the super-Young tableaux, and then make contact with the H_3 Hecke algebra.

On the mathematical side, the representations R_{PM} have some useful properties for the analysis of $H_n(q)$ at q a root of unity. In essence, they serve to break up the problem in a new way, by introducing a new nesting of quotient algebras well defined for all q (i.e. those for which R_{PM} is faithful). Some of these new quotients (see section 5) are potentially simpler to analyse than the conventional ones[17], the simplest of which is the Temperley-Lieb algebra (which is presently the only one for which the structure is completely known). The new quotients also have some nice Loewy decomposition properties[18] (coming from the faithfulness of R_{PM}). Considerable progress has been made in analysing $H_n(q)$ by noting Morita equivalences[19] among towers of quotient algebras[20]. In our notation these quotients correspond to $(P/M) = (N/0)$. The question is, do our other quotients have similar useful properties?

2 Irreducible content of (P/M) chains via super-Young tableau

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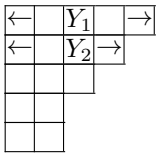
In this section we will consider the chains (P/M) in the case $q = 1$, and just mention the modifications for $q \neq 1$ (here we assume q generic). Let us first consider the chains $(N/0)$, which are $SU(N)$ symmetric. If we denote by $\square$ the N dimensional fundamental representation, then the content of $SU(N)$ irreducibles for a chain with L sites is obtained in the following way[14]. We consider a Young tableau (Y) with L boxes and with at most N rows, and Y_i boxes in row i:



We distribute the numbers 1,2,..,N inside the L boxes such that they are nondecreasing if we move from the left to the right within a row (this corresponds to the total symmetrisation in a row) and increasing if we move downwards inside a column (this corresponds to the antisymmetrisation inside a column).

The number of ways we can repeat this distribution gives the dimension of the irreducible representation $d(\lambda)$ (see appendix B for an example). In the tensor product

$$\square \otimes \square \otimes .... \otimes \square$$

(L times) we will obtain this irreducible $D(Y)$ times (where $D(Y)$ is the dimension of the symmetric group representation corresponding to the same tableau). For example, for $L = 4$ the tableau (3,1) will appear 3 times since this tableau corresponds to the 3 dimensional irreducible representation of S_4. Notice that we have used $U(N)$ tableaux for $SU(N)$ since they emphasise the simple correspondence between $d(Y)$ and $D(Y)$. Of course the dimension $d(Y)$ depends only on the differences $\lambda_i = Y_i - Y_N$ $(i = 1, 2, .., N-1)$. Let us stress that using Young tableaux allows us to organise the states into irreducibles without using Clebsch-Gordan coefficients.

Let us now consider the (P/M) case. Again we have to consider the tensor product

$$\square \otimes \square \otimes .... \otimes \square$$

(L times). Now the box represents the fundamental representation for $SU(P/M)$. It is most convenient to take the first P states bosonic and the remaining ones fermionic. In order to find the content we will use super-Young tableaux[15]. We take rows of length $Y_1, Y_2, .., Y_L$ such that

$$\sum_i Y_i = L. \tag{7}$$

In each box we distribute the numbers 1,2,..,N (remember $N = P + M$) in the following way. In each row the numbers are non-decreasing if they are bosonic and increasing if they are fermionic (this corresponds to a symmetrisation for the bosonic-bosonic, or bosonic-fermionic couples, and an antisymmetrisation for fermionic couples). In each column the numbers are increasing if they are bosonic and non-decreasing if they are fermionic (this corresponds to the 'antisymmetrisation' of the column). From this rule we learn that for the (P/M) chain the Young tableaux have at most P rows and at most M columns. Like the $SU(N)$ case the number of ways we can distribute the numbers 1,2,..,N in the super-Young tableau gives the dimension of the irreducible. For example, if the super-Young tableau is just a row with L boxes the dimension $d(Y)$ is

$$d((L)) = \sum_{k=0}^{L} \frac{M!}{k!(M-k)!} \frac{(P+L-k-1)!}{(L-k)!(P-1)!} \tag{8}$$

and if the tableau is just a column with L boxes the dimension of the irreducible is

$$d((1^L)) = \sum_{k=0}^{L} \frac{P!}{k!(P-k)!} \frac{(M+L-k-1)!}{(L-k)!(M-1)!} \tag{9}$$

Notice that equation 9 is obtained from equation 8 by interchanging P and M. As in the $SU(N)$ case a given super-Young tableau Y will appear $D(Y)$ times, where again $D(Y)$ is the dimension of the irreducible representation of S_L corresponding to the Young tableau Y. Up to now we have considered the case $q = 1$. For $q \neq 1$ (but generic) one can introduce the q-analogue of the Young symmetriser (see later) in such a way that the overall picture with respect to multiplicities and dimensions is unchanged.

Using the information presented in the next section, coming from the representation theory of Hecke algebras, we will have the following result. Take a given (P/M) chain with L sites. Draw all the super-Young tableau with L boxes (keep in mind that they have at most P rows and M columns). To each Young tableau corresponds a block of dimension $D(Y) \times D(Y)$ in the Hamiltonian. Look at all the chains you have computed before. If you already met the tableau Y you use the spectra already computed. If the tableau is new you diagonalise the block, and can then use the spectra for the present chain and for all subsequent chains which contain this tableau. Conversely, since the $SU(2/0)$ and $SU(1/1)$ spectra are known[3,16], the physical properties of the tableau types $\lambda = (S + T, S)$ and $(S, 1^T)$ (for S, T any natural numbers) are known and can be used for other chains.

We will see in the next section that in general if we have two chains (P/M) and (P'/M') of the same length then the spectrum of the former is entirely contained in the spectrum of the later (up to multiplicities) if and only if $P' > P$ and $M' > M$. Furthermore, if $P' > P$ and $M > M'$ then barring accidental

degeneracies (e.g. at q a root of unity) the intersection of the two spectra is precisely the spectrum of the (P, M') chain.

3 Irreducible content of (P/M) chains via Hecke algebras

In this section we first rewrite the Hamiltonian representation in such a way as to exhibit a block diagonal structure, and then analyse the content of the blocks.

3.1 Hamiltonian representations of H_n

Let P, M be non-negative integers such that $N = P + M$ is positive. Let $V_N = \{1, 2, .., N\}$ be shorthand for the standard ordered basis for $\mathbb{C}^N$, and I_N be the $N \times N$ identity matrix, and R the $N^2 \times N^2$ matrix with action on $u \otimes v \in V_N^2$ given by

$$R \quad u \otimes v = 0 \qquad if \quad u = v \leq P \tag{10}$$

$$R \quad u \otimes v = x \quad u \otimes v \qquad if \quad u = v > P \tag{11}$$

and otherwise, with $p = sign(v - u)$,

$$R \quad u \otimes v = q^p \quad u \otimes v \; + \; v \otimes u. \tag{12}$$

Then for $N < n$ and V the space spanned by V_N^{n+1} we can check by direct computation that there is a representation $R_{PM} : H_n(q) \mapsto End_{\mathbb{C}}(V)$ given by

$$R_{PM}(U_i) = I_N \otimes I_N \otimes ... \otimes R \otimes ... \otimes I_N \tag{13}$$

where M appears in the i^{th} position in the product. This is the *same* representation as that defined in equation 2, i.e.

$$R_{PM}(U_i) = U_i^{(P/M)}.$$

3.2 On q-Young symmetrisers

From the Hecke representation theory perspective we can deduce the irreducible content of R_{PM} as follows:

To make contact with the representation theory of the symmetric group we must review q-Young symmetrisers[21]. The following passage follows closely the presentation in ref.17.

Definition 1 (Idempotents) *For each $m = 1, 2, 3, .., n+2$ define an idempotent $E_m \in H_n(q)$ by*

$$E_1 = E_2 = 1$$

and then

$$E_m \in H_{m-2}(q) \subset H_n(q)$$

and

$$E_m E_m = E_m$$

and for $i = 1, 2, .., m-2$

$$E_m U_i = U_i E_m = 0.$$

There can be at most one such element, since if $E_m, E'_m \in H_{m-2}(q)$ both have the above properties then $E_m E'_m = E_m = E'_m$.

Let us consider the existence of such an element. We need

Definition 2 *For each positive integer n define k_n, a function of x, by $k_1 = 0$ and*

$$k_{n+1} = 1/(x - k_n).$$

Definition 3 *Define $I[m-2] \in H_{m-2}(q)$ by $I[0] = 1$ and*

$$I[m-2] = I[m-3](1 - k_{m-1}U_{m-2})I[m-3]$$

The existence of $I[m-2]$ for a given value of q is guaranteed unless some k_n required in its construction has a pole at that point.

Proposition 1 (see ref.17) *If $I[m-2]$ exists then*

$$E_m = I[m-2]$$

For example

$$E_4 = 1 + \frac{U_1U_2 + U_2U_1 - x(U_1 + U_2)}{x^2 - 1} + \frac{U_1 - U_1U_2U_1}{x(x^2 - 1)}.$$

Under the automorphism $D : H_n(q) \to H_n(q)$ defined by

$$U_i \mapsto x - U_i$$

we have another idempotent

$$D(E_m) = F_m.$$

For $X \in H_n(q)$ we define $X^{(t)} \in H_{n+t}(q)$ by the translation

$$U_i^{(t)} = U_{i+t}.$$

Definition 4 *For $n \geq b \geq a > 0$ and $c = 3 - a + b$ define $F_{ab} \in H_n(q)$ by the translation*

$$F_{ab} = F_c^{(a-1)}.$$

Consequently, if $a \leq i \leq b$ then

$$U_i \; F_{ab} = F_{ab} \; U_i = x \; F_{ab}. \tag{14}$$

For example,

$$F_{11} = \frac{U_1}{x}$$

$$F_{12} = \frac{U_1 U_2 U_1 - U_1}{x(x^2 - 1)}.$$

We similarly define (with $c = 3 - a + b$ as before)

$$E_{ab} = E_c^{(a-1)}$$

so that

$$U_i \; E_{ab} = E_{ab} \; U_i = 0$$

if $a \leq i \leq b$. For example $E_{1\,-1} = 1$.

Note that, as with E_{ij}, F_{ij} may not be well defined for all x (consider our examples). However, it follows from the definition that for each problematic x there exists a well defined function $f = f(x)$ such that fF_{ij} is finite at this x, although nilpotent. Then clearly

$$U_i \;\; fF_{ij} = x \;\; fF_{ij}. \tag{15}$$

The numbers f are computable, but we will not need the details here (see ref. 20, for example).

3.3 On q-permutation representations

Definition 5 *For $A \in H_a(q)$ but not in any $H_c^{(t)}(q)$ with $c < a$, and $B \in H_b(q)$ similarly, we define*

$$A \otimes B \in H_{a+b+1}(q)$$

as the product of A with the translation of B into $H_b^{(a+1)}(q)$.

Note from the definition above that our representation R_{PM} is block diagonal with blocks labelled by partitions of $n+1$, corresponding to fixed numbers of 1's, 2's, 3's,...,N's in the basis vectors. Suppose we label these partitions by Young tableau $\lambda = (\lambda_1, \lambda_2, ..)$ with (as before) i^{th} row length λ_i giving the number of i's, then our blocks are isomorphic to the left H_n modules generated by

$$\Lambda_\lambda(P, M) = \quad \left(\otimes_{i=1}^{P} E_{\lambda_i + 1}\right) \otimes \left(\otimes_{i=P+1}^{P+M} F_{\lambda_i + 1}\right).$$

Outline proof: Consider the action of the generators $\{U_i\}$ on the basis vector

$$1111..11222..2333..3.....NN..N$$

in R_{PM} and compare with the action on $\Lambda_\lambda(P, M)$. The isomorphism follows from an analogous treatment to the $M = 0$ case in Martin ref.17 (specifically in the limit of independent large imaginary x_i, in the notation of chapter 9 therein).

By analogy with the $q = 1$ case these representations are induced from the Young symmetrisers/antisymmetrisers for smaller algebras - so we call them (q) permutation representations.

3.4 Irreducible content of q-permutation modules

The generic irreducible content of such an induced module may be determined by continuity with the symmetric group case as follows (a trusty old reference is Hamermesh, ref. 22):

1. write all the 1's in a row (assuming $P \neq 0$);

2. for each number $i = 2, 3, .., P$ take the λ_i copies and add them one at a time to the existing diagram in such a way that each stage is a standard shape Young diagram with non-decreasing rows and columns, and such that no two i's appear in the same column;

3. for each subsequent number $i = P + 1, .., N$ take the λ_i copies and add them to the diagram such that no two appear in the same row.

Each diagram constructed in this way gives the tableau shape for an irreducible component of the block, which we will describe in the next section. The procedure may be checked by looking at the restriction from n to $n - 1$ in the Bratelli diagrams for the irreducible and permutation modules.

For example, consider the case $P = 2, M = 1$ and λ given by

$$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & 3 & \end{array}$$

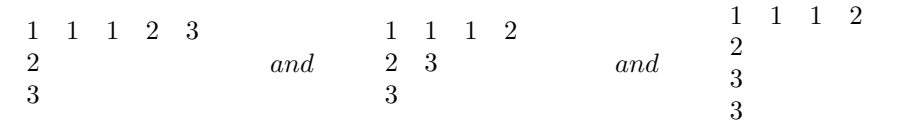
(since the basis is obtained by writing the numbers here in all possible orders this is a 7.6.5.4/2.2=210 dimensional representation). We have, after step 2.:

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \end{array} \quad \textit{and} \quad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \end{array} \quad \textit{and} \quad \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \end{array}$$

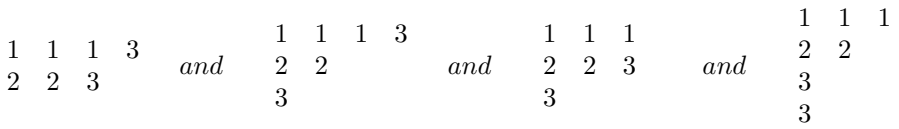
giving, after step 3.,

$$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 3 \\ 3 & & & & & \end{array} \quad \textit{and} \quad \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 3 & & & & \\ 3 & & & & \end{array} \quad \textit{and} \quad \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & & & \end{array}$$

and



and



We will give a means to check that all the diagrams have been found (by dimension counting) shortly. Notice that in this example $\lambda = (5, 1, 1)$ occured as output with multiplicity 2. Summing over all the permutation blocks we would get the total multiplicity of the corresponding irreducible representation in the Hamiltonian, and hence the dimension of an irreducible representation of the centralising quantum group.

In general, working over the whole of the representation R_{PM} (equation 10), every possible partition is realised by this procedure except those *involving* the rectangular diagrams of height $P + 1$ and width $M + 1$. This cannot appear as a subdiagram in any diagram, since the first box in the last row must contain at least a $P + 1$ by rule 2., and then each subsequent number in the row must be at least 1 higher than its predecesor, so the last is $P + M + 1$, which is a contradiction. To see that all other diagrams *do* appear note that if the original (permutation module) diagram satisifies the rules in the three steps then this diagram will appear in the output from the three steps.

We are now in a position to check that $(L - 1)x$ is the largest eigenvalue for any non-zero P and M and real q. First note that the one dimensional irreducible representation associated to $\lambda = (1, 1, .., 1)$ appears in all such chains (and has this eigenvalue - for an explicit demonstration of this see the next section). Now for given L consider the $(1, 1, .., 1)$ q-permutation module associated to the $(L, 0)$ chain. Clearly no irreducible representation is excluded in this case (an $(L + 1) \times 1$ diagram is impossible with L boxes) and in fact this is the regular representation of the whole Hecke algebra. This means, of course, that one dimensional irreducibles appear with multiplicity one. In the Hamiltonian basis at $q = 1$ this representation of the Hamiltonian has unique largest eigenvalue $(L - 1)x$ ($x = 2$) with eigenvector v defined by $v_i = 1$. On the other hand, for q a positive indeterminate, some power of H in this representation is a positive matrix (see the definition). Then by the Perron-Frobenius theorem the largest magnitude eigenvalue of H is unique for all positive q. This means that there can be no crossings of magnitudes of eigenvalues involving the largest magnitude eigenvalue, so $(L - 1)x$ is the largest throughout this region.

4 Building the irreducible representations of H_n

We have shown how to determine the generic irreducible content of R_{PM}, identifying each irreducible representation by a Young diagram λ. Now we show how to work out matrices $R_\lambda(U_i)$ representing the generators for each diagram, and hence the relevant block of the Hamiltonian.

Here we present 2 distinct procedures, one designed to be well defined for all q, and one not, but with a highly algorithmic construction (suitable for putting on a computer). The search for a basis with both qualities goes on!

For a given diagram λ we take λ_i to be the length of the i^{th} row as before, and λ'_i the length of the i^{th} column (examples will follow shortly). For the algebraically minded we will first define the representation $R_\lambda(U_i)$ in a robust way, i.e. by giving a basis for the representation in the algebra itself. The explicit matrices may then be deduced from the defining relations, as we will see.

The representation λ is associated to the left H_n module generated from the one dimensional subspace

$$W_o = \left(\otimes_i F_{\lambda'_i+1}\right) H_n \left(\otimes_j E_{\lambda_j+1}\right)^{(-)}$$

where the $(-)$ signifies that the term in the bracket is reflected in the generators (i.e. $U_i \leftrightarrow U_{n+1-i}$). An elementary introduction to this construction may be found in, for example, Martin, ref. 17.

It is known how to compute the dimensions of irreducibles. There follows a brief review of the procedure involved. The interested but inexperienced reader should consult Hamermesh, ref.22. The dimension of the representation R_λ is given by the 'hook length' rule

$$dim(R_\lambda) = (n+1)! \; / \; \prod_i h_i$$

where the product is over boxes in λ, and h_i is the hook length for box i, defined as the number of boxes traversed in first reaching box i horizontally from the far right and then leaving box i vertically to the bottom. For example, the shape $(3,2,1,1)$ gives hook lengths (positioned in the boxes to which they corrrespond)

$$\begin{array}{lll} 6 & 3 & 1 \\ 4 & 1 & \\ 2 & & \\ 1 & & \end{array}$$

and hence dimension 7!/(1.3.1.6.4.2.1)=35.

This dimension is also the number of standard Young tableau for λ - which is the number of ways of putting the numbers 1 to n in the boxes of the diagram so that each row and column is increasing (examples shortly). These standard

tableau are thus a convenient list of labels for the basis vectors. A good basis is any linear combination of the set of $dim(\lambda)$ elements $W_s \in H_n$ constructed iteratively in the following way:

Starting from an initial basis vector W_o as defined above (corresponding to a given standard tableau, which we will define shortly), and then for s and t standard Young tableaux, we obtain new basis vectors by

$$W_s = U_i W_t$$

if $t \to s$ on interchanging the numbers i and $i+1$ in the tableau (the correspondence with the definition in Martin, ref.17, is obtained by replacing tableau rows with orthogonal directions on the hypercubical lattice).

We are now only missing a starting tableau (o) for this iteration. To obtain this note that the iteration defines a partial order on standard tableau, with that tableau exhibiting unit increases down the columns, e.g.

$$\begin{array}{cccc} 1 & 5 & 9 & 11 \\ 2 & 6 & 10 & \\ 3 & 7 & & \\ 4 & 8 & & \end{array}$$

the unique first; and unit increases across the rows, e.g.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & 9 & & \\ 10 & 11 & & \end{array}$$

the unique last. The first standard tableau $s = o$ corresponds to the first basis element, W_o.

4.1 Examples

4.1.1 The canonical basis

In the case $n = 4$ there are 5 irreducible representations corresponding to the partitions

$$(4), (3,1), (2,2), (2,1,1), (1^4).$$

These have dimensions 1,3,2,3, and 1 respectively. For example, the $(3,1)$ representation has standard tableau

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array} \quad and \quad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array} \quad and \quad \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \end{array} .$$

The corresponding basis states are, altogether,

$$B_{(4)} = \{E_{13}\}$$

$$B_{(3,1)} = \{U_1E_{23},\ U_2U_1E_{23},\ U_3U_2U_1E_{23}\}$$
$$B_{(2,2)} = \{U_1U_3U_2E_1E_3,\ U_2U_1U_3U_2E_1E_3\}$$
$$B_{(2,1,1)} = \{F_{12}E_3,\ U_3F_{12}E_3,\ U_2U_3F_{12}E_3\}$$
$$B_{(1^4)} = \{F_{13}\}.$$

Writing x for $q + q^{-1}$, the representations associated to these basis states are then

$$R_{(4)}(U_i) = 0$$
$$R_{(3,1)}(U_1) = \begin{pmatrix} x & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; R_{(3,1)}(U_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & 0 \\ 0 & 1 & 0 \end{pmatrix}; R_{(3,1)}(U_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{pmatrix}$$
$$R_{(2,2)}(U_1) = \begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix}; R_{(2,2)}(U_2) = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix}; R_{(2,2)}(U_3) = \begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix}$$
$$R_{(2,1^2)}(U_1) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ x & 1 & 0 \end{pmatrix}; R_{(2,1^2)}(U_2) = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{pmatrix}; R_{(2,1^2)}(U_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & 0 \\ -x & 1 & x \end{pmatrix}$$
$$R_{(1^4)}(U_i) = x.$$

The Hamiltonian subblocks are thus

$$(0), \begin{pmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix}, \begin{pmatrix} 2x & 1 \\ 2 & x \end{pmatrix}, \begin{pmatrix} 2x & 1 & 0 \\ 0 & 2x & 1 \\ 0 & 2 & 2x \end{pmatrix}, (3x)$$

respectively, with eigenspectra

$$(0), (x, x \pm \sqrt{2}), ((3x \pm \sqrt{x^2+8})/2), (2x, 2x \pm \sqrt{2}), (3x).$$

Notice that in the large x limit the largest and smallest eigenvalues must be associated to $\lambda = (1^L)$ and (L) respectively. This result obviously holds as long as there are no spectrum crossings, and in general there are no spectrum crossings until $x = 2$.

In the case $n = 5$ there are 7 irreducible representations corresponding to the partitions

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1^5).$$

Let us restrict attention here to $(3,1,1)$. The dimension is 5!/(5.2.2)=6. The first tableau is

$$o = \begin{array}{ccc} 1 & 4 & 5 \\ 2 & & \\ 3 & & \end{array}$$

and the only exchange possible here (consistent with a standard tableau output) is $3 \leftrightarrow 4$, so with $W_o = F_{12}E_{34}$ we have $W_s = U_3W_o$. The new tableau is

$$s = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{array}$$

which may be modified either with $i = 2$ or $i = 4$. Carrying on in this way we get the basis:

$$\{W_o,\ U_3W_o,\ U_4U_3W_o,\ U_2U_3W_o,\ U_2U_4U_3W_o,\ U_3U_2U_4W_o\}.$$

From this we can read off (by application of the defining relations) the representation

$$R_{(3,1,1)}(U_1) = \begin{pmatrix} x & & & & & \\ 0 & x & & & & \\ 0 & 0 & x & & & \\ x & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; R_{(3,1,1)}(U_2) = \begin{pmatrix} x & & & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & x & & \\ 0 & 0 & 0 & 0 & x & \\ -x & 0 & 0 & x & 1 & 0 \end{pmatrix}$$

$$R_{(3,1,1)}(U_3) = \begin{pmatrix} 0 & 1 & & & & \\ 0 & x & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & \\ -x & 1 & 0 & x & & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}; R_{(3,1,1)}(U_4) = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & x & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & x & 0 \\ x & 0 & 0 & -x & 1 & x \end{pmatrix}$$

and the Hamiltonian

$$H = \begin{pmatrix} 2x & 1 & & & & \\ 0 & 2x & 1 & 1 & & \\ 0 & 1 & 2x & 0 & 1 & \\ 0 & 2 & 0 & 2x & 1 & \\ 0 & 0 & 1 & 0 & 2x & 1 \\ 0 & 1 & 0 & 0 & 2 & 2x \end{pmatrix}$$

with spectrum

$$\{2x, 2x, 2x \pm 1, 2x \pm \sqrt{5}\}.$$

If we write the basis states out according to the partial order on standard tableau then the matrix entries above the diagonal will always be either zero or one. Unfortunately those below the diagonal require application of the defining relations. We can avoid this by working with another basis:

4.1.2 The Hoefsmit (Young normal) basis

For those requiring a more algorithmic construction we have a basis $\{|s>:$ *s standard in* $\lambda\}$ (Hoefsmit, ref. 23) for the same representations with the following action:

$$R(U_i)|s>= k_g|s> +\sqrt{k_g/k_{g+1}}|s^i>$$

where $g = g_i(s)$ is the difference in row position of the numbers i and $i+1$ minus the difference in column position, k_g comes from definition 2, and s^i is

the tableau obtained by interchanging i and $i+1$ in s (note that the coefficient is zero every time this is not standard).

We can reproduce one of our previous examples as follows: For $\lambda = (2,2)$ we have standard tableau

$$o = \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} ; \quad s = \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} .$$

For U_1 we have $i = 1$, so $g_1(o) = -1$ and $g_1(s) = 1$, giving

$$U_1|o> = \overbrace{k_{-1}}^{x} |o> + 0$$

and

$$U_1|s> = \overbrace{k_1}^{0} |s> + 0$$

while for U_2 we have $g_2(o) = -2, g_2(s) = 2$, giving

$$U_2 = \begin{pmatrix} 1/x & \sqrt{\frac{x^2-1}{x^2}} \\ \sqrt{\frac{x^2-1}{x^2}} & \frac{x^2-1}{x} \end{pmatrix}$$

and $g_3(o) = -1, g_3(s) = 1$ so $U_3 = U_1$. The Hamiltonian is

$$\begin{pmatrix} 2x + 1/x & \sqrt{\frac{x^2-1}{x^2}} \\ \sqrt{\frac{x^2-1}{x^2}} & \frac{x^2-1}{x} \end{pmatrix}$$

which the reader will readily verify has the same spectrum as before. It should be clear already from this example, however, that algebraic diagonalisation of H is typically somewhat harder than before. This is the price paid for a more agorithmic construction.

4.2 On computing the spectrum

The partial order associated to the standard tableau of λ is in fact a lattice (in the algebraic sense, see e.g. MacLane and Birkoff, ref.24). This lattice may be arranged in levels corresponding to the number of steps required from the lowest standard tableau. The non-zero elements of the Hamiltonian in the Hoefsmit basis correspond to the nodes (diagonal elements) and bonds (off-diagonal elements) of the Hasse diagram for the lattice. Their value must be computed by the procedure given above, but the level property means that there are at most n non-zero elements in each row, and frequently much fewer. This and the lattice property means that diagonalisation of the Hamiltonian requires the solution of far fewer simultaneous equations than might be expected.

In the canonical basis the picture is similar on and above the diagonal, but much simpler. All the non-zero elements above the diagonal are 1, the diagonal elements are just the number of integers i in the standard tableau for which i is in a higher row than $i+1$. In the $(P/M) = (1/1)$ case this number is always a constant, and it always seems to be possible to symmetrise the matrix without affecting the upper triangle. The resultant matrix can be diagonalised by a kind of Bethe ansatz (our $(3,1,1)$ example illustrates all these points nicely). Unfortunately below the diagonal is complicated in general, although finite lattice tests suggest that it can be tidied up relatively easily in many cases. Once again, you pay your money and you take your choice.

5 The quotient algebras $(P,M)H_n(q)$

This section is intended for those interested in the structure of the Hecke algebra at q a root of unity.

We define a sequence of quotient algebras of $H_n(q)$ as follows. The quotient $(P,M)H_n(q)$ is obtained by imposing the quotient relations

$$(P,M) = \left(\otimes^{M+1} F_{P+2}\right) H_{PM+N} \left(\otimes^{P+1} E_{M+2}\right) = 0.$$

For example:

(i) with $x \neq 0$ then $(1,0)H_n(q)$ is obtained by putting $F_{11} = U_1/x = 0$, whilst with $x = 0$ the quotient relation is $x\, F_{11} = U_1 = 0$ (the first expression is, of course, purely formal at $x = 0$);

(ii) the case $(2,0)$ corresponds to the Temperley-Lieb algebra;

(iii) the case $(1,1)$ has quotient relations given by

$$(1,1) = (U_1 U_3) U_2((x - U_1)(x - U_3)) = 0.$$

In this case the reader will readily confirm that, among the complete list of generic irreducible representations of H_4 given in section 4.1.1 for example, only $R_{(2,2)}$ violates this relation (and so can not be a representation of the $(1,1)$ quotient). Furthermore, writing $(1,1)^{op}$ for $(1,1)$ with the factors written out in reverse order, we find that in $H_n(q)$

$$(1,1)^{op} U_2 (1,1)^{op} = (x^4 - x^2)\,(1,1)^{op}$$

so generically $(1,1) = 0$ implies $(1,1)^{op} = 0$; and $(1,1)\ U_2$ is (generically) idempotent up to a finite factor.

We conjecture that $R_{PM}H_n(q)$ is a faithful representation of $(P,M)H_n(q)$ for all q, as has been proved in the $M = 0$ case in Martin ref.18, and for q generic in this paper.

6 Conclusions and discussion

We have classified the structure of the chains (P/M), but much work remains to be done. One obvious question is to study the thermodynamical limit $(L \to \infty)$ and to classify the different excitations according to the super-Young tableaux. This implies the development of a proper Bethe ansatz formalism. A very important further step has to be made in the case where q is on the unit circle. This implies on one side understanding the representation theory of $U_q SU(P/M)$ superalgebras, which will allow the derivation of the corresponding restricted models (see ref.16 for the (2/0) chain); and on the other side use of the non-semisimple Hecke algebras, in order to classify the structure of the part of the spectrum which does not enter the restricted models.

From the mathematical point of view, results such as the (1/1) spectrum in appendix A can be used to examine the break up of generically irreducible representations into smaller representations at q root of unity. This is because of the property that if one representation becomes an invariant subspace of another, then its Hamiltonian spectrum must be contained in that of the other. Thus, although the spectrum is not a sensitive enough probe to prove that an invariant subspace appears (because of the possibility of accidental degeneracies), it can often prove that one does not appear, and when one may appear it gives all the candidates.

Another natural question to ask is whether there are models corresponding to the partition of N into more than 2 parts. This implies, on the one hand para-fermion-like statistics, and on the other hand a higher order local relation for the quotient of the braid group (relation 4 is quadratic). Some candidates may be found in ref.25.

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Appendix A: The (1/1) chain

This chain is described by the Hamiltonian implied by equation 3. As shown in ref.3b, the Hamiltonian can be diagonalised through a Jordan-Wigner transformation and brought into the form:

$$H = \epsilon_0 \eta_0^\dagger \eta_0 + \sum_{k=1}^{L-1} \left(x - 2\cos(\pi k/L)\right) a_k^\dagger a_k \tag{16}$$

where

$$\{a_m^\dagger, a_n\} = \delta_{m,n}, \qquad \{a_m, a_n\} = 0 \qquad (m, n = 1, 2, .., L-1) \tag{17}$$

$$\{\eta_0^\dagger, \eta_0\} = 1, \qquad \{\eta_0^\dagger, a_m\} = \{\eta_0, a_m^\dagger\} = \{\eta_0, \eta_0\} = 0 \tag{18}$$

and $\epsilon_0 = 0$ (null mode). The existence of a null fermionic mode makes all the irreducible representations 2 dimensional: $d(Y) = 2$ irrespective of Y. We can now give the relation between the super-Young tableau and the fermionic description. In the (1/1) case the allowed tableaux for a chain with L sites are

only of the type $(r, 1^s)$ with $r + s = L$. The $D(r, 1^s)$ dimensional vector space is built with the help of s creation operators $a_k^\dagger$:

$$a_{k_1}^\dagger \quad a_{k_2}^\dagger \quad ... \quad a_{k_s}^\dagger |0> . \tag{19}$$

Note that the tableau (1^L) has to be written $(1, 1^{L-1})$. We will give as an example the case $L = 4$. Since the Hamiltonian 16 is already in the diagonal form we can read off the eigenvalues directly:

(4) (no fermions): 0

$(3, 1)$ (one fermion): $(x, x \pm \sqrt{2})$

$(2, 1^2)$ (two fermions): $(2x, 2x \pm \sqrt{2})$

$(1, 1^3)$ (three fermions): $3x$.

As expected we obtain the same results as for the (2/1) chain for the same tableau (see section 4).

We would like to mention that the special chain described by equation 1 (one has in general

$$H = \sum_{i=1}^{n} g_i U_i^{(P/M)}$$

with g_i variable) can have degeneracies larger than $d(Y)$ even for generic q ! For the (1/1) chain this can easily be seen by noticing that the two fermion excitations $a_k^\dagger a_{L-k}^\dagger |0>$ give the same contribution $(2x)$ to the spectrum, independent of k.

Appendix B: The (2/1) chain

We consider for simplicity the case $q = 1$. We also regard the problem from a different viewpoint, namely that of the general representation theory[26], which does not use super-Young tableau but rather branching rules and Clebsch-Gordan coefficients. This approach might be useful since the Clebsch-Gordan coefficients are known analytically for $q = 1$ and can easily be extended to $q \neq 1$.

The superalgebra is defined through eight generators, which satisfy commutation (anti-commutation) relations:

$$[I^3, I^\pm] = \pm I^\pm, \qquad [I^+, I^-] = 2I^3, \qquad [B, I^\pm] = [B, I^3] = 0$$

$$[I^3, V^\pm] = \pm 1/2 \;\; V^\pm, \qquad [I^3, W^\pm] = \pm 1/2 \;\; W^\pm,$$

$$[I^\pm, V^\mp] = V^\pm, \qquad [I^\pm, W^\mp] = W^\pm,$$

$$[I^\pm, V^\pm] = [I^\pm, W^\pm] = 0, \qquad [B, V^\pm] = 1/2 \;\; V^\pm,$$

$$[B, W^\pm] = -1/2 \;\; W^\pm,$$

$$\{V^{\pm},V^{\pm}\}=\{V^{\pm},V^{\mp}\}=\{W^{\pm},W^{\pm}\}=\{W^{\pm},W^{\mp}\}=0$$
$$\{V^{\pm},W^{\pm}\}=\pm I^{\pm},\qquad \{V^{\pm},W^{\mp}\}=-I^{3}\pm B. \tag{20}$$

The Cartan subalgebra contains two generators B and I^3 which we take diagonal:

$$B|b,I>=b|b,I>;\qquad I^{3}|b,I>=I|b,I>\ .$$

In the chain (2) one takes on each site a copy of the nontypical irreducible representation $[1/2]_+$. A nontypical representation $[I]_+$ contains $4I+1$ states, with eigenvalues in the Cartan subalgebra given by:

$$|I,I>,|I,I-1>,...,|I,-I>,$$

$$|I+1/2,I-1/2>,|I+1/2,I-3/2>,...,|I+1/2,-I+1/2>\ .$$

In the tensor product

$$[1/2]_{+}\otimes[1/2]_{+}\otimes...\otimes[1/2]_{+}$$

one finds not only the nontypical representations $[I]_+$ but also the typical representations $[b,I]$. A typical representation contains $8I$ states, with eigenvalues in the Cartan subalgebra given by:

$$|b,I>,|b,I-1>,...,|b,-I>,$$

$$|b,I-1>,|b,I-2>,...,|b,-I+1>,$$
$$|b+1/2,I-1/2>,...,|b+1/2,-I+1/2>,$$
$$|b-1/2,I-1/2>,...,|b-1/2,-I+1/2>\ .$$

The tensor product of two nontypical representations is:

$$[I_1]_{+}\otimes[I_2]_{+}=[I]_{+}\oplus\left(\bigoplus_{n=0}^{2I_{min}-1}[(I+1/2),(I-1/2-n)]\right) \tag{21}$$

where

$$I=I_1+I_2,\qquad I_{min}=min(I_1,I_2).$$

The tensor product of a typical and a nontypical representation is:

$$[b_1,I_1]\otimes[I_2]_{+}=\theta(I_1-I_2)[b,|I_1-I_2|]\oplus\left(\bigoplus_{n=0}^{2I_{min}-1}[b,I-n]\right)$$
$$\oplus\left(\bigoplus_{n=0}^{2I_{min}-1}[b+1/2,I-1/2-n]\right) \tag{22}$$

where

$$b=b_1+I_2.$$

Let us consider as an example the (2/1) chain with 4 sites. Using equations 21 and 22 we find

$$[1/2]_+ \otimes [1/2]_+ \otimes [1/2]_+ \otimes [1/2]_+ \\ = [2]_+ \oplus 3.[5/2, 3/2] \oplus 2.[5/2, 1/2] \oplus 3.[3, 1] \oplus [7/2, 1/2]. \tag{23}$$

We can easily check that there is the following correspondence between the super-Young tableau and the irreducible representations described above:

$$(4) = [2]_+, (3, 1) = [5/2, 3/2], (2, 2) = [5/2, 1/2], (2, 1, 1) = [3, 1], (1^4) = [7/2, 1/2].$$

Equation 23 describes the results of section 4 (page 13) from a different perspective.