The type-1,1 Rittenberg algebra and the Pascal–Rittenberg triangle

P P Martin

Mathematics Department, City University,
Northampton Square, London EC1V 0HB, UK.

Abstract

In these informal notes we update progress on Rittenberg and Deguchi's programme (initiated in [18, 5]) for studying quantum super spin chains and corresponding quotients of Hecke algebras, at roots of unity. (N.B., We assume you have read [18, 5].) By solving the restricted 'James recursion' (see e.g. [13, 10]) we obtain Gram determinants for Specht modules on the 1,1 Rittenberg quotient. We hence determine the dimensions of the 1,1 Rittenberg algebras (a step towards one of the Rittenberg–Deguchi conjectures).
(Caveat: Not all arguments are written out!)

1 Preamble

We start by summarizing from [18]. In [18] quotients of the Hecke algebra $H_n(q)$ denoted $(P, M)H_n(q)$ and $H_n^{PM}(q)$ are introduced, over a ground field k containing an element q; and it is conjectured that in case $k = \mathbb{C}$ these algebras are isomorphic: $(P, M)H_n(q) \cong H_n^{PM}(q)$, for all choices of $q \in \mathbb{C}$. (It is straightforwardly shown in [18] that the conjecture holds excluding the cases in which q is a root of unity. And it is shown in [17] that the conjecture holds for all q in case $M = 0$.)

Let $P, M, n \in \mathbb{N}_0$. Fix k a field and let $q \in k$. The type-P, M Rittenberg algebra $H_n^{PM}(q)$ is the k-algebra of local operators in the P, M-super q-spin chain Hamiltonian [18] (cf. e.g. [4, 5] and references therein). That is, it is the algebra generated by operators σ_i defined as follows.

Set $N = P + M$. Let $\{e_1, e_2, ..., e_N\}$ be the standard basis of k^N. We write $|ij\rangle = e_i \otimes e_j \in (k^N)^{\otimes 2}$ and so on. We define the action of an operator σ on $(k^N)^{\otimes 2}$ by

$$\sigma|ii\rangle = |ii\rangle \quad \text{if } i \leq P \tag{1}$$
$$\sigma|ii\rangle = -q^2|ii\rangle \quad \text{if } i > P \tag{2}$$
$$\sigma|ij\rangle = (1-q^2)|ij\rangle + q|ji\rangle \quad \text{if } i < j \tag{3}$$
$$\sigma|ij\rangle = q|ji\rangle \quad \text{if } i > j \tag{4}$$

Then σ_i acts on $(k^N)^{\otimes n}$ by

$$\sigma_i = 1_N \otimes 1_N \otimes ... \otimes 1_N \otimes \sigma \otimes 1_N \otimes ... \otimes 1_N$$

with σ in the i-th position. This gives a representation of the braid group, and hence Hecke algebra with appropriate parameter, as noted by Deguchi–Akutsu [4].

Aside: As a quotient of the Hecke algebra $H_n(q)$ the algebra $H_n^{PM}(q)$ inherits the Zariski-topological notions of *generic*-ness from that setting. In its simplest form this involves thinking of k initially rather as a commutative ring such as $\mathbb{C}[q]$. We observe that, passing to the field of fractions, (I) $H_n(q)$ is semisimple; (II) arithmetic for finite n will involve finite sets of finite polynomials in the denominator. Thus specific evaluations of q have open neighbourhoods where q can be varied continuously. And thus for example continously-varying but integer-valued functions must in fact be constant here. See later.

Let $\varrho(P,M)$ denote the integer partition

$$\varrho(P,M) = \ (M+1)^{P+1} \ \vdash (M+1)(P+1)$$

(or the corresponding rectangular Young diagram). Generically $H_n^{PM}(q)$ is isomorphic to $(P,M)H_n(q)$, the quotient of the ordinary Hecke algebra $H_n(q)$ [1] by a certain element $Y_{\varrho(P,M)} \in H_{(M+1)(P+1)}(q) \hookrightarrow H_n(q)$ (via the obvious inclusion) such that

$$H_{(M+1)(P+1)} Y_{\varrho(P,M)} = \Delta_{\varrho(P,M)}$$

— the corresponding Specht module. (This result is given in [18, §3.4]. As noted, 'generically' means in essence that we work over $\mathbb{C}[q]$, passing to the field of fractions of $\mathbb{C}[q]$, and that the property holds on open subsets with respect to the Zariski topology. The proof in [18] uses outer-product rules for the symmetric group — essentially the complex irreducible restriction rules — and generic continuity.)

As quotients of $H_n(q)$, the algebras $H_n^{PM}(q)$ are generically semisimple — the structure for all n for fixed P, M is obtained from the Young graph (regarded as a poset [14, 8, 7]) by deleting the ideal generated by the partition $(M+1)^{P+1}$.

For general q, the composition factors of any Hecke Specht module may be determined algorithmically by Kazhdan–Lusztig theory [21, 20] (or of any *given* modules in principle using methods of James et al [12, 6, 10, 11], but see later). Thus the same holds for $H_n^{PM}(q)$. For $M = 0$ this programme is essentially complete — see [20] (although there are open questions on structure beyond composition factors for large P — NB the case $P = 2$ is the Temperley–Lieb algebra by [17]). A key ingredient is the local thermodynamic limit derived from the spin chain (case $P = 2$ is the important XXZ spin chain). But for $P, M > 0$ there is not a canonical thermodynamcic limit and computations are correspondingly harder. Here we report on the 1,1 case.

One question is what is the dimension of $H_n^{PM}(q)$ in the non-generic cases. Generically the dimension is given, using the Artin–Wedderburn Theorem, by the sum of squares from the appropriate row of the Pascal triangle (itself of course giving a famous sequence as n varies). In the non-semisimple cases things are a bit more complicated as we shall see.

[1] Caveat: our notation $H_n(q)$ corresponds to notation $H_{n-1}(q)$ in [18]. Note typo in [5] Def.11.

2 Introduction

A general tool for Hecke representation theory is the Gram determinant of the Specht modules with respect to the Young/Hoefsmit form [9, 13, 12, 6]. [2] This Gram determinant is hard to compute in general. There is a recursive method due to James–Murphy-Mathas [13, 10]. And closed forms are available for the Specht modules restricted to the weights corresponding to $P, 0$ for small P.
(Remark: The recursive method provided by James et al is a wonderful tool, the use of which lies at the heart of our approach here. It is an interesting feature that this comes very much from the 'combinatorial' toolkit of James that is driven by symmetric group representation theory. This toolkit (containing many other wonderful tools as well, but none-the-less) eventually runs into the issue that it addresses very hard problems! But it is interesting to hold aspects of it up to the 'light' cast by more Lie-theoretic perspectives, and this note can be seen as an instance of this tactic. Albeit one where the Lie-theory input is itself unclear.)

Note that the form is natural for the Hecke algebra viewed holistically, but it is not so for P, M quotients, so it is an interesting question to interpret it in these settings. Here we give a closed form for $1, 1$ and discuss consequences for representation theory.

The following Fig.1 is a mild extension of the corresponding table in [19], to bring it up to the first interesting $p = 2$ cases.

(The second version of the fig below (now suppressed) is the in-line one in the tex file (I think it is just here temporarily for comparison cf. the input one above):)

In [20] it was shown how category theoretic machinery can be combined with some simple combinatorial data to determine the characteristic 0 representation theory of Hecke algebras for q a root of unity. (In particular it exhibits the role of alcove geometry on weight space.) This characteristic 0 representation theory is also accessible by other means [11, **?**, **?**].

In §3 we solve the recurrence. In §4 we analyse the solution in terms of representation theory. ...

[2](The form itself is a deformation of the restriction of the delta-function form on Young modules to 'corresponding' Specht modules. This procedure is beautiful but not entirely canonical. In particular it treats $\lambda = (1^n)$ and (n) hugely differently; and introduces some overall scalars into the form in general.)

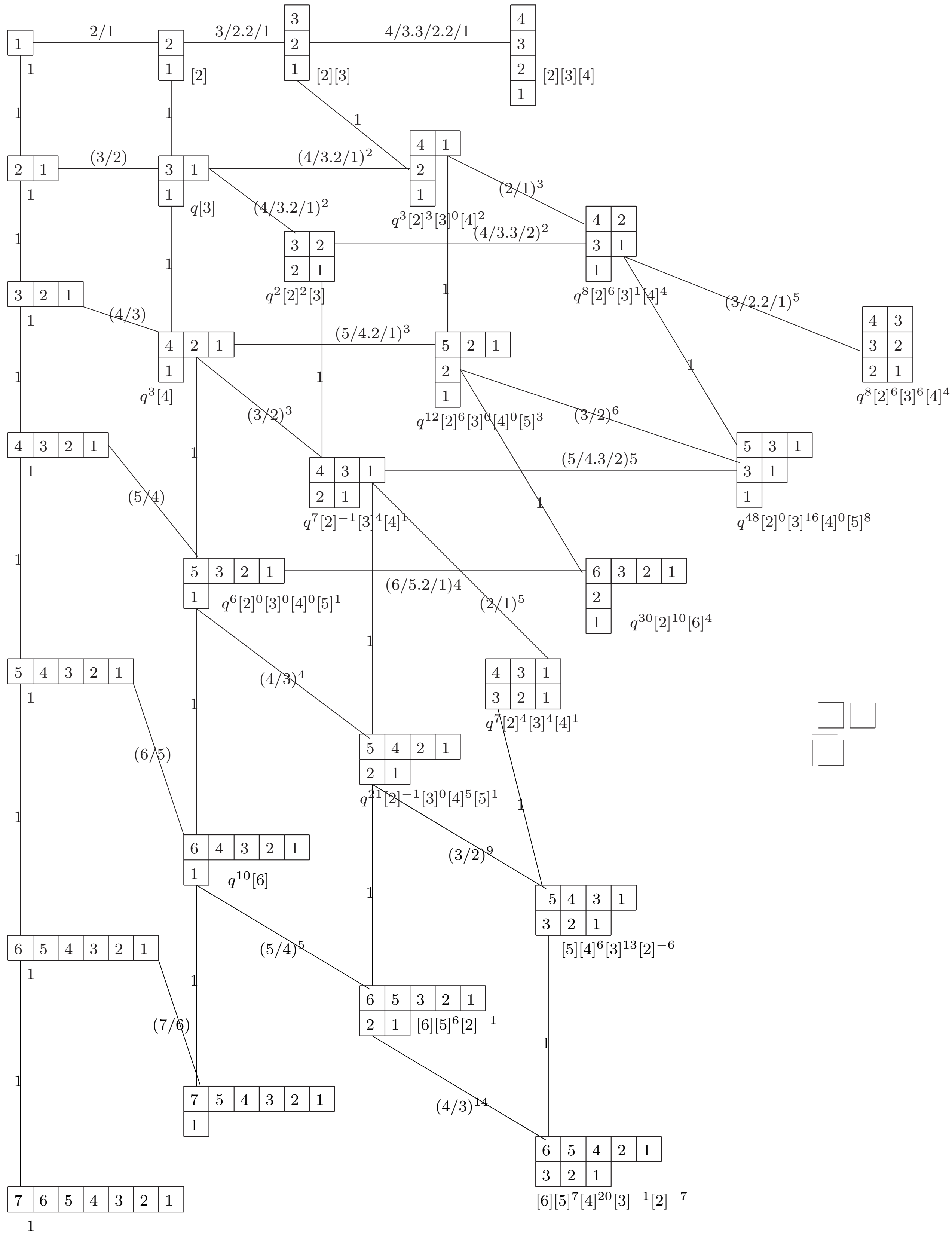

Figure 1: Gram data on Young graph (unfinished). NB this fig may work in dvi and not in pdf.

3 The Pascal–Rittenberg triangle solves the James recurrence

The 1,1 part of the table has the structure of a Pascal triangle, but containing representation theory data for the Rittenberg algebra (hence Pascal–Rittenberg). It starts:

$$\begin{array}{ccccccccccc}
&&&&&1&&&&&\\
&&&&[2]&&1&&&&\\
&&&[2][3]&&[3]&&1&&&\\
&&[2][3][4]&&[2]^3[4]^2&&[4]&&1&&\\
&[2][3][4][5]&&[2]^4[3]^4[5]^3&&[2]^6[5]^3&&[5]&&1&\\
[6]!&&[2]^5[3]^5[4]^5[6]^4&&[2]^{10}[3]^{10}[6]^6&&[2]^{10}[6]^4&&[6]&&1
\end{array}$$

NB we assume q is invertible throughout, and omit the overall factors of q here (they are discussed in [19]). ... And here 'tidied up' a little:

$$\begin{array}{ccccccccccccc}
&&&&&&1^1&&&&&&\\
&&&&&1^1[2]^1&&1^1&&&&&\\
&&&&[2]^1[3]^1&&1^2[3]^1&&1^1&&&&\\
&&&[3]!^1[4]^1&&[2]^3[4]^2&&1^3[4]^1&&1^1&&&\\
&&[4]!^1[5]^1&&[3]!^4[5]^3&&[2]^6[5]^3&&1^4[5]^1&&1^1&&\\
&[5]!^1[6]^1&&[4]!^5[6]^4&&[3]!^{10}[6]^6&&[2]^{10}[6]^4&&1^5[6]^1&&1^1&\\
[6]!^1[7]^1&&[5]!^6[7]^5&&[4]!^{15}[7]^{10}&&[3]!^{20}[7]^{10}&&[2]^{15}[7]^5&&1^6[7]^1&&1^1
\end{array} \qquad (5)$$

The pattern will be clear. ... See (7) below. We call the array continuing this pattern the Pascal–Rittenberg triangle, or just the Rittenberg array.

It will also be useful to have to hand the ordinary Pascal triangle $\mathcal{P}$:

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&&\\
&&&&&1&&1&&&&&\\
&&&&1&&2&&1&&&&\\
&&&1&&3&&3&&1&&&\\
&&1&&4&&6&&4&&1&&\\
&1&&5&&10&&10&&5&&1&\\
1&&6&&15&&20&&15&&6&&1
\end{array} \qquad (6)$$

Observe in particular that both sets of exponents in the Rittenberg array (5), denoted $\mathcal{R}$, come from this table, with the set of form $[n+1]^e$ offset by one step 'south-west'.

If we coordinatize the triangle with rows labelled 0, 1, 2, ... from the top; and sw-ne diagonals labelled 0,1,2,... from the left; then the entry of $\mathcal{P}$ in row i and diagonal j is $\mathcal{P}(i,j) = \binom{i}{j}$. NB row i corresponds to $n = i+1$ for our rank n.

The $\mathcal{R}$ array satisfies the James recurrence. Thus the form's gram determinant $gram_J(\lambda)$ is:

$$gram_J(j+1, 1^{i-j}) \;=\; \mathcal{R}(i,j) = \; [i-j]!^{\binom{i}{j}} \; [i+1]^{\binom{i-1}{j}} \qquad (7)$$

— note the vanishing convention for chooses outside the bound.

4 Representation theory

The form 'corresponds' in a suitable sense to morphisms from the contravariant dual of the Specht module into the Specht module — this connection being defined up to an overall factor. Thus the specific form is generically of full rank and the determinant is not zero. Conversely if the determinant is zero then (with some caveats regarding the overall factor — see later) it is not of full rank, and the head of the dual is mapped 'below' the head of the Specht module. In other words the Specht module is not simple. ...

Let us try to address the issue with the overall factor. If we rescale a form by an overall factor κ then the determinant will be rescaled by κ^d where d is the dimension. Thus if a determinant has such a factor then it may scale away. In our case we see comparing (**??**) and (6) that we have such a factor in essentially every position. Putting this aside, then, what remains is the factor $[i+1]^{\binom{i-1}{j}}$. From this we read the following.

If $[n] = [i+1] \neq 0$ then the rank of the (rescaled) form is maximal in every case and the Specht module is simple and the algebra $H_n^{11}(q)$ is semisimple.

If $[n] = [i+1] = 0$ then the rank of the form is

$$\binom{i}{j} - \binom{i-1}{j} \tag{8}$$

Note that $\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}$ (for example $\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$). From this, (8), and the restriction rules for $H_{n-1} \hookrightarrow H_n$, we see that we have a pattern of Loewy structures of Specht modules for each row n when $[n] = 0$ like the following, which is $n = 7$:

$$1' \quad \frac{1'}{5'} \quad \frac{5'}{10'} \quad \frac{10'}{10} \quad \frac{10}{5} \quad \frac{5}{1} \quad 1 \tag{9}$$

(primes simply distinguish equal dimensions for non-isomorphic modules) ...

JOBS: Look at Young modules, dimensions vs ranks... For example the 1111112 Young module has dim 7 and simple content 6+1. We know from other quotients that this is non-split. See below.

A (possibly somewhat special) specific case to look at is $[2] = 0$. In this case the 'trivial' and 'alternating' representations coincide.

The case(s) we focus on next are ...

4.1 The case $n = 3$

In case $n = 3$ the 1,1-spin chain rep is 8-dimensional (as a representation). The charge sectors have bases

$B_{(3,0)} = \{111\}$, $B_{(2,1)} = \{112, 121, 211\}$, $B_{(1,2)} = \{122, 212, 221\}$ and $B_{(0,3)} = \{222\}$.

It might be good to write out the reps explicitly...

(Remark: Note that case $n = 3$ is too small for the element defining the quotient to exists, so $(1,1)H_3(q) = H_3(q)$. In other words the isomorphism conjecture says that the spin-chain rep is faithful on $H_3(q)$.)

The charge-blocks of the rep (the Young modules) are denoted R_x where x is a composition of n. Thus:

$$R_{(3,0)}(\sigma_1) = R_{(3,0)}(\sigma_2) = 1$$

- obviously this is the trivial module.

$$R_{(2,1)}((12)) = \begin{pmatrix} 1 & & \\ & 1-q^2 & q \\ & q & 0 \end{pmatrix}; \qquad R_{(2,1)}((23)) = \begin{pmatrix} 1-q^2 & q & \\ q & 0 & \\ & & 1 \end{pmatrix}$$

- this has a Specht filtration consisting of the trivial $Sp_{(3)}$ and $Sp_{(21)}$. We have characters

$$\chi_{R_{21}} = \chi_{(3)} + \chi_{(21)}$$

so $\chi_{R_{21}}(\sigma_1) = 2 - q^2$ giving $\chi_{(21)}(\sigma_1) = 1 - q^2$.

Consider case $[3] = 0$ ($q^6 = 1$?). Here the Specht module $Sp_{(21)}$ is reducible (with the trivial module in the socle) and not isomorphic to its contravariant dual. Since the Young module R_{21} *is* isomorphic to its dual, we see that we have a non-split extension of the trivial module over $Sp_{(21)}$. bla bla contravariant duality, symmetry ... Meanwhile the Young module

$$R_{12} \cong \; Sp_{(21)} + Sp_{(1^3)},$$

and again this must be non-split, with the $Sp_{(21)}$ over the $Sp_{(1^3)}$, by the contravariant self-duality property.

Finally we have

$$R_{(0,3)}(\sigma_1) = R_{(0,3)}(\sigma_2) = -q^2$$

We can do a dimension count. Generically we have 2 1-d simples and a 2-d simple, giving dimension 6. For our $[3] = 0$ case We have 2 1-d simples, and each is glued over the other; and each also self-extends. Thus 2+2+2=6. Note that all the 'glue' for the simple L_3 is in R_{21}; and all for $L_{21} = L_{1^3}$ is in R_{12}. This implies that R is indeed faithful.

It should be straightforward to extend this to general n in the $PM = 11$ case...

4.2 ...And beyond

Figures 2-3 are an extract from [16] giving the structure of projective modules for TL in a couple of indicative root of unity cases. We borrow them since (they are interesting for comparison, but also) we can use the same latex template for the corresponding tables for 1,1 Rittenberg.

The TL case is arguably much more 'beautiful' than the present case since it engages alcove geometry – the p labels in the figures give the position in the dominant alcove. ...But maybe we are just looking at things from the wrong perspective here!

Let's try to make roughly analogous figures for the present case. So firstly we have underlying the full Pascal rather than the truncated Pascal. Then in each row we should (perhaps, aiming for efficiency of exposition) select just the interesting cases – i.e. we will present a different q in each row, such that in the row corresponding to n we have $[n] = 0$.

Note that the algebra is defined for a Brauer modular system [3, 2]. I.e. the defining representation is defined over a suitable integral ring that passes both to the cases of interest (roots of unity) and to a case over the field of fractions that is semisimple. Note that the Specht modules work as the generic irreducibles of this setting (NB dual Specht modules, say, would also work). Thus the Cartan decomposition matrix is

$$C = D^T D \tag{10}$$

where D is the Specht decomposition matrix. Specifically D gives the simple content of Specht modules — we can use (with care) the same labelling scheme, of suitable integer partitions, for both sets of isomorphism classes. For definiteness let us start with the trivial module label (n) and then $(n-1,1)$ and so on. In case $[n]=0$ the 'alternating' Specht module with label (1^n) is then superfluous — the corresponding simple is actually the head of $(2,1^{n-2})$. The Specht decomposition matrix is

$$D = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

(it is easy to attach filtration layer data here; but it is not generally clear how this might be carried to the Brauer-modular analysis, since that works also for dual Specht modules etc). Note that C follows immediately (again with a caveat about filtration data in general; however it appears that the projective filtrations are forced in this relatively simple case).

It is now a combinatorial exercise to check that the dimension is the same as for the generic case (albeit by a very different route) in all cases. For example

$$0.1+1.5+3.10+3.10+1.5+0.1 \;=\; 1.1+4.4+6.6+4.4+1.1 \;=\; 70$$

The general pattern is the same.

5 Comments

The Hecke algebra has a symmetry under 'exchanging the roles of the two eigenvalues of the local operator', which has a manifestation as Young diagram conjugation. Note that the inner product used for the James setup is very asymmetrical.

A feature of transfer matrix and Hamiltonian algebras is that they form towers of recollement corresponding to a thermodynamic limit. For example the TL algebra at fixed q has this feature. For general spin chains of the type relevant here is it much less clear how to take the thermodynamic limit, and correspondingly it is less clear whether to expect, and how to look for, towers of recollement. This remains an interesting problem however ...

The following section is an extract from [**?**].

5.1 Ring Arithmetic

Set $\mathcal{A} = \mathbb{Z}[x, x^{-1}]$, $q = x^2$ and, for integer n,

$$[n] = \frac{x^n - x^{-n}}{x - x^{-1}}.$$

$p=$ 1	2	3	4	5	6	7	8	9
	1							
1.		1						
	1. 1		1 1. 1					
1. 1		3		1 1. 1				
	1. 4		4 1. 4		1			
1. 4		9		4 1. 1 4		1 4 1		
	1. 13		13 1. 1 13		6		1 13 1	
1. 13		28		13 1. 7 13		7 13 7		1

Figure 2: Structure of projective modules for the TL case $q+q^{-1}=1$.

$p=$ 1	2	3	4	5	6	7	8	9
	1							
1.		1						
	2		1					
2		2 1		1 2 1				
	4 1		4		1 4 1			
4 1		4 5		4 5 4		1 4 1		
	8 6		14		6 8 6		1	
8 6		8 20		20 8 20		6 8 1 6		1 6 1

Figure 3: Structure of projective modules for the TL case $q + q^{-1} = 2$.

				$p =$ 1	2	3	4	5	6	7	8
					1						
				0 1		1 1					
			0 1.		1. 1 1.		1 1. 1				
		0 1.		1. 0 2 1.		2 1. 1 2		1 2 1			
	0 1.		1. 0 3. 1.		3. 1 3 3.		3 3. 1 3		1 3 1		
0 1.		1. 0 4. 1.		4. 1. 6 4.		6 4. 4 6		4 6 1 4		1 4 1	
	1. 0 5. 1.		5. 1. 10. 5.		10. 5. 10 10.		10 10. 5 10		5 10 1 5		1 5 1

Figure 4: Structure of projective modules for OUR 1-1 case. The q values shown are different in each layer (essentially satisfying $[n] = 0$ in layer n — see main text). Thus in all cases restriction from n to $n-1$ changes from the shown pattern to the *generic* pattern of the plain Pascal triangle (and $n+1$ to n has a complementary configuration). –WORK in PROGRESS!–

It is trivial (but useful) to note that if $n = pm$ then

$$x^n - x^{-n} = (x^p - x^{-p})(x^{(m-1)p} + x^{(m-3)p} + ... + x^{(1-m)p})$$

(N.B., the second factor contains m terms). Thus

$$\frac{[mlp]}{[hl]} = \frac{(x^{(m-1)lp} + x^{(m-3)lp} + ... + x^{(1-m)lp})}{(x^{(h-1)l} + x^{(h-3)l} + ... + x^{(1-h)l})}(x^{(p-1)l} + x^{(p-3)l} + ... + x^{(1-p)l})$$

and indeed

$$\frac{[mlp^j]}{[hlp^k]} = \frac{(x^{(m-1)lp^j} + x^{(m-3)lp^j} + \dots x^{(1-m)lp^j})}{(x^{(h-1)lp^k} + x^{(h-3)lp^k} + \dots x^{(1-h)lp^k})} \frac{(x^{(p^j-1)l} + x^{(p^j-3)l} + \dots x^{(1-p^j)l})}{(x^{(p^k-1)l} + x^{(p^k-3)l} + \dots x^{(1-p^k)l})}$$

Specialising so that $x^l = 1$ we have

$$\frac{[mlp^j]}{[hlp^k]} \equiv \frac{m}{h} p^{j-k}. \tag{11}$$

(**5.1**) For $\nu \vdash n$ define

$$\dim_q R_\nu \ = \frac{[n]!}{\prod_i [\nu_i]!}$$

(**5.2**) Consider the quantum–plane coordinates $y_2 y_1 = q y_1 y_2$ [15]. Each word w in in the expansion of $(y_1 + y_2)^n$ may be *straightened* with factor $q^{l(w)}$. The overall coefficients in the straightened basis are given by a q–binomial theorem:

$$(y_1 + y_2)^n = \sum_{i=0}^{n} \left(\sum_{w \in B_{(n-i,i)}} q^{l(w)} \right) y_1^{n-i} y_2$$

NB, noting from prop. **??** that $B_{(n-i,i)} = \{w1 \mid w \in B_{(n-i-1,i)}\} \cup \{w2 \mid w \in B_{(n-i,i-1)}\}$ and that if $w \in B_{(n-i-1,i)}$ then $l(w1) = l(1w) + i$, the q–Pascal triangle rule is:

$$\sum_{w \in B_{(n-i,i)}} q^{l(w)} = q^i \sum_{w \in B_{(n-i-1,i)}} q^{l(w)} + \sum_{w \in B_{(n-i,i-1)}} q^{l(w)}$$

(see for example [1]). More generally, we may define a corresponding notion of q–multinomial coefficients depending on the shape ν of the word w. It is easy to see that

Proposition 1 *The 'q–dimension' $\dim_q R_\nu$ coincides with the q–multinomial coefficient for shape ν. Thus in particular it is a polynomial, i.e. it lies in $\mathcal{A}$.*

Note that $[m]$ lies in the $\mathcal{A}$–ideal $\mathcal{A}[l]$ if and only if l divides m; and that $[m]$ never lies in $\mathcal{A}[l]^2$. It follows that no q–dimension which is a q–choose (i.e. with ν of form (ν_1, ν_2)) lies in $\mathcal{A}[l]^2$ (for if $[lm][lm-1]..[l(m-i)]$ lies in the numerator then at least i factors of $[l]$ also appear in the denominator).

These observations motivate the following definition of a formal 'valuation' on q–numbers.

Fix $l \in \mathbb{N}$ and a prime p. Let $\mathbb{Z}[x, x^l = 1]$ denote the extension of $\mathbb{Z}$ by a primitive l^{th} root of 1. Then $\mathbb{Z}[x, x^l = 1]$ is an $\mathcal{A}$–algebra in the obvious way. Let $f(d)$ be the image of $d \in \mathcal{A}$ in $\mathbb{Z}[x, x^l = 1]$. Let d_l be the prime polynomial in $\mathcal{A}$ such that $f(d_l) = 0$. If $f(d) = 0$ then d_l divides d in $\mathcal{A}$. Let $v(d)_1 = i \in \mathbb{N}$ be maximal such that d_l^i divides d. Then $f_i(d) = f(d/d_l^i)$ is finite (or $d = 0$). Let $v(d)_2 = j \in \mathbb{N}$ be maximal such that $f_i(d) \in p^j\mathbb{Z}[x, x^l = 1]$. The formal l, p–valuation of d is $v(d) = (i, j)$.

If $l = 1$ then j is is just the usual p–adic valuation on $\mathbb{Z}$. Otherwise, for example, $v([l]) = (1, 0)$, $v([lp]) = (1, 1)$, $v([p]) = (0, 0)$, $v([lp]/[l]) = (0, 1)$.

We say that (i, j) *dominates* (i', j') (and write $(i, j) > (i', j')$) if $i \geq i'$, $j \geq j'$, and at least one of these relations is strict inequality.

References

[1] G E Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, vol. 2, Addison Wesley, 1976.

[2] D J Benson, *Representations and cohomology I*, Cambridge, 1995.

[3] R Brauer, Annals of Mathematics **38** (1937), 857.

[4] T Deguchi and Y Akutsu, J Phys A **23** (1990), 1861.

[5] T Deguchi and P P Martin, Int J Mod Phys A **7 suppl.1A** (1992).

[6] R Dipper and G D James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. **54** (1987), 57–82.

[7] S Donkin, *On Schur algebras and related algebras I*, J Algebra **104** (1986), 310–328.

[8] ______, *The q-Schur algebra*, LMS Lecture Notes Series, vol. 253, Cambridge University Press, 1998.

[9] P N Hoefsmit, *Representations of Hecke algebras of finite groups with BN pairs of classical type*, Ph.D. thesis, University of British Columbia, 1974.

[10] G. James and A. Mathas, *A q-analogue of the Jantzen-Schaper theorem*, Proc. London Math. Soc. **74** (1997), 241–274.

[11] G D James, *Representations of general linear groups*, LMS Lecture Notes 94, Cambridge UP, 1984.

[12] G D James and A Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.

[13] G D James and G E Murphy, *The determinant of a Gram matrix for a Specht module*, J. Algebra **59** (1979), 222–235.

[14] I Macdonald, *Symmetric functions and Hall polynomials*, Oxford, 1979.

[15] Yu I Manin, *Quantum groups and noncommutative geometry*, Universite de Montreal, Centre de Rechercher Mathematiques, Montreal, 1988.

[16] P P Martin, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[17] ———, *On Schur-Weyl duality, A_n Hecke algebras and quantum $sl(N)$*, Int J Mod Phys A **7 suppl.1B** (1992), 645–674.

[18] P P Martin and V Rittenberg, *A template for quantum spin chain spectra*, Int J Mod Phys A **7 suppl.1B** (1992), 707–730.

[19] P P Martin and B W Westbury, *On quantum spin–chain spectra and the representation theory of Hecke algebras by augmented braid diagrams*, J Phys A **30** (1997), 5471–5495.

[20] P P Martin and D Woodcock, *On quantum spin–chain spectra and the structure of Hecke algebras and q–groups at roots of unity*, J Phys A **31** (1998), 10131–10154.

[21] W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation Theory **1** (1997), 83–114.

I have deleted the appendices for the moment, since they are even less ready!