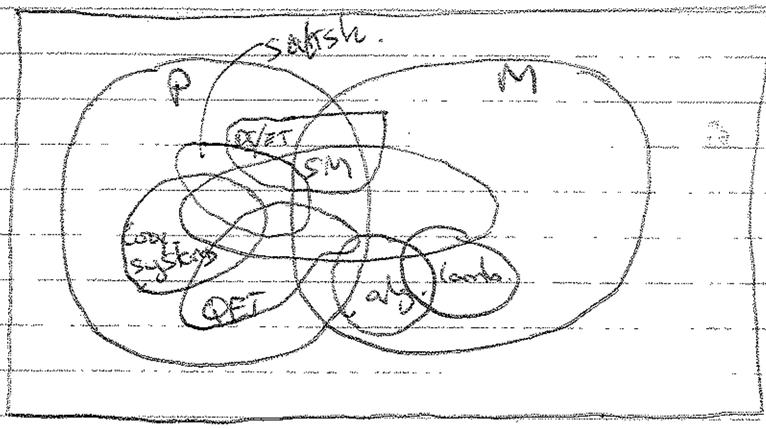
Statistical Mechanics I. Jeanne Scott 9/10/08. (1)



Dynamical systems
Ergodic theory.

Satsh: non-equi phenomena.

Jeanne Scott.
Archive.

Alan Sokal: "Some variants of the exponential formula with applications to the (& A.D. Scott) (multivariate) Tutte polynomial."

"An introduction to the dimer model," R. Kenyon.
"A fermionic field theory for spanning hyperforests," A. Bedini
"A little statistical mechanics for the graph theorist," L. Beaudin
R. Shrock
J. Ellis-Monaghan

Potts Model

Γ finite graph (unoriented)
– collection of vertices $\{1, 2, \ldots, n\}$, together with a collection of edges $\mathcal{E}$.

Allow: loops , multiple edges 

Approximate a piece of substance using the graph

Endow Γ with a "weighting" – an assignment of parameters j_e to edges $e \in \mathcal{E}$ in the graph, i.e.

$$j : \mathcal{E} \to \mathbb{C}.$$
$$e \mapsto j_e.$$

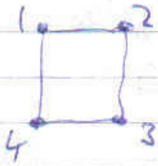
(weightings correspond to "fugacities").

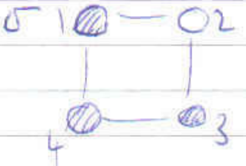
q: positive integer ≥ 2.

Symbols $\{z_1, \ldots, z_q\}$ "states" a node can have

A "state" σ of Γ is an assignment of symbols $\{z_1, \ldots, z_q\}$ to vertices

$\sigma : \Gamma \to \{z_1, \dots, z_q\}$, function.

e.g. $\Gamma =$  $q = 2$. $z_1 \longleftrightarrow$ ○, $z_2 \longleftrightarrow$ ●



<u>Defn</u> If $\sigma : \Gamma \to \{z_1, \dots, z_q\}$ is a state, its energy is

$$H_1(\sigma) = - \sum_{\substack{i \overset{e}{—} j \\ e \in \mathcal{E}}} j_e \, \delta(\sigma_i, \sigma_j) . \qquad \delta = \begin{cases} +1 & \text{if } \sigma_i = \sigma_j \\ 0 & \text{if not.} \end{cases}$$

$$H_2(\sigma) = \sum_{\substack{i \overset{e}{—} j \\ e \in \mathcal{E}}} j_e \left(1 - \delta(\sigma_i, \sigma_j) \right) = |\mathcal{E}| j + H_1(\sigma),$$

where $j = \frac{1}{|\mathcal{E}|} \sum_{e \in \mathcal{E}} j_e$.

K fixed constant.

T temperature

Partition function(s) $P_i(\Gamma) = \sum_{\sigma} \exp\left(\frac{-K}{T} H_i(\sigma) \right)$.

$\sigma \longmapsto \dfrac{\exp\left(-\frac{K}{T} H_i(\sigma) \right)}{P_i(\Gamma)}$ gives a probability measure on the space of states.
- depends on temperature.

As $T \to \infty$, assuming all $j_e > 0 \; \forall e \in \mathcal{E}$, $\exp\left(\frac{-K}{T} H_1(\sigma) \right) \to 1$.

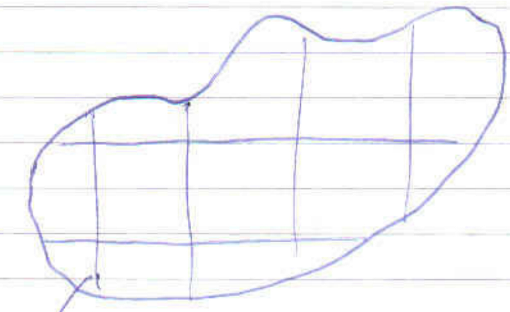
e.g.  $= \Gamma$ $\quad j_e = j \;\; \forall e \in \mathcal{E}, \; q = 2$.

4 ways. 2 ways. 4 ways. ~~four ways~~

$H_1 = -4j$ $\quad H_1 = -2j$ $\quad H_1 = 0$ $\quad H_1 = -2j$. $\quad H_1 = -2j$. $\quad H_1 = -4j$

$P_1 = 2e^{4jK/T} + 12^{2jK/T} + 2$.

(Mathematical) problem:



nested sequence of finite graphs

$\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$

Vertices embed.
Edges embed. Γ_i is the induced subgraph.

~~$\log P_1(\Gamma_\Lambda)$~~ $\lim_{\Lambda \to \infty} \frac{1}{|\Gamma_n|} \log P_1(\Gamma_\Lambda)$. Phase changes are governed by this.

internal heat
specific ~