Stat Mech (3) - Paul Martin. The 1-dimensional Ising Model (6)

$f = \frac{1}{N} \ln Z_N$. $Z_N = \text{poly}$ ← collection of graphs for the q-state Potts model.

N-stable properties of collection of functions wrt graphs.

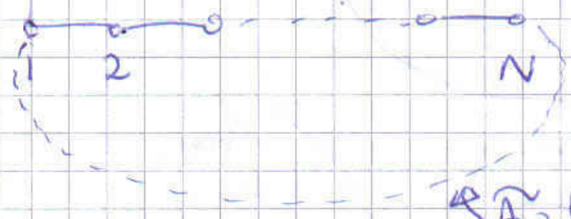
$\lim_{N\to\infty} f$

N-stable properties of SM associated to chain graphs.

much loved of string theorists

We will see boundary conditions.

A_N :



(or $\tilde{A}_N$ - periodic if you like)

← $\tilde{A}_N$ has this

Physics - not a significant difference between them.

write

Z_N for $Z(A_N)$; and seek

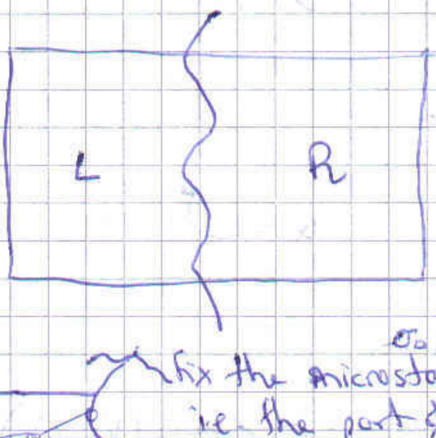
(note P_i's are the same up to a constant).

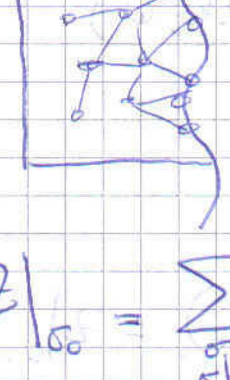
$$\lim_{N\to\infty} f_N = \lim_{N\to\infty} \frac{1}{N} \ln Z_N$$

"Solve" - compute all Z_N in such form that lim can be extracted.

For computation, generalise to "graph with boundary," and hence generalise the partition function to partition vector.

Idea





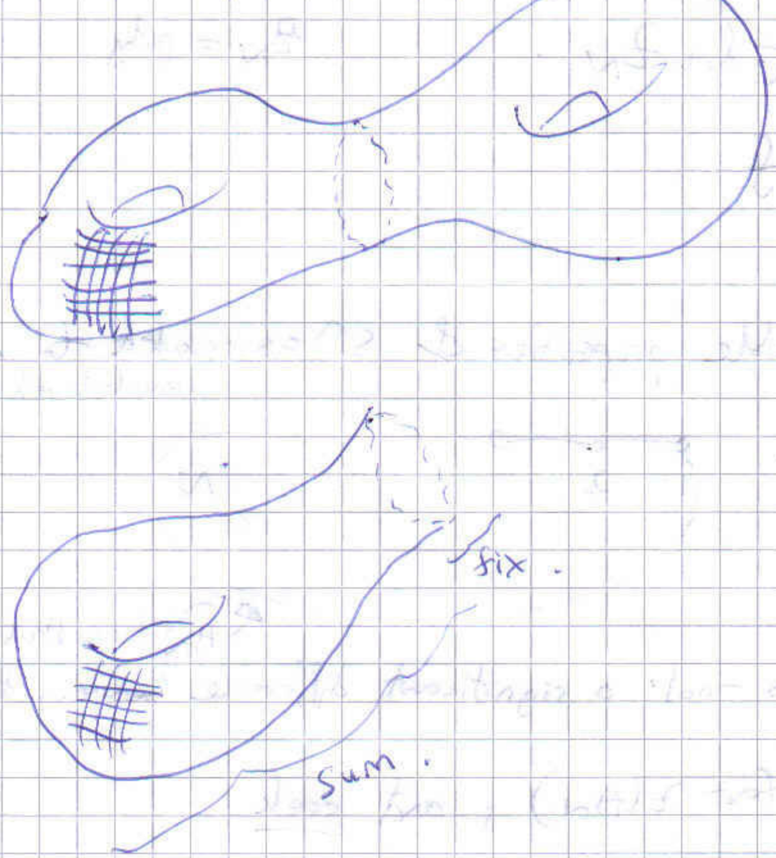
σ_0

fix the microstate on the boundary ie the part of the microstate on the boundary.

$$Z = \sum_{\sigma} \exp(\beta H(\sigma)).$$

$$Z|_{\sigma_0} = \sum_{\sigma / \sigma|_b = \sigma_0} \exp(\beta H(\sigma)) \quad \text{(partial sum)}$$

$$Z = \sum_{\sigma_0} z\Big|_{\sigma_0}.$$



Suggestive idea:

Boundary has some geometric/topological meaning

Specifics. Γ: set of graphs $\underline{Q} = \{1, 2, \ldots, Q\}$.

S a set, $Q^S = \mathrm{Hom}(S, \underline{Q})$ - set of maps from S to $\underline{Q}$.

- each map $f \in Q^S$ assigns state q to vertex s by $f(s) = q$.

Q^S = set of microstates. (S will be vertex set)

$H : \Gamma \longrightarrow \mathrm{Hom}(-, \mathbb{Z})$ ← often take $\mathbb{R}$ here.

$G \longmapsto \underset{H_G}{\mathrm{Hom}(Q^{V_G}, \mathbb{Z})}$. $\qquad x = e^{\beta}$

Fix Q (eg $Q = 2$). $z : \Gamma \longrightarrow \mathbb{Z}[x]$

Bounded graph is (G, V_0), $V_0 \subseteq V_G$. $\sigma_0 \in \mathrm{Hom}(V_0, Q)$

$$z\Big|_{\sigma_0} = \sum_{\substack{\sigma \in \mathrm{Hom}(V, \underline{Q}) \\ \text{s.t. } \sigma|_{V_0} = \sigma_0}} \exp(\beta H(\sigma)) \qquad H(G)(\sigma)$$

$z(G, V_0)$ is a $\mathrm{Hom}(V_0, \underline{Q})$-indexed vector s.t. $z(G, V_0)_i = z|_i$.

This is a $Q^{|V_0|}$-tuple

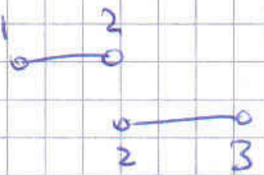
Note no weights in the graphs here.

"dim"$(Z(G, V_0)) = Q^{|V_0|}$

i.e. # of entries

A cover of a graph G is a set of graphs $\{G_i\}$ s.t.

$$\bigcup_i E_{G_i} = E_G \quad \left(\Rightarrow \bigcup \text{vertex sets} = \text{vertex set of } G \right).$$

eg  covers A_3.

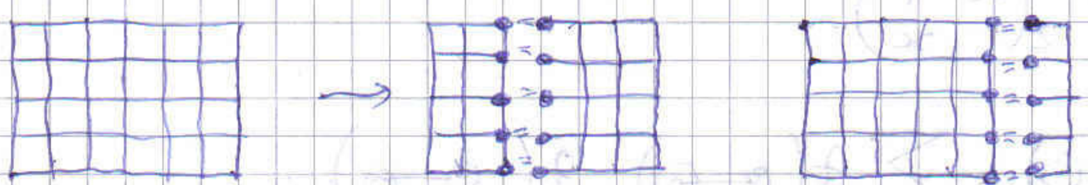
A separation of a graph G is a pair of graphs G_1, G_2, such that $\{G_i\}$ covers G and $E_{G_1} \cap E_{G_2} = \emptyset$ (as is the given example).

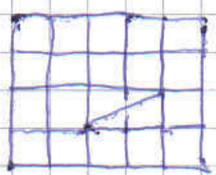
Thm: $\{G_1, G_2\}$ a separation of G (with $V_{G_1} \cap V_{G_2} = V_0$)

$$\Rightarrow \boxed{Z(G) = Z(G, \emptyset) = \sum_{\sigma_0 \in \mathrm{Hom}(V_0, \underline{Q})} \underbrace{Z(G_1, V_0)_{\sigma_0} Z(G_2, V_0)_{\sigma_0}}_{\sigma_0\text{'th component}}}$$

~~Chapman-Kolmogorov Theorem~~

Trivial Chapman Kolmogorov Theorem – not relevant to talk.



 not so easy – split into two "roughly equal halves".

eg $Q=2$. $H = H_1$ from week 1 — all edge weights are 1. $\qquad H=\sum_{(i,j)\in E}\delta_{\sigma_i\sigma_j} \qquad x=e^{\beta}$.

$$Z(\circ\!-\!\!-\!\circ) = 2x+2$$

	δ
11	1
12	0
21	0
22	1

Note can model (>0 integer) weights using multiple edges.

$$Z(\circ\!-\!\!-\!\circledcirc) = \begin{array}{l} 1 \\ 2 \end{array}\begin{cases} x+1 \\ x+1 \end{cases}. \qquad Z(\circ\!-\!\!-\!\circledcirc) = \begin{pmatrix} x+1 \\ x+1 \end{pmatrix}$$

$$Z(\circ\!-\!\!-\!\circ\!-\!\!-\!\circ) = \sum_i Z(\circ\!-\!\!-\!\circledcirc)_i\, Z(\circledcirc\!-\!\!-\!\circ)_i$$

$$= (x+1 \;\; x+1)\begin{pmatrix} x+1 \\ x+1 \end{pmatrix} = 2(x+1)^2$$

$$Z(\circledcirc\!-\!\!-\!\circledcirc)_{11} = x$$

$$Z(\circledcirc\!-\!\!-\!\circledcirc)_{12} = 1$$

$$Z(\circledcirc\!-\!\!-\!\circledcirc) = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix} \end{array} \quad Z(\circledcirc\!-\!\!-\!\circledcirc)_{12} \text{ etc}$$

$$Z(\circledcirc\!-\!\!-\!\circ\!-\!\!-\!\circledcirc)_{i,j} = \sum_k Z(\circledcirc\!-\!\!-\!\circledcirc)_{ik}\, Z(\circledcirc\!-\!\!-\!\circledcirc)_{k,j}$$

$$= \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^2.$$

$$Z(\underbrace{\circledcirc\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circledcirc}_{N-1}) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^{N-1}$$

$$\text{So } \; Z(A_N) = \overset{Z(\circledcirc)}{\overbrace{(1 \;\; 1)}}\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^{N-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \overbrace{(x \pm 1)}^{\lambda_1,\ \lambda_2}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

– does not often happen that we get an eigenvector.

So $\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^N \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (x+1)^N \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore Z(A_N) = 2(x+1)^{N-1} = 2\lambda_1^{N-1}$.

So $\lim f = \ln \lambda_1$

$$\left(\begin{aligned} f &= \frac{1}{N} \ln \lambda_1^{N-1} \\ &= \frac{N-1}{N} \ln \lambda_1 \end{aligned} \right).$$