Stat Mech 5 (Paul Martin) 6/11/08 10

$$Z_N^P = (x+1)^N + (x-1)^N = \mathrm{tr}\left(\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^{N-1}\right)$$

Last time:

In the physical region (of $x = e^{\beta}$),

$$\lim_{N\to\infty} f = \ln \lambda_0$$ [RJM: I think f is now $\frac{\ln Z}{N}$]

↳ largest magnitude eigenvalue of the transfer matrix

$Z \sim y^N + 1 \qquad y = \frac{x+1}{x-1}$

$y = \exp(\beta')$

$$Z = 2y^{N/2}\frac{\left(y^{N/2} + y^{-N/2}\right)}{2}$$

$$f = \frac{\ln Z}{N} = \frac{1}{N}\left(\ln 2 + \frac{N}{2}\ln y + \ln\left(\cosh \frac{N}{2}\beta'\right)\right)$$

So, $U = -\frac{\partial\left(\frac{1}{N}\ln Z\right)}{\partial \beta'} \sim -\tanh\left(\frac{N}{2}\beta'\right)$

↑ changes fast for large N near $\beta' = 0$.

Evidently, $Z \sim y^N + 1$ has zeros lying on a circular locus, for any N.

$$\lim_{N\to\infty} f \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} a(y)\ln(\beta' + iy)\,dy$$

(line density of zeros (=1 in our case!)

$$\Rightarrow U \sim \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(y)\,dy}{y - i\beta'}$$

Integrand has a simple pole at $y = i\beta'$.

$\Rightarrow$ integral changes by $2\pi a(0)$ as β' changes sign (as it passes through the point on the real line with value i).

$\Rightarrow a(0) \neq 0$ gives an internal energy discontinuity, hence a 1st order phase transition.

In practice (e.g. in 2D), can get line density $a(y) = |y|^{1-p}$ $(0 \le p \le 1)$
In many cases, the transfer matrix is reducible to a 2×2 polynomial (in x) matrix. As before,

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathcal{Z} = \lim_{N\to\infty} \frac{1}{N} \ln\left(\lambda_+^N + \lambda_-^N\right) = \ln \lambda_+$$

on real axis.

What happens to zeros in this (slightly) more general case?
Writing $C^N + D^N = \prod_n \left(C + \exp\left(\frac{2\pi i n}{N}\right) D\right)$

$$= \prod_n{}' \left(C^2 + D^2 + \cos\left(\frac{2\pi n}{N}\right) CD\right).$$

$$\ln(\lambda_+) = \frac{1}{2\pi}\int_0^{\infty} \ln\left(2(A^2+B) + 2\cos y\,(A^2-B)\right) dy.$$

$\lambda_\pm = A \pm \sqrt{B}$
For Ising, $\lambda_\pm = x \pm 1$.

where $CD = A^2 - B$
$C^2 + D^2 = A^2 + B$.

What we have here is $\log$ of an ∞-polynomial with lines of zeros on loci $|\lambda_+| = |\lambda_-|$.

N.B. Makes sense of limit is that zeros of finite N lie on same loci (up to some boundary choices (be careful)).