Stat Mech 9, ~~Robert Marsh~~.

Concatenating Graphs

Recall our notation (G, V_0) for a graph G with boundary $V_0 \subseteq V$ (V=vertices).

We have the partition function $Z = \sum_{\substack{\sigma \\ \text{microstate}}} \exp(\beta H(\sigma))$.

H = Hamiltonian.

$$Z\big|_{\sigma_0} = \sum_{\substack{\sigma \in \mathrm{Hom}(V, \underline{Q}) \\ \text{s.t. } \sigma|_{V_0} = \sigma_0}} \exp(\beta H(\sigma)) \quad \text{for } \sigma_0 \in \mathrm{Hom}(V_0, Q)$$

$Z(G, V_0)$ is a $\mathrm{Hom}(V_0, \underline{Q})$-indexed vector s.t. $Z(G, V_0)_{\sigma_0} = Z\big|_{\sigma_0}$.

We are usually writing it as a matrix by writing $V_0 = V_0^{in} \dot\cup V_0^{out}$,

$\sigma_0^{in} = \sigma_0|_{V_0^{in}}$, $\sigma_0^{out} = \sigma_0|_{V_0^{out}}$, and setting

note, still ok even if union not disjoint.

$$Z_G\big|_{\sigma_0^{in}, \sigma_0^{out}} := \underbrace{Z_G\big|_{\sigma_0 = (\sigma_0^{in}, \sigma_0^{out})}}_{\text{i.e. defined by } (*)}.$$

We get a matrix with rows indexed by $\mathrm{Hom}(V_0^{in}, \underline{Q})$ and columns indexed by $\mathrm{Hom}(V_0^{out}, \underline{Q})$.

Defn (Tensor product of matrices).
Let A be a matrix indexed by I (rows) and J (columns).
Let A' " " " " " I' " " J' ".
Then $A \otimes A'$ is the matrix with rows indexed by $I \times I'$ and columns indexed by $J \times J'$, defined by

$$(A \otimes A')_{(i,i'),(j,j')} = A_{ij} A'_{i'j'}.$$

Since I and J are totally ordered, we can totally order $I \times I'$ and $J \times J'$ using the usual lexicographic ordering.

eg. $$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & a_{11} & a_{12} \\ 2 & a_{21} & a_{22} \end{array} \otimes \begin{array}{c|cc} & 1' & 2' \\ \hline 1' & a'_{11} & a'_{12} \\ 2' & a'_{21} & a'_{22} \end{array} =$$

	$11'$	$12'$	$21'$	$22'$
$11'$	$a_{11}a'_{1'1'}$	$a_{11}a'_{1'2'}$	$a_{12}a'_{1'1'}$	$a_{12}a'_{1'2'}$
$12'$	$a_{11}a'_{1'2'}$	$a_{11}a'_{2'2'}$	$a_{12}a'_{2'1'}$	$a_{12}a'_{2'2'}$
$21'$	$a_{21}a'_{1'1'}$	$a_{21}a'_{1'2'}$	$a_{22}a'_{1'1'}$	$a_{22}a'_{1'2'}$
$22'$	$a_{21}a'_{2'1'}$	$a_{21}a'_{2'2'}$	$a_{22}a'_{2'1'}$	$a_{22}a'_{2'2'}$

.

<u>Lemma</u>

If $A^1, \ldots, A^k$ are matrices with rows indexed by $I_1, \ldots, I_k$, cols. indexed by $J_1, \ldots, J_k$, then $A^1 \otimes \cdots \otimes A^k$ has rows indexed by $I_1 \times \cdots \times I_k$, cols. indexed by $J_1 \times \cdots \times J_k$, with

$$(A^1 \otimes \cdots \otimes A^k)_{(i_1, \ldots, i_k),(j_1, \ldots, j_k)} = A^1_{i_1 j_1} \cdots\cdots A^k_{i_k j_k}.$$

We have the following (easy) theorem:

<u>Theorem</u> Suppose that $H = G \dot{\cup} G'$ is the disjoint union of two graphs (i.e. totally disconnected, no edges between them either), with vertex sets V, V', boundaries $V_0 \subseteq V$, $V'_0 \subseteq V'$.

Then $Z_{G \amalg G'}(V_0 \amalg V'_0) = Z_G(V_0) \otimes Z_{G'}(V'_0)$.

<u>Proof</u> $Z_{G \amalg G'}\big|_{(\sigma_{in}, \sigma'_{in}),(\sigma_{out}, \sigma'_{out})} =$

$$\sum_{\substack{\tilde{\sigma} \in \mathrm{Hom}(V \amalg V', \underline{Q}) \\ \tilde{\sigma}|_{V_0 \amalg V'_0} = (\sigma_{in}, \sigma_{out}, \sigma'_{in}, \sigma'_{out})}} \exp(\beta H(\tilde{\sigma}))$$

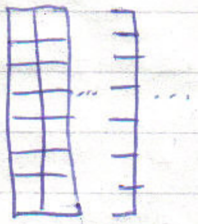
$$= \sum_{\substack{\sigma \in \mathrm{Hom}(V, \underline{Q}),\ \sigma' \in \mathrm{Hom}(V', \underline{Q}) \\ \sigma|_{V_0} = (\sigma_{in}, \sigma_{out}),\ \sigma'|_{V'_0} = (\sigma'_{in}, \sigma'_{out})}} \exp(\beta H(\sigma)) \exp(\beta H(\sigma'))$$

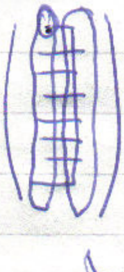
noting $H(\tilde{\sigma}) = H(\sigma) + H(\sigma')$ as the graphs are disjoint.

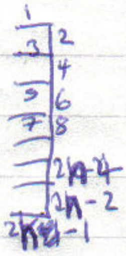
$$= \sum_{\substack{\sigma \in \mathrm{Hom}(V, \underline{Q}) \\ \sigma|_{V_0} = (\sigma_{in}, \sigma_{out})}} \exp(\beta H(\sigma)) \sum_{\substack{\sigma' \in \mathrm{Hom}(V', \underline{Q}) \\ \sigma'|_{V'_0} = (\sigma'_{in}, \sigma'_{out})}} \exp(\beta H(\sigma')) = Z_G\big|_{(\sigma_{in}, \sigma_{out})} Z_{G'}\big|_{(\sigma'_{in}, \sigma'_{out})}.$$

$\blacksquare$

We can now apply this theorem to the following case:

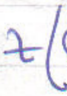


We need to compute $Z\left(\right.$$\left.\right)$.



n vertices.

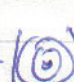
We first compute $Z\left(\right.$$\left.\right)$ where e is a vertical edge.

We know $Z\left(\right.$$\left.\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $Z\left(\right.$$\left.\right) =$

$$\begin{array}{c|cccc} & 11 & 12 & 21 & 22 \\ \hline 11 & x & 0 & 0 & 0 \\ 12 & 0 & 1 & 0 & 0 \\ 21 & 0 & 0 & 1 & 0 \\ 22 & 0 & 0 & 0 & x \end{array}$$

- note that, by making the off-diagonal entries equal to zero, we can deal with the case where the two boundaries overlap - we get no contribution from the cases where the states on the boundaries are different.

~~So if e joins vertices i~~ So if $e = 2r$, we obtain:

$t_e := Z\left(\right.$$\left.\right) = Z\left(\right.$$\left.\right)^{\otimes(r-1)} \otimes Z\left(\right.$$\left.\right) \otimes Z\left(\right.$$\left.\right)^{\otimes(n-r-1)}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes(r-1)} \otimes \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes(n-r-1)}.$$

From the lemma, we obtain:

$$\boxed{(t_e)_{(\sigma_1,\dots,\sigma_n),(\sigma_1',\dots,\sigma_n')} = \begin{cases} x & \text{if } (\sigma_1,\dots,\sigma_n) = (\sigma_1',\dots,\sigma_n') \text{ and } \sigma_{\frac{1}{2}r} = \sigma_{\frac{1}{2}r+1} \\ 1 & \text{if } (\sigma_1,\dots,\sigma_n) = (\sigma_1',\dots,\sigma_n') \text{ and } \sigma_{\frac{1}{2}r} \neq \sigma_{\frac{1}{2}r+1} \\ 0 & \text{else.} \end{cases}}$$

$\frac{2n-1-(2r+1)}{2}$

$\frac{2n-2r-2}{2}$

And if $v = 2r+1$, we obtain

$$t_v := z\Bigg($$  $$\Bigg) = z(\odot\ \odot)^{\otimes r} \otimes z(\odot\!-\!\odot) \otimes z(\odot\ \odot)^{\otimes(n-r-1)}$$

$$t_v = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{\otimes r} \otimes \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{\otimes(n-r-1)}.$$

Therefore $$\boxed{(t_v)_{(\sigma_1,\dots,\sigma_n),(\sigma'_1,\dots,\sigma'_n)} = \begin{cases} x & \text{if } \sigma_{r+1} = \sigma'_{r+1} \\ 1 & \text{else} \end{cases}}.$$