Potts Models and Related Problems in Statistical Mechanics

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Chapter 4 (Draft 2nd Edition)

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Contents

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Chapter 2

Transfer Matrices

2.11 Exercises and so on

(**2.11.1**) Exercise. Explain the sense in which two local interaction Hamiltonians differing only by a multiplicative and an additive constant may be considered to give rise to equivalent models.

Solution. A rescaling of the Hamiltonian may be absorbed by rescaling the temperature units. Additive constants in the Hamiltonian cancel out of normalised Boltzmann weights.

(**2.11.2**) Using the Ising variables $\sigma_j \in \{1, -1\}$ on the sites j of a lattice, we may write the homogeneous Ising model Hamiltonian as

$$H_{\text{Ising}} = \sum_{\langle jk \rangle} \sigma_j . \sigma_k$$

where the sum is over nearest neighbours pairs of sites. If we use the Potts spin variables $s_j \in \{1, 2\}$ we may write this as

$$H_{\text{Ising}} = \sum_{\langle jk \rangle} (2\delta_{s_j, s_k} - 1)$$

(**2.11.3**) Exercise. Recall the single-bond transfer matrices $t_i(y)$ for the $Q = 2$-state Potts model, given in Section 2.1. Write down the single-bond transfer matrices $t'_i(y)$ for the equivalent form given by the Ising model Hamiltonian.

Solution. With either kind of spin variable we have (for a transfer matrix layer of N sites)

$$t'_{1.} = \begin{pmatrix} y & y^{-1} \\ y^{-1} & y \end{pmatrix} \otimes 1_2^{\otimes N-1} = y^{-1} \begin{pmatrix} y^2 & 1 \\ 1 & y^2 \end{pmatrix} \otimes 1_2^{\otimes N-1}$$

$$t'_{2.} = 1_2 \otimes \begin{pmatrix} y & y^{-1} \\ y^{-1} & y \end{pmatrix} \otimes 1_2^{\otimes N-2}$$

$$t'_{12} = \begin{pmatrix} y & & & \\ & y^{-1} & & \\ & & y^{-1} & \\ & & & y \end{pmatrix} \otimes 1_2^{\otimes N-2} = y^{-1} \begin{pmatrix} y^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & y^2 \end{pmatrix} \otimes 1_2^{\otimes N-2}$$

and so on. In other words these matrices differ from the Potts versions by an overall scalar, and $y \to y^2$.

(**2.11.4**) EXERCISE. Verify that for *given* bond i the single-bond transfer matrices $\{t_i(y)\}_{y\in\mathbb{C}}$ for the $Q = 2$-state Potts model obey $[t_i(y), t_i(y')] = 0$, and hence can be simultaneously diagonalised for all y.

Solution. Case $t_{jk}(y)$ is diagonal, so this is trivial. In case $t_{j.}(y)$ we see that

$$t_{j.}(y) = (y-1)1_- + M$$

$$t'_{j.}(y) = y^{-1}((y^2-1)1_- + M)$$

where 1_- is the unit matrix and M is a constant real symmetrix matrix. Thus $t_{j.}(y)$ and $t_{j.}(y')$ differ by a scalar multiple of the unit matrix, and are always diagonalisable (and similarly for the Ising formulation t'_i).

(**2.11.5**) EXERCISE. Note that, fixing a bond j (of either type) and β, the single-bond transfer matrix $t'_j(y = \exp(\beta))$ for the Ising model is either a scalar multiple of the identity matrix, or has two distinct eigenvalues of equal multiplicity. Determine alternative coupling variables γ (for each of the two types of bond) so that the ratio of these eigenvalues is $\exp(i2\gamma)$.
Show that there are scalar functions $k(y), k_.(y)$ and a rescaling of each transfer matrix $t''_{j.}(y) = k_.(y)t'_{j.}(y)$, $t''_{jk}(y) = k(y)t'_{jk}(y)$ so that they obey

$$t''_j(f(\gamma))\, t''_j(f(\gamma')) \;=\; t''_j(f(\gamma+\gamma'))$$

and $t''_j(f(0)) = t''_j(f(2\pi)) = 1$.
Solution. We require to find a reparameterisation such that the eigenvalues of $t'_j(y = f(\gamma))$ can be expressed in the form $\lambda_\pm = k(y)\exp(\pm\sqrt{-1}\gamma)$ for some $k(y)$. For $t'_{jk}(y)$ the eigenvalues are $\{y, y^{-1}\}$, so a possible solution is $y = \exp(i\gamma)$. For $t'_{j.}(y)$ the eigenvalues are $\{y+y^{-1}, y-y^{-1}\}$, so set $y \pm y^{-1} = k(y)\exp(\pm i\gamma)$ giving $y^{\pm 1} = \frac{k(y)}{2}(\exp(i\gamma) \pm \exp(-i\gamma))$ thus

$$y^2 = \cos(\gamma)/(i\sin(\gamma)) = \frac{1}{i\tan(\gamma)}$$

giving

$$y \pm y^{-1} = \frac{1}{\cos(\gamma)\sqrt{i\tan(\gamma)}}\exp(\pm i\gamma)$$

It follows that $t''_j = \cos(\gamma)\sqrt{i\tan(\gamma)}t'_j$ has eigenvalues $\exp(\pm i\gamma)$. By Exercise 2.11.4 we can work in a diagonal basis, and the second result then also follows.

2.12 Background for subsequent Chapters

2.12.1 Preliminaries: Notations and basic linear algebra

It will be convenient to be able to perform various simple operations on matrices and vector spaces, beyond the basic products and sums. We introduce notations for these now.

For S, T sets write $\mathrm{Hom}(S,T)$ for the set of functions $f : S \to T$.

(**2.12.1**) For $L \in \mathbb{N}$ write $\underline{L} = \{1, 2, ..., L\}$ (as for example in Green [**?**]). Thus $\mathrm{Hom}(\underline{N}, \underline{L})$ is the set of functions $i : \underline{N} \to \underline{L}$. Write $I(L,N)$ for the set of these functions encoded as sequences $i = (i_1, ..., i_N)$ with $i_j = i(j) \in \underline{L}$ (again following Green). Write $I'(L,N)$ for the subset of such functions that are injective.

If $L \in \mathbb{N}$ is given, write

$$\mathbf{e}_j = (0, 0, ..., 0, 1, 0, ..., 0)^T$$

for the L-component elementary column vector with 1 in the j-th position.

For V the vector space with basis $\{\mathbf{e}_1, ..., \mathbf{e}_L\}$ and $i \in I'(N,L)$ write V_i for the subspace with ordered basis $\{\mathbf{e}_{i_1}, ..., \mathbf{e}_{i_N}\}$.

(**2.12.2**) Write $S = S_L$ for the cyclic shift operator on V, defined by $S\mathbf{e}_i = \mathbf{e}_{i-1}$ for $i > 1$ and $S\mathbf{e}_1 = \mathbf{e}_L$. Note that if V is equipped with an *ordered* basis, as here, then S may be considered as a specific $L \times L$ matrix.

If M is an $L' \times L'$ matrix with $L' \leq L$ and $i = (i_1, i_2, ..., i_{L'})$ a sequence of pairwise distinct elements from $1, 2, ..., L$, then define

$$M_i = M_{i_1, i_2, ..., i_{L'}} \tag{2.1}$$

to act on V like M on V_i, via the ordered basis, and trivially otherwise. For example, for R a 2×2 matrix, $R_{12} = R \oplus 1_{L-2}$ and

$$S\ R_{ij}\ S^{-1} \ = \ R_{i-1\ j-1}$$

(with the cyclic rule on indices understood).

(**2.12.3**) If V is complex, define the *fourier transform* basis $\{f_0, f_1, ..., f_{L-1}\}$ by

$$f_j \ = \ \sum_{k=1}^{L} \exp(i2\pi kj/L)\ \mathbf{e}_k$$

(note that we may set $f_L = f_0$). Then

$$S\ f_j \ = \ \exp(i2\pi j/L)\ f_j \tag{2.2}$$

Define $L \times L$ matrix

$$F'_L = (f_0, f_1, ..., f_{L-1})$$

Note that this is a unitary matrix: $F'_L (F'_L)^\dagger = 1$; and that $f_j = F'_L \mathbf{e}_{j+1}$.

(**2.12.4**) LEMMA. Suppose that a matrix $A \in M_L(\mathbb{C})$ has a complete set of eigenvectors v_i, with associated eigenvalues α_i (not necessarily distinct). Write $V(\alpha)$ for the α-eigenspace. Then if $[M, A] = 0$ we have $AMv_i = MAv_i = \alpha_i Mv_i$ so that

$$Mv_i \in V(\alpha_i). \tag{2.3}$$

Note that this holds for every $v_i \in V(\alpha_i)$.
Note also that the subset of matrices in the associative algebra $M_L(\mathbb{C})$ commuting with A is a subalgebra; and that (2.3) holds for every element of this subalgebra. Thus for each α_i the action of the subalgebra on $V(\alpha_i)$ provides a representation (see Chapter 5 for more on this).

(**2.12.5**) LEMMA. We say a matrix M is cyclic translation invariant if $[S, M] = 0$. Then (it follows immediately from (2.3) and (2.2) that) the fourier basis is a complete set of eigenvectors for M.

2.12.2 Preliminaries: On rotations

Define

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Fix N and $V = \mathbb{R}^{2N}$, and define $R_{ij}(\theta)$ as in (2.1) above, i.e. as the $2N \times 2N$ matrix that acts like $R(\theta)$ on the i, j-coordinates of a vector and leaves the rest alone.

Thus with $N = 2$, for example,

$$R_{12}(\theta) = R(\theta) \oplus 1_2$$

and

$$R_{12}(\theta)R_{34}(\theta) = R(\theta) \oplus R(\theta)$$

and

$$R_{23}(\theta)R_{41}(\theta) = \begin{pmatrix} \cos(\theta) & 0 & 0 & -\sin(\theta) \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ \sin(\theta) & 0 & 0 & \cos(\theta) \end{pmatrix}$$

(**2.12.6**) Note that $R_{T,2} := R_{12}(\theta')R_{34}(\theta')R_{23}(\theta)R_{41}(\theta)$ is nowhere zero. However defining

$$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes 1_2$$

we have that

$$FR_{23}(\theta)R_{41}(\theta)F^{-1} = R(\theta) \oplus R(\theta)^T$$

i.e. it is 2×2-block diagonal. Further, $[F, R_{12}(\theta)R_{34}(\theta)] = 0$, and hence $FR_{T,2}F^{-1}$ has the same block-diagonal structure. This means that we could easily go on and work out the eigenvalues of $R_{T,2}$.

(**2.12.7**) More generally, define

$$R_{T,N} = \left(\prod_{i=1}^{N} R_{2i-1\ 2i}(\theta') \right) \left(\prod_{i=1}^{N} R_{2i\ 2i+1}(\theta) \right)$$

(with the understanding that $R_{2N\ 2N+1}$ means $R_{2N\ 1}$).

It will be useful (in Section **??**) to know the spectrum of this matrix. We compute this next.

Note that S_{2N}^2 commutes with $R_{T,N}$. On the other hand,

$$S_{2N}^2 = 1_2 \otimes S_N$$

and hence has N 2-dimensional eigenspaces. By Lemma (2.12.5) this means that $R_{T,N}$ only mixes up the column vectors in $1_2 \otimes F'_N$ in pairs. So, once we have worked out this action, we will be able to compute the eigenvalues of $R_{T,N}$.

Define $f_z = (1, z, z^2, ..., z^{L-1})^T$ and

$$v_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes f_z \qquad v'_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes f_z$$

Note that every column vector in $1_2 \otimes F'_N$ is of this form. As noted, these vectors span a closed subspace of the action of $R_{T,N}$:

$$R_{T,N} \begin{pmatrix} v_z \\ v'_z \end{pmatrix} = \begin{pmatrix} cc' + z^{-1}ss' & zcs' - sc' \\ sc' - z^{-1}cs' & cc' + zss' \end{pmatrix} \begin{pmatrix} v_z \\ v'_z \end{pmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. We want to express the eigenvalues in the form

$$\lambda_\pm = e^{\pm l_z}$$

so we have

$$\cosh(l_z) = cc' + \frac{z + z^{-1}}{2} ss'$$

Chapter 4

On exactly solved cases

Among the thermodynamic limit Potts models in two or more dimensions only $Q = 2, d = 2$ has been solved for arbitrary couplings (cf. chapter **??**). Beside some results for two-dimensional models at criticality discussed in chapter **??** some other non-trivial special cases, equivalent to colouring problems, have been solved (see, for example, Baxter 1987). Three dimensions remains uncharted territory (although plenty of exact finite lattice results are available)!

The eight-vertex model, equivalent to a general 2-valued site variable model of interactions round a square face, and the hard hexagon model, have been solved by Baxter (see his book, 1982). Other solved models appear in Andrews, Baxter and Forrester 1984, Date *et al* 1987, Akutsu *et al* 1986, and several other of the references at the end of this book. Although these solutions do not pertain directly to the Potts models away from criticality, they are interesting in their own right.

The two-dimensional Ising model has been solved in many different ways. A comparison between some of the more elegant methods of solution provides an exellent way of reviewing the mathematical tools potentially at our disposal for addressing other Potts models.

4.1 The two-dimensional Ising model

The next few sections describe various very disitinctive methods of solution. We work either with the Ising model itself, or equivalently in the $Q = 2$-state Potts model formulation.

The method discussed in Section 4.1.1 is essentially Kaufman's (1949) method. The second method is that of Schultz, Mattis and Lieb (1964), which is related to Onsager's (1944) method, and the third is due to Kac and Ward (1952) and Vdovichenko (1965). After looking at those it is worth having another look at Baxter's commuting transfer matrix method, which was discussed in chapter **??**.

4.1.1 Solution by rotations

The single bond transfer matrices $t_i(\exp(\beta))$ for Potts models may be thought of as giving representatives of rotations in a plane, in the sense (at least for i lateral) that the effect of applying such a matrix twice is to count the same bond interaction again, or equivalently to double the coupling constant on that bond. In other words the coupling constant corresponds to some multiple of the rotation angle.

In the case of the Ising model ($Q = 2$, so $r = 4$), for example, we note that the fourth power of the braid generator

$$g_i \mapsto t_i(-q) = 1 - qU_i$$

is unity, so we associate this matrix with the rotation through the real angle $\pi/2$.

This is not as facile an observation as it might seem. In 3 dimensions a rotation through $\pi/2$ about the x-axis followed by a rotation through $\pi/2$ about the y-axis followed by a rotation through $\pi/2$ about the x-axis is equivalent to the same operation with the x and y axes interchanged. In other words, extending to D dimensions, and (arbitrarily) assigning a total order to D orthogonal axes, we have a realisation of the braid group as follows: For a, b axes adjacent in the order let w_{ab} signify a rotation in the ab plane, then

$$w_{ab}w_{bc}w_{ab} = w_{bc}w_{ab}w_{bc}. \tag{4.1}$$

In fact we have much more than this, since *any* pair of planes give a braid relation, but we will not need this property (see the chapter on graph Temperley-Lieb algebras).

It is easy to see that equation 4.1 extends to a solution of the Yang-Baxter equations, although the relationships between the rotation angles and the spectral parameters u of chapter **??** are rather complicated (see later). This means that we can think of the strings of our braid as corresponding to orthogonal axes, and the braid generators as rotations in the plane of their associated pairs of axes.

The representation of the braid group here just comes from the transfer matrix in the usual way. Only the interpretation is new. However, associated with this interpretation comes another representation, in terms of the D dimensional orthogonal matrices. Instead of our bond transfer matrix $t_i(y)$ we can write the corresponding rotation matrix $w_{ab}(\theta) \in End(V_D)$, which acts trivially except in the 2 dimensional subspace associate with the axes a and b, where it acts as

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{4.2}$$

The axes a and b are determined by the interaction i (i and $i+1$ have an axis in common, i and $i+2$ do not), and the relationship between $y = \frac{e^\beta - 1}{\sqrt{Q}}$ and θ

is determined by noting that $t_i^4(-q) = 1$ implies that

$$\beta = i\pi/2$$

corresponds to

$$\theta = \pi/2.$$

The easiest way to see that we have a solution of the Yang-Baxter equations at the level of rotations is to note that it is sufficient to check the relation

$$w_{ab}(\theta)w_{bc}(\psi)w_{ab}(\phi) = w_{bc}(\phi)w_{ab}(\psi)w_{bc}(\theta)$$

for all θ, ϕ (with ψ determined) for any triple of axes a, b, c. In this case we can use the quaternionic realisation of 3 dimensional rotations. Defining the quaternionic generators $1, i, j, k$ by the relations

$$ii = jj = kk = -1 \quad ij = -ji \;\; ik = -ki \;\; jk = -kj = -i$$

we must check that

$$(\cos(\theta/2) + i\sin(\theta/2))(\cos(\psi/2) + j\sin(\psi/2))(\cos(\phi/2) + i\sin(\phi/2))$$

$$= (\cos(\phi/2) + j\sin(\phi/2))(\cos(\psi/2) + i\sin(\psi/2))(\cos(\theta/2) + j\sin(\theta/2))$$

where each factor corresponds to a rotation through twice the anglular argument shown about the x-axis (i) or the y-axis (j) (or the z-axis (k), which we didn't need).

After some algebra we obtain

$$\tan(\theta/2) = \frac{\sin(\frac{\psi-\phi}{2})}{\cos(\frac{\psi+\phi}{2})}.$$

Note that the braid limit is the angle $\pi/4$ (i.e. $\theta = \psi = \phi = \pi/2$).

It is remarkable that this relationship between the Temperley-Lieb algebra and rotations should give rise to a new algebra, again related to the braid group. This new algebra has a local relation which is higher order than quadratic, determined by the lowest order multiplicative relation at the transfer matrix level - in this case the quartic relation

$$t_i^4 = 1.$$

Multiplicative relations are necessarily carried across to the new solution, but additive ones are not.

We note that this coincidence is apparently uniquely associated with the Ising model, since the condition $w_{ab}^4(\pi/2) = 1$ is uniquely associated with this model.

We note further that there is a relationship between the eigenvalues of the Transfer Matrix and those of the corresponding rotation matrix. If we take a single rotation in the plane of one of our pairs of axes the eigenvalues of the

transfer matrix are, up to multiplicity, $\exp(\pm i\theta/2)$, while those of the rotation matrix are 1 or $\exp(\pm i\theta)$. If we take several commuting plane rotations by θ_j ($j = 1, ..., n$ say, with $n \leq D/2$, or equivalently $n = D/2$ with some $\theta_j = 0$) then the eigenvalues are

$$\exp\left((i/2)\sum_j \pm\theta_j\right) \tag{4.3}$$

(for all 2^n possible choices of the signs) and

$$\exp(\pm i\theta_1),\ \exp(\pm i\theta_2), ...$$

($2n$ distinct possibilities) respectively.

In order to compute the spectrum of a full transfer matrix it is thus only necessary to compute the spectrum of the corresponding rotation matrix, provided that this may be put in the above form. Since the rotation matrix is exponentially smaller than the transfer matrix this is a potentially vast simplification of the problem.

Arranging the m-site layer transfer matrix in the symmetric form of equation **??**, that is, in the form

$$T = V_1^{1/2} V_2 V_1^{1/2} \tag{4.4}$$

we can write the factor $V^{1/2}$ in the corresponding rotation matrix in the form $J(\theta) \otimes 1_m$ where, from equation 4.2 and below

$$J(\theta) = \begin{pmatrix} \cosh(\theta) & i\sinh(\theta) \\ -i\sinh(\theta) & \cosh(\theta) \end{pmatrix}. \tag{4.5}$$

Let us introduce a matrix S, which achieves a single cyclic shift of basis states, for example

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & .. & 0 \\ 0 & 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 0 & 1 & .. & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & .. & 1 \\ 1 & 0 & 0 & 0 & .. & 0 \end{pmatrix}.$$

If required, S could also introduce some sign changes at the boundary, which we have assumed periodic. Provided we stay away from the boundary it will not be necessary to specify the precise boundary conditions in order to illustrate the method, as we will see. We can then write the second factor in equation 4.4, up to an overall scalar, as the statistical mechanical *dual* squared of equation 4.5 (with a different angle parameter), that is,

$$S(J(2\phi) \otimes 1_m)S^{-1}.$$

The angles θ and ϕ are determined by the usual horizontal and vertical lattice coupling parameters β_1 and β_2. We note, by comparing the definition of the $Q = 2$ state Potts transfer matrix from chapter **??** with the definition of $J(\theta)$ above, that

$$\phi = \beta_2/2 \tag{4.6}$$

and

$$\theta = \tanh^{-1}(e^{-\beta_1}).$$

The resultant matrix for T is clearly invariant under double cyclic shifts of basis states (conjugation by S^2), so its diagonalisation can be greatly facilitated by the (fourier) similarity transformation

$$F.[(J(\theta) \otimes I_m).\left(S(J(2\phi) \otimes I_m)S^{-1}\right).(J(\theta) \otimes I_m)].F^{-1}$$

where, with $z = e^{2i\pi/m}$,

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & z & z^2 & ... & z^{n-1} \\ 1 & z^2 & z^4 & ... & z^{2n-2} \\ \vdots & & & & \\ 1 & z^{n-1} & z^{2n-2} & ... & z^{(n-1)^2} \end{pmatrix}.$$

After a modicum of algebra we find that the resultant 2×2 blocks are each of the form

$$\begin{pmatrix} \cosh(2\phi)\cosh(2\theta) - & -i\cosh(2\phi)\sinh(2\theta) \\ (z^k + z^{-k})[\sinh(2\phi)\sinh(2\theta)] & +(z^k)[i\sinh(2\phi)\sinh^2(\theta)] \\ & +(z^{-k})[i\sinh(2\phi)\cosh^2(\theta)] \\ \\ i\cosh(2\phi)\sinh(2\theta) & \cosh(2\phi)\cosh(2\theta) - \\ -(z^{-k})[i\sinh(2\phi)\sinh^2(\theta)] & (z^k + z^{-k})[\sinh(2\phi)\sinh(2\theta)] \\ -(z^k)[i\sinh(2\phi)\cosh^2(\theta)] & \end{pmatrix} \tag{4.7}$$

where the complete set of m blocks is given by $k = 0, 1, 2, ..., m-1$.

Notice that the determinant of each block is 1, as required, so the spectrum *will* have the form

$$l_{\pm k} = e^{\pm x_k}$$

suitable for reading off the spectrum of the transfer matrix (cf. equation 4.3). In fact it must have this form, since an arbitrary rotation can be written as a product of commuting rotations. To see this note that a real matrix has eigenvalues which are either real or in complex conjugate pairs.

By taking the trace of each block we deduce that the exponents x_k are just the solutions of

$$\cosh(x_k) = \cosh(2\phi)\cosh(2\theta) - (z^k + z^{-k})[\sinh(2\phi)\sinh(2\theta)].$$

Each of these is positive in the physical region, so the largest eigenvalue of the *transfer matrix* from equation 4.3, i.e. the one appropriate for obtaining the free energy, will be

$$\lambda_0 = \exp\left((1/2)\sum_{k=1}^{m} x_k\right)$$

giving, in the thermodynamic limit,

$$f \sim \lim_{m\to\infty}(1/m)\ln(\lambda_0) = \lim_{m\to\infty}(1/2m)\left(\sum_{k=1}^{m} x_k\right)$$

The equivalence '$\sim$' here is to remind us that we have ignored a regular additive contribution. This comes from neglecting a scalar factor in the transfer matrix, which can be eliminated in any case by introducing a trivial additive term in the Hamiltonian.

Noting equation 4.6 we find that in the *isotropic* case ($\beta = \beta_1 = \beta_2$) we have

$$\cosh(x_k) = \cosh(\beta)\coth(\beta) - \cos(k\pi/m).$$

Using the integral representation

$$x_k = (1/\pi)\int_0^{\pi} dy \ln(2(\cosh(\beta)\coth(\beta) - \cos(k\pi/n)) - 2\cos(y))$$

we can thus write the limit free energy as

$$f \sim \frac{1}{2\pi^2}\int_0^{\pi} dy \int_0^{\pi} dy' \ln(2(\cosh(\beta)\coth(\beta) - \cos(y')) - 2\cos(y)) \qquad (4.8)$$

or equivalently, after some manipulation,

$$f \sim \frac{1}{2\pi}\int_0^{\pi} dz \ln\left(\cosh(\beta)\coth(\beta)\left(1 + \sqrt{1 - \left(\frac{2\cos(z)}{\cosh(\beta)\coth(\beta)}\right)^2}\right)\right).$$

To get the last form we used the identity

$$\frac{1}{2}\int_{-\pi}^{\pi} dw \ln(1 + t\cos(w)) = \int_0^{\pi} dw \ln(1 + t\cos(w)) = \pi \ln(1 + \sqrt{1 - t^2}).$$

The dominant contribution to the specific heat close to the critical point $k = 1$, where

$$2/k = \cosh(\beta)\coth(\beta),$$

is thus given by

$$S(\beta) \sim \int_0^{\pi/2} \frac{dz}{\sqrt{1 - k^2 sin^2(z)}}$$

which has a logarithmic singularity at the critical point, that is $\alpha = 0$.

What has happened here is that the complexities of the Ising representation of the braid group and the Yang-Baxter equation have been miraculously pulled through onto a smaller and more manageable realisation. This has happened in such a way that the spectrum of the full model may be recovered. In general there is no obvious reason why different realisations should have such closely related spectra (unless they are actually isomorphic representations). It is intriguing to wonder if any of the multitude of other tensored space (and hence exponentially growing) representations occuring in statistical mechanics may be similarly mimicked.

MORE TO FOLLOW.