REPRESENTATION THEORY meets STATISTICAL MECHANICS

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15/4/08

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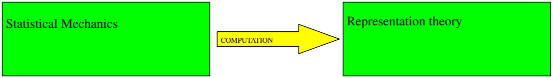
Everything at once!



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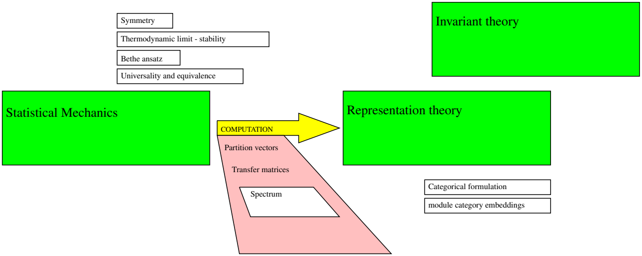

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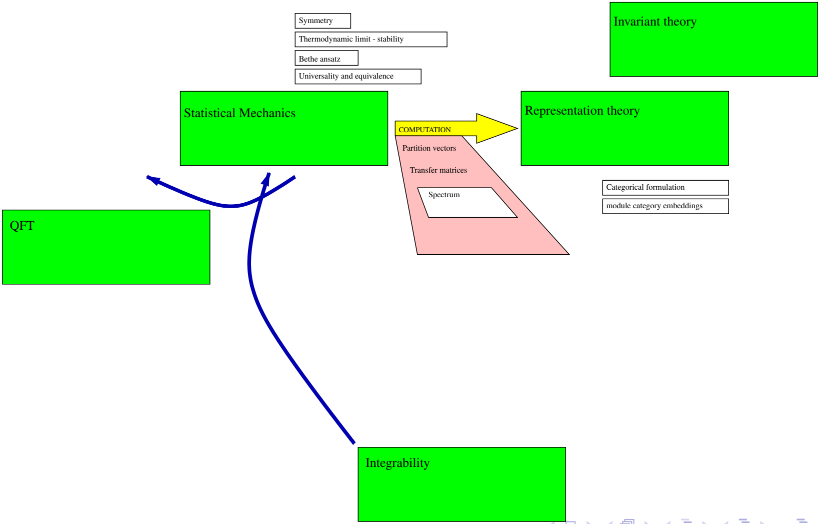

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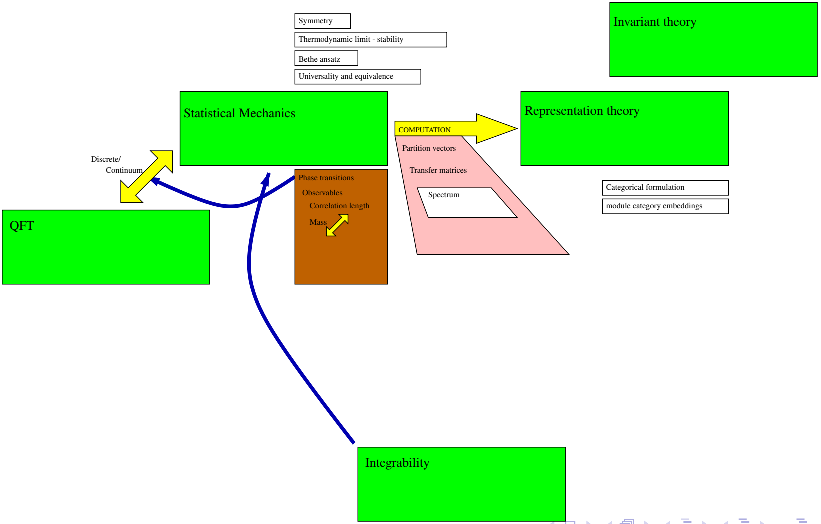

Aims

Compute observables in statistical mechanics
e.g. QCD mass gap

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Determine representation theory of Brauer algebra
e.g. representation theory of the symmetric group

Objectives

Talk aim: relate SM and RT in such a way that they significantly inform each other.
Our first job is to unpack the terms (in the title) sufficiently.

- (Micro course in) Statistical Mechanics
- Representation theory
- meets

Our summary of SM cannot be entirely superficial, or we won't have any concepts to pass over to the other side.
Allow 10 minutes.

Statistical mechanics tries to model bulk properties
of large collections of interacting microscopic components,
given a model for the microscopic interactions
(such as might come from electromagnetics and quantum mechanics).

Note that a very complete understanding of quantum physics (a complete string theory, say) is of little practical use to humans. It is not in the business, for example, of telling us that ice will melt.[1] This is not a reductive but a cooperative phenomenon.

We ignore microdynamics
instead say probability of finding system in equilibrium in microstate s
depends only on 'energy' $H(s)$ of that state, and 'temperature'$\propto 1/\beta$

(no time for heat-bath arguments here).

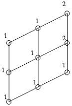
$$P(s) = \frac{e^{\beta H(s)}}{Z_H(\beta)} \qquad where \qquad Z = \sum_s e^{\beta H(s)}$$

$$\langle \mathcal{O} \rangle = \sum_s \mathcal{O} e^{\beta H(s)} / Z$$

[1] Never mind what consciousness is, say.

For now, fine details of microscopic interactions are unimportant. We can stylise the degrees of freedom of our ‘atoms’, and their interactions.

Let possible state of each of our atoms be labelled by
$\underline{Q} := \{1, 2, .., Q\}$
and let them interact, pairwise, if they are sufficiently close.

Let adjacency be determined by adjacency on a graph. 
(In practice this graph would have very special properties, but it is convenient to sustain this level of generality for now.)

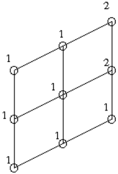
Notation: S, T sets, $\qquad T^S := hom(S, T)$

For each Q and graph G there is a Q-state 'Potts Hamiltonian':

$$H : \underline{Q}^{V_G} \rightarrow \Re \quad (1)$$

$$\sigma \mapsto \sum_{(i,j)\in E_G} \delta_{\sigma(i),\sigma(j)} \quad (2)$$

Case $Q = 2$ is Ising Model on G.

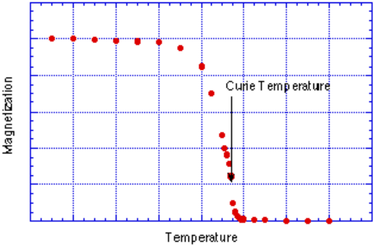
E.g. 

has $H = 9$

$$Z(\beta) = \sum_{\sigma \in \underline{Q}^{V_G}} \exp(\beta \sum_{\{i,j\} \in E_G} \delta_{\sigma(i),\sigma(j)})$$

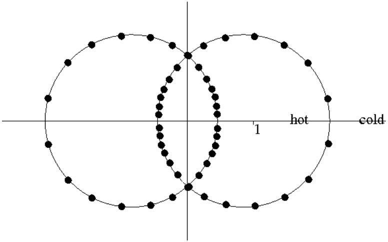
This is the Potts model partition function.
Idea: ordered states have biggest H, so biggest individual weight;
but many more, typically disordered, states give lower H values.
The winner in this ENERGY/ENTROPY battle for $\langle \mathcal{O} \rangle$ will depend on β (inverse temperature).
This seems roughly right. How good is it?

Our Z is a polynomial in e^{β}
but we need to model things like:

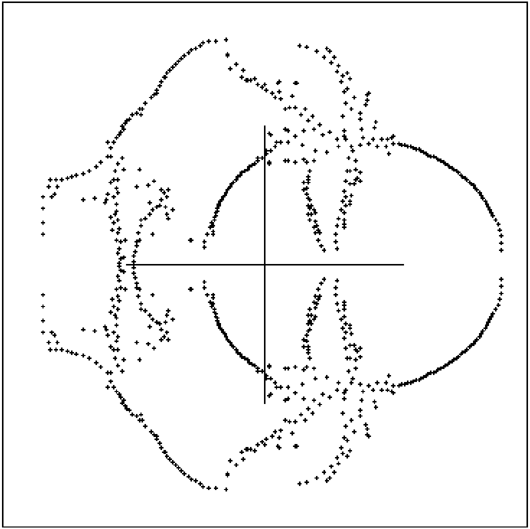


Very hard to measure close to Curie point experimentally
(critical slowing down),
but this result on Avogadro's number of atoms is best modelled by something non-analytic in the thermodynamic limit...

On a finite square grid the zeros of Z might be distributed like

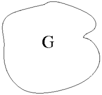


while in the limit these become continuous distributions, pinching the real axis at the phase-transition point.
We are interested in computational formalism rather than results today, but a couple more interesting finite lattice cases follow.



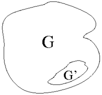
It looks like it works.
How compute?

Fixing Q and H, we have a polynomial for each graph:

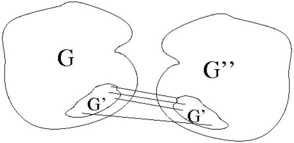


$\rightsquigarrow Z_G$

Introduce *relative* Z: 'partition vector'

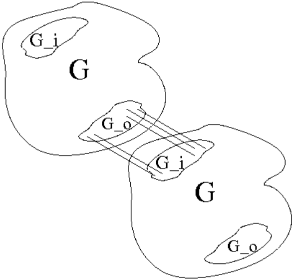


$\rightsquigarrow$ fix configuration $s' \in \underline{Q}^{V_{G'}}$ on subset of vertices $V_{G'}$
call this $(Z_G)_{s'}$.
Vector $Z_{G|G'} := ((Z_G)_{s'})_{s' \in \underline{Q}^{V_{G'}}}$



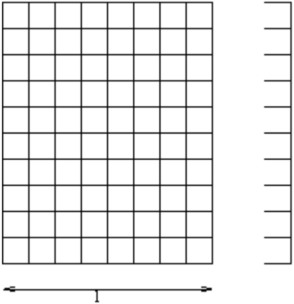
$$Z_{GG''} = \sum_{G'} Z_{G|G'} Z_{G''|G'}$$

Further



— data now organised as matrix: iterated composition.

Example: simple 2D crystal lattice:



$$Z = \langle | T^l | \rangle$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing T — and stability with respect to this growth too.)

Typical observable is *correlation function*: dependence of correlation between states of 2 separated atoms on separation
— normally exponential with some decay rate ‘correlation length’, that can depend on temperature.

So $Z \sim T^l$.
But T is $+$ve; so Peron-Frobenius theorem applies;
so large l limit controlled by largest eigenvalue of T.
— gap between this and next (or appropriate) lower eigenvalue determines a correlation length (and so on).

Upshot: want spectrum of T.

(See later for some necessary refinements, such as lateral thermodynamic limit.)

Computing spectrum T hard (integrable or not).
Can sometimes express

$$T = R(t)$$

representation matrix of (β-dependent) element t of some algebra, in some big representation R. (No time to explain why — not always the case.)
(NB still holding lateral graph fixed here — will need a new algebra for each larger lateral size.)

Idea: decompose

$$R = +_i R_i$$

(R_i smaller representations) gives very helpful block diagonalisation.
Helps computationally. Also helps physically — labels i label correlations!
(Masses in Field Theory.)

TO DO: Universality; Equivalence of models; Examples; dichromatic polynomials; Effect of Phase Transition; Connection to QFT; lateral thermodynamic limit;...
...quantum case (e.g. quantum spin chain); quantum group; renormalisation group; fusion; boundary conditions;...
...but anyway, we are interested now in the Representation Theory of the Transfer Matrix algebra.

What is the TMA?
Depends on the model.

What does the physical context tell us about the TMA?

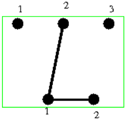
- TMA is sequence of algebras including the lateral thermodynamic limit.
- labels for simple modules should be associated to correlation lengths (and hence have some metricity)
  coherently through the whole sequence
  (Once an observable is defined, it makes sense irrespective of the size of the system.)
  ...suggests functors between module categories for algebras in sequence.

  Core properties cf. weight theory and invariant theory in Lie theory.

Category theory construction

$m, n \in \mathbb{N}_0$ $\quad\quad$ $\underline{m} \coprod \underline{n} := \underline{m} \times \{1\} \cup \underline{n} \times \{0\}$

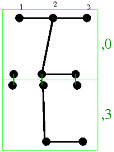
$\mathbb{P}(S)$ partitions of S

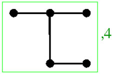
e.g. 

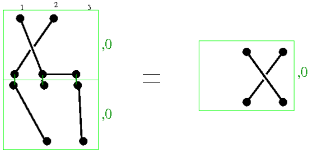
$\in \mathbb{P}(\underline{3} \coprod \underline{2})$ $\quad\quad$ $\{\{(1,1)\}, \{(2,1), (1,0), (2,0)\}, \{(3,1)\}\}$

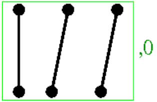
Consider triple $\quad\quad$ $C_{\mathbb{P}} = (\mathbb{N}_0, hom_{\mathbb{P}}(-,-), *)$

$hom_{\mathbb{P}}(m, n) = \mathbb{P}(\underline{m} \coprod \underline{n}) \times \mathbb{N}_0$

E.g. $A * B =$ 

$=$ 



$C_{\mathbb{P}}$ category, with $1_n =$ 

(this is case $n = 3$).

K a ring
$KC_{\mathbb{P}}$ K-linear category
$\delta \in K$, define relation $\sim_\delta$ on $Khom_{\mathbb{P}}(m,n)$

$$A \sim_\delta B \qquad \textit{if} \qquad \delta^{A_2}(A_1, 0) = \delta^{B_2}(B_1, 0)$$

This is congruence, so for each δ, quotient

$$C_{\mathbb{P}} = (\mathbb{N}, Khom_\delta(-,-), *)$$

K-finite category.
'Partition category', $End(n) = hom(n,n)$ is n-th partition algebra, P_n.
NB, $Khom_\delta(m,n)$ is left End(m) right End(n)-bimodule
so get lots of functors between *module* categories.

$$\begin{aligned} F : P_n - mod &\rightarrow P_m - mod &(3)\\ M &\mapsto hom(m,n) \otimes_{P_n} M &(4)\end{aligned}$$

(if δ a unit, the ascending ones are full embeddings — thermodynamic limit)

Write $hom^l(m,n)$ for the image $*(hom(m,l) \times hom(l,n))$ in $hom(m,n)$. This is a sub-bimodule. Easy to see that

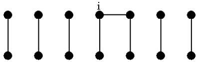
$$hom(n,n)/hom^{n-1}(n,n) \cong KS_n$$

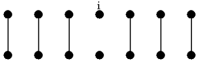
Thus simple modules of P_n indexed (for δ a unit) by simple modules of collection of symmetric groups.
$hom^l(n,l)/hom^{l-1}(n,l)$ is left P_n right S_l module, and projective as S_l-module, so

$$M(\lambda) = hom^l(n,l)/hom^{l-1}(n,l) \otimes_{S_l} \Delta(\lambda)$$

is cellular inflation of S_l cell module, hence P_n cell module.

Physics: Set

$(i : i{+}1)$:= 

$(i.)$:= 

$\in P_n$

(these have $n = 7$). Then

$$t = c \prod_i (1 + v(i.)) \prod_i (v + (i : i{+}1))$$

where $v = \frac{x-1}{\delta}$, c scalar,
is t for 2D crystal lattice, δ^2-state Potts model. (Now choose a representation.)

Subcategories

$hom_{\mathbb{B}}(m,n) \subset hom_{\mathbb{P}}(m,n)$ — subset such that partition part is a *pair partition*
NB closed under *:
$C_{\mathbb{B}} = (\mathbb{N}, hom_{\mathbb{B}}(-,-), *)$
δ-quotient:
Brauer category/subalgebra.

$hom_{\mathbb{T}}(m,n) \subset hom_{\mathbb{P}}(m,n)$ — subset such that partition part is *planar*:
Temperley-Lieb subcategory.

(Aside: Gram matrices for contravariant forms on cell modules give access to simple modules — and can sometimes be calculated by integrable methods...)

Schur-Weyl duality and representations

What is the representation R?

For $N \in \mathbb{N}$ let $V = K\{e_1, e_2, ..., e_N\}$. Then we have the following collection of pairs of commuting (indeed centralizing) actions:

$$\begin{tikzcd}
GL(V) \arrow[dr] & & S_n \arrow[dl] \arrow[d, hook] \\
O(V) \arrow[u, hook] \arrow[r] & V^{\otimes n} & B_n(N) \arrow[l] \arrow[d, hook] \\
S_N \arrow[u, hook] \arrow[ur] & & P_n(N) \arrow[ul]
\end{tikzcd}$$

Fix a field k. Then recall that Vect is the category of k-spaces. For G a group and V a G-module then $\mathrm{Vect}_{G,V}$ is the subcategory with objects

$$k, V, V^2, V^3, \ldots$$

and homs commuting with the diagonal action of G, i.e.

$$f : V^m \to V^n$$

such that

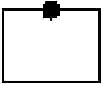
$$f \sigma v = \sigma f v \qquad \forall \sigma \in G$$

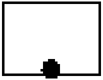
This inherits the tensor structure from Vect.

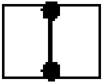
The following functor

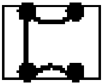
$$F_N : C_{\mathbb{P}(N)} \to \mathrm{Vect}_{S_N, V}$$

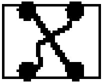
is a representation of $C_{\mathbb{P}}$. We begin by giving the images of some elements (in case $N = 2$):

$\hom_{\mathbb{P}}(\underline{1}, \underline{0}) \ni$  $\mapsto \begin{pmatrix} 1 & 1 \end{pmatrix}$

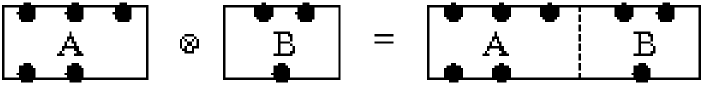
$\hom_{\mathbb{P}}(\underline{0}, \underline{1}) \ni$  $\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\hom_{\mathbb{P}}(\underline{1}, \underline{1}) \ni$  $\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hom_{\mathbb{P}}(\underline{2}, \underline{2}) \ni$  $\mapsto \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$

$$\hom_{\mathbb{P}}(\underline{2},\underline{2}) \ni$$  $$\mapsto \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

Note that all the images are invariant under the appropriate S_2 action. We conclude by noting that $C_{\mathbb{P}}$ is a tensor category with

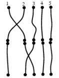


and that the examples given above (respectively their direct generalisations to other N) generate.

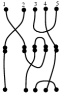
Thus we have constructed representation for all the partition algebras simultaneously. This gives the representation R for any given n.

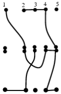
Let $T = V^{\otimes n}$ with $V = K\{e_1, e_2, ..., e_N\}$ for some K.
For physics $K = \mathbb{C}$ but often a lattice over both.

$$\begin{tikzcd}
GL(V) \arrow[dr] \arrow[d, phantom, "\cup"] & & S_n \arrow[dl, "1927"'] \arrow[d, phantom, "\cap"] \\
O(V) \arrow[r] \arrow[d, phantom, "\cup"] & T & B_n(N) \arrow[l] \arrow[d, phantom, "\cap"] \\
S_N \arrow[ur] & & P_n(N) \arrow[ul, "1991"]
\end{tikzcd}$$



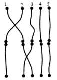
(significant Leeds contribution)

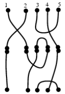


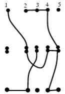


Let $T = V^{\otimes n}$ with $V = K\{e_1, e_2, ..., e_N\}$ for some K.

$$\begin{tikzcd}
GL(V) \arrow[ddr] & & S_n \arrow[ddl, "1927"'] & T_n(N) \arrow[ddl, hook] \\
\cup & & \cap & \\
O(V) \arrow[r] & T & B_n(N) \arrow[l] & \\
\cup & & \cap & \\
S_N \arrow[uur] & & P_n(N) \arrow[uul, "1991"] &
\end{tikzcd}$$







$T_n(N)$, $B_n(N)$ and $P_n(N)$ all make sense with N replaced by an indeterminate scalar (see later).

Finite dimensional algebra over K

Aims:

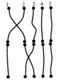
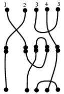
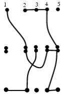
1. classify indecomposable projectives (simples)
2. describe blocks
3. give projective decomposition matrices / socle series /...

Often there are intermediate modules
(cell, standard, Specht, over $\mathbb{Z}$)

$$A = \Sigma_\lambda dim(\Delta_\lambda)\Delta_\lambda = \oplus_\lambda dim(L_\lambda)P_\lambda$$

Combinatorics / Physics
Diagram algebra methods (see later).

$$
\begin{tikzcd}
GL(V) \arrow[d, phantom, "\cup"] \arrow[dr] & & S_n \arrow[dl, "1927"'] \arrow[d, phantom, "\cap"] & & T_n(N) \arrow[dll, hook'] \\
O(V) \arrow[r] \arrow[d, phantom, "\cup"] & T & B_n(N) \arrow[l] \arrow[d, phantom, "\cap"] & & \\
S_N \arrow[ur] & & P_n(N) \arrow[ul, "1991"] & &
\end{tikzcd}
$$

The physical background

- Roughly: Objects on the left are "symmetries" of lattice models; objects on the right are building blocks ("Transfer Matrix Algebras") of lattice models.
- $T_n(N)$ is "Transfer Matrix Algebra" for many important 2D lattice models.
  Includes ones which satisfy criterion of realism — e.g. predict phase transitions, observable critical exponents etc. (no time to show how); and integrable ones (YBE — it is quotient of braid group).
  Physically vital question: what is corresponding algebra for 3D (4D) lattice models?
  Answer: a subalgebra of P_n (which is 'high D' case).

For X a set, let Q^X be the set of functions from X to $\{1, 2, ..., Q\}$ — the set of *colourings* of X from Q colours. For each graph G (vertex set V_G, edge set E_G) there is a function

$$Z_G(\beta) = \sum_{\sigma \in Q^{V_G}} \exp(\beta \sum_{\{x,y\} \in E_G} \delta_{\sigma(x),\sigma(y)})$$

This is the Potts model partition function.

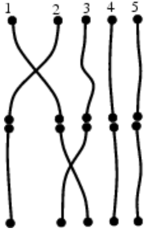
What we did

Armed with the physical picture...

- For TL: determined 1-3 in characteristic zero.
  (char.p done by Arkhipov)
- discovered partition algebra
  determined 1-3 in characteristic zero.
  (char.p defeated all comers so far)
- For Brauer: (1 known since Brauer, Brown 50s) PPM with Cox, Devisscher, determined blocks in characteristic zero;
  3 is interesting problem.

Return

S_n:



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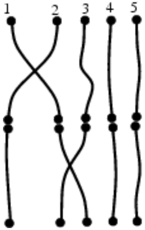

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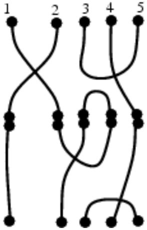

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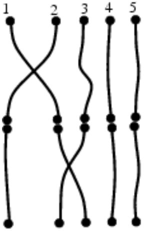
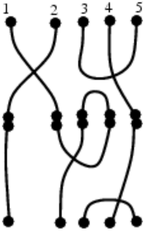
S_n: 

$B_n(N)$: 

$P_n(N)$: