

A large iceberg floating in the ocean. The top part of the iceberg is above the water surface, and the much larger bottom part is submerged. The sky is blue with some clouds, and the water is a deep blue. A small ship is visible on the horizon line.

# REPRESENTATION THEORY

inspired by

# COMPUTATIONAL STATISTICAL MECHANICS

Paul Martin

15/6/08

# Table of contents

Preamble

Statistical mechanics

Transfer matrix algebra

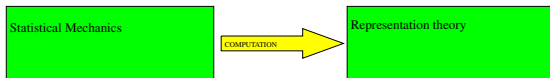
Representation theory

Decomposition matrices

Category theory

Schur-Weyl duality

# Everything at once!



## Overview/Aims/Objectives

Aim: Give a quick flavour of the physical setting which seems to be helpful in my research in Representation Theory (and how it helps).

relate Representation Theory and Statistical Mechanics in such a way that they significantly inform each other.

Leeds has strong groups in:

Algebra

Integrable systems

Polymers

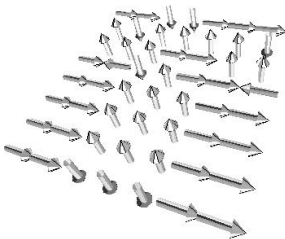
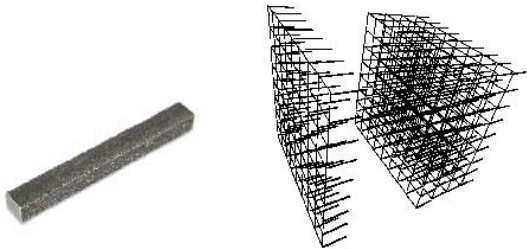
Logic

and several others all directly (or potentially directly) relevant to this activity.

**Statistical mechanics** models bulk properties of large collections of interacting microscopic components given a model for microscopic interactions



**Statistical mechanics** models bulk properties of large collections of interacting microscopic components given a model for microscopic interactions





Idea: Ignore microdynamics

probability of finding system in equilibrium in microstate  $s$   
depends on

'energy'  $H(s)$  of state, and 'temperature'  $\propto 1/\beta$

$$P(s) = \frac{e^{\beta H(s)}}{Z_H(\beta)} \quad \text{where} \quad Z = \sum_s e^{\beta H(s)}$$

Expected result of observation: weighted average

$$\langle O \rangle = \sum_s O \frac{e^{\beta H(s)}}{Z}$$



Idea: Ignore microdynamics

probability of finding system in equilibrium in microstate  $s$   
depends on

'energy'  $H(s)$  of state, and 'temperature'  $\propto 1/\beta$

$$P(s) = \frac{e^{\beta H(s)}}{Z_H(\beta)} \quad \text{where} \quad Z = \sum_s e^{\beta H(s)}$$

Expected result of observation: weighted average

$$\langle O \rangle = \sum_s O \frac{e^{\beta H(s)}}{Z}$$

What is  $H$ ?...

Use stylised degrees of freedom of ‘atoms’,  
and interactions.

Possible state of each atom labelled by  
 $Q := \{1, 2, \dots, Q\}$

$\mathcal{S}$ : Set of all possible states of whole system.

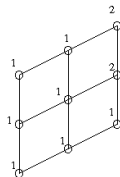
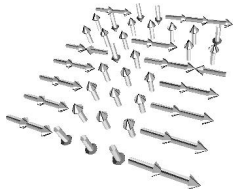
let atoms interact, pairwise, if sufficiently close.  
(adjacency described by adjacency on graph  $G$ )

$$H : \mathcal{S} \rightarrow \mathfrak{R}$$

Example

$$H : \sigma \mapsto \sum_{(i,j) \in E_G} \delta_{\sigma(i), \sigma(j)}$$

...bigger probability for ordered states.



$$H = 9$$

Altogether

$$\langle \mathcal{O} \rangle = \frac{1}{Z(\beta)} \sum_{\sigma \in \mathcal{S}} \mathcal{O} \exp(\beta \sum_{\{i,j\} \in E_G} \delta_{\sigma(i), \sigma(j)})$$

'Ising/Potts model'

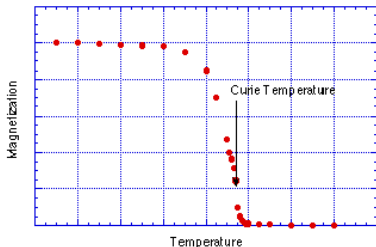
Idea:

ordered states have biggest  $H$ , so biggest individual weight  $e^{\beta H}$ ;  
but many more, typically disordered, states give lower  $H$  values.

The winner in this **ENERGY/ENTROPY** battle for  $\langle \mathcal{O} \rangle$   
will depend on  $\beta$  (inverse temperature).

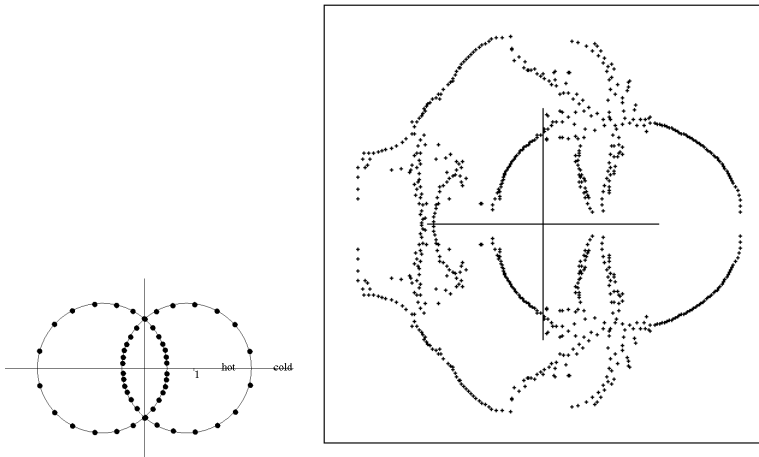
This seems roughly right. How good is it?

Our  $Z$  is polynomial in  $e^\beta$   
but recall need to model things like:



Very hard to measure close to Curie point experimentally  
(critical slowing down),  
but this result on **Avogadro's number** of atoms best modelled by  
something non-analytic in thermodynamic limit...

On finite grid, complex zeros of  $Z$  distributed like

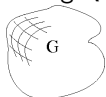


**in the limit** these become continuous distributions, pinching real axis at phase-transition point.

So it works!  
How compute?

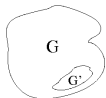
# Computation

Fixing  $Q$  and  $H$ , we have a polynomial  $Z$  in  $e^\beta$  for each graph  $G$ :



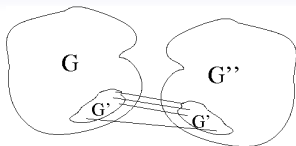
$$\rightsquigarrow Z_G = \sum_{s \in \mathcal{S}} e^{\beta H}$$

Introduce *relative*  $Z$ : 'partition vector'



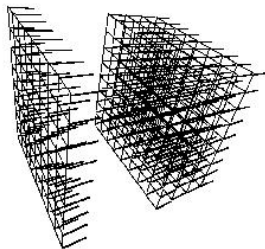
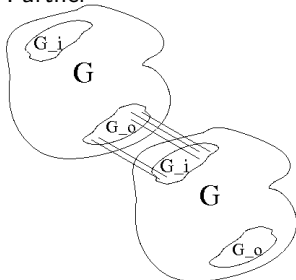
$\rightsquigarrow$  fix configuration  $s' \in \mathcal{S}'$  on subset of vertices  $V_{G'}$ ,  
sum only over configurations in  $\mathcal{S}$  having this subconfiguration,  
call this partial sum  $(Z_G)_{s'}$ .

Vector  $Z_{G|G'} := ((Z_G)_{s'})_{s' \in \mathcal{S}'}$



$$Z_{GG''} = \sum_{G'} Z_{G|G'} Z_{G''|G'}$$

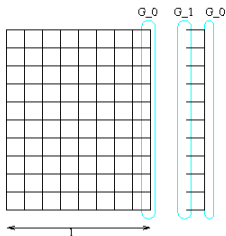
Further



— data now organised as matrix: iterated composition.



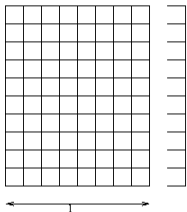
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

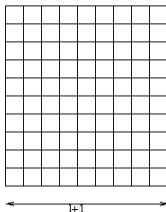
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

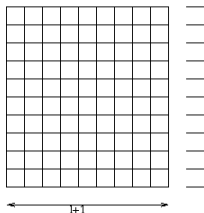
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

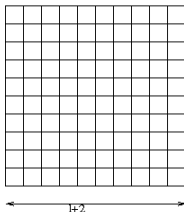
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

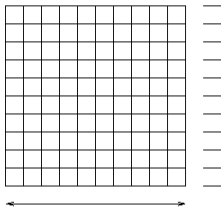
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

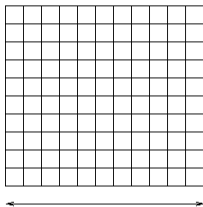
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

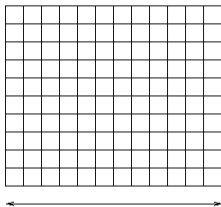
Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)

Example: simple 2D crystal lattice:



$$Z = \langle |T^l| \rangle = \sum_i \alpha_i \lambda_i^l = \alpha_0 \lambda_0^l \left( 1 + \sum_{i>0} \frac{\alpha_i}{\alpha_0} \left( \frac{\lambda_i}{\lambda_0} \right)^l \right)$$

(Note that this grows the graph transversely but not laterally — will eventually need a separate growth in lateral direction, thus changing  $T$  — and stability with respect to this growth too.)



In summary,  $Z$  computed by computing  $T^l$ .

But  $T$  is +ve;

so Peron-Frobenius theorem applies;

so large  $l$  limit of Free Energy Density

$$\frac{1}{l.m} \ln(Z)$$

controlled by largest eigenvalue  $\lambda_0$  of  $T$ .

— gap between this  $\lambda_0$  and next (or appropriate) lower eigenvalue determines a correlation length<sup>1</sup> (and so on).

**Upshot:** want spectrum of  $T$ .

---

<sup>1</sup>Typical observable is *correlation function*: dependence of correlation between states of 2 separated atoms **on** separation

— normally exponential with some decay rate 'correlation length', that can depend on temperature.



Computing spectrum  $T$  hard.

Computing spectrum  $T$  hard.

Can sometimes express

$$T = R(t)$$

representation matrix of ( $\beta$ -dependent) element  $t$  of some algebra  $A$ ,  
in some big representation  $R$ .

Computing spectrum  $T$  hard.

Can sometimes express

$$T = R(t)$$

representation matrix of ( $\beta$ -dependent) element  $t$  of some algebra  $A$ ,  
in some big representation  $R$ .

(NB still holding lateral graph fixed here — will need a new algebra for  
each larger lateral size.)



Computing spectrum  $T$  hard.

Can sometimes express

$$T = R(t)$$

representation matrix of ( $\beta$ -dependent) element  $t$  of some algebra  $A$ ,  
in some big representation  $R$ .

(NB still holding lateral graph fixed here — will need a new algebra for  
each larger lateral size.)

Idea: decompose

$$R = \oplus_j R_j$$

( $R_j$  smaller representations) gives very helpful block diagonalisation.



Computing spectrum  $T$  hard.

Can sometimes express

$$T = R(t)$$

representation matrix of ( $\beta$ -dependent) element  $t$  of some algebra  $A$ ,  
in some big representation  $R$ .

(NB still holding lateral graph fixed here — will need a new algebra for  
each larger lateral size.)

Idea: decompose

$$R = \sum_i R_i$$

( $R_i$  smaller representations) gives very helpful block diagonalisation.

Helps computationally.

Also helps physically — labels  $i$  label correlations!

(Masses in Field Theory.)

TO DO: Universality; Equivalence of models; Examples; dichromatic polynomials; Effect of Phase Transition; Connection to QFT; lateral thermodynamic limit;...

...quantum case (e.g. quantum spin chain); quantum group; renormalisation group; fusion; boundary conditions;...

TO DO: Universality; Equivalence of models; Examples; dichromatic polynomials; Effect of Phase Transition; Connection to QFT; lateral thermodynamic limit;...

...quantum case (e.g. quantum spin chain); quantum group; renormalisation group; fusion; boundary conditions;...

...but anyway, we are interested now in the Representation Theory of the **Transfer Matrix algebra**.

What is the TMA?

Depends on the model.





What does the physical context tell us about the TMA?

- TMA is sequence of algebras including the lateral thermodynamic limit.
- labels for simple modules should be associated to correlation lengths (and hence have some metricity) coherently through the whole sequence (Once an observable is defined, it makes sense irrespective of the size of the system.)  
...suggests functors between module categories for algebras in sequence.

Core properties cf. weight theory and invariant theory in Lie theory.

## Summary:

- Statistical Mechanics gives us lots of algebras to study together with lots of representations for each.  
(As we'll see, if there is time.)
- Some of these algebras are new; some are established objects for study.
- The new representations give us new ways to study these algebras. In some cases they give us access to a rather complete picture. What does this mean?...

## Recall group/algebra representation theory

Group  $G$ :

$$\rho : G \rightarrow GL(V)$$

is representation if it preserves multiplication.

(example of algebra (i.e. associative algebra) is group algebra over field  $(\mathbb{C})$  — space with basis  $G$  and multiplication linearly extending  $G$ -multiplication)

- Given rep  $\rho$ , conjugating all images by invertible matrix  $M$  gives another 'isomorphic' rep  $\rho_M$ .
- Given  $\rho, \rho'$ , then  $\rho \oplus \rho'$  (corresponding to matrix direct sum) is also rep.  
but one with invariant subspaces.
- Irreducible representation  $L_i$  is one with no proper invariant subspace.

For any  $t$

$$\text{spec}(\rho \oplus \rho'(t)) = \text{spec}(\rho(t)) \cup \text{spec}(\rho'(t))$$

$$\text{spec}(\rho(t)) = \text{spec}(\rho_M(t))$$

$$\text{spec}(R(t)) \subseteq \cup_i \text{spec}(L_i(t))$$

Consider  $\text{spec}$  to encode both spectrum and multiplicities (i.e. as characteristic polynomial), then even

$$\text{spec}(R(t)) = \cup_i m_i(R) \text{spec}(L_i(t))$$

multiplicities  $m_i(R)$  indep of  $t$ .

Called simple composition multiplicities of  $R$ .

Big challenge RT1: classify representations  $L_i$  up to isomorphism.

Big challenge SM1: classify spectrum of  $T$ .

We now see that these are closely related!

Finite dimensional algebra  $A$  is acted on by itself by (say) left multiplication

hence induces rep, called regular representation  $R$ .

Let  $R = \bigoplus_i P_i$  be complete direct-sum decomposition of  $R$ .

For group algebra over  $\mathbb{C}$  the sets  $\{P_i\}$  and  $\{L_i\}$  coincide up to isomorphism.

More generally, matrix

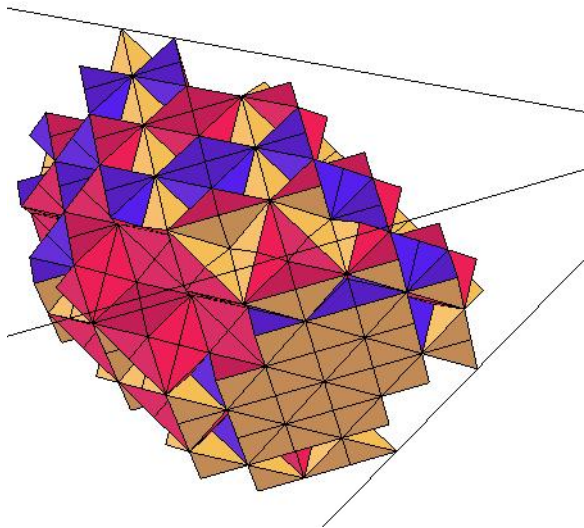
$$C = (m_i(P_j))_{ij}$$

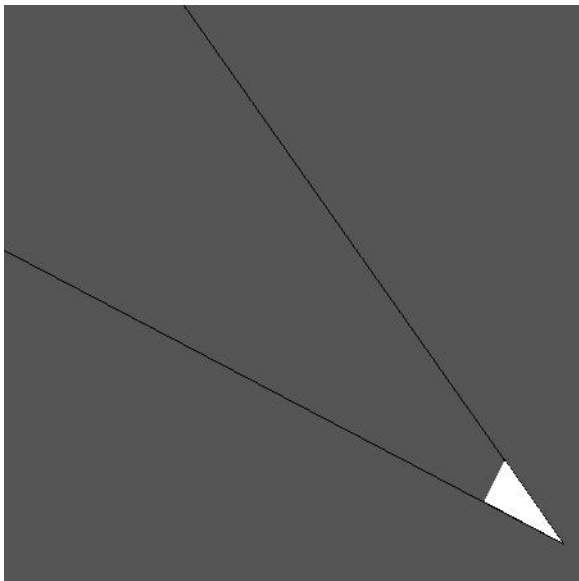
core data of algebra.

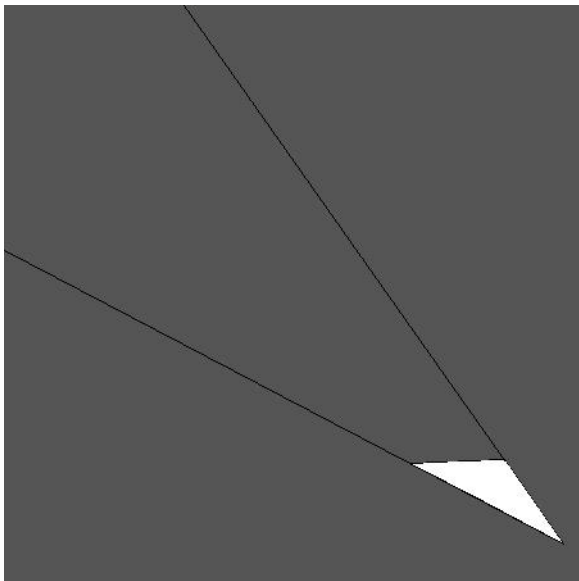
(For various reasons) index set for irreducible representations of  $A$  often a set of lattice points in some space ( $\mathbb{Z}$  in  $\mathbb{R}$ , say).

Thus for each point  $i$  there is subset of lattice points  $j$  s.t.  $C_{ij} \neq 0$ .

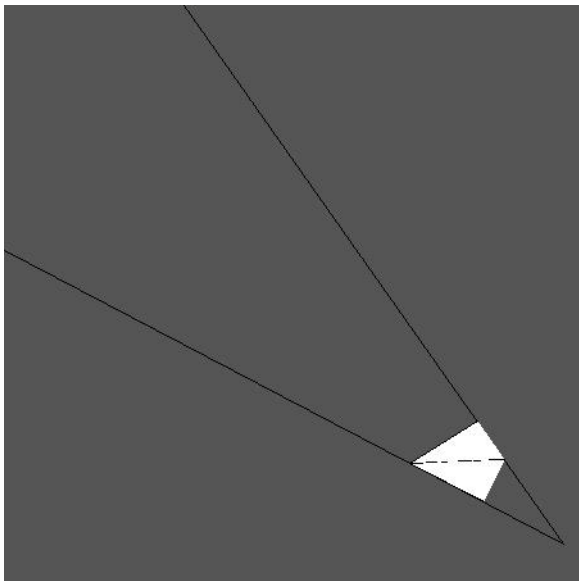
Shape of this subset in space is one way to visualise core data.

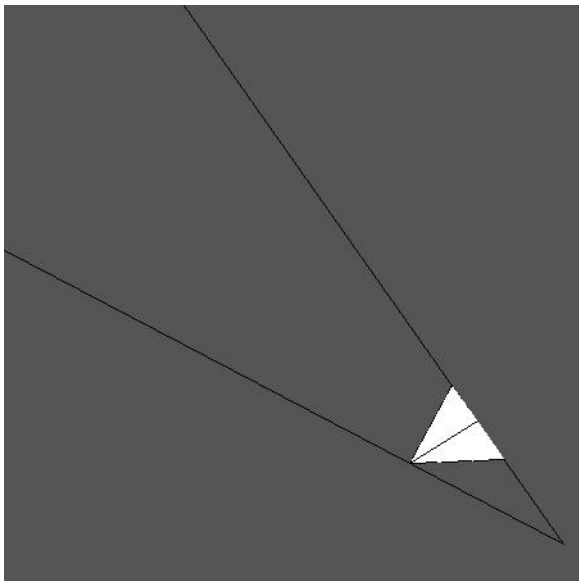


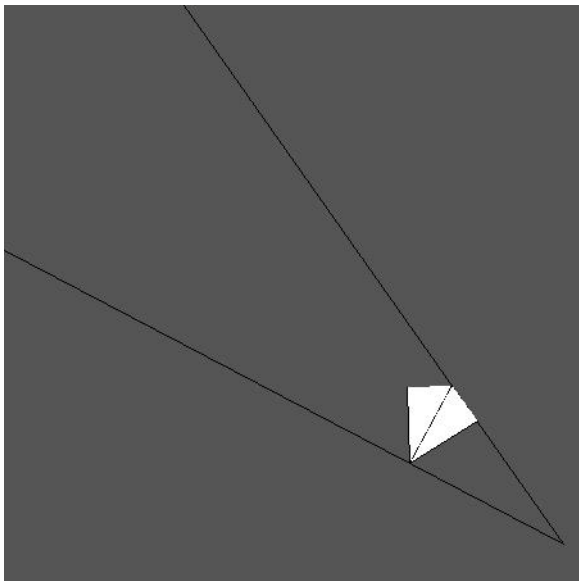




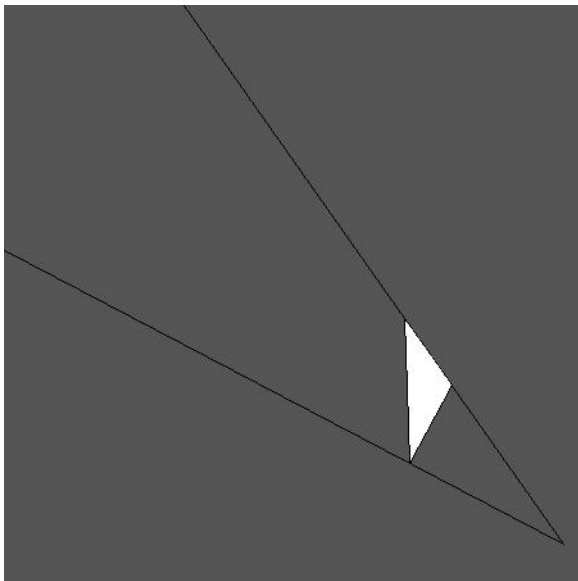


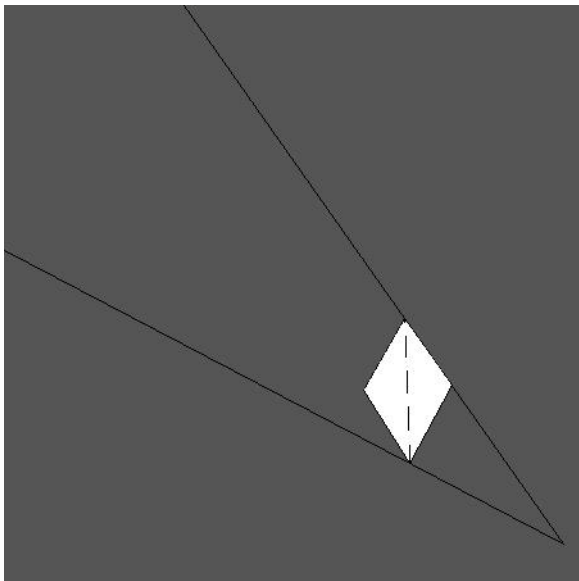


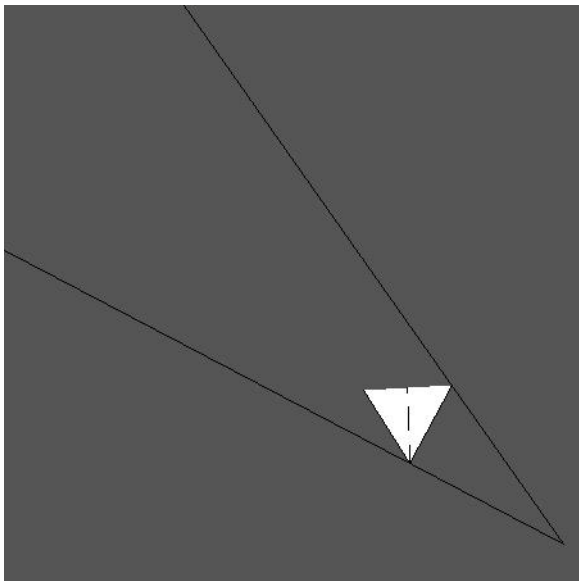


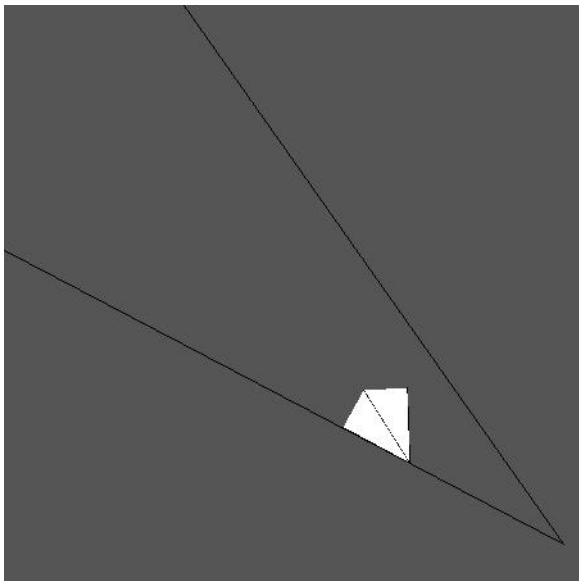


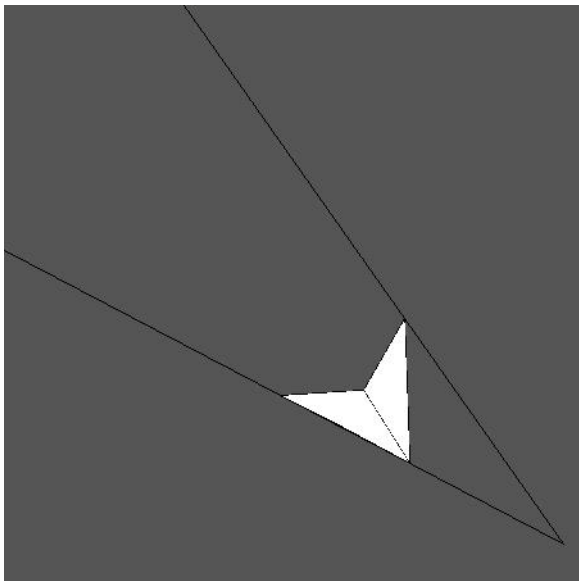
○○○

○○●○○  
○○○○○  
○○○

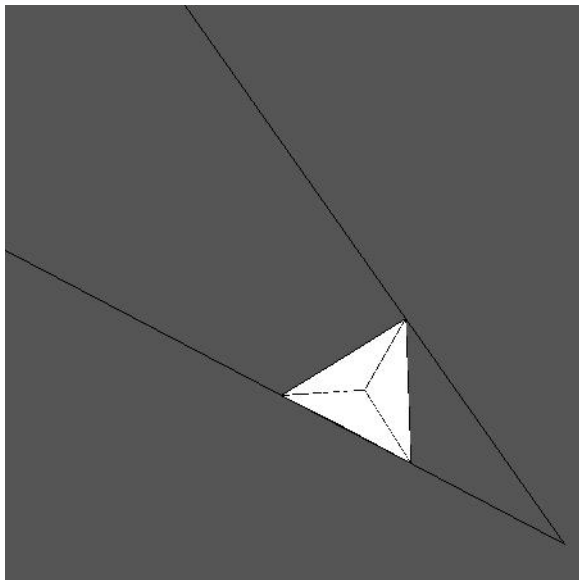


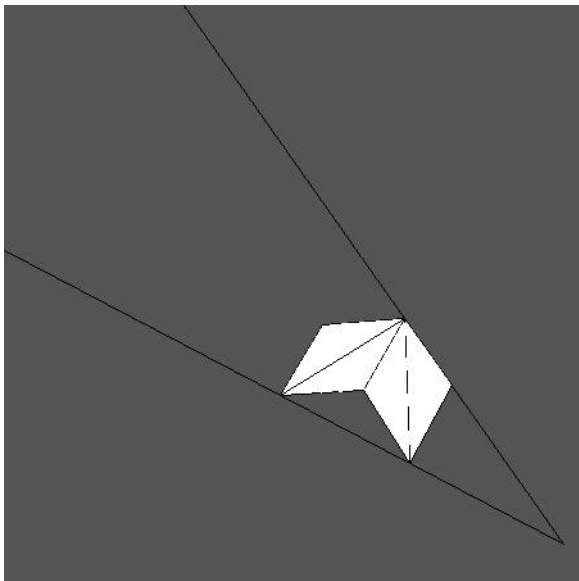


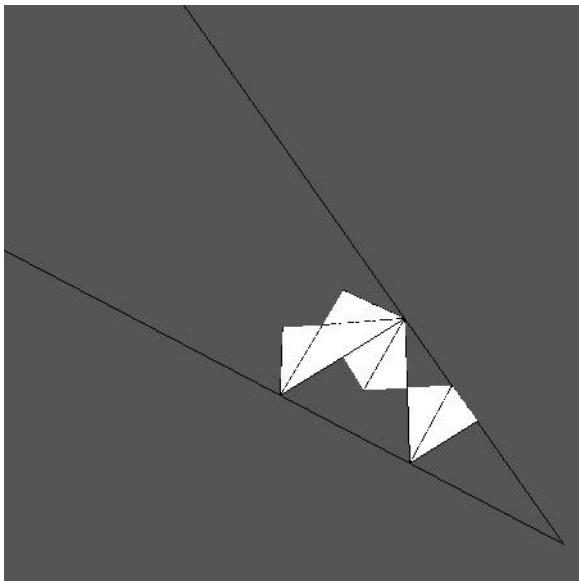


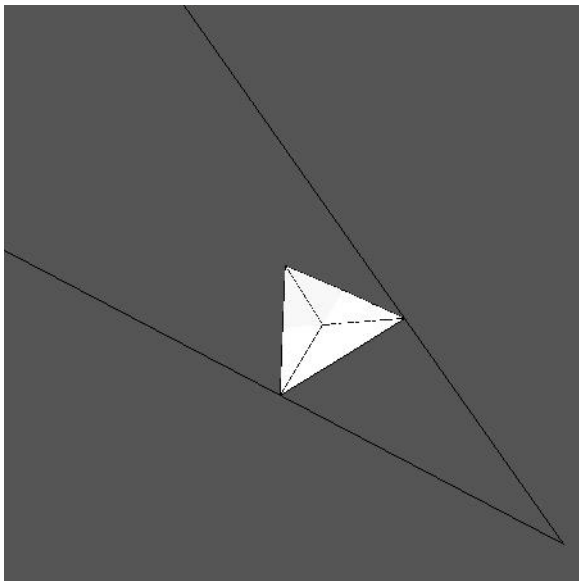


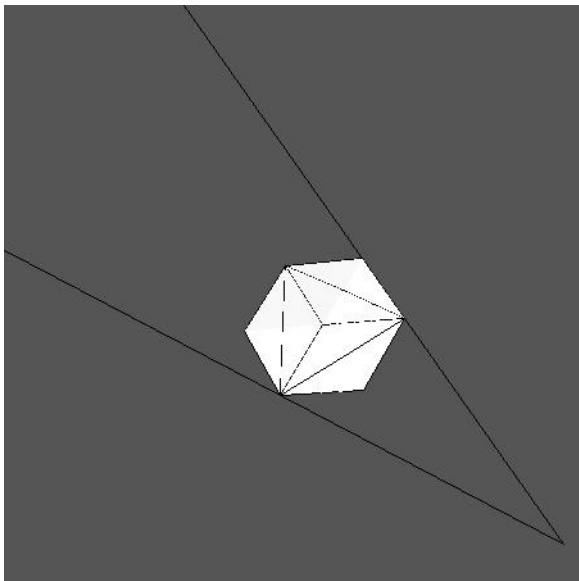


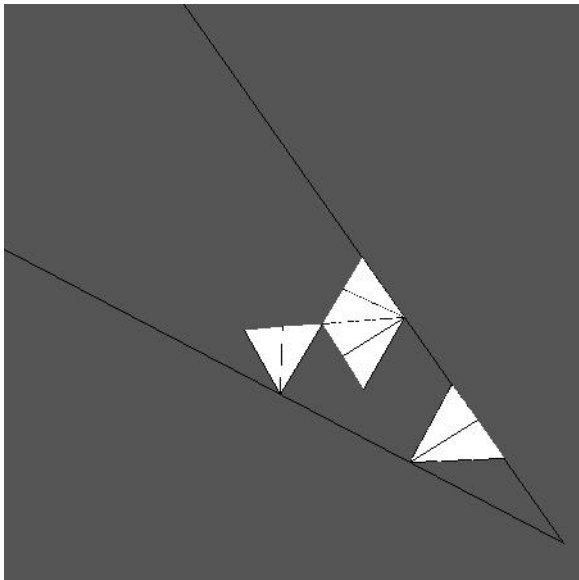


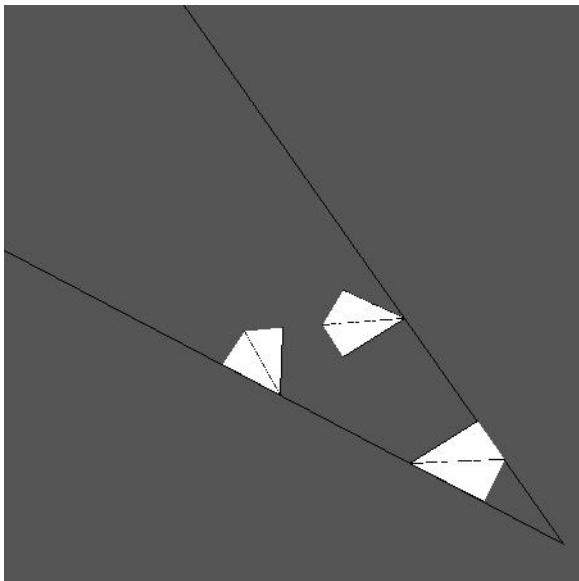


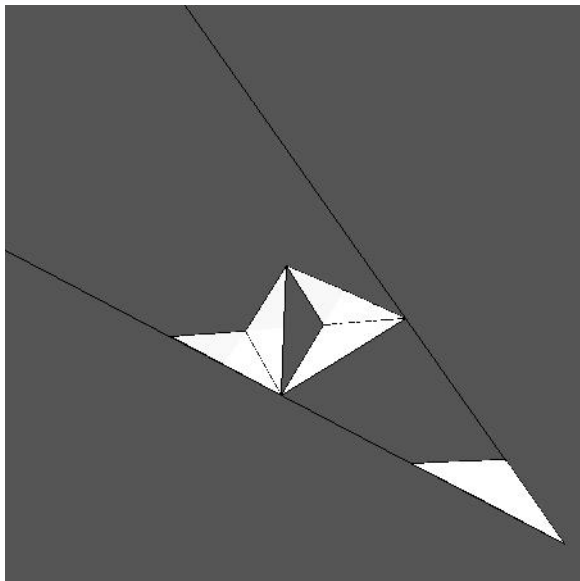




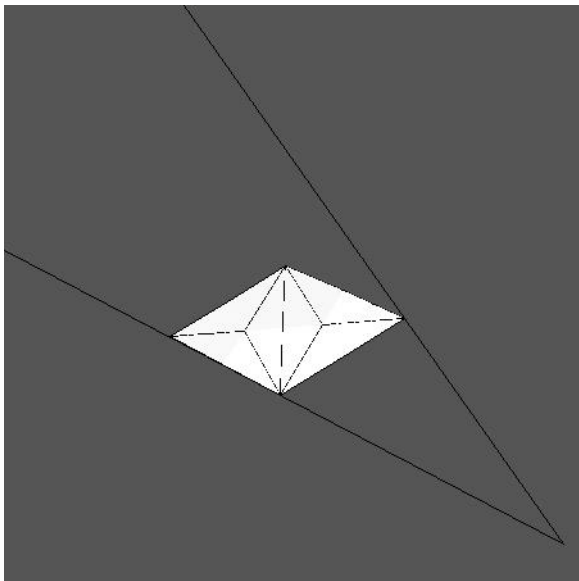


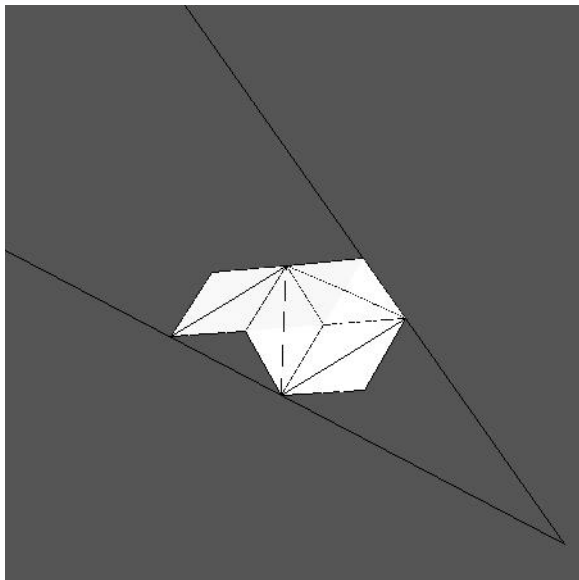


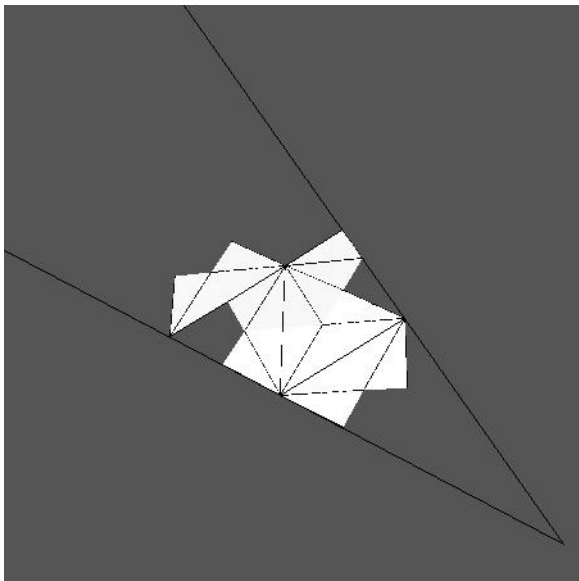












also see webpage...

To compute  $C$  we now go back to Statistical Mechanics.

Algebra  $A$  associated to model for given system — say  $10^{13}$  atoms wide.

(A bit more if it is an iceberg!)

Physically expect  $10^{13} + 1$  atom system similarly behaved.

...but different algebra of course.

so guess these algebras nicely related.

To be specific: group  $S_{10^{13}}$  of permutations of  $10^{13}$  objects site nicely in

$S_{10^{13}+1}$  — just don't perm the last atom.

unfortunately this relation rather weak in representation theory

not enough to unify physics OR determine irreducible representations.

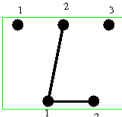
Need another relation — leads to algebras naturally embracing changing

$n$  — called linear categories. (no time for this now)

# Category theory construction

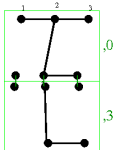
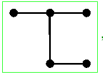
$$m, n \in \mathbb{N}_0 \quad \underline{m} \amalg \underline{n} := \underline{m} \times \{1\} \cup \underline{n} \times \{0\}$$

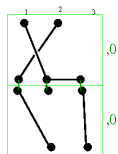
$\mathbb{P}(S)$  partitions of  $S$

e.g.   $\in \mathbb{P}(\underline{3} \amalg \underline{2}) \quad \{ \{(1, 1)\}, \{(2, 1), (1, 0), (2, 0)\}, \{(3, 1)\} \}$

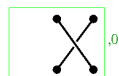
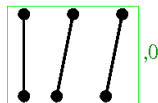
Consider triple  $\mathcal{C}_{\mathbb{P}} = (\mathbb{N}_0, \text{hom}_{\mathbb{P}}(-, -), *)$

$$\text{hom}_{\mathbb{P}}(m, n) = \mathbb{P}(\underline{m} \amalg \underline{n}) \times \mathbb{N}_0$$

E.g.  $A * B =$    $=$  



=


 $C_{\mathbb{P}}$  category, with  $1_n =$ 
(this is case  $n = 3$ ).

$K$  a ring

$K\mathbb{C}_{\mathbb{P}}$   $K$ -linear category

$\delta \in K$ , define relation  $\sim_{\delta}$  on  $K\text{hom}_{\mathbb{P}}(m, n)$

$$A \sim_{\delta} B \quad \text{if} \quad \delta^{A_2}(A_1, 0) = \delta^{B_2}(B_1, 0)$$

This is congruence, so for each  $\delta$ , quotient

$$\mathbb{C}_{\mathbb{P}} = (\mathbb{N}, K\text{hom}_{\delta}(-, -), *)$$

$K$ -finite category.

'Partition category',  $\text{End}(n) = \text{hom}(n, n)$  is  $n$ -th partition algebra,  $P_n$ .

NB,  $K\text{hom}_{\delta}(m, n)$  is left  $\text{End}(m)$  right  $\text{End}(n)$ -bimodule

so get lots of functors between *module* categories.

$$F : P_n - \text{mod} \rightarrow P_m - \text{mod} \quad (1)$$

$$M \mapsto \text{hom}(m, n) \otimes_{P_n} M \quad (2)$$

(if  $\delta$  a unit, the ascending ones are full embeddings — thermodynamic limit)



Write  $\text{hom}^l(m, n)$  for the image  $*(\text{hom}(m, l) \times \text{hom}(l, n))$  in  $\text{hom}(m, n)$ . This is a sub-bimodule. Easy to see that

$$\text{hom}(n, n)/\text{hom}^{n-1}(n, n) \cong KS_n$$

Thus simple modules of  $P_n$  indexed (for  $\delta$  a unit) by simple modules of collection of symmetric groups.

$\text{hom}^l(n, l)/\text{hom}^{l-1}(n, l)$  is left  $P_n$  right  $S_l$  module, and projective as  $S_l$ -module, so

$$M(\lambda) = \text{hom}^l(n, l)/\text{hom}^{l-1}(n, l) \otimes_{S_l} \Delta(\lambda)$$

is cellular inflation of  $S_l$  cell module, hence  $P_n$  cell module.

Physics: Set

$$(i : i+1) := \begin{array}{ccccccc} & & & i & & & \\ & & & \text{---} & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ | & | & | & | & | & | & | \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

$$(i.) := \begin{array}{ccccccc} & & & i & & & \\ & & & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ | & | & | & | & | & | & | \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \in P_n$$

(these have  $n = 7$ ). Then

$$t = c \prod_i (1 + v(i.)) \prod_i (v + (i : i+1))$$

where  $v = \frac{x-1}{\delta}$ ,  $c$  scalar,  
is  $t$  for 2D crystal lattice,  $\delta^2$ -state Potts model. (Now choose a representation.)

## Subcategories

$hom_{\mathbb{B}}(m, n) \subset hom_{\mathbb{P}}(m, n)$  — subset such that partition part is a *pair partition*

NB closed under  $*$ :

$$C_{\mathbb{B}} = (\mathbb{N}, hom_{\mathbb{B}}(-, -), *)$$

$\delta$ -quotient:

Brauer category/subalgebra.

$hom_{\mathbb{T}}(m, n) \subset hom_{\mathbb{P}}(m, n)$  — subset such that partition part is *planar*.

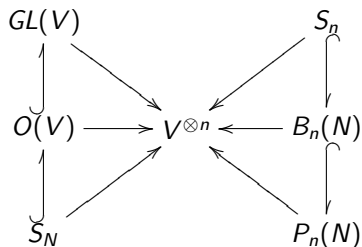
Temperley-Lieb subcategory.

(Aside: Gram matrices for contravariant forms on cell modules give access to simple modules — and can sometimes be calculated by integrable methods...)

# Schur-Weyl duality and representations

What is the representation  $R$ ?

For  $N \in \mathbb{N}$  let  $V = K\{e_1, e_2, \dots, e_N\}$ . Then we have the following collection of pairs of commuting (indeed centralizing) actions:



Fix a field  $k$ . Then recall that  $\text{Vect}$  is the category of  $k$ -spaces. For  $G$  a group and  $V$  a  $G$ -module then  $\text{Vect}_{G,V}$  is the subcategory with objects

$$k, V, V^2, V^3, \dots$$

and homs commuting with the diagonal action of  $G$ , i.e.

$$f : V^m \rightarrow V^n$$

such that

$$f\sigma v = \sigma f v \quad \forall \sigma \in G$$

This inherits the tensor structure from  $\text{Vect}$ .

The following functor

$$F_N : C_{\mathbb{P}(N)} \rightarrow \text{Vect}_{S_N, V}$$

is a representation of  $C_{\mathbb{P}}$ . We begin by giving the images of some elements (in case  $N = 2$ ):

$$\text{hom}_{\mathbb{P}}(\underline{1}, \underline{0}) \ni \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \mapsto \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\text{hom}_{\mathbb{P}}(\underline{0}, \underline{1}) \ni \begin{array}{|c|} \hline \\ \hline \bullet \\ \hline \end{array} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{hom}_{\mathbb{P}}(\underline{1}, \underline{1}) \ni \begin{array}{|c|} \hline \bullet \\ \hline \bullet \\ \hline \end{array} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{hom}_{\mathbb{P}}(\underline{2}, \underline{2}) \ni \begin{array}{|c|} \hline \bullet \quad \bullet \\ \hline \bullet \quad \bullet \\ \hline \end{array} \mapsto \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \\ & & & 1 \end{pmatrix}$$

$$\text{hom}_{\mathbb{P}}(\underline{2}, \underline{2}) \ni \begin{array}{|c|} \hline \text{Diagram} \\ \hline \end{array} \mapsto \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

Note that all the images are invariant under the appropriate  $S_2$  action. We conclude by noting that  $C_{\mathbb{P}}$  is a tensor category with

$$\begin{array}{|c|} \hline \text{Diagram A} \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \text{Diagram B} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram A} \quad \text{Diagram B} \\ \hline \end{array}$$

and that the examples given above (respectively their direct generalisations to other  $N$ ) generate.

Thus we have constructed representation for all the partition algebras simultaneously. This gives the representation  $R$  for any given  $n$ .

---