On diagram algebras and statistical mechanics

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On diagram algebras and statistical mechanics

Idea: try to discuss relationship

structures of towers of diagram algebras

$\longleftrightarrow$

statistical mechanical models of many body problems (in which they arise as computational tools)

Aim to discuss in a way which informs problems in representation theory with something like 'physical intuition'. (E.g. ice cubes and icebergs melt at same temperature $\rightsquigarrow$ sequences of algebras have global limits.)

To this end, have to start by recalling physical context of statistical mechanics.

Stat Mech tries to model equilibrium many body problems by positing a probability distribution on set of possible states of system under consideration (Gibbs),
such that relative probability of 'finding' system in state s in some epoch of time takes form

$$\exp(-\beta H(s))$$

where $\beta = \frac{1}{kT}$,
$H : \{states\} \to \mathbb{R}$ is 'Hamiltonian' (absolute probability

$$\frac{\exp(-\beta H(s))}{Z}$$

$Z = \sum_t \exp(-\beta H(t))$).
Depending on axiomatic framework H may have interpretation of Energy (ergodic hypothesis)
(as in, the more energy required to sustain a state s, the less likely we are to be in it — idea being that high energy states are less damped when $T >> 0$, and there is a lot of kinetic energy about).

Expectation values of observables are then

$$\langle \mathcal{O} \rangle \ := \ \frac{\sum_s \mathcal{O} \exp(-\beta H(s))}{Z}$$

Phenomena amenable to Stat Mech - arguably Two types:

I.	II.
I. microscopic constituents of system practically non-interacting	II. strong cooperative interaction (assumes macroscopic significance at some T_c)
$\Rightarrow$ thermodynamic functions of macroscopic system *follow from* energy levels of individual microscopic constituents	$\Rightarrow$ energy levels of macroscopic system *not* directly related to microscopic constituents - small changes in T can produce *huge* changes in thermodynamic functions
e.g. specific heat of gas, solid condensation of ideal Bose gas spectrum of black-body radiation elementary electron theory of metals paramagnetism	e.g. condensation of gas melting of solid (anti)ferromagnetism order-disorder transitions in alloys normal material $\rightarrow$ superconductor lattice spacing dependent LGT $\rightarrow$ QFT
...most thermodynamic functions smooth continuous (in e.g. T)	...discontinuities/singularities in thermodynamic functions. Formidable mathematical problems.

(NB Inter atomic interaction important in solids, but positions don't vary much so go to normal coordinates and treat as practically noninteracting SHOs.)

type II

How model microscopically?

How compute macroscopically in model?

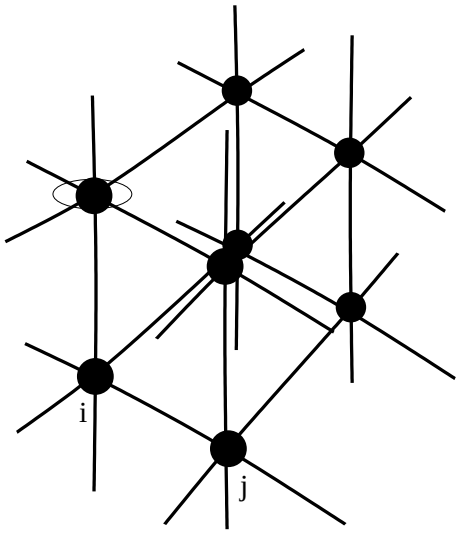
What do we learn?

> OK to simplify model a little (1. necessary; 2. still qualitatively interesting; 3. still quantitatively interesting!)

Example

Ferromagnetism `(directly ports to many other settings)`

N site crystal lattice (3D, maybe 2D)



atom is effective magnetic dipole with magnetic moment μ of magnitude

$$g\mu_B j$$

Lande g-factor

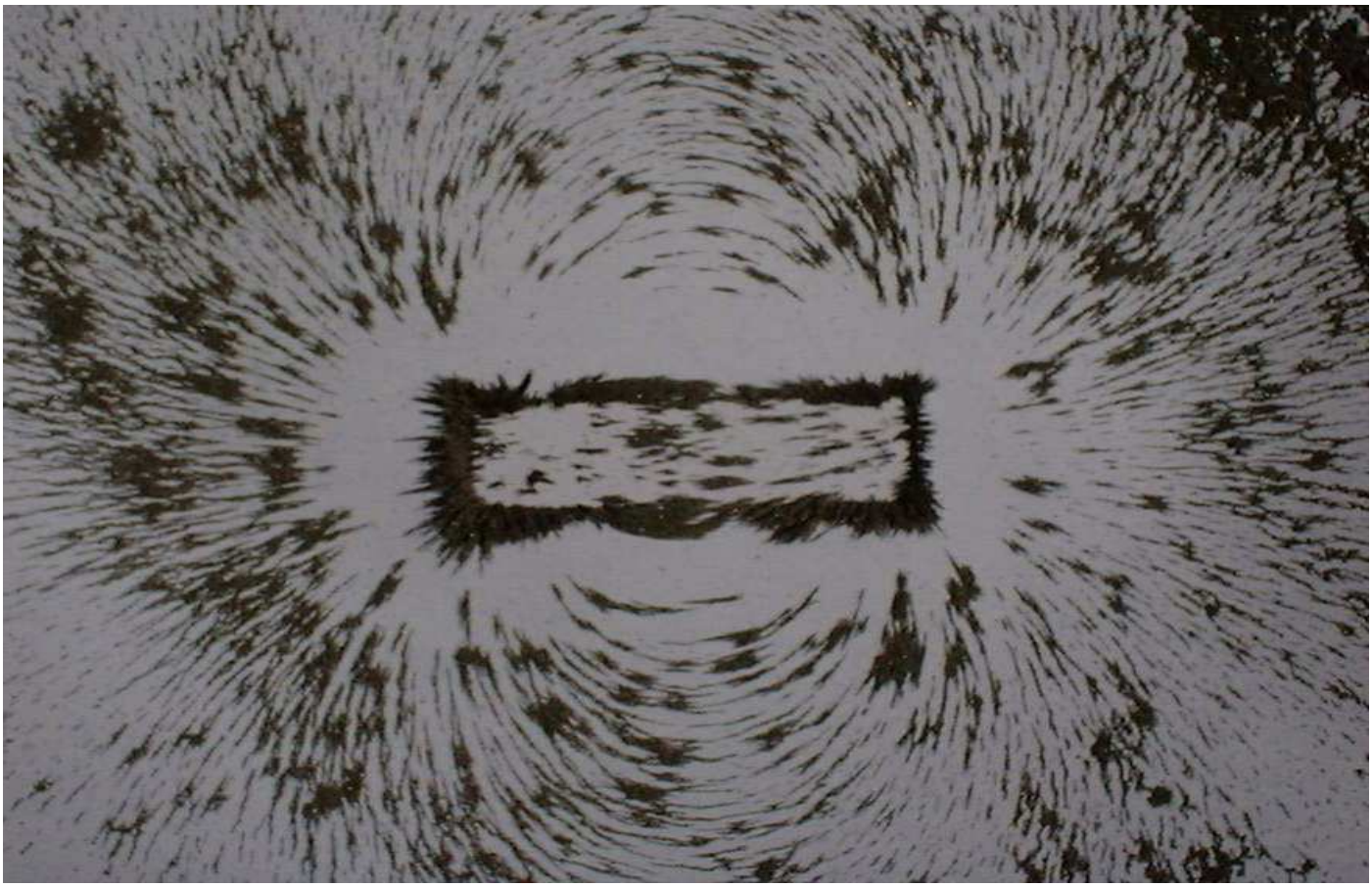
Bohr magneton $= e\hbar/2m_e c$

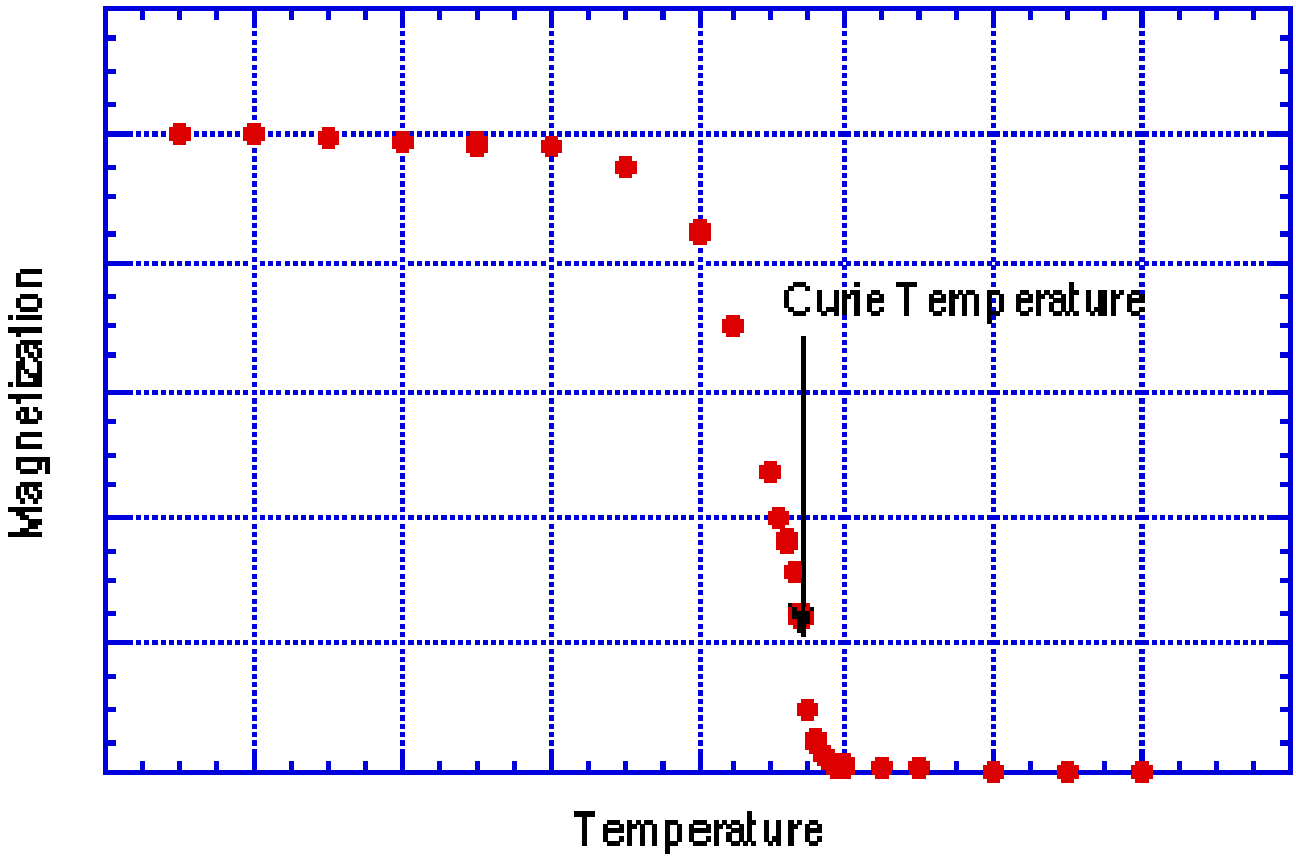
$j \in \frac{1}{2}\mathbb{N}$, state quantised to one of $2j+1$ orientations in space. (Via Dirac (1928) treatment of electron spin and Pauli exclusion principle.) [a]

This μ is very small and yet we have a macroscopic effect.

[a] Iron, nickel, cobalt spontaneous magnetisations fit with $j = 1/2$ `(via e.g. Weiss domain theory (1907)).`

Ferromagnetism





MODELS:

Pair interaction of spins i, j: quantum Hamiltonian

$\epsilon_{ij} = J_{ij}\underline{s_i}.\underline{s_j}$ so overall $H \sim \sum_{ij} J_{ij}\underline{s_i}.\underline{s_j}$

'Exchange coupling' J_{ij} drops off quickly with distance so consider only $J = J_{ij}$ nearest neighbours (n.n.) on crystal lattice.

$\rightarrow$ Heisenberg model of ferromagnetism (1928) (back to this later)

OR quantise along (say) z-axis to get diagonal part

$$H \sim J \sum_{nn} \sigma_i \sigma_j$$

($\sigma_i \in \{1, -1\}$)

$\rightarrow$ Ising–Lenz model of ferromagnetism (1920s)

$$Z = \sum_s \exp(\beta J \sum_{nn} \sigma_i \sigma_j)$$

Can we compute and analyse Z? ... for 10^{23} atoms.

Disregard kinetic energy of atoms on vertices (not relevant for phase transitions).

Hamiltonian in config $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$

$$H(\{\sigma_i\}) = -J \sum_{nn} \sigma_i \sigma_j - \mu B \sum_i \sigma_i$$

(includes effect of external magnetic field B).

Partition function

$$Z(B,T) = \sum_{configs\ \{\sigma\}} \exp(-\beta H)$$

Helmholtz free energy

$$A(B,T) = -kT \ln(Z)$$

Internal energy

$$U(B,T) = kT^2 \frac{\partial \ln(Z)}{\partial T} = \frac{\sum H \exp(-\beta H)}{Z}$$

Specific heat

$$C(B,T) = -T \frac{\partial^2 A}{\partial T^2}$$

net magnetisation

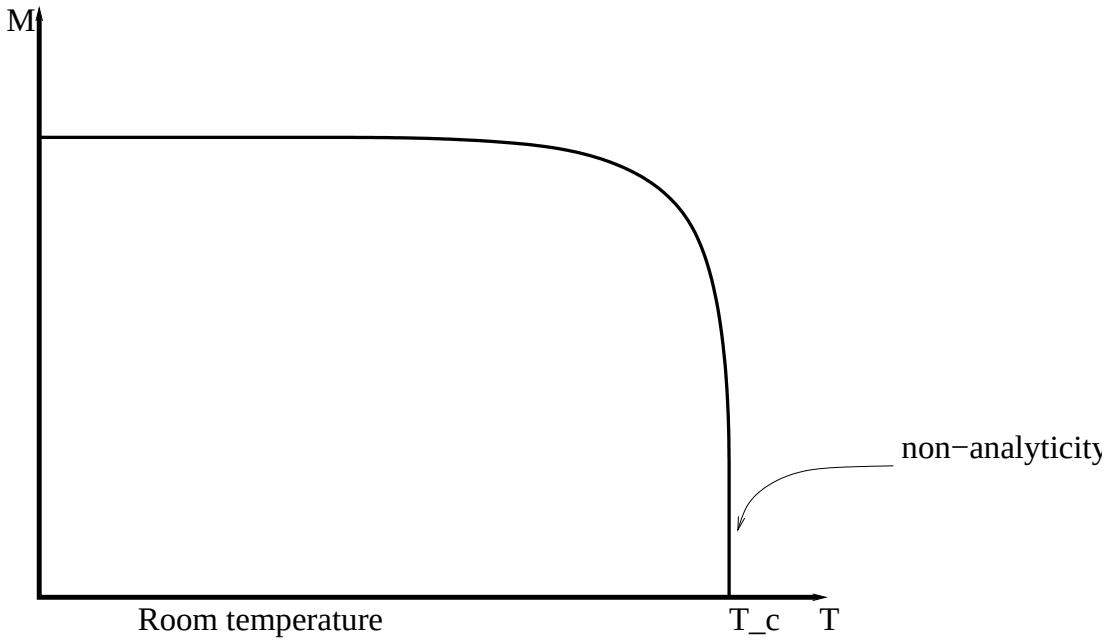
$$\overline{M}(B,T) = -\frac{\partial A}{\partial B}$$

Put $Z = Z(0, T)$ for now. Put $J = 1$.
Finite system Z positive integral polynomial in e^{β}.

NB, Ferromagnet is $\sim 10^{23}$ atoms; LGT can be unboundedly large. $N = \infty$ likely to be a simplification!

What kind of limit makes sense?
What do we expect to happen?
How interpret?

spontaneous magnetisation



nonanalyticity of this function

$\Rightarrow$

need something dramatic to happen to Z in crossing over from energy dominated to entropy dominated phase.

This is not crystal melting point.

Tiny microscopically irrelevant change in T produces huge macroscopic change.

Presumably a cooperative effect.

Does Z model it?

Energy vs Entropy

At very low T ($\beta >> 0$) Gibbs probability $e^{-\beta H}$ is much higher for 'ordered' states,
so (given spontaneous symmetry breaking) a large net magnetisation is plausible.
At very high T all states equally likely,
but there are many more 'disordered' states (Entropy of disorder is higher) so their contribution dominates the state sum in Z.
A low net magnetisation is inevitable.

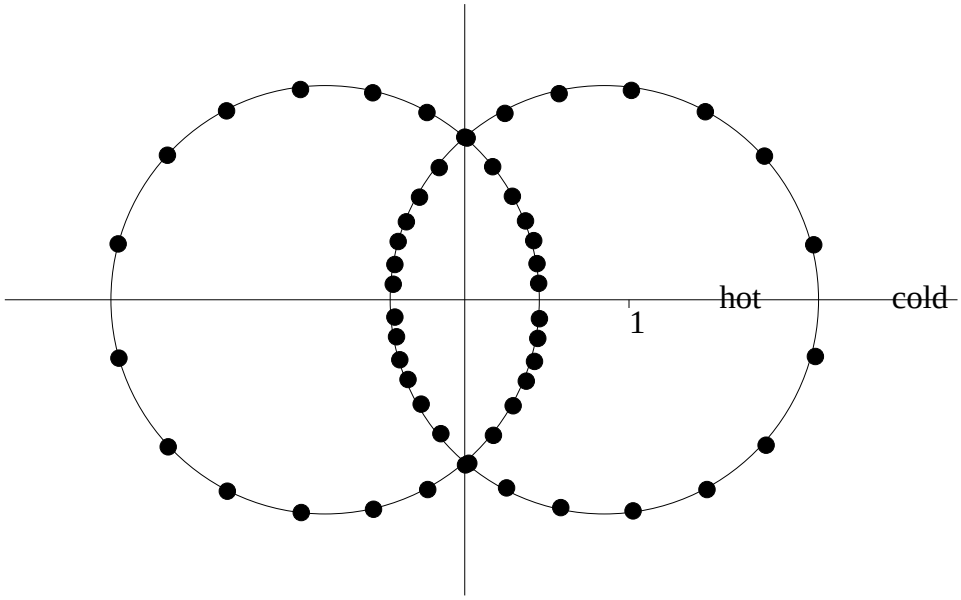
Our question is about the nature of the crossover between the 'energy dominated' and 'entropy dominated' regions.
Smooth or abrupt?

Perform $\sum_s$ (2D case, see later): $x = e^{2\beta}$

$$Z \sim \prod_{k=0}^{N} \left((x + 1/x)^2 - 4(x - 1/x)\cos(2\pi k/N) \right)$$

(Caveat: expression is approximate in many senses! See references in [Martin91] for something usable.)

Polynomial: only analytic features are zeros.



Set $\kappa = \frac{2\sinh(2\beta)}{\cosh^2(2\beta)}$.
Point where zeros pinch axis is $\kappa = 1$.

Internal Energy density (measures amount of correlation of spins)

$$\frac{U}{N} = -2\tanh(2\beta) - \frac{\sinh^2(2\beta) - 1}{\sinh(2\beta)\cosh(2\beta)} \left(\frac{2}{\pi} K(\kappa) - 1 \right)$$

where $K(\kappa)$ is complete elliptic integral of first kind.
I.e. energy$\sim (T - T_c) \ln |T - T_c|$ close to $\kappa = 1$.
$\Rightarrow$ slope of energy curve (hence heat capacity) infinite.

Unfortunately very few 2D models 'solved' this way, none in 3D! (No LGT etc.)
The paradigm has capability to represent critical phenomena,
but much computation to be done.

How compute?
0. Brute force — no! $2^{10^{23}}$ states.
1. Formalism
1.1 then brute force
1.2 'integrable' cases.

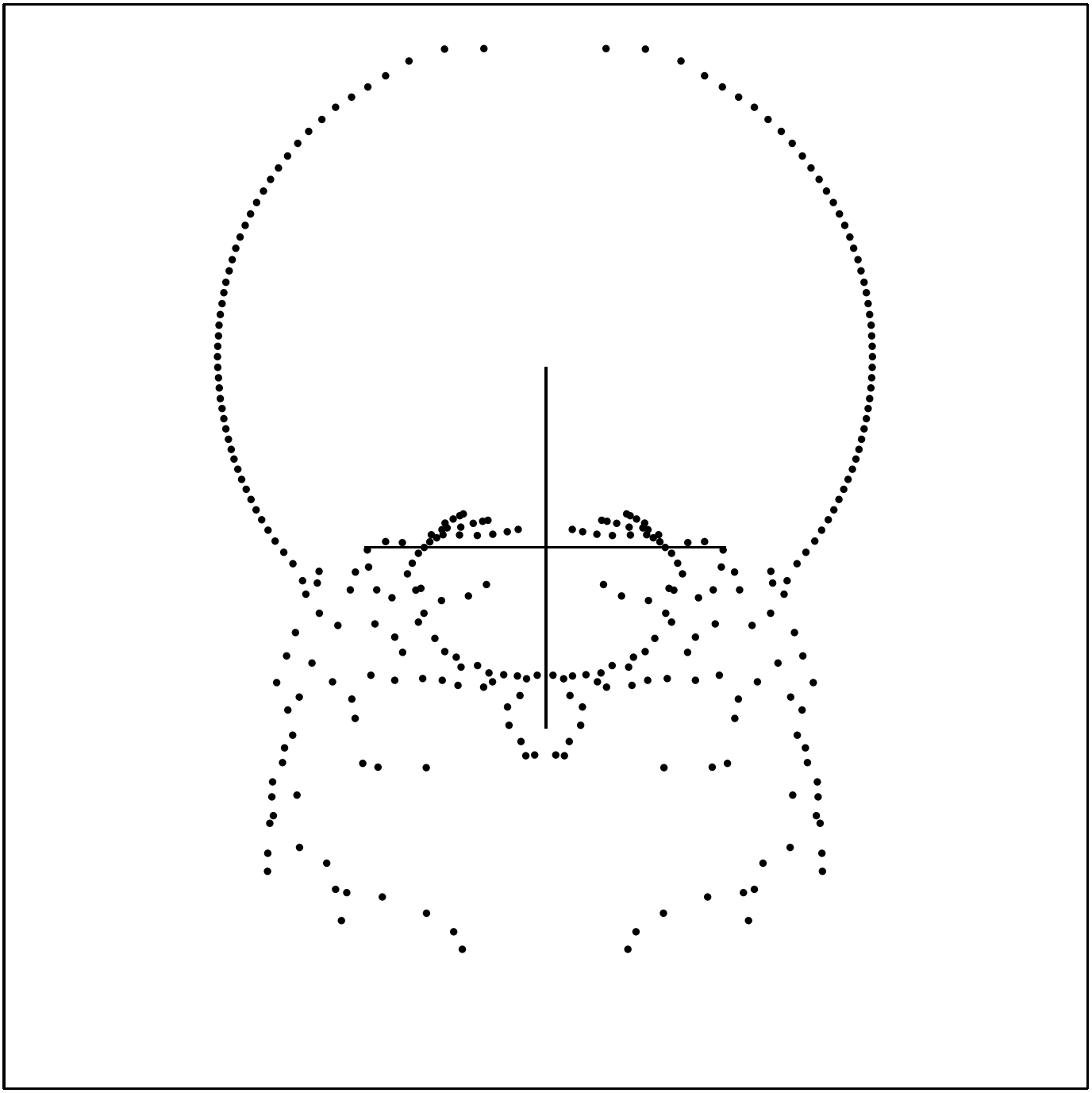
Formalism: depends on interaction, but suppose e.g.

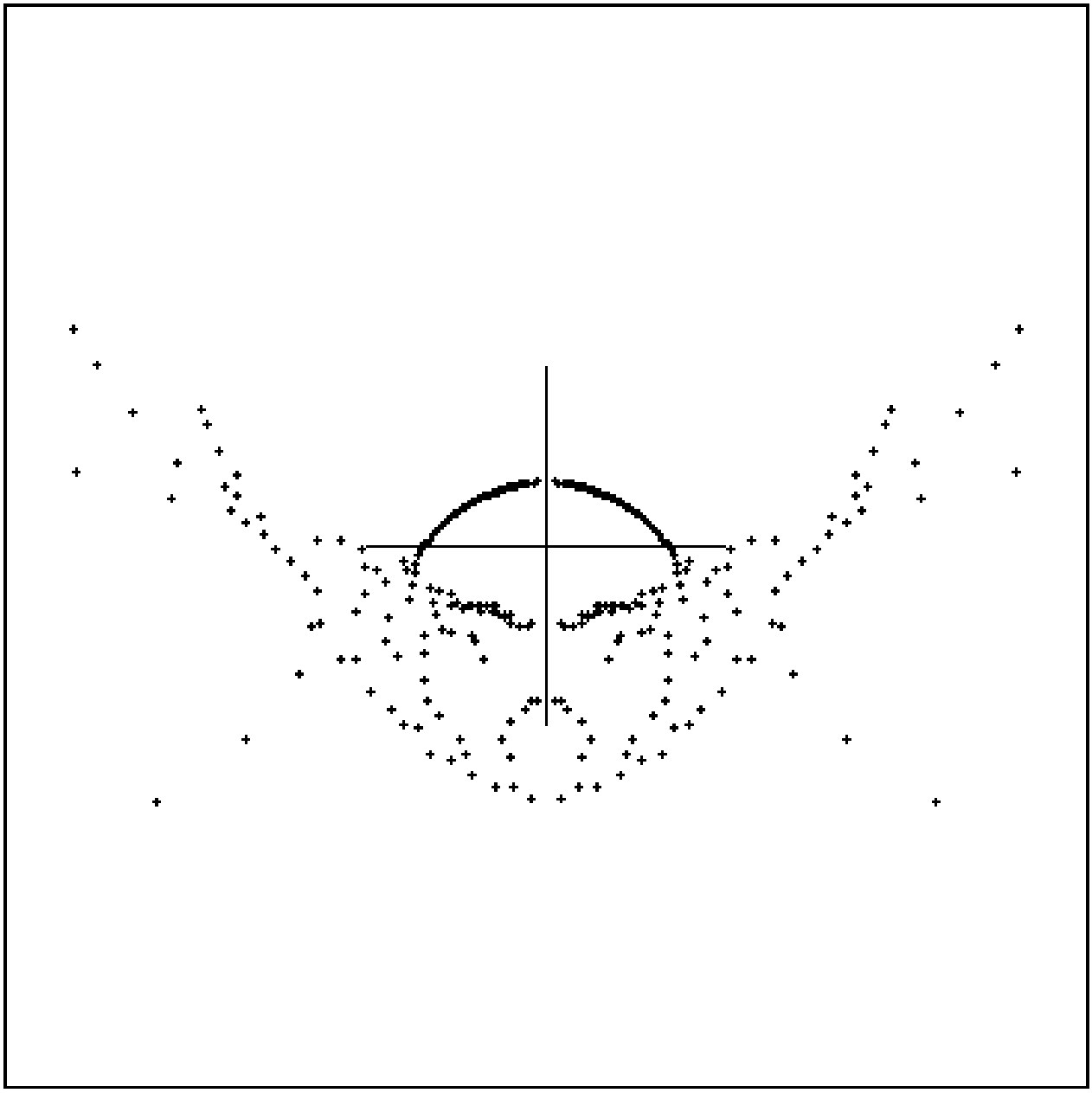
$$H_{n.n.} = \delta_{\sigma_i,\sigma_j} = \begin{cases} 1 & \sigma_i = \sigma_j \in \{1, 2, ..., Q\} \\ 0 & o/w \end{cases}$$

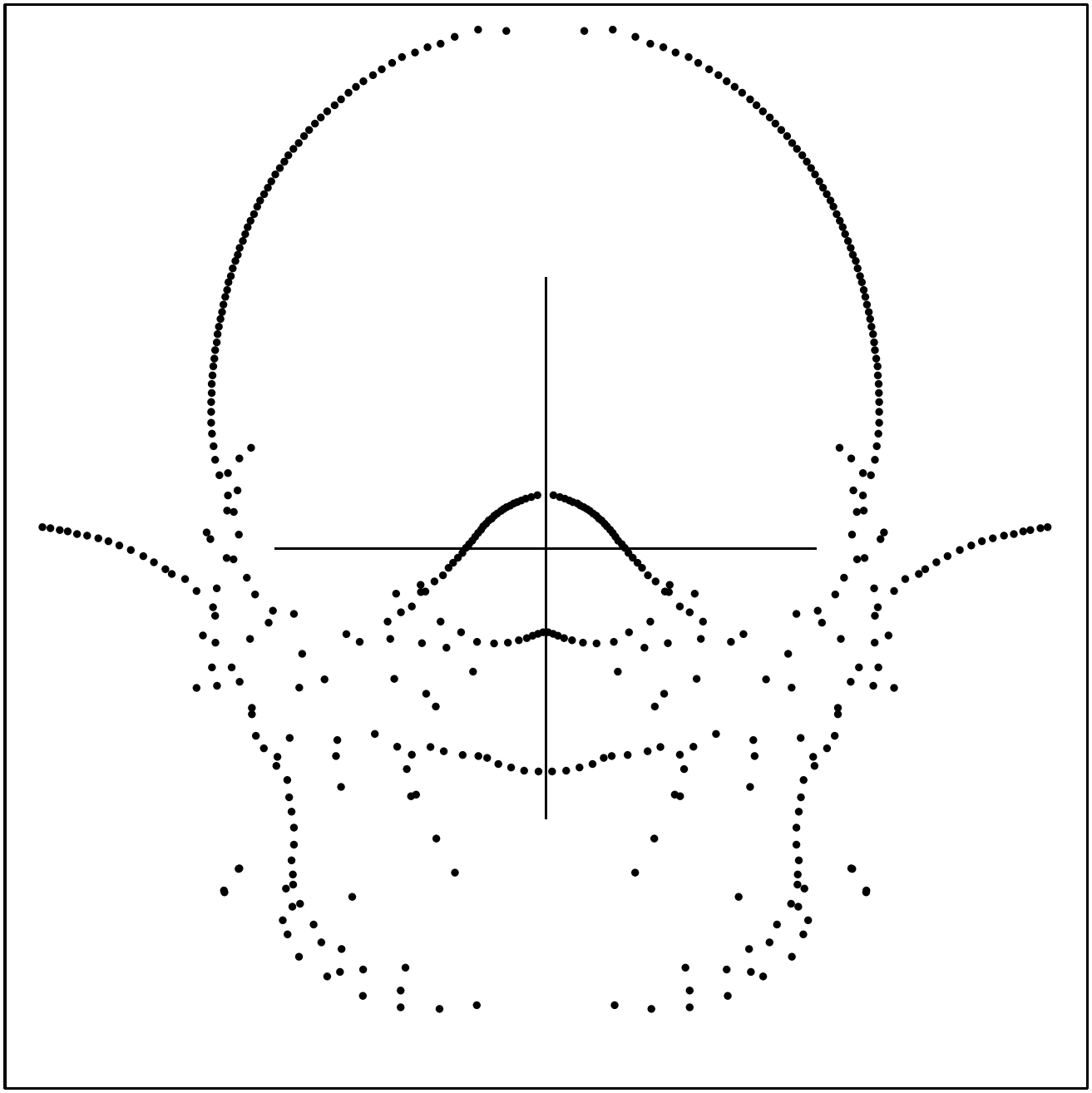
(Q state Potts model (Domb, Potts 1950s)).
Then have map from graphs G to polynomials Z_G.
Physically we're interested in sequences of graphs s.t. corresponding sequence of polynomials settles down to a limit (in above sense of limit analytic structure).

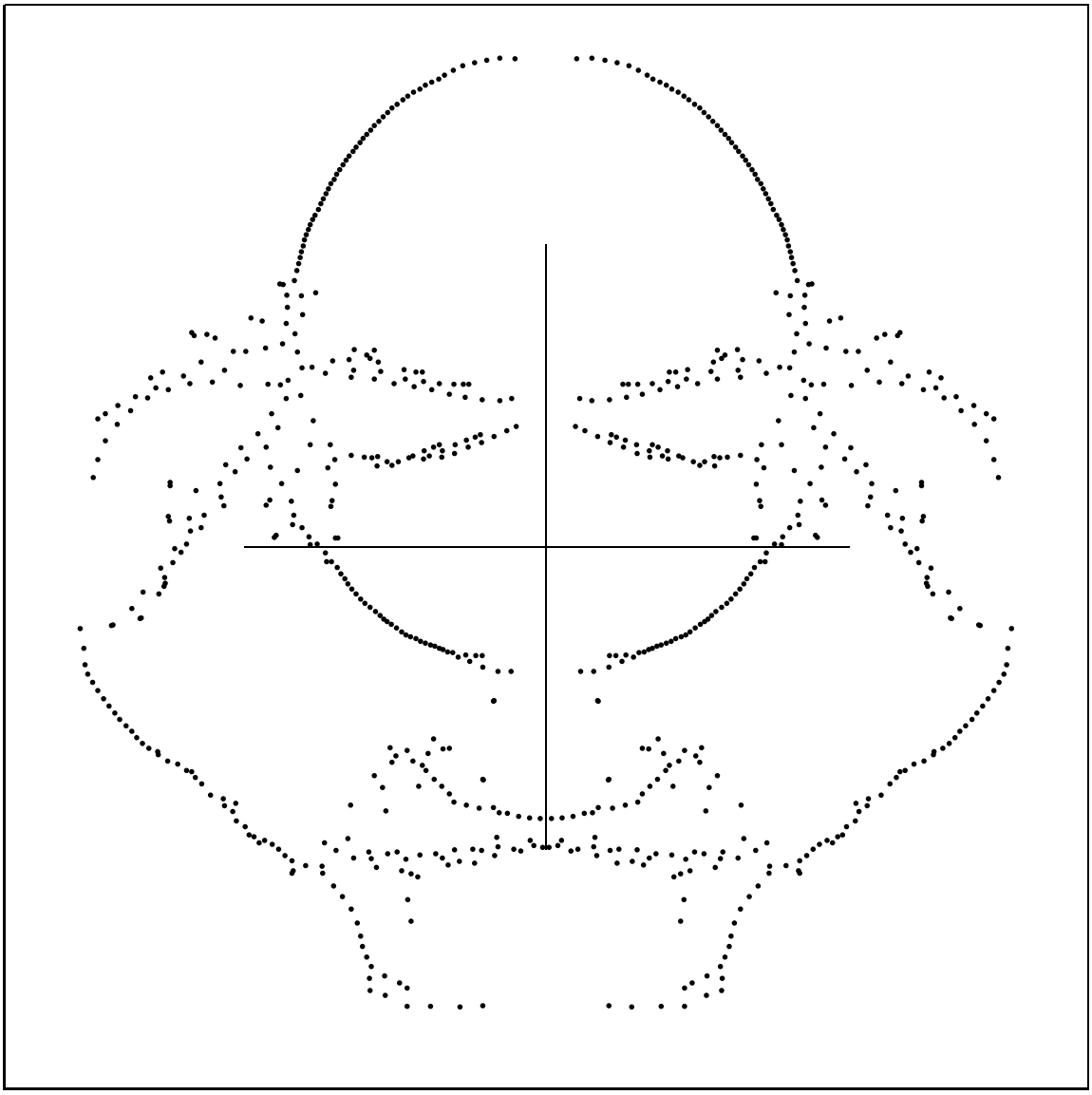
Aside: Here we're interested in computational formalism, but before getting into this, here are some examples of results — the Physics payoff from the mathematical formalism...

The figures show the roots of the polynomial partition function Z_G in $x = \exp(\beta)$ for:
$Q = 3$–state Potts model on 12×13 square lattice.
$Q = 3$–state Potts model on 12×13 square lattice (plotted in $\exp(-\beta)$).
$Q = 3$–state Potts model on 12×13 triangular lattice.
$Q = 4$–state (3,1,0) clock model (see [Martin91] for notation) on 10×13 square lattice.
In each case the scale is set by unit length of the positive axis, so the first figure (for example) provides evidence of a phase transition for positive temperature.









(`$G$ is crystal lattice; $Z_G$ is partition function.`)

Hereafter if G is a graph then V_G is vertex set.

For any $V \subseteq V_G$ let x denote a particular configuration of 'spins' on V.

Let $Z_G|_x$ be Z_G but with V *fixed* to x.

$$Z_G|_x = \sum_{s \ s.t. \ state \ s|V=x} \exp(-\beta H)$$

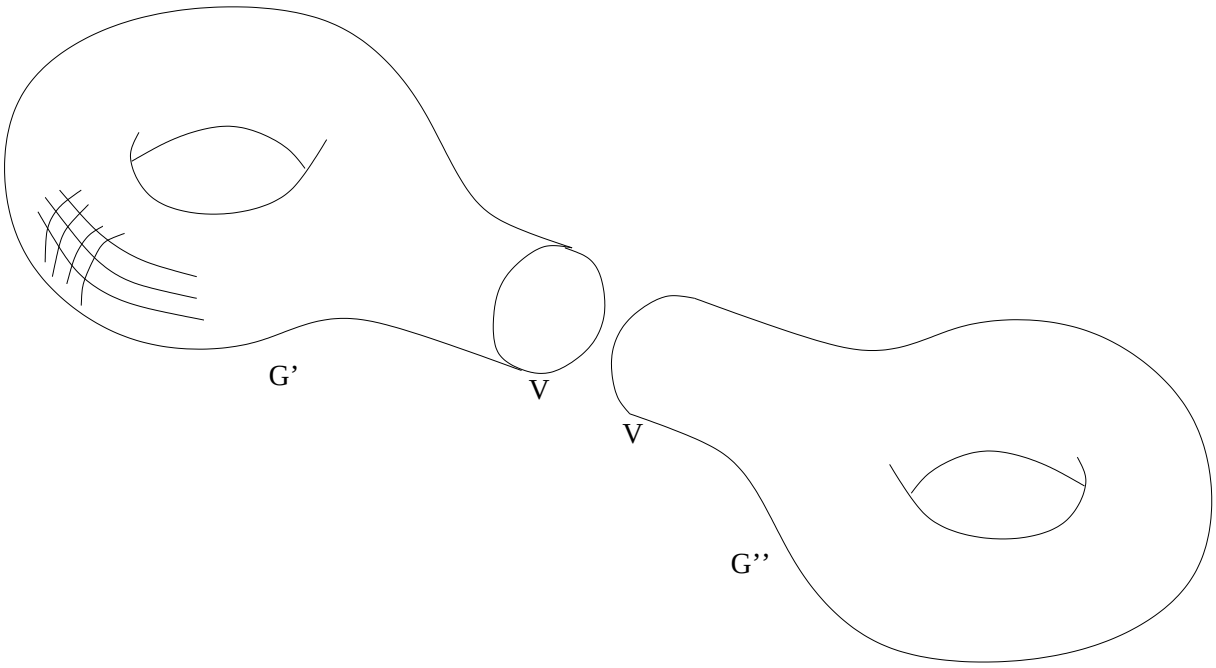
$Z_G(V)$ is a vector indexed by configurations of V, whose x-th entry is $Z_G|_x$.

'Partition vector'.

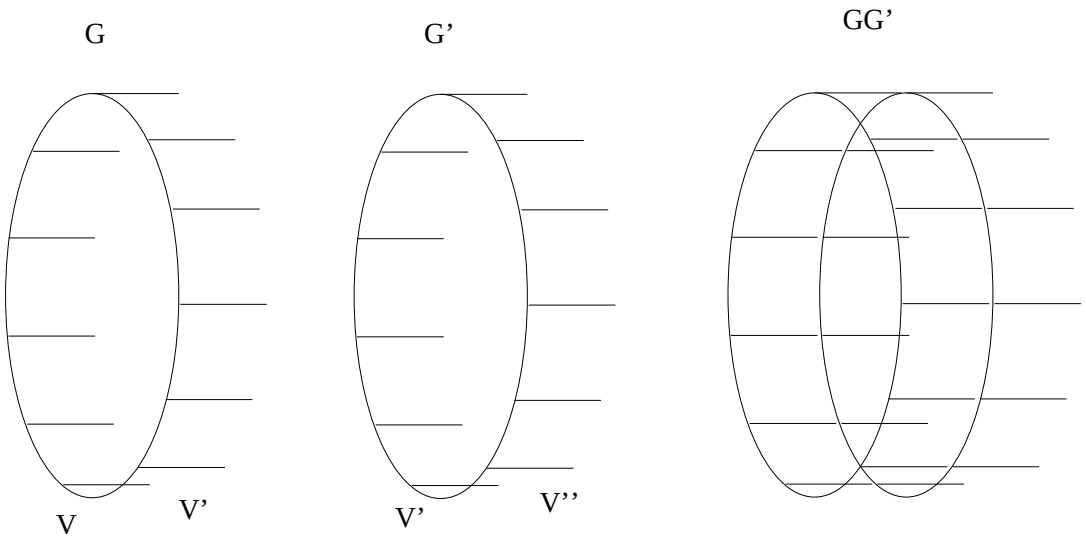
If $G = G' \cup G''$ where $V_{G'} \cap V_{G''} = V$

$$Z_G = \sum_x (Z_{G'}(V))_x (Z_{G''}(V))_x$$

Typically G has topological properties, V is a boundary:



Example



$$Z_G(V,V')Z_{G'}(V',V'') = Z_{GG'}(V,V'')$$

'Transfer Matrix' $\mathcal{T}$.

Typical physical system has at least 1 direction translation symmetry.
$G = G_1G_2G_3...G_M$

$$Z_G(V,V') = \mathcal{T}^M$$

$$Z_G = \sum_{s,t}(\mathcal{T}^M)_{st}$$

or, say

$$= Tr(\mathcal{T}^M) = \sum_i \lambda_i^M$$

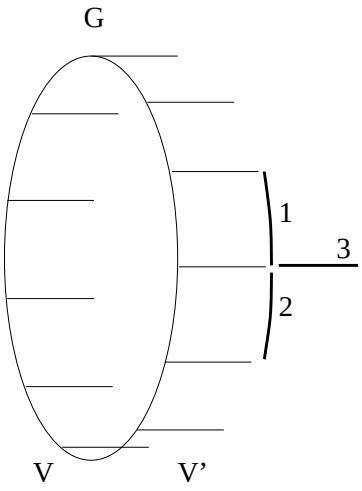
λ_i eigenvalues of $\mathcal{T}$.

$\mathcal{T}$ `equiv to Hermitian, indeed real symmetric, matrix.`

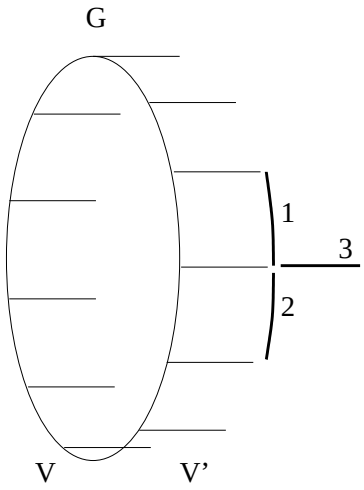
NB Real β, finite size
$\Rightarrow$! λ_0 of largest magnitude (+ve)
so Helmholtz free energy $\sim \frac{1}{N} \ln(Z) \sim \ln(\lambda_0)$.

NB Correlation functions (see later) require other eigenvalues.
So investigate spectrum $\mathcal{T}$.

Next step is to break up $\mathcal{T}$ into local factors — matrices accounting for effect of one local interaction at a time:



NB. Large scale structure of G just here for translation invariance. Could be open chain, ...or arbitrary transverse graph.



$$\mathcal{T} = \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 ...$$

$$\mathcal{T}_1 = \begin{pmatrix} e^\beta & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & e^\beta & \\ & & & & \ddots \end{pmatrix}$$

$$\mathcal{T} = \mathcal{T}_1 \begin{pmatrix} e^\beta & & & & \\ & e^\beta & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} e^\beta & 0 & 1 & & \\ 0 & e^\beta & 0 & 1 & \\ 1 & 0 & e^\beta & & \\ & 1 & 0 & e^\beta & \\ & & & & \ddots \end{pmatrix} ...$$

Sanity check: rows and columns indexed by configuration of spins in respective lattice layer. Write $\langle 111...|$ for row position corresp. to spin a in state 1, spin b in state 1, etc.. Then

$$\langle 111...|\mathcal{T}_1|111...\rangle = \exp(\beta)$$

since when spins at ends of edge 1 both in state 1 get contribution $\beta\delta_{1,1}$ to H.

$v = \frac{e^{\beta}-1}{\sqrt{Q}}$ $(Q = 2)$

$$\mathcal{T} = \underbrace{(1 + vE_{12})}_{\mathcal{T}_1}(1 + vE_{23})\sqrt{Q}\,(v1 + E_{2.})$$

$$E_{12} = \sqrt{Q}\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \end{pmatrix} \quad E_{2.} = \frac{1}{\sqrt{Q}}\begin{pmatrix} 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & & \ddots & \end{pmatrix}$$

$R(U_-) = E_-$ representation of Temperley–Lieb algebra (1971) $T_{2n}(q)$ $(q + q^{-1} = \sqrt{Q})$.
I.e. $\mathcal{T} = R(X)$, $X \in T_{2n}(q)$.
$\Rightarrow$ spectrum $\mathcal{T}$, correlation functions etc.
at least partially indexed by simple modules of TLA.

`(NB. Independently of temperature.)`

(This is elementary. Any ST putting $\mathcal{T}$ in upper block triangular form partitions its spectrum accordingly. If R reducible such a ST manifests this.)

Change the model — change the algebra (or rep.)

$Z_G = \sum_s \exp(\beta \sum_{n.n.,ij} \delta_{\sigma_i,\sigma_j})$

$$= \sum_s \prod_{n.n.,ij} \left(\exp(\beta \delta_{\sigma_i,\sigma_j})\right)$$

$$= \sum_s \prod_{n.n.,ij} \left((e^\beta - 1)\delta_{\sigma_i,\sigma_j} + 1\right)$$

$$= \sum_s \sum_{e \in P(E_G)} \prod_{edges \in e} (e^\beta - 1)\delta_{\sigma_i,\sigma_j}$$

($P(E_G)$ power set of set of edges of G)

$$Z_G = \sum_{e \in P(E_G)} Q^{c(e)} (e^\beta - 1)^{|e|}$$

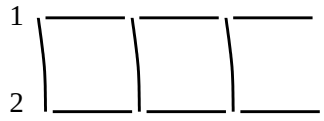
($c(e)$ number of connected components of e)

'dichromatic' (or Whitney-Tutte) polynomial.

$e^\beta = 0$ `case is` $T = -0$ $\rightsquigarrow$ `adjacent sites must be coloured differently: chromatic polynomial (Beraha 1970s etc.)`

NB, well defined for any Q.

NB, require to write this in Transfer Matrix formalism.
(Consider $e^{\beta} = 0$ case.)
Consider graphs built from Transfer Matrix layer of 2 sites



(for simple concrete example).

Basis should be possible connectivities between first and last layers of such a graph.
NB, if these layers disconnected then extending graph cannot connect them so submodule with basis of connectivities (partitions) of a single layer of 2 vertices: $\{\{\{1\},\{2\}\},\{\{1,2\}\}\}$.
I.e. either connected or not.
Focus on this 2d submodule for simplicity.
$\sqrt{Q}R(U_{1.})$ is the 'hot' operator, so

$$\sqrt{Q}R_c(U_{1.}) = \begin{pmatrix} Q & 0 \\ 1 & 0 \end{pmatrix}$$

And similarly for the cold operator.

actually prefer convenient base renormalisation to

$$R(U_{1.}) = R(U_{2.}) = \begin{pmatrix} \sqrt{Q} & 0 \\ 1 & 0 \end{pmatrix} \qquad R(U_{12}) = \begin{pmatrix} 0 & 1 \\ 0 & \sqrt{Q} \end{pmatrix}$$

$$\mathcal{T}_c = (-1+\sqrt{Q}R_c(U_{1.}))(-1+\sqrt{Q}R_c(U_{2.}))(1-\frac{1}{\sqrt{Q}}R_c(U_{12}))$$

$$= \begin{pmatrix} Q-1 & 0 \\ \sqrt{Q} & -1 \end{pmatrix} \begin{pmatrix} Q-1 & 0 \\ \sqrt{Q} & -1 \end{pmatrix} \begin{pmatrix} 1 & -1/\sqrt{Q} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (Q-1)^2 & -(Q-1)^2/\sqrt{Q} \\ \sqrt{Q}(Q-2) & -(Q-2) \end{pmatrix}$$

To extract chromatic polynomial need good boundary conditions!

$QR(U_{1.}U_{2.})$ completely isolates graph.

Thus $R(U_{1.}U_{2.})\mathcal{T}_c QR(U_{1.}U_{2.}) = Z_G R(U_{1.}U_{2.})$

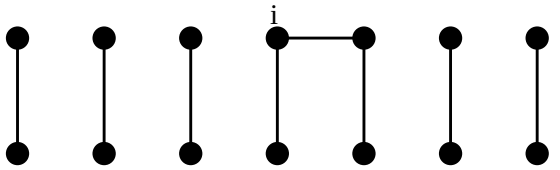
$$\begin{pmatrix} Q & 0 \\ \sqrt{Q} & 0 \end{pmatrix} \begin{pmatrix} (Q-1)^2 & -(Q-1)^2/\sqrt{Q} \\ \sqrt{Q}(Q-2) & -(Q-2) \end{pmatrix} \begin{pmatrix} Q & 0 \\ \sqrt{Q} & 0 \end{pmatrix}$$

$$= (Q(Q-1)^2 - (Q-1)^2) \begin{pmatrix} Q & 0 \\ \sqrt{Q} & 0 \end{pmatrix}$$

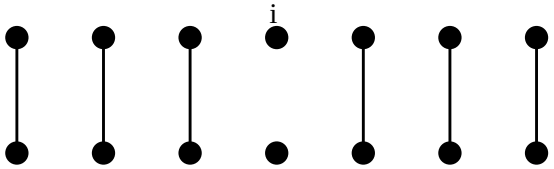
$Z_G = Q(Q-1)^3$

Hence "Whitney diagrams" (as in Whitney polynomials)

$(i\ i{+}1)$ $=$



$(i.)$ $=$

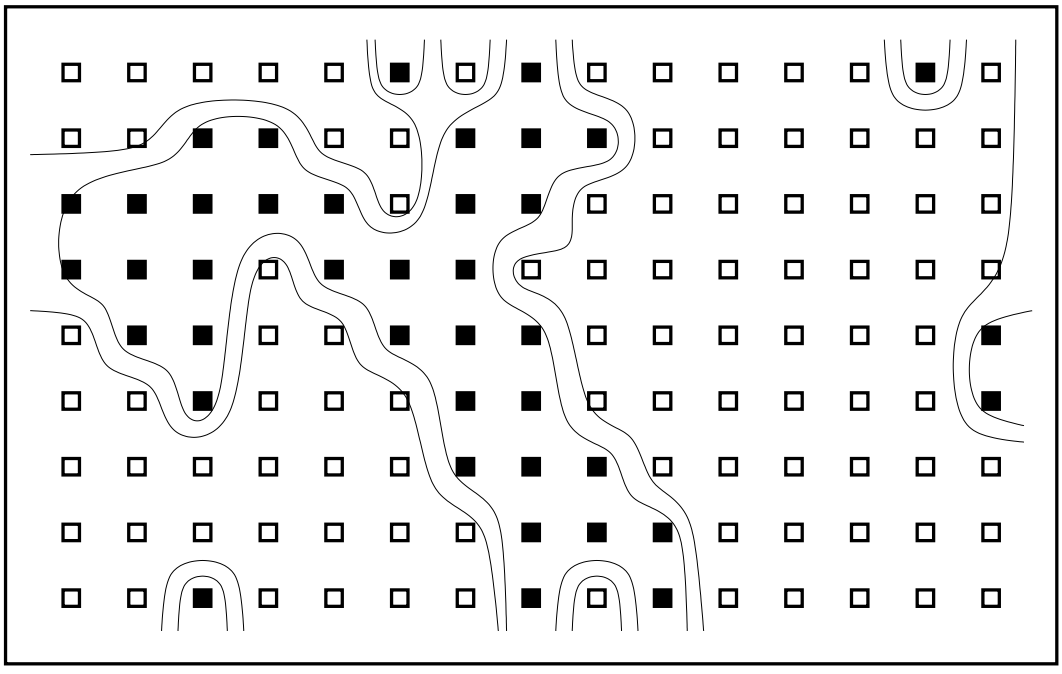


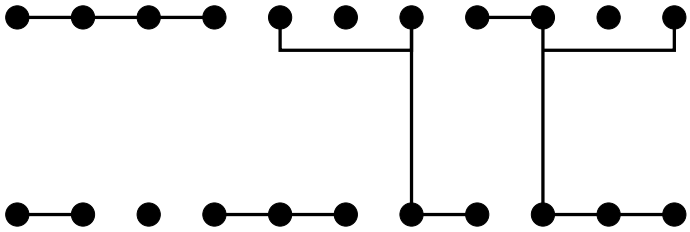
pictures of partitions of $2n$ objects.

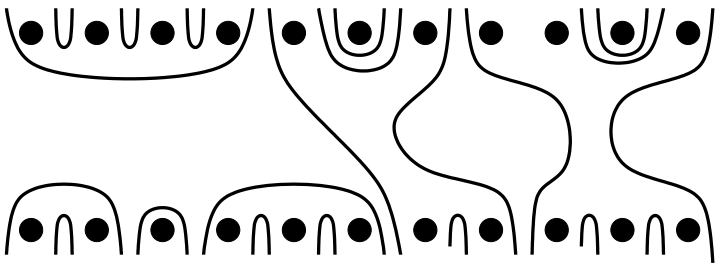
`rotated through` 90^o `cf. previous to save page space.`

These are 'dual', in physical usage, to "boundary diagrams".

Consider Potts spin system (e.g. Ising)







(Planar) Connected clusters : Boundary diagrams (as in boundaries of regions of aligned spins).

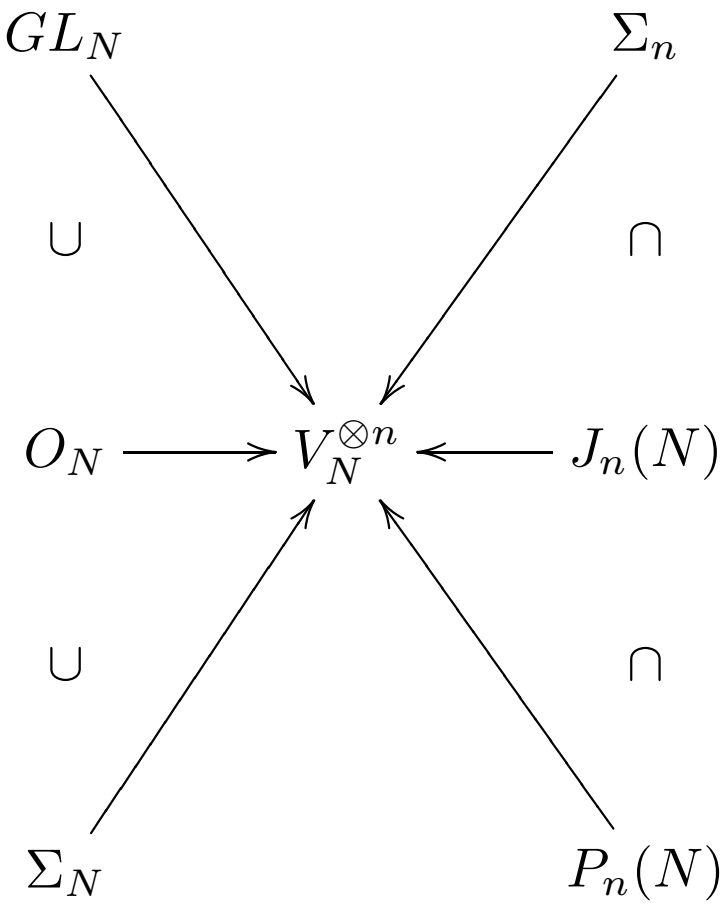
Very many generalisations possible.

---think about boundaries $\longrightarrow$ blob algebra

---think about 3D...

Example: formalism works for arbitrary transverse graphs.
Every algebra arising is subalgebra of partition algebra (introduced to help address 3D models).
Whitney diagrams generalise immediately.
Boundary diagrams do not!

Transverse graph in 2D is simple chain.
In 3D it is square grid. Harder to treat than complete graph case.
(This has some lovely properties.)



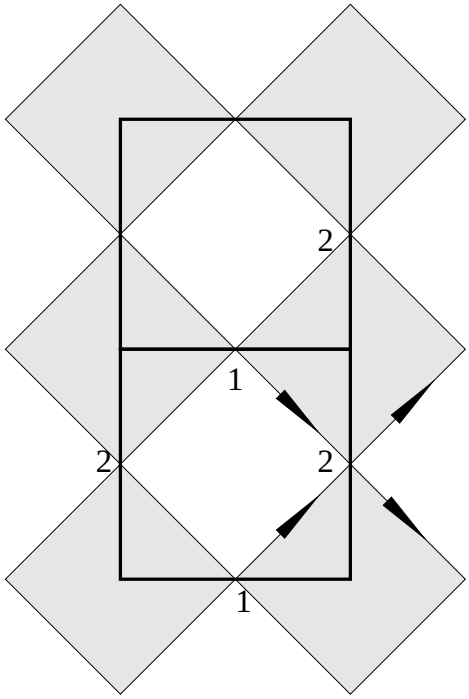
(here $P_n(N)$ acts in ‘‘Potts’’ way.)

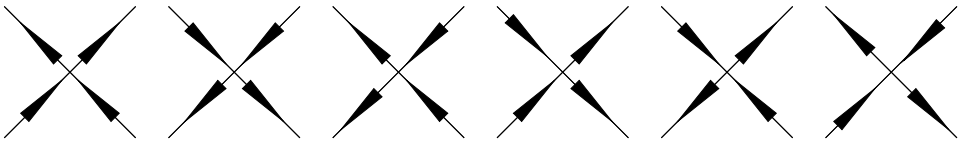
6-vertex model (Ice-type model)

(Slater 41, TL 71)

In ice Oxygen atom lattice has coordination number 4. Between each pair a H ion, nearer one end or the other, s.t. for each O, 2 Hs are close. In 2d (!)...

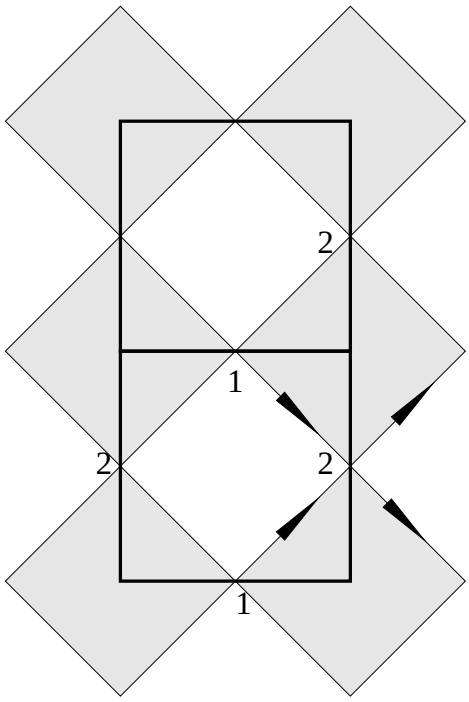
Medial lattice

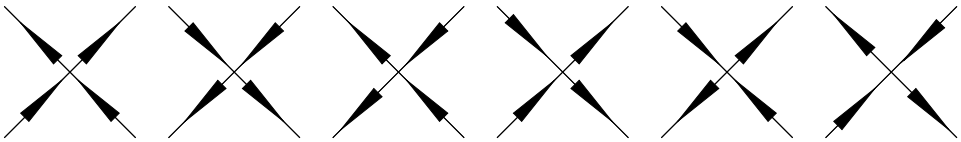




degrees of freedom live on edges of medial lattice

Boltzmann weights:...

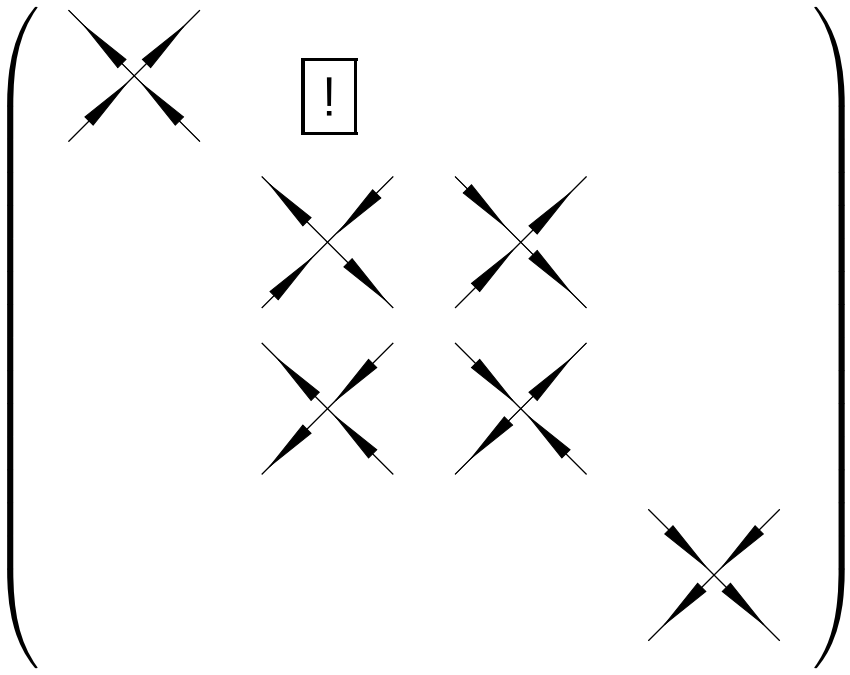




Type 1: $1 \quad 1 \quad x_1 \quad x_1 \quad 1+\dfrac{x_1}{q} \quad 1+x_1 q$

Type 2: $x_2 \quad x_2 \quad 1 \quad 1 \quad x_2+\dfrac{1}{q} \quad x_2+q$

Boltzmann weights.



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{x}{q^{-1}} & x & 0 \\ 0 & x & 1+xq^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 + xU$$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

...Another representation of $T_{2n}(q)$
due to TL (1971). Acts on $V_2^{\otimes 2n}$ for any q.

$$H_n(q) \rightarrow V_N^{\otimes n} \leftarrow U_q sl_N$$

NB. No straightforward generalisation to higher D.

Observables II

A signal of a phase transition is correlation (via local cooperation) of "spins" over long distances.

Experimentally, degree of correlation of spins

$$\langle \sigma_i \sigma_{i+r} \rangle \sim \exp(-r/\rho)$$

away from T_c
— length scale $\rho(T)$ measured in terms of lattice spacing.

As $T \to T_c$, $\rho \to \infty$ (decay of correlations becomes power law).

[Crucial in Lattice Field Theory — dont want lattice spacing to provide bogus length scale: $\rho \to \infty$, $a \to 0$ allows keeping ρa fixed
$\frac{1}{\rho a}$ = particle mass (Euclidean $\to$ Minkowski)]

How compute correlation lengths ρ in TM formalism?

$$\begin{aligned}
\langle \sigma_i \sigma_{i+r} \rangle &\sim \frac{(\mathcal{T}^{N_1} \hat{\sigma} \mathcal{T}^r \hat{\sigma} \mathcal{T}^{N_2})_.}{(\mathcal{T}^{N_1+r+N_2})_.} \\
&\sim \left(\frac{\lambda_\sigma}{\lambda_0} \right)^r \\
&= \exp(-r(\underbrace{\ln(\lambda_0) - \ln(\lambda_\sigma)}_{\frac{1}{\rho}}))
\end{aligned}$$

$\Rightarrow$ want λ_σ and others.

NB. Labelled by operator content *not* N.
Shouldn't depend on N for large N.

NB. this is tricky. See e.g. PPM J Phys A (2000) for some details.

Tower properties

Even fixing the physical model, there is a $\mathcal{T}$ for each N $\Rightarrow$ a TMA for each N.

But observables, and hence 'leading' spectrum components, are defined essentially independently of N. For given N spectrum components are (partly) indexed by simple modules, thus these can be indexed independently of N.

Thus "expect" a global limit algebra, and localisation functors picking out fibres of "physically equivalent" modules.

$$U_i A_n U_i \cong A_{n-1}$$

`see Cox's talk for more.`

Bethe Ansatz

Mild generalisation of Heisenberg ferromagnet (in 1D) is

$$H = \sum_i R_{ice}(U_i)$$

which has the same spectrum as

$$\mathcal{O} = \sum_i U_i$$

Consider $T_n(q)$ module with basis

$$\{C_1, C_2, C_3, C_4\} = \{\cup\cdots, \cdot\cup\cdot\cdot, \cdot\cdot\cup\cdot, \cdots\cup\}$$

$$U_i C_m = (\delta_{i,m-1} + \sqrt{Q}\delta_{i,m} + \delta_{i,m+1})C_i$$

Suppose $\mathcal{O}v = \lambda v$ with $v = \sum a_j C_j$

$$a_{j-1} + \sqrt{Q}a_j + a_{j+1} = \lambda a_j \qquad 1 \le j \le n-1$$

$(a_0 = a_n = 0)$

→ particle moving in vacuum

$\sim$ translation invariance

$\rightsquigarrow$ ansatz for v:

$$a_j = \exp(ijk) - \exp(-ijk) \qquad \text{some } k$$

$$a_0 = 0$$

$$a_n = 0 \qquad \Rightarrow \qquad \exp(2ink) = 1$$

$$\exp(-ik)\exp(ijk) - \exp(ik)\exp(-ijk)$$

$$+\sqrt{Q}\exp(ijk) - \sqrt{Q}\exp(-ijk)$$

$$+\exp(ik)\exp(ijk) - \exp(-ik)\exp(-ijk)$$

$$= \lambda\exp(ijk) - \lambda\exp(-ijk)$$

$$\lambda = 2\cos(k) + \sqrt{Q}$$

"dispersion relation"

relation between energy and momentum of our particle.

We are interested in many (interacting) particles

Some references to this point:

R Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press 1982.

P Martin, Potts models and related problems in Statistical Mechanics, World Scientific 1991.

R Pathria, Statistical Mechanics, Pergamon 1972.

G Wannier, Statistical Physics, Wiley 1966.

Recall **Yang–Baxter equations:**

system of equations leading to "integrability" of corresponding 2D Statistical Mechanical systems.

One form: Operators $R_i(\theta) \quad (i = 1, 2, ...)$

$$R_i(\theta_1)\, R_{i+1}(\theta_1{+}\theta_2)\, R_i(\theta_2) = R_{i+1}(\theta_2)\, R_i(\theta_1{+}\theta_2)\, R_{i+1}(\theta_1)$$

$$R_i(\theta_1)\, R_j(\theta_2) = R_j(\theta_2)\, R_i(\theta_1) \qquad\qquad i{-}j \neq \pm 1$$

Typically $R_i(\theta)$ is matrix encoding Boltzmann weights of site i in 1D transfer matrix (layer of 2D system)

$$T(\theta) = \prod_i R_i(\theta)$$

YB tells us that transfer matrices with different θ can be arranged to commute. (Hence simultaneously diagonalisable — Bethe ansatz.)

How so? and Where do the pictures come in?

Suppose $\lim_{\theta\to\infty} R_i(\theta)$ defined, invertible ($=: g_i$)

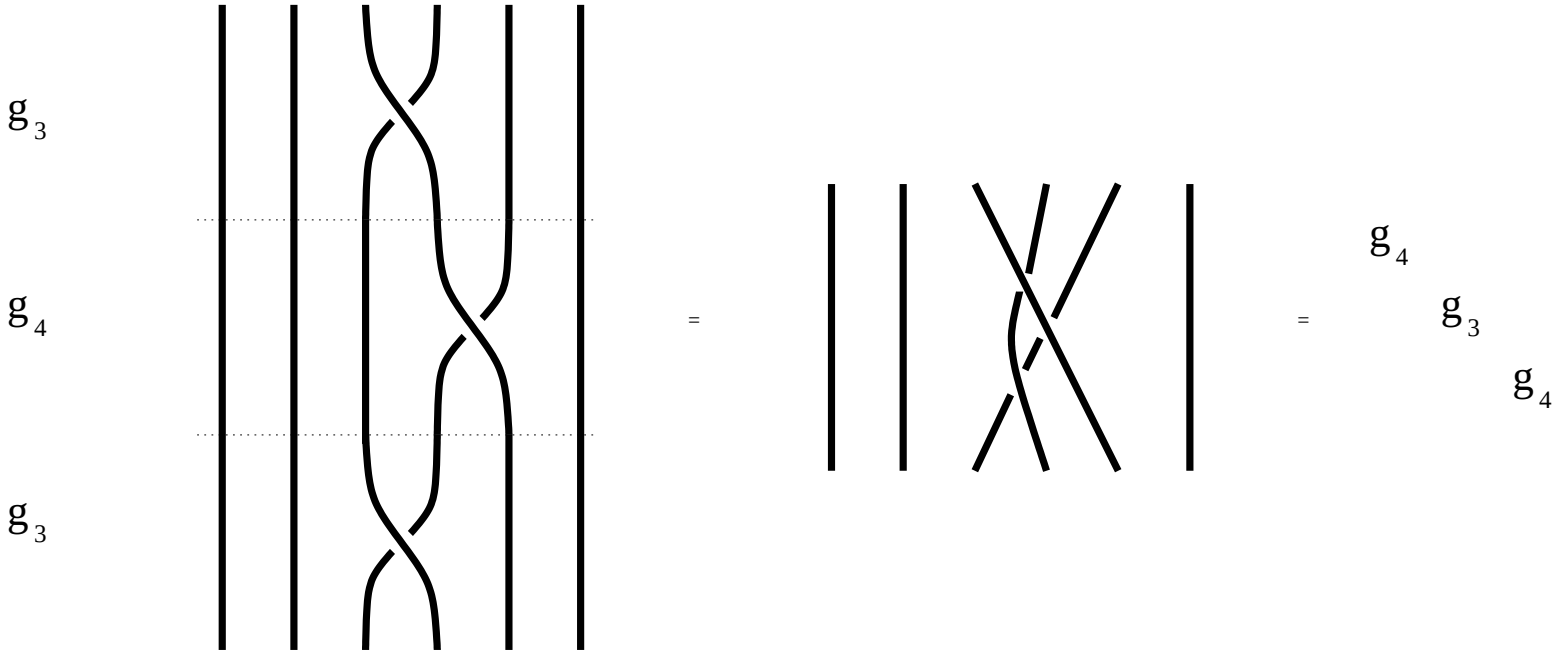
then YB $\to$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

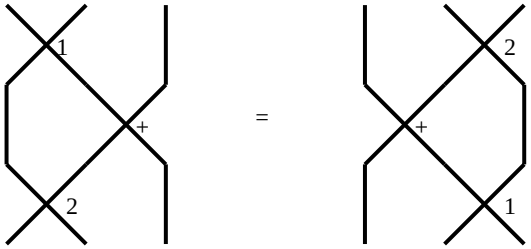
$$g_i g_j = g_j g_i \qquad\qquad i - j \neq \pm 1$$

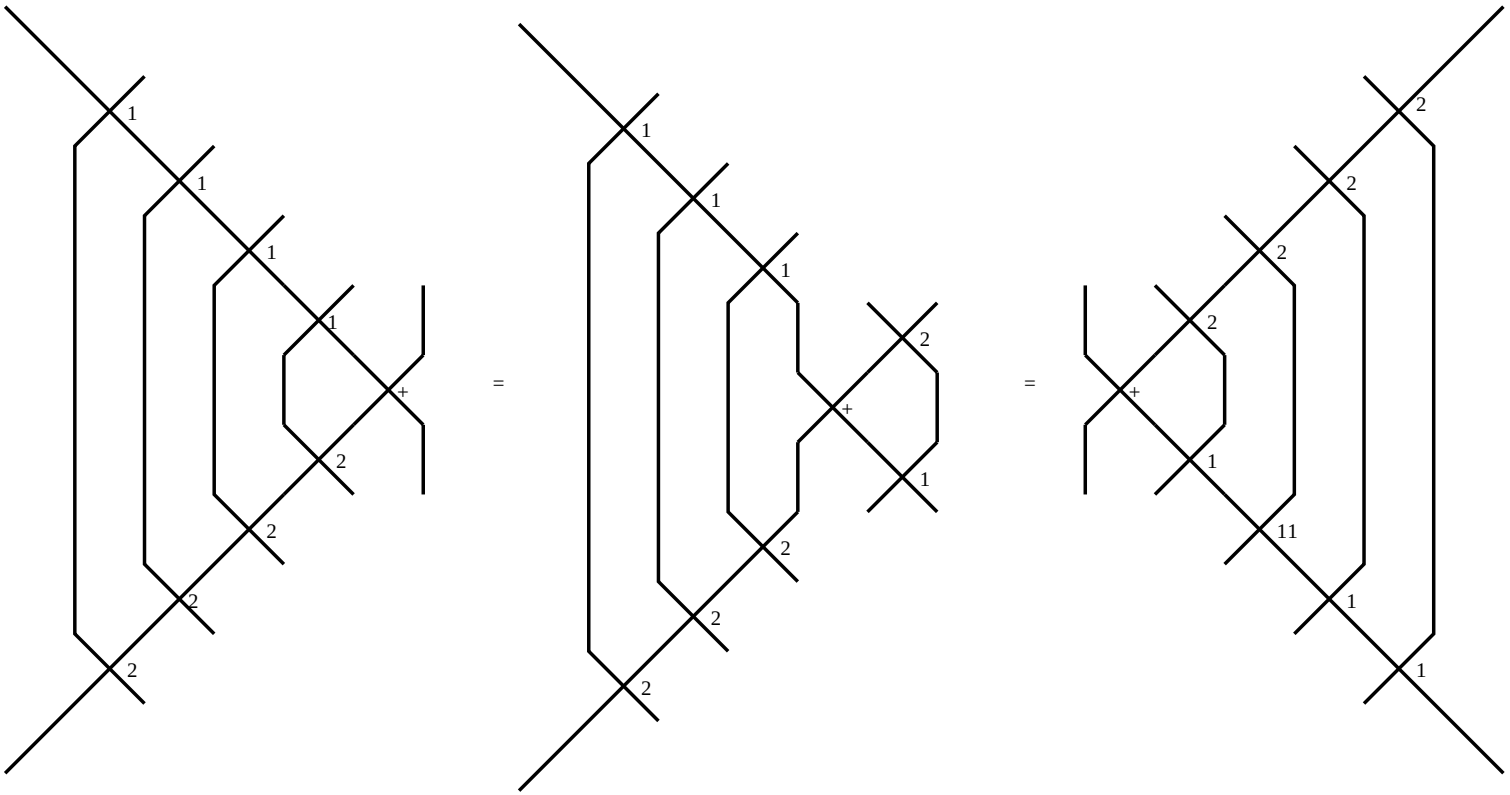
so $\mathbb{C}\langle 1, g_i^{\pm 1} \mid i = 1, 2, ..., n-1\rangle = \mathbb{C}\mathscr{B}_n$

Braid group algebra.



YB:





so suitably tracing over boundaries we get $[T'(\theta_1), T'(\theta_2)] = 0$.

Thus transfer matrices may be simultaneously diagonalised, and may use Bethe ansatz for eigenvectors.

Hence find some eigenvalues

(corresponding to free energy and some correlation functions of the physical system).

Example: $H_n(q) = \mathbb{C}\mathscr{B}_n/(g_i - q)(g_i + q^{-1})$

set $U_i = g_i + q^{-1}$, $q = e^{i\mu}$

$$R_i(\theta) = \sinh(\mu(\theta + i))1 + \sinh(\mu\theta)U_i$$

is meta solution to YB.

That is, every representation of H_n is a solution.

Passing to irreducible representations corresponds to a partial diagonalisation of T.

Thus equivalence classes of irreducible representations (and momentum) label physical observables.

Thus want to study representation theory.

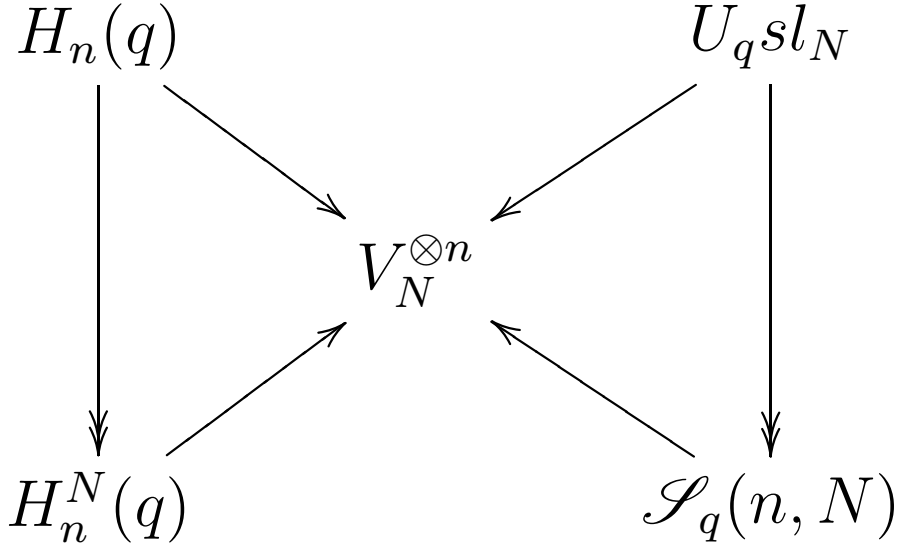
NB. Most "observables" not sensitive to whether system size is $n = 10^{26}$ or $n + 1$ (and certainly can make the same kind of observation on both — e.g. melting point)

thus physics relates representation theory at different n,

and implies stable global limit.

This is not a property of H_n (or $\mathbb{C}\mathscr{B}_n$), so physics picks special *quotients*.

Examples:



$N = 2$: Temperley–Lieb algebra.

$\mathcal{T}_n$:

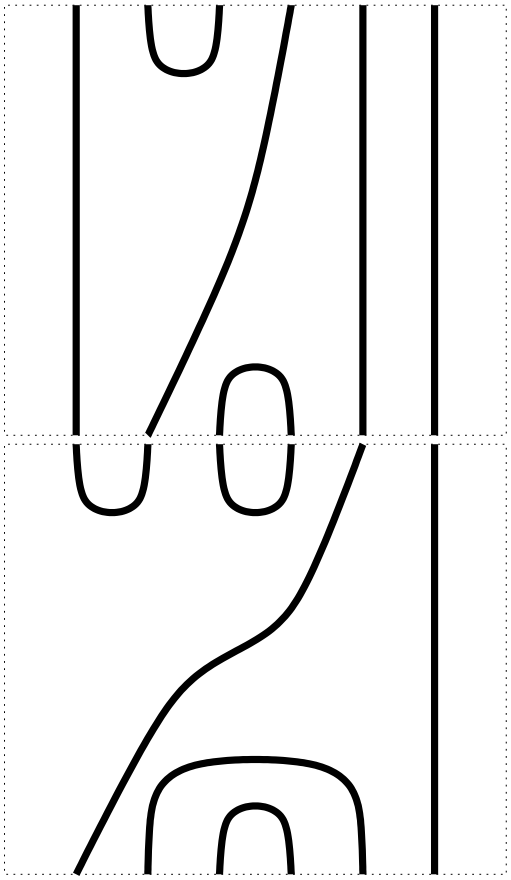
$$\langle 1, U_i \mid i = 1, 2, .., n-1 \rangle$$

$$U_i U_i = [2] U_i$$

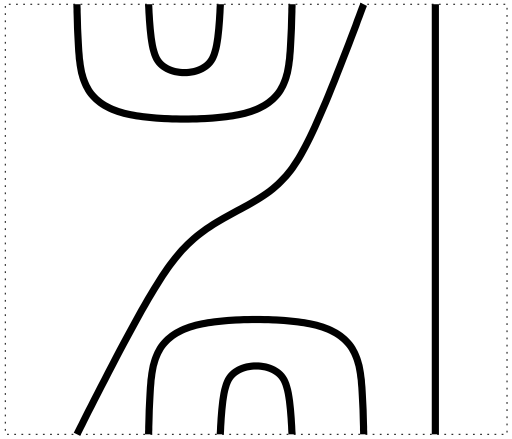
$$U_i U_{i \pm 1} U_i = U_i$$

$$U_i U_j = U_j U_i \qquad i - j \neq \pm 1$$

Basis:

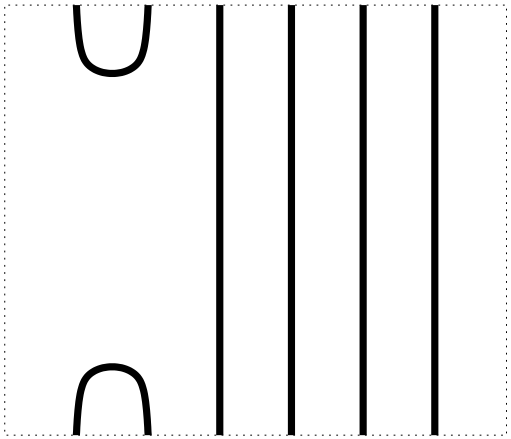


= [2]



NB Number of "propagating" lines non–increasing in any composition.
Thus with

$U_1 \mapsto$



we have $\mathcal{T} \supset \underbrace{\mathcal{T}U_1\mathcal{T}}_{\leq n-2 \text{ prop. lines}} \supset \mathcal{T}U_1U_3\mathcal{T}...$

Recall A an algebra, $ee = e \in A$, $B \cong eAe$, then have functors

$$\begin{array}{rcl} \mathsf{mod}B & \xrightarrow{F} & \mathsf{mod}A \\ M & \mapsto & Ae \otimes_{eAe} M \\ \mathsf{mod}A & \xrightarrow{G} & \mathsf{mod}B \\ N & \mapsto & eN \end{array}$$

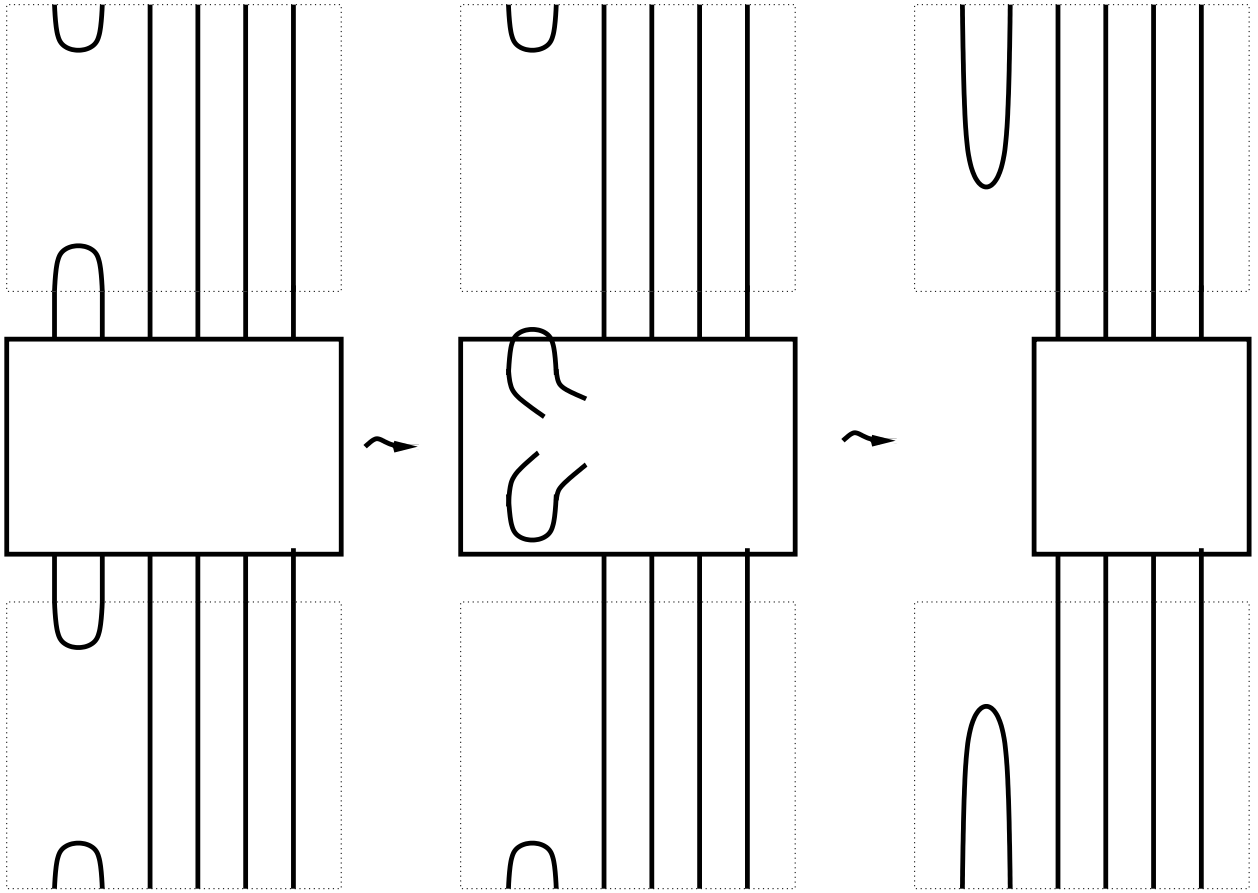
$$M \overset{F}{\mapsto} Ae \otimes M \overset{G}{\mapsto} eAe \otimes M \cong M$$

$$N \overset{G}{\mapsto} eN \overset{F}{\mapsto} AeN$$

If N is simple AeN is either N or 0.

Thus $\{\text{simples of } A\} \leftrightarrow \{\text{simples of } B\} \cup^{\cdot} \{\text{simples of } A/AeA\}$.

Proposition. $[2] \neq 0, \quad U_1 \mathcal{T}_n U_1 \cong \mathcal{T}_{n-2}$.



Proof:

Let $\Lambda_n = \{i \in \mathbb{N} \mid i \leq n,\ i \equiv n \text{ mod.2}\}$

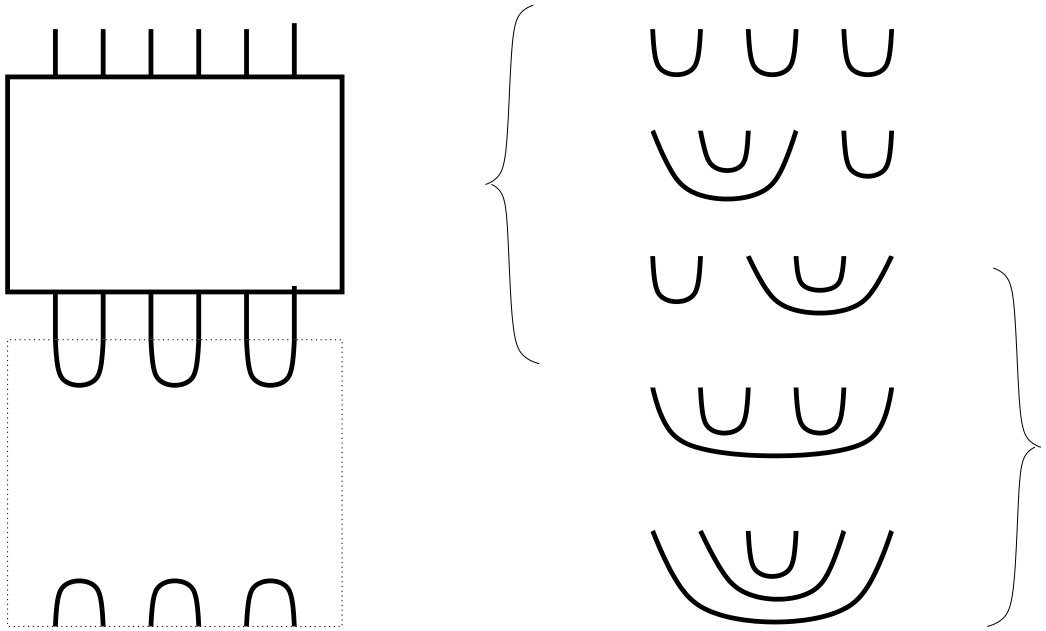
Proposition. Irreducible representations of $\mathcal{T}_n$ indexed by Λ_n.

Proof: by induction. True for $\mathcal{T}_1 = \mathbb{C}1$.
Suppose true for $\mathcal{T}_{n-2}$, then enough to show $|\Lambda_n \setminus \Lambda_{n-2}| = 1$ simples in $\mathcal{T}_n/\mathcal{T}_n U_1 \mathcal{T}_n$. But since only basis elements with $> n-2$ (and hence exactly n) propagating lines survive the quotient we have $\mathcal{T}_n/\mathcal{T}_n U_1 \mathcal{T}_n = \mathbb{C}1$. Done.

Entire $q^r = 1$ and even char.p representation theory may be computed this way.

Example: Simple with label 0 is

$\Delta_0 =$



This gives *free energy*.

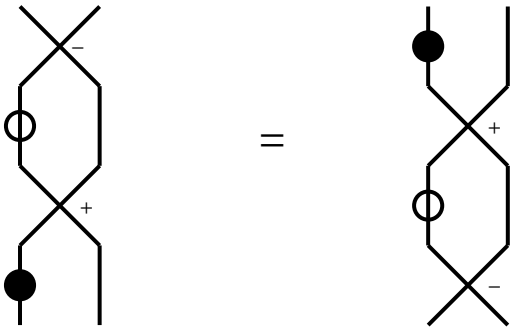
Newer Stuff

YB *comes with* a certain specific treatment of boundaries. Can get control of these with Reflection Equation:

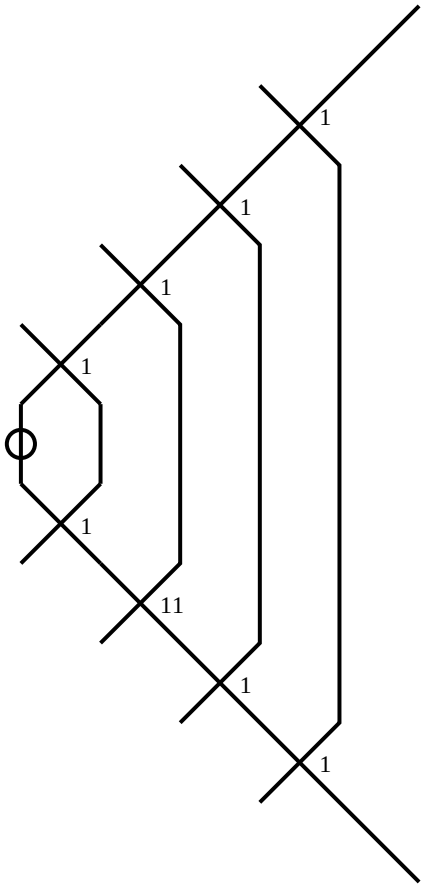
$$R_1(\theta_1 - \theta_2)\; K(\theta_1)\; R_1(\theta_1 + \theta_2)\; K(\theta_2)$$

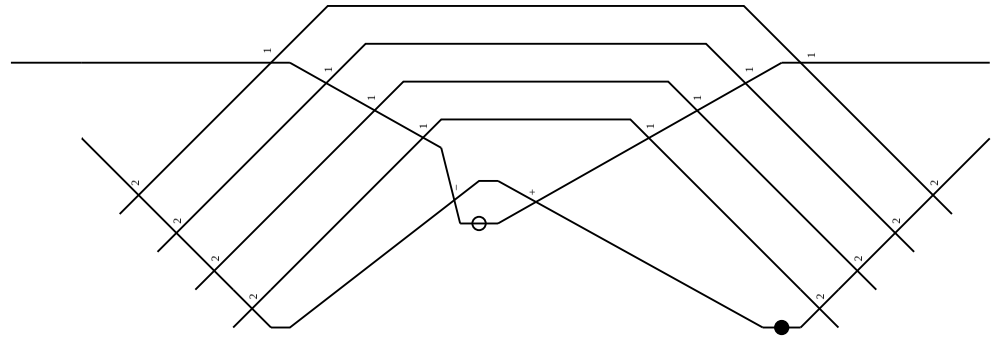
$$= \; K(\theta_2)\; R_1(\theta_1 + \theta_2)\; K(\theta_1)\; R_1(\theta_1 - \theta_2)$$

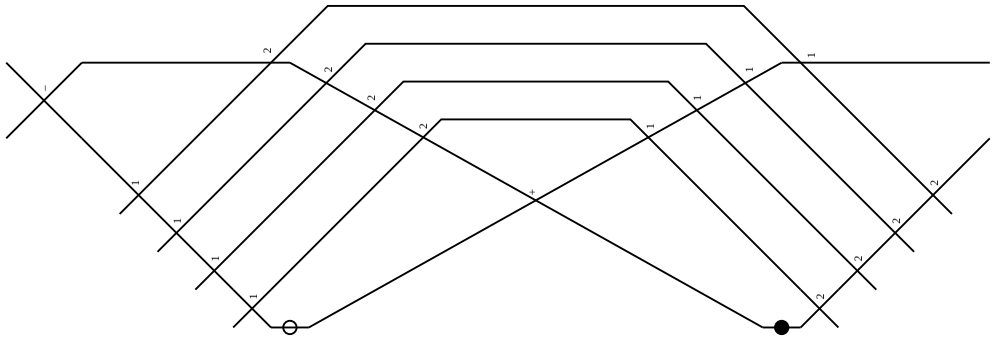
$$R_i(\theta_1)\; K(\theta_2) = K(\theta_2)\; R_i(\theta_1) \qquad\qquad i > 1$$

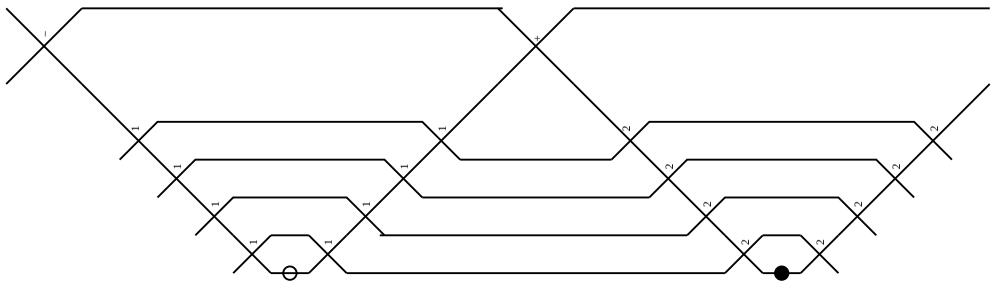


Set $T(\theta_1) =$









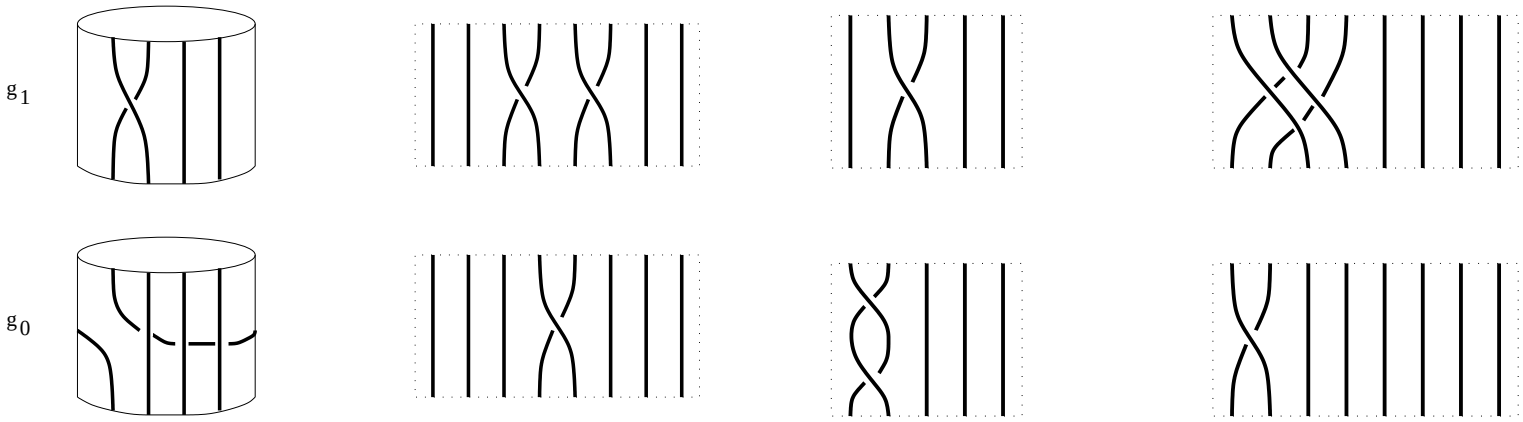
Same $\lim_{\theta\to\infty}$ process gives $\mathscr{B}_n^\circ$:

$$g_1 g_2 g_1 = g_2 g_1 g_2$$

$$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0$$

so we want to study this in the same way.

Realisations:

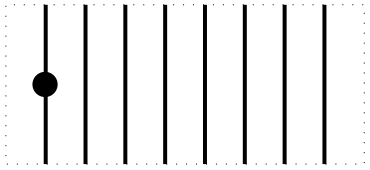


Affine Hecke: $\mathscr{B}_n^\circ/(g_1 - q)(g_1 + q^{-1})$

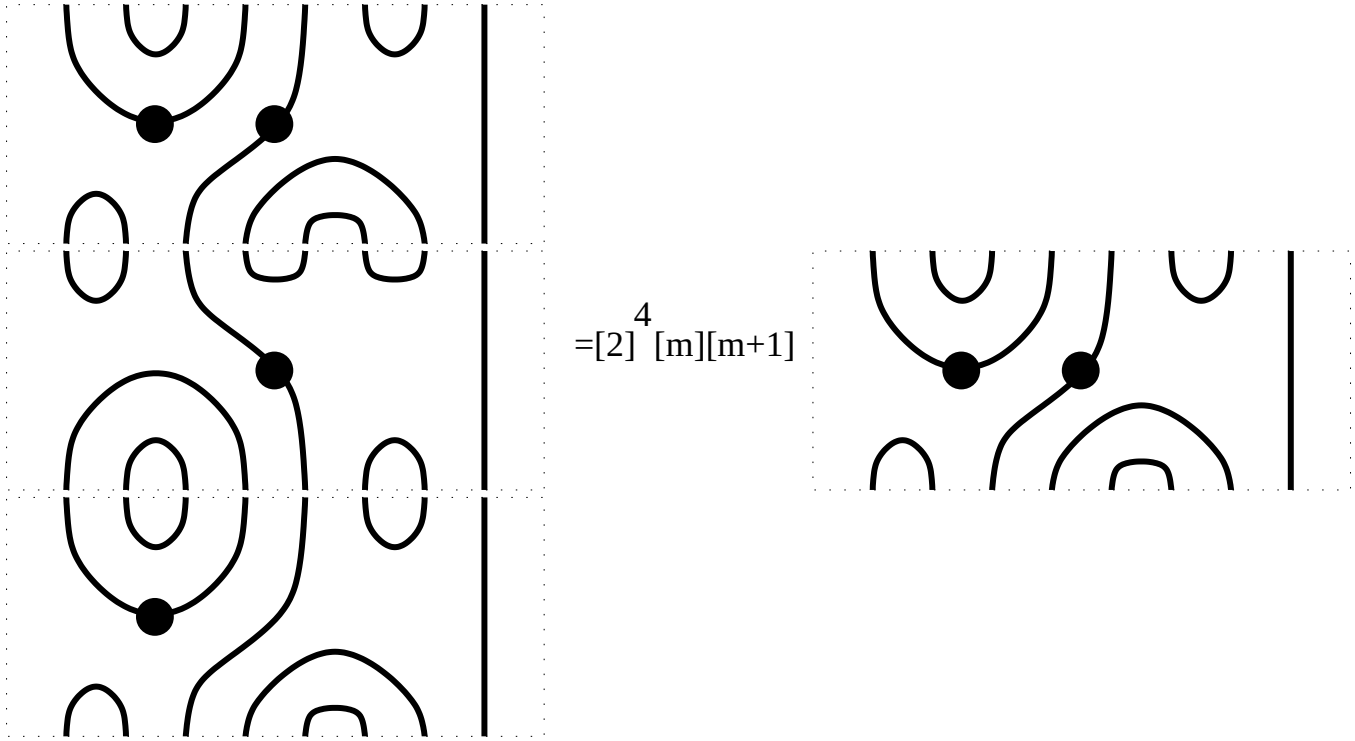
↓

Cyclotomic Hecke, $H(n,d)$: $\prod_{i=1}^{d}(g_0 - \lambda_i) = 0$

↓

Blob, b_n : add $e =$  to $\mathcal{T}_n$.

Basis of $\mathcal{T}$–diagrams as before, but now allow decoration (by blob) of any line which is "exposed" to left hand edge:



This solves RE with $K(\theta) = 1 + b(\theta)e$ (suitable b).

How are the simples indexed?

- New perspective on (and generalisation of) Soergel's procedure for analysing tilting modules (and hence q-group representation theory) via alcove geometry and parabolic Kazhdan–Lusztig polynomials.
- Partition algebra — generalisation to higher dimensional Statistical Mechanics and higher dimensional "pictures".
- Representation theory of affine Hecke algebras.