On Diagram Categories, Representation Theory and Statistical Mechanics

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Abstract

We explain how various categories arising in statistical mechanics may be used as tools in algebraic representation theory.

1 Introduction

The idea of a diagram category has not been precisely defined, but it is (for us) a K-linear category whose object class is naturally a poset, in a way we describe later; and whose hom sets have bases of certain 'diagrams'. Diagrams may not be planar, but are amenable to physical operations such as juxtaposition and reversal (related to their role in describing physical configurations in Statistical Mechanics — see later). The composition of two suitable diagrams may be computed in a way facilitated by juxtaposition; while the reversal operation gives a self contravariant equivalence. By the K-linear property a diagram category thus contains a poset of (diagram) algebras as its end(omorphism)-sets. The hom-sets are therefore bimodules for pairs of diagram algebras (sometimes the same one). These bimodules may be used to construct functors between the categories of (left) modules for the corresponding diagram algebras. These functors can be a powerful tool in representation theory, passing structural data up the poset order. Here we aim to show how to use this machinery in representation theory, conveniently unifying and generalising a number of examples in the literature. Indeed we are interested generally in the utility in algebraic representation theory of constructing collections of algebras as end-sets in a diagram category.

The structure of the paper is as follows. Our approach is guided, informed and motivated (in part) by problems in computational Statistical Mechanics. Accordingly it is appropriate to make some effort to explain this connection. Our aim in the first part of the paper is to do this. We explain by example the key ideas of partition function, correlation function, thermodynamic limit and transfer matrix algebra. In the second part we replace the Physical framework with a corresponding, but free-standing, abstract categorical setting. In the final part we introduce some specific categories (again coming from Statistical Mechanics), and use the tools developed in the previous section to analyse part

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of their representation theory. This brings in further ideas from Schur-Weyl duality, alcove geometry and monoidal categories.

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1.1 Basic notations

For S a set then $\mathbf{E}(S)$ is the set of partitions of S, and $\mathcal{P}(S)$ the power set.

If S, T are sets we write $T^S := \hom(S,T)$ (so if both are finite $|T^S| = |T|^{|S|}$). We think of $f \in T^S$ as a T-valued 'vector' with components indexed by S. If $T = \underline{Q} := \{1, 2, .., Q\}$ we abbreviate T^S slightly to Q^S.

A map $f \in Q^S$ we call a *colouring* of S by Q colours. Thus there is a map from $Q^S \to \mathbf{E}(S)$ given by $s \sim_f t$ if $f(s) = f(t)$. Note that the symmetric group S_Q acts on Q^S by $w \circ f(s) = w(f(s))$, and that this action commutes with the map to partitions.

If G is a simple undirected graph then V_G denotes its vertex set; and E_G its edge set, represented as a set of pairs of vertices. Examples: Define graph A_l by $V_{A_l} = \underline{l}$ and $E_{A_l} = \{\langle i, i+1 \rangle \mid 1 \leq i \leq l-1\}$. For $l > 2$ define $\hat{A}_l$ as the extension of A_l by a further edge $\langle 1, l \rangle$.

Let G, G' be undirected simple graphs. Then graph $G \times G'$ is defined by $V_{G \times G'} = V_G \times V_{G'}$ and $\langle (v_{11}, v_{12}), (v_{21}, v_{22}) \rangle \in E_{G \times G'}$ if $\langle v_{11}, v_{21} \rangle \in E_G$ and $v_{12} = v_{22}$ or $\langle v_{12}, v_{22} \rangle \in E_{G'}$ and $v_{11} = v_{21}$.

Example: $A_{l,m} := A_l \times A_m$ is a rectangular grid.

Suppose G, G' are two such graphs, then $G \backslash_\Gamma G'$ denotes the graph obtained from G by omitting any edges that it has in common with G'.

Example: $C_l := A_{l,2} \backslash_\Gamma A_{l,1}$ is a comb (a ladder with one main strut removed). More generally, for G a graph as before define

$$C(G) = (G \times A_2) \backslash_\Gamma (G \times A_1).$$

Thus $C(A_l) = C_l$.

2 Physics background

For reasons that we shall not fully axiomatise here, most of our diagram categories come from, or have close connections with, computational statistical mechanics. It is not essential fully to understand this setting to understand diagram categories, but it is certainly useful to understand some of its mathematics. Accordingly we begin with a brief review by example (which the reader may skip if desired).

It is appropriate to concentrate on the mathematical aspects, and leave aside such issues as the realm of validity of the basic assumptions of statistical mechanics (but see [21, 5, 24] for example).

2.1 The Potts model

In a classical equilibrium statistical mechanical system one computes the expectation value of an observable as a certain weighted average of its value over the set Σ of possible states of the system. The weighting is determined by the *Hamiltonian* $H : \Sigma \to \mathbb{R}$, which thus defines the model; and the system temperature. Specifically, if $\mathcal{O} : \Sigma \to \mathbb{R}$ is an observable then the expectation value is

$$\langle \mathcal{O} \rangle(\beta) \; := \; \sum_{f \in \Sigma} \mathcal{O}(f) \frac{\exp(\beta H(f))}{Z(\beta)} \tag{1}$$

where

$$Z(\beta) = \sum_{f \in \Sigma} \exp(\beta H(f))$$

the *partition function*; and β is an inverse temperature variable (strictly, if T is temperature then $\beta = \frac{1}{kT}$, where k is Boltzmann's constant).

The example we shall use is the Potts model [4, 9, 24]. Let G be a graph with vertex set V_G and edge set E_G (we shall assume that G is undirected, simple, so elements of E_G can be represented simply as pairs of vertices). The Potts Hamiltonian for G may then be introduced as follows.

Fix $Q \in \mathbb{N}$. We associate to each $i \in V_G$ a *Q-state Potts variable* σ_i, called a *spin*. This is a variable taking values in $\underline{Q}$. Thus the set of all possible configurations of the Potts variables on G is $\underline{Q}^{V_G}$, where for $f \in Q^{V_G}$ we have $\sigma_i(f) = f_i$. Formally then we have Potts Hamiltonian $H_G : Q^{V_G} \to \mathbb{R}$, given by

$$H_G = J \sum_{<ij> \in E_G} \delta_{\sigma_i, \sigma_j} + h \sum_{i \in V_G} \delta_{\sigma_i, 1}$$

Here we shall take coupling constant $J = 1$ and magnetic field parameter $h = 0$. Thus for example if f_+ is the configuration in which every variable takes the value $1 \in \underline{Q}$ we have $H_G(f_+) = |E_G|$.

The partition function is now

$$Z_G(\beta) \; = \; \sum_{\{\sigma_i\}} \exp(\beta H_G) \; := \; \sum_{f \in Q^{V_G}} \exp(\beta H_G(f)) \tag{2}$$

Note that Z_G can be viewed as defining a map from graphs to polynomials in $\exp(\beta)$. However, only certain types of graph are physically interesting, as we shall see later.

With this Hamiltonian the weighted sum (1) models statistically the competing effects of entropy and energetic factors (respectively the sum and the weighting) on the outcome of an observation. For example, this might be the *internal energy*

$$U(T) := kT^2 \frac{\partial ln(Z_G)}{\partial T} = \frac{\sum_{\{\sigma_i\}} H_G \exp(\beta H_G)}{Z_G}$$

Roughly speaking this works as follows.
When β is large (low temperature) the sum will be dominated by states such as f_+, with the largest possible value of H_G. In this sense the model system

appears in an ordered or 'frozen' state.
When β is small (high temperature) all states contribute to the sum essentially equally. Neighbouring variables agree only by chance in a random state, and hence with probability $\frac{1}{Q}$, so the typical value of H_G is proportionally smaller. Thus the system appears in a disordered or 'hot' state.
(In practice one is particularly interested in the *transition* between the cold and hot phases, but this need not concern us here.)

Other important observables include the spontaneous magnetisation, and correlation functions (see [5, 24] for more details). But these will only be definable once we have restricted to suitable types of graph.

The partition function Z_G is thus a fundamental component of any physical computation. The remainder of our physical discussion is motivated purely by the practicalities of computing Z_G, noting that even with $Q = 2$, for a macroscopic system the sum in (2) is of order 2^{α_A} terms, where α_A is Avogadro's number (roughly 10^{27}).

2.2 Computational formalism

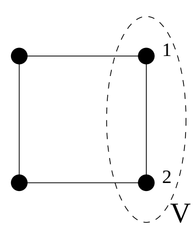
Fix Q, and set $x = \exp(\beta)$. Note that Z_G is an element of $\mathbb{Z}[x]$, since our H_G only takes values in the natural numbers. Let V be a subset of V_G (we shall call it the *external* subset), so that Q^V is the set of configurations of this subset of vertices. For each G, V we may define a 'vector' Z_G^V — an element of $\mathbb{Z}[x]^{(Q^V)}$ whose f-th component is

$$(Z_G^V)_f = \sum_{g \in Q^{V_G} \text{ s.t. } g|_V = f} \exp(\beta H_G)$$

where $g|_V = f$ means that g agrees with f on the subset V. Thus

$$Z_G = \sum_{f \in Q^V} (Z_G^V)_f$$

(**2.1**) Example. Set $Q = 2$ and consider the graph



where subset V is indicated, and its vertices have been labeled. Then

$$Z_G^V = \big((Z_G^V)_{\sigma_1=1,\sigma_2=1}, (Z_G^V)_{\sigma_1=1,\sigma_2=2}, (Z_G^V)_{\sigma_1=2,\sigma_2=1}, (Z_G^V)_{\sigma_1=2,\sigma_2=2}\big)$$

where

$$(Z_G^V)_{\sigma_1=1,\sigma_2=1} = x^4 + 3x^2$$

and so on.
(Remark: Q^V is sometimes called a configuration space or state space, although it is just a set of configurations. It *will* have a role as a basis for a

vector space.)
Further, suppose that graph G may be decomposed into graphs G' and G'' with vertices V in common, but no edges in common. Then

$$Z_G = \sum_f (Z^V_{G'})_f (Z^V_{G''})_f$$

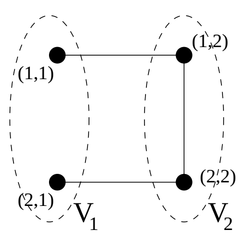
The computational utility of this simple *partition vector* formalism is that we may compute Z_G by 'sewing' smaller systems together. Suppose that we partition the set of *external* vertices V of a graph G into two parts: $V = V_1 \cup V_2$. Then index f in $(Z^V_G)_f$ becomes, trivially, a two-component index ($f = (f_1, f_2)$, say), and we can think of organising our partition vector as a matrix (i.e. a two index tensor): $(Z^V_G)_{f_1,f_2}$. This is a trivial reorganisation of the data, but now we can grow partition functions for larger graphs by iterated sewing:

$$(Z^{V_{13}}_G)_{f_1,f_3} = \sum_{f_2} (Z^{V_{12}}_{G_1})_{f_1,f_2} (Z^{V_{23}}_{G_2})_{f_2,f_3} \tag{3}$$

(**2.2**) EXAMPLE. Consider the comb graph $C_m := A_{m,2} \setminus_\Gamma A_{m,1}$, and partition the complete set of vertices as $V_{C_m} = V_1 \cup V_2$ where

$$V_i := \{(v, i) \mid v \in A_m\}$$

In particular for $m = 2$ we have



and with $Q = 2$, and states ordered as 11,12,21,22:

$$Z^{V_{12}}_G = \begin{pmatrix} x^3 & x & x & x \\ x^2 & x^2 & 1 & x^2 \\ x^2 & 1 & x^2 & x^2 \\ x & x & x & x^2 \end{pmatrix}$$

The graph $C'_m = A_{m,3} \setminus_\Gamma A_{m,2}$ is isomorphic to C_m, differing only in the second coordinates of the labels. Thus

$$Z^{V_1 \cup V_2}_{C_m} = Z^{V_2 \cup V_3}_{C'_m}$$

and we have the identity

$$Z^{V_1 \cup V_3}_{A_{m,3} \setminus_\Gamma A_{m,1}} = Z^{V_1 \cup V_2}_{C_m} . Z^{V_2 \cup V_3}_{C'_m} = (Z^{V_1 \cup V_2}_{C_m})^2$$

and indeed

$$Z^{V_1 \cup V_l}_{A_{m,l} \setminus_\Gamma A_{m,1}} = (Z^{V_1 \cup V_2}_{C_m})^{l-1}$$

and

$$Z_{A_m \times \hat{A}_l} = Tr((Z_{C_m}^{V_1 \cup V_2})^l) \tag{4}$$

(**2.3**) Not every graph decomposes into subgraphs in such a way as to make this approach useful. However not every graph corresponds to an interesting physical system either. In practice it is, fortuitously, the graphs that *are* amenable to this approach that are among those of greatest physical interest. In particular we may consider that the graph represents the crystal lattice, say. That is, the vertices represent a large regular array of molecules in physical space; and the edge terms in H_G represent nearest neighbour interactions between them. The objective here, then, is to compute Z_G for large 'translationally regular' graphs.

Indeed, the graphs of interest are so *very* large, that in practice, in most cases, one must look for stable properties of certain *sequences* of increasingly large graphs in a limit of large graphs. The graph of interest is in this sequence, and one assumes that it is in the stable region. We shall give a concrete explanation of this process shortly. (It is vital to the relationship between representation theory and statistical mechanical observation.)

It is easy to see that our example above generalises directly to cases where G has ('time') translation symmetry, i.e. $G = G_0 \times A_l$ or $G = G_0 \times \hat{A}_l$ for some graph G_0 and natural number l. That is

$$Z_{G_0 \times \hat{A}_l} = Tr((Z_{C(G_0)}^{V_1 \cup V_2})^l) \tag{5}$$

Let us set $\mathcal{T}_{G_0} = Z_{C(G_0)}^{V_1 \cup V_2}$ for simplicity. Then depending on the boundary conditions

$$Z_G = \langle\!\langle \mathcal{T}_{G_0}^l \rangle\!\rangle := \mathcal{V} \mathcal{T}_{G_0}^l \mathcal{V}' \tag{6}$$

where $\mathcal{V}, \mathcal{V}'$ are suitable vectors (or Z_G is given by a trace as above).

It can be shown that (for real x) $\mathcal{T}_{G_0}$ is similar to a real-symmetric matrix, so it is diagonalisable and has a complete orthonormal set of left and right eigenvectors. We write these as:

$$\langle\!\langle i | \, \mathcal{T}_{G_0} = \langle\!\langle i | \, \lambda_i \qquad\qquad \mathcal{T}_{G_0} \, | i \rangle\!\rangle = \lambda_i \, | i \rangle\!\rangle \tag{7}$$

Thus

$$Z_G = \sum_i \alpha_i \lambda_i^l$$

where α_i are some coefficients depending on the boundary conditions. Note by the Perron–Frobenius theorem that $\mathcal{T}_{G_0}$ has a unique positive eigenvalue of largest magnitude. Let us label it as λ_0. Then

$$Z_G = \alpha_0 \lambda_0^l (1 + \sum_{i \neq 0} \frac{\alpha_i}{\alpha_0} \left(\frac{\lambda_i}{\lambda_0} \right)^l)$$

We can now give an example of a stable property in a limit of large graphs. Suppose we consider the sequence of graphs $(G_0 \times A_l \mid l = 1, 2, 3, ...)$. The *free energy* is defined as

$$F_G = \frac{1}{|V_{G_0}| l} \ln(Z_G)$$

so

$$\lim_{l\to\infty} F_G = \frac{1}{|V_{G_0}|} \ln \lambda_0 \tag{8}$$

A detailed illustration of the relationship between λ_0 and Z_G for large but finite l is given in [24]. For our present purposes we simply observe that we have passed from the study of a statistical mechanical model, to the study of the spectrum of a matrix.

Note also, however, that G_0 was held fixed in this exercise. For physics purposes one would require this to grow also, as l does. Thus there is another limit to come, and we will end up studying the stable properties of the spectra of a sequence of matrices. (These matrices will be associated to algebraic representations; and the stability to functors between the module categories for these algebras.)

For convenience of reference we shall call the limit in (8) the Hamiltonian limit ([23] we will not justify the name here); and the overall limit the thermodynamic limit.

(**2.4**) For physical computation the matrix organisation of the data described above is the most useful (it is the *transfer matrix* formalism, see below). However we shall also see later that the following 'tensor' generalisation is of interest (see also [24]).

Let U be a universe of graph vertex labels (so that every $V_G \in \mathcal{P}(U)$). For any graph G, each partition p of a subset of U restricts to a partition p_G of a subset of V_G. For any partition p write $\overline{p} = \cup_i p_i$ (the flattening of p). For given p, G the set $\overline{p_G}$ is called the set of *external* vertices of G. The partition tensor Z_G^p is simply the organisation of the partition vector $Z_G^{\overline{p_G}}$ such that the index f is a multi-index, with one component for each part in p_G. A surgery generalising (3) pertains in the obvious way.

2.3 Correlation functions

As we have seen, we are interested physically in graphs embedded in metric spaces, so that there is a notion of distance. An important observable is then the dependence of the *correlation* of two or more spins on their separation.

In the transfer matrix formalism above the simplest notion of separation on $G_0 \times A_l$ is to separate the spins with respect to the A_l-coordinate (using the obvious notion of distance on A_l). Thus we can define an observable function

$$c(r) = \delta_{\sigma_{a,i},\sigma_{a,i+r}}$$

(labeling spins by graph coordinates, with $a \in V_{G_0}$ any vertex). This example also serves to explain the notion of correlation (in general it depends on the nature of the interactions in the Hamiltonian — the delta function corresponds to the delta functions in H).

So how do we compute expectations in the transfer matrix formalism? Suppose for example that we want to compute the expectation of $\delta_{\sigma_{a,i},1}$. That is, the expectation that we will find the spin at the vertex with coordinates (a, i) taking value 1. (The answer here is obvious on symmetry grounds, but the mechanics of the example will serve.) In the notation of (6) we have

$$\langle \delta_{\sigma_{a,i},1} \rangle = \frac{\mathcal{V}\mathcal{T}_{G_0}^{i} X_a \mathcal{T}_{G_0}^{l-i} \mathcal{V}'}{\mathcal{V}\mathcal{T}_{G_0}^{l} \mathcal{V}'}$$

Here X_a is a diagonal matrix with diagonal entry 1 if the layer configuration has spin with label a taking value 1; and zero otherwise.

The computation of separated correlations is rather more subtle in general, but in essence we compute objects of the form

$$\langle c(r) \rangle \sim \frac{\mathcal{V}\mathcal{T}_{G_0}^{l_1} X_a \mathcal{T}_{G_0}^{r} X_a \mathcal{T}_{G_0}^{l-l_1-r} \mathcal{V}'}{\mathcal{V}\mathcal{T}_{G_0}^{l} \mathcal{V}'} \tag{9}$$

Using (7) we can expand

$$X_a = \sum_{ij} \alpha_{ij} \, |\imath\rangle\langle\jmath|$$

so

$$\mathcal{T}_{G_0}^{l_1} X_a \mathcal{T}_{G_0}^{r} X_a \mathcal{T}_{G_0}^{l-l_1-r} = \sum_{ijk} \alpha_{ij}\alpha_{jk} \lambda_i^{l_1} \lambda_j^{r} \lambda_k^{l-l_1-r} \, |\imath\rangle\langle k|$$

Now recall that $\lambda_0 > \lambda_{i\neq 0}$. It follows that, unless $\alpha_{00} = 0$, then in the limit of large l this sum is dominanted by $i = j = k = 0$, i.e. by a term like λ_0^l. Since the denominator in (9) is also like λ_0^l there is no non-trivial dependence on r. Thus we are interested in X_a such that $\alpha_{00} = 0$. Then supposing that $\alpha_{01} = \alpha_{10} \neq 0$ we get

$$\langle c(r) \rangle \sim \left(\frac{\lambda_1}{\lambda_0} \right)^r = \exp(-r/\eta)$$

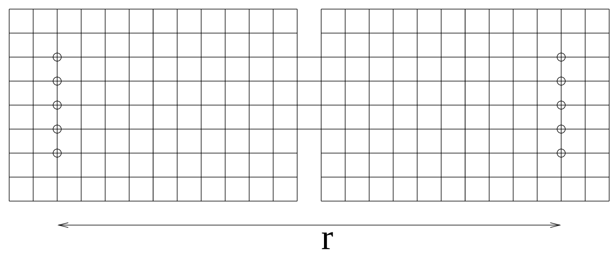
where $\eta = 1/(\ln(\lambda_0) - \ln(\lambda_1))$. Explicit examples can be found in [27], but for our purposes the point is that the observable decay length scale (the correlation length) η depends on the gap between λ_0 and a subsequent element of the spectrum. Once again then, we may bypass the Hamiltonian limiting process (as in (8)) by attending to the spectrum directly.

We shall see shortly that $\mathcal{T}_{G_0}$ can be expressed as a representation matrix for an element of an algebra, A_{G_0} say. The irreducible decomposition of this representation is thus part of the spectral decomposition of $\mathcal{T}_{G_0}$, and hence tied to the correlation length observables of the model. That is, we may label the spectrum of correlation lengths, at least in part, by the irreducible representations of A_{G_0} (or if you prefer, we may label the irreducibles by correlations).

The correlation length η (and other such) should again have a stable limit as G_0 is taken suitably large, and it should certainly be possible to define a given correlation function (the correlation of a single spin with a single spin, say) throughout the sequence. By the correspondence above this tells us to expect that the sequence of algebras is unified by having *fibres* of irreducible representations running through it. That is, a fibre picks out the representation from each algebra associated to a given correlation.

In Section 6.1 we shall see that there are functors between the module categories for these algebras which precisely fix such fibres, and lead to beautiful global limit algebras.

The kind of different correlation functions that arise turn out to come from observing the r-dependence in correlations involving multiple different spins in each layer (cf. the one spin in each layer case in (9)) [27]. Further structure is then revealed by noting that a trivial difference between the thermodynamic limit and a finite 'width' lattice is that, on a finite lattice there *are* only finitely many spins in a layer to be observed — so only finitely many spin correlations can be observed. Later we shall consider what we expect to observe on composite lattices (of varying width) such as



where a many-spin correlation can be observed, but the bottleneck prevents this from being independent of lower correlations; and give this a representation theoretic and categorical interpretation.

2.4 The Potts model/dichromatic polynomial paradigm

We now need to recast the partition function in a different form. Expanding the exponential

$$Z_G(\beta) \;=\; \sum_{\{\sigma_i\}} \prod_{<ij>\in E_G} \exp(\beta\delta_{\sigma_i,\sigma_j}) \;=\; \sum_{\{\sigma_i\}} \prod_{<ij>\in E_G} (1+v\delta_{\sigma_i,\sigma_j})$$

where $v = \exp(\beta) - 1$. Expanding this we have

$$Z_G(\beta) \;=\; \sum_{\{\sigma_i\}} \sum_{G'\in\mathcal{P}(E_G)} \prod_{<ij>\in G'} v\delta_{\sigma_i,\sigma_j} \;=\; \sum_{G'\in\mathcal{P}(E_G)} v^{|G'|} Q^{\#(G')} \qquad (10)$$

where $|G'|$ is the number of edges and $\#(G')$ is the number of connected components of G' regarded as a subgraph of G in the obvious way. Example: Figure 1(i) shows a subgraph G' on a square lattice, with $\#(G') = 12$.

Equation (10) holds for any *given* Q, but we can now consider the RHS of (10) in its own right, as a 'dichromatic' polynomial in variables v and Q. Example:

$$Z_{A_2} = Q^2 + vQ$$

The objective now is to compute Z_G in *this* form, for the same kind of large graphs as before. The exercise, therefore, is to construct a *transfer matrix* formulation in which to compute it, analogous to the fixed Q example above. That is, we seek a matrix $\mathcal{T}$ such that

$$Z_G(\beta) \;=\; Tr(\mathcal{T}^l)$$

generalising (4) and (5). Passing to $\mathcal{T}$, where possible, allows us to study Z_G by studying eigenvalues of $\mathcal{T}$:

$$Tr(\mathcal{T}^l) \;=\; \sum_i \lambda_i^l$$

(There are several reasons for casting the partition function in the dichromatic form. For our purposes the point is that it *has* a $\mathcal{T}$, and a particularly mathematically interesting one.)

We also require, for Z_G to be physically interesting, that G embeds in some Euclidean space and that its edges, and hence the Potts interactions, are local. That is, the terms $\delta_{\sigma_i,\sigma_j}$ in H_G connect near neighbour vertices in the Euclidean embedding. (NB, This is exemplified by the graph in equation (4), with $A_m \times \hat{A}_l$ embeddable in $\mathbb{R}^2$ in an obvious way (perhaps using cylindrical boundary conditions).) It is this locality which moderates the size of the state space Q^V. However even this local graph embedding is not enough to make the interactions in the dichromatic polynomial formulation local, since $\#(G')$ is not local. Instead we need to introduce an entirely different state space (cf. Q^V).

Although the restriction is not necessary, for the sake of simplicity we will describe this by considering the example of the m-site wide square lattice: the graph $A_m \times A_n$.

In adding an extra layer to this graph/lattice, i.e. going from graph $A_m \times A_n$ to $A_m \times A_{n+1}$, say, we are adding $2m-1$ edges. As ever in a transfer matrix formalism, the problem is to find a set of states which keep enough information about the old lattice G to determine $\#(G')$ for the new one. It will be evident that each state must record which of the last layer of vertices in G are connected to each other (by some route in G — cf. Figure 1 (i), (ii) and (iii)). Neither the details of the connecting routes nor any other information is needed, thus our state set is simply contained in the set of partitions of the last layer of vertices (Figure 1(iii)). It is straightforward to see that (in the square, or otherwise plane, lattice case) precisely the set of 'plane' partitions are needed. These are the partitions realisable by noncrossing paths in the interior when the vertices are arranged around the edge of a disk.

In other words the partition vector for graph G and exterior vertex set V, which we shall denote $Z_G^V(Q)$, has entries $(Z_G^V(Q))_c$, where c is a partition of the vertices in V. The c-th entry

$$(Z_G^V(Q))_c \;=\; \sum_{G' \in \mathcal{P}(E_G) \,|\, G' \sim c} v^{|G'|} Q^{\#_c(G')}$$

is a relative version of $Z_G(Q)$ (in the final form in (10)), including only summands in which vertices in V appear in the same connected component precisely when they are in the same part in c; and where $\#_c$ is a version of $\#$ that does not count the components involving vertices in V.

(**2.5**) Example. The labelled graph in Example (2.1) now has a 2-component state set: v_1, v_2 connected; v_1, v_2 not connected. The (now Q-dependent) partition vector relative to this set is

$$Z_G^V(Q) \;=\; (v^4 + 4v^3 + 3v^2 Q + vQ^2, \quad 3v^2 + 3vQ + Q^2)$$

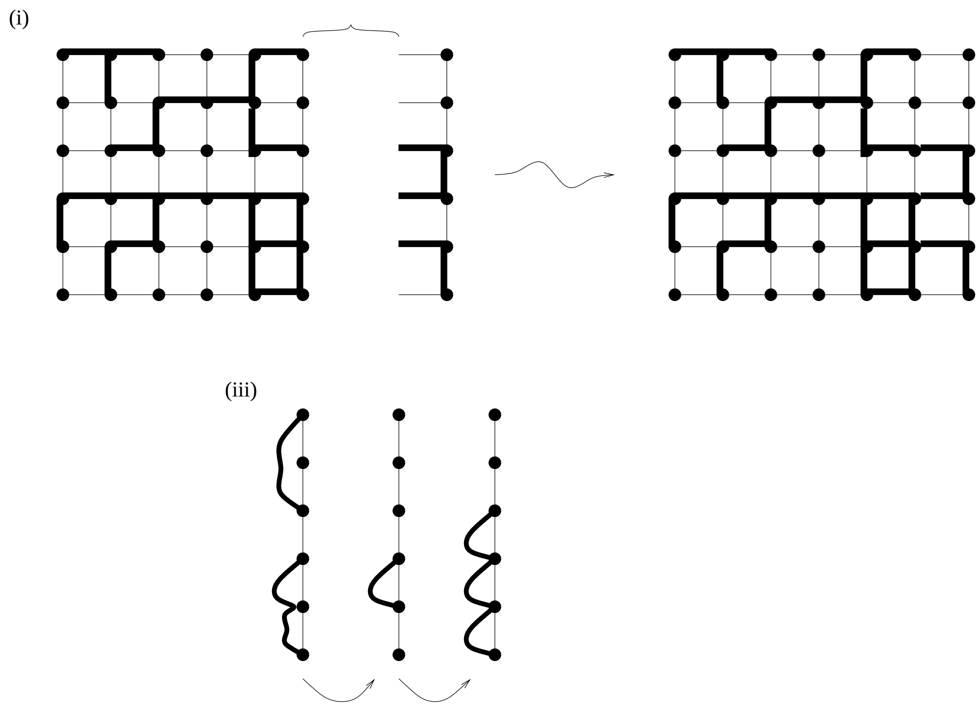


Figure 1: (i) A subgraph of a square lattice and an extra layer. (ii) The corresponding new subgraph. (iii) A sequence showing: the connectivity of the original subgraph (running $\# = 12$); the connectivity after adding the new horizontal edges (running $\# = 12+3$); the connectivity after adding the new vertical edges (running $\# = 12+3-2$).

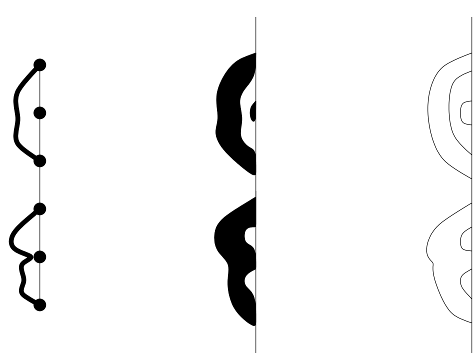
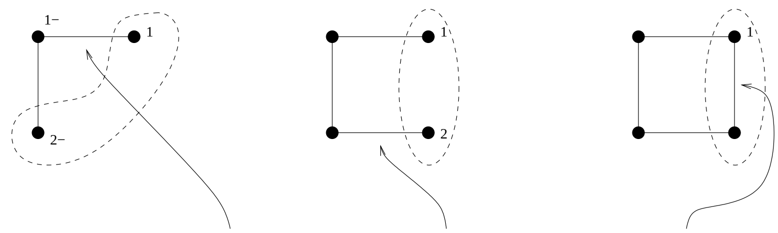


Figure 2: Mapping planar Whitney diagram to TL diagram.

where the first element is Z_G with the condition that v_1, v_2 are connected to each other, and we do not yet count this connected component in $\#(G)$; and the second element is Z_G with the condition that v_1, v_2 are not connected to each other, and we do not yet count either connected component in $\#(G)$. In other words

$$Z_G(Q) = Z_G^V(Q) \begin{pmatrix} Q \\ Q^2 \end{pmatrix}$$

We can even grow the graph one edge at a time:

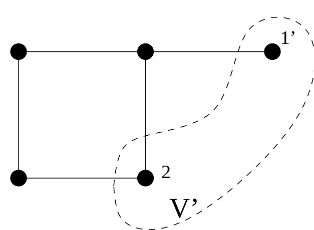


$$\underbrace{\begin{pmatrix} v & 1 \end{pmatrix} \begin{pmatrix} v & 1 \\ 0 & v+Q \end{pmatrix}}_{\begin{pmatrix} v^2 & 2v+Q \end{pmatrix}} \begin{pmatrix} v & 1 \\ 0 & v+Q \end{pmatrix} \begin{pmatrix} v+1 & 0 \\ v & 1 \end{pmatrix}$$

$$\begin{pmatrix} v^2 & 2v+Q \end{pmatrix} \begin{pmatrix} Q \\ Q^2 \end{pmatrix} = Qv^2 + 2vQ^2 + Q^3$$

Here the first (vector) factor is associated to the first vertical edge, with the column position determining whether the two vertices are connected or not. The second (matrix) factor is for the first horizontal edge. In this matrix the 11 position is Z_G with the contraint that both v_{1-}, v_{2-} (the vertices at the 'trailing' end of the graph) and v_1, v_{2-} (the vertices at the 'leading' end of the graph) are connected; and so on.
Note that any partial computation may be completed to give a partition function by post-multiplying by the appropriate column vector to take account of the components in $\#(G)$ not included in $\#_c(G)$. Thus the final equality above computes $Z_G(Q)$ for the leftmost of the graphs shown.

The labelled graph

$G' =$ 

has

$$Z_{G'}^{V'}(Q) \;=\; Z_G^V(Q).Z_{edge}^{V \cup V'}(Q)$$

$$= (v^4 + 4v^3 + 3v^2Q + vQ^2, \quad 3v^2 + 3vQ + Q^2) \begin{pmatrix} v & 1 \\ 0 & v+Q \end{pmatrix}$$

$$= (v(v^4 + 4v^3 + 3v^2Q + vQ^2), \quad v^4 + 4v^3 + 3v^2Q + vQ^2 + (v+Q)(3v^2 + 3vQ + Q^2))$$

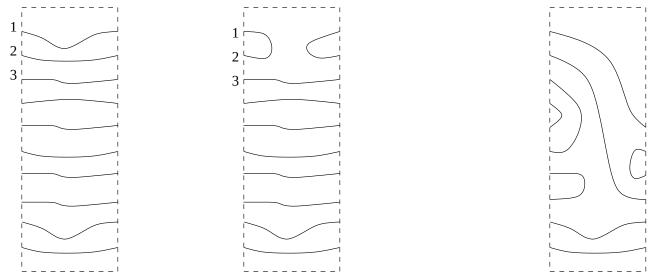


Figure 3: TL identity diagram, diagram D_1, and a diagram with different numbers of in and out-vertices.

(**2.6**) Pictures of partitions as collections of paths as in Figure 1(iii) are called Whitney diagrams [24]. If instead we represent plane partitions by boundaries of connected regions (see Figure 2 for an example) these diagrams become Temperley–Lieb (or boundary) diagrams on the disk. Note that these are plane *pair* partitions (of double the number of vertices). Note that the original lattice itself has all but disappeared from the state space (replaced by a topological/combinatorial construct).

(**2.7**) Finally we note that in order to compute correlation functions some further information must be retained (essentially the details of connections also with the vertices on the left-hand side of the graph in Figure 1). This corresponds to Temperley–Lieb diagrams on the rectangle – i.e. with both in-vertices and out-vertices. See Figure 3 for examples — these are, specifically, two $(10, 10)$-diagrams followed by a $(10, 6)$-diagram. These diagrams may be composed by juxtaposition at one edge of the rectangle when the number of states agrees. With an appropriate reduction rule for interior loops (replace by a factor $\delta = \sqrt{Q}$) this becomes the Temperley–Lieb algebra (indeed category, indeed monoidal category — see later).

NB, casting the state space in this form is certainly beautiful and computationally convenient (see [27]), but it is not the same as integrability. Since the Potts model is integrable under certain conditions solutions to the Yang–Baxter equations can be constructed using Temperley–Lieb diagrams, but such exercises will not be our focus in the present paper.

(**2.8**) The following set of Temperley–Lieb diagrams generate the Temperley–Lieb algebra on n vertices (i.e. n in- and n out-vertices). The identity diagram is the rectangle in which each in-vertex is connected to the corresponding out-vertex. The diagram D_i is like the unit except that in-vertices i and $i+1$ are connected, and out-vertices i and $i+1$ are connected. (See Figure 3.) The generators are $D_1, .., D_{n-1}$. As already noted, composition $B \circ C$ is by juxtaposition so that the out-vertices of B meet the in-vertices of C (becoming internal points in the new diagram). Thus for example

$$D_i \circ D_i = \sqrt{Q} D_i \tag{11}$$

The state space we have constructed induces a representation R of these elements. The transfer matrix is then

$$\mathcal{T} = \prod_i (1 + \frac{v}{\sqrt{Q}} R(D_{2i}) \prod_i (\frac{v}{\sqrt{Q}} + R(D_{2i-1}))$$

and

$$Z(\beta) = Tr(\mathcal{T}^n)$$

Finally, the trace can be decomposed into the irreducible representations in R (amongst other partial diagonalisations). The close relationship this engenders between representation theory and correlation functions (see e.g. [27]) survives passage from our chosen example up to a considerable degree of generality.

Since we need to be able to understand correlation functions stably in the thermodynamic limit we need to be able to understand representation theory in an analogous limit. This leads us to consider towers of algebras with suitable stable limits. One (algebraic) notion of stability here is provided by functors between module categories built from bimodules (see later).

A natural setting in which we find towers of algebras and bimodules is K-linear categories, such as diagram categories. Accordingly we are now ready to introduce and study some more general diagram categories.

3 General category notations

We assume familiarity with some category theory basics. See [1, 3, 15, 20, 29]. In this section however we recall a few points, in order to establish some general notation. In section 3.2 we develop one or two notions specific to diagram algebras.

(**3.1**) A category $C = (\mathbb{O}_C, \hom_C, \circ)$ is a triple consisting of a class of objects $\mathbb{O}_C$; a class of homs consisting of a set $\hom_C(s,t)$ for each pair s,t of objects; and for each triple s,t,u of objects a composition $\circ$,

$$\circ : \hom_C(s,t) \times \hom_C(t,u) \quad \to \quad \hom_C(s,u) \tag{12}$$

$$(f,g) \quad \mapsto \quad f \circ g \tag{13}$$

obeying

$$f \circ (g \circ h) = (f \circ g) \circ h$$

and such that every $(\hom_C(s,s), \circ)$ is a monoid and $f \circ 1_s = f$, $1_t \circ f = f$ whenever defined [3, §0.11].

(N.B. Here we use *diagram* rather than *function* order for the objects labeling a hom set in this notation. This suits diagram categories, where composition is by diagram juxtaposition, but not necessarily categories whose homs are set maps. In practice it will be clear from context which notation is being used.)

We sometimes write $\mathrm{End}_C(s)$ for $\hom_C(s,s)$, and write 1_s for the identity element in $\mathrm{End}_C(s)$.

We shall assume (merely for notational simplicity) that all our categories are small.

(**3.2**) EXAMPLE. Category $\mathbb{S}$ is the category of sets, set maps and map composition [3, §0.11Ex(1)] (we say that there is a unique map from $\emptyset$ to any set). Category $\mathbb{S}_{Fin}$ is the full subcategory on the class of finite sets.

(**3.3**) A functor $F : A \to B$ is

- full (resp. faithful) if all hom-set maps
$$F : \hom_A(S,T) \to \hom_B(FS, FT)$$
are surjective (resp. injective);
- isomorphism dense if for every object T in B there is an object S in A such that $F(S)$ is isomorphic to T;
- an embedding if injective on homs;
- an equivalence if it is full, faithful and isomorphism dense [1].

(**3.4**) DEFINITION. A skeleton for a category is a full, isomorphism dense subcategory in which no two objects are isomorphic [20, Ex4.1][1].

(**3.5**) EXAMPLE. The assembly of sets in $\mathbb{S}_{Fin}$ into cardinality classes induces a corresponding set of isomorphisms between hom sets.

$$f_S : S \xrightarrow{\sim} S'$$

$$\begin{array}{rcl} f : \hom(S,T) & \to & \hom(S',T') \\ g & \mapsto & f_T \circ g \circ f_S^{-1} \end{array} \tag{14}$$

Associate a representative element of each class to each cardinality ($\underline{n}$ to n, say). We may then construct a category whose objects are the set $\mathbb{N}$ of finite cardinals, and with $\mathrm{Hom}(m,n) = \hom(\underline{m}, \underline{n})$. The functor which takes object n to $\underline{n}$ and identifies the corresponding hom sets is obviously isomorphism dense and full. This is thus a full subcategory of $\mathbb{S}_{Fin}$, from which the rest of $\mathbb{S}_{Fin}$ may be constructed. We have:

PROPOSITION. *This* $(\mathbb{N}, Hom(-,-), \circ)$ *is a skeleton for* $\mathbb{S}_{Fin}$. □

(**3.6**) Let K be a ring (respectively a field). A K-linear category is a category in which each hom set is a K-module (respectively a K-vector space) and the composition map is bilinear.
A *basis* for a K-linear category $C = (\mathbb{O}_C, \hom_C(-,-), \circ)$ is a subset $\hom_C^o$ of $\hom_C$ such that
$$\hom_C^o(m,n) = \hom_C^o \cap \hom_C(m,n)$$
is a basis for $\hom_C(m,n)$. (For flexibility we may sometimes write $\hom_C^o$ when C may be K-linear or not. In case it is *not* we shall intend $\hom_C^o = \hom_C$.)
Any category C extends K-linearly to a K-linear category KC.
Let R be a ring in K. An *R-calculus* for a K-linear category is a basis such that every composition has structure constants in R, with at most one non-zero.
EXAMPLE. I) Fix a ring (respectively field) K. Then $K-\mathrm{mod}$ (respectively Vect) is the category of left K-modules (respectively K-vector spaces).
EXAMPLE. II) Fix K and $\delta \in K$. Category
$$\mathcal{C}_{T(\delta)} = (\mathbb{N}, \hom_{T(\delta)}(-,-), \circ)$$

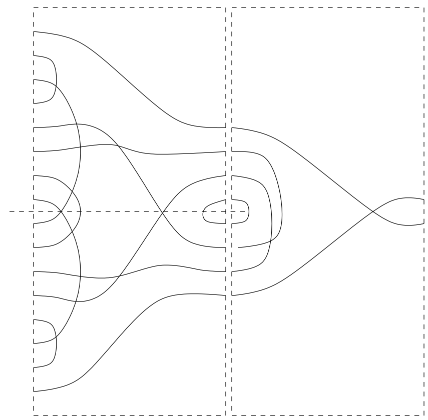


Figure 4: Composition by concatenation of a (16,8)-diagram and an (8,2)-diagram in the LR Brauer category.

is the K-linear category with a calculus of TL diagrams as discussed in Section 2. Here $\hom_{T(\delta)}(m,n)$ is the K-space with basis the set of (m,n)-Temperley–Lieb diagrams (without loops).

Example. III) An (m,n)-Brauer diagram is a diagram of a pair partition of $m+n$ vertices like a Temperley–Lieb diagram except that crossings are allowed (that is, the restriction to plane pair partitions is removed). The composition of Temperley–Lieb diagrams generalises in the obvious way to Brauer diagrams, giving rise to the Brauer category $\mathcal{C}_{B(\delta)}$.

Example. IV) An LR Brauer diagram is a Brauer diagram that is invariant under reflection in a line connecting the edges of the diagram on which the vertices reside (such as in figure 4). Such diagrams generate a subcategory $\mathcal{C}_{B'(\delta)}$ of $\mathcal{C}_{B(\delta)}$.

(**3.7**) If C is a K-linear category then each $End_C(s)$ is a K-algebra. Further $\hom_C(s,t)$ is a left $\mathrm{End}_C(s)$-module and a right $\mathrm{End}_C(t)$-module. Thus for each pair of objects s,t we may construct functors

$$\mathfrak{F}_{s,t} : \mathrm{End}_C(s) - \mathrm{mod} \;\to\; \mathrm{End}_C(t) - mod$$

$$M \mapsto \hom_C(t,s) \otimes_{\mathrm{End}_C(s)} M$$

Note [13] that $\mathfrak{F}_{s,t}$ is right exact, and exact if $\hom_C(t,s)$ is a flat (e.g. projective) $\mathrm{End}_C(s)$-module.

This idea has been used for studying diagram categories for some time (see [24, §9.5] for example). On a more basic (but still useful) level studying the modules ${}_sM_t = \hom_C(s,t)$ directly is itself a way of studying the structure of $\mathrm{End}_C(s)$. We have an $\mathrm{End}_C(s)$-module for each object, and in particular if there is an object ω such that $\mathrm{End}_C(\omega)$ is scalar then we have an inner product on $\hom_C(\omega,s)$ via $\hom_S(\omega,s) \times \hom_C(s,\omega) \to \hom_C(\omega,\omega)$ (and the opposite isomorphism). If the latter map is surjective (as it usually is — see later) the $\mathrm{End}_C(s)$-module ${}_sM_\omega$ is even indecomposable projective.

We can generalise this considerably as follows.

(**3.8**) Let $\mathcal{C}, \mathcal{C}'$ be categories. The *functor category* $(\mathcal{C}, \mathcal{C}')$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{C}'$, and whose homs are natural transformations.
EXAMPLE. Note for any category C and object F in C that the hom functor $\hom_C(F,-)$ takes objects to hom sets, and so is an object in the functor category $(C, \mathbb{S})$. The action of $\hom_C(F,-)$ on a hom $f \in \hom_C(A,B)$ say is to take it to a set map $\hom_C(F,-)f$ in $\hom_{\mathbb{S}}(\hom_C(F,A), \hom_C(F,B))$ given by:

$$\hom_C(F,-)f(F \xrightarrow{u} A) = A \xrightarrow{f} B \ \ F \xrightarrow{u} A = F \xrightarrow{fu} B$$

Note that homs is C, such as u, f, are not necessarily set maps, but we have used function notation for the sake of definiteness.

Recall that the Yoneda embedding (see e.g. [20]) identifies C^{op} with the category of hom functors within $(C, \mathbb{S})$. This is via the functor h_- given by $h_-(F) = \hom_C(F,-)$ and, for $f \in \hom_{C^{op}}(F,G)$ say

$$h_-(f) : \hom_C(F,-) \to \hom_C(G,-)$$

is given by

$$h_-(f)(A) = \hom_C(-,A)f$$

that is

$$h_-(f)(A) : \hom_C(F,A) \to \hom_C(G,A)$$

$$u \mapsto uf$$

regarding f as being in $\hom_C(G,F)$.

By the preceding remark, if C is K-linear (so its hom sets are K-modules) then the embedding $h_- : C^{op} \to (C, \mathbb{S})$ is actually into $(C, K-\text{mod})$.

Specifically consider $\mathcal{C}_T$. An object F is mapped to a functor $\hom_{\mathcal{C}_T}(F,-)$. And a hom d in $\hom_{\mathcal{C}_T}(F,G)$, such as a diagram: for each object A this maps to a K-module (indeed right $\mathrm{End}(A)$-module) morphism constructed by attaching the diagram f to the F 'end' of each diagram in $\hom(F,A)$.

3.1 Representation theory

(**3.9**) By the remarks at the end of the previous section we are interested in the representation theory of certain K-linear categories. Let $\mathcal{C}$ be such a category. Then in the most general case we are interested in $(\mathcal{C}, \mathcal{C}')$, where $\mathcal{C}'$ is some other category. Again from the previous section we are interested in particular in the simple representations, over various fields, of the algebras of endomorphisms in $\mathcal{C}$ (what might be called reductive representation theory). In practice, to gain access to these representations, it is useful to use the methods of K-orders and K-lattices [13] where K is a ring 'common' to all the fields of interest. That is we are interested in the functor category $(\mathcal{C}, K\text{–mod})$. Here functors between K-linear categories will be assumed to be K-linear, so that they only need to be defined on generators.

(One could analogously assume that if $\mathcal{C}$ is a tensor category [10] then functors are monoidal, but this is too restrictive for reductive representation theory. However monoidal functors do play an interesting role, as we shall see later.)

EXAMPLE. As noted, the subcategory of $\mathcal{C}$ with a single object N, call it $\mathcal{C}_{|N}$, will be a K-algebra, and the restriction $(\mathcal{C}_{|N}, K{-}\text{mod})$ will be the category of representations of this algebra. That is, each functor has a single K-module as object image, and a set of endomorphisms of this module as hom (i.e. algebra element) images. (And the natural transformations will be intertwiners between such $K\mathcal{C}_{|N}$-modules.)

(**3.10**) Let $d \in \hom_C(x, y)$ a hom in some category C. A factorisation of d in C is any composition $d = d_1 \circ d_2 \circ ... \circ d_l$. The set of all homs d_i appearing in factorisations of d in this way are the *factors* of d in C. The ideal in C generated by d is the set of homs containing d as a factor.

EXAMPLE. In the TL category $\mathcal{C}_{T(\delta)}$ homomorphism $D_i \in \text{End}(l)$ (as defined above) is not a factor of the identity diagram $1_{l'}$ in any $\text{End}(l')$ with $l' \geq l$.

(**3.11**) Note that if $\mathcal{F} \in (\mathcal{C}, K-\text{mod})$ takes a hom d to the zero morphism then it takes the entire ideal generated by d to (the various) zero morphisms. If $\mathcal{C}$ is K-linear then the relation on homs given by $d \sim d'$ if $\mathcal{F}(d - d') = 0$ is a congruence. In this way every $\mathcal{C}$-representation may be associated to a quotient category — a category that the representation functor $\mathcal{F}$ factors through. (Of course other representations factor through the same quotient, including those whose images are direct sums of copies of the original image module.)

Suppose we have a representation of only the part of a K-linear category associated to a single object T, say, (i.e. of a single end-set – a single algebra). This defines a *local kernel*, that is, the collection of end(omorphism)s d such that $\mathcal{F}(d) = 0$. This kernel in turn generates an ideal in the category, and congruence modulo this ideal defines a category congruence, and hence a quotient category. The ideal in the category may intersect the original end-set in an algebra ideal larger than the original local kernel, but if not then any $\mathcal{C}$-representation that factors through the quotient (possibly restricting at T, as it were, to the original representation) is called a $\mathcal{C}$-*extension* of the original representation.

(**3.12**) A K-*ideal* $\mathcal{I}$ in a K-linear category $\mathcal{C}$ is a collection of homs that is closed in the obvious sense under category composition; *and* that intersects each hom-set in a K-submodule.

The relation of congruence modulo $\mathcal{I}$ is a congruence on the category $\mathcal{C}$, and hence defines a quotient $\mathcal{C}/\mathcal{I}$.

If $\mathcal{I}$ and $\mathcal{I}'$ are ideals then so is $\mathcal{I} \cap \mathcal{I}'$, so that there is a well defined smallest ideal containing any given collection of homs. If X is a collection of homs (or a single hom) we write $\mathcal{I}_X$ for the smallest ideal containing X. For F an object define

$$\mathcal{I}_F = \mathcal{I}_{1_F}$$

EXAMPLE. In the TL category $\mathcal{I}_0$ is the ideal spanned by diagrams with no propagating lines.

3.2 Some more terminology of our own

By historical convention there is no formal definition of *diagram category*. Here we shall consider the underlying idea of correlation functions, explained in Section 2, to be fundamental to the *notion* of diagram categories (along with the closely related idea of a diagram calculus). Accordingly, following [24, 27], we shall give the following useful partial axiomatisation the handy name of *propagating category*.

(**3.13**) For any poset $(T, \leq)$ and map $f : \hom_C \to T$ we say that C is filtered by f if for each composable pair of homs D, D' we have $f(D \circ D') \leq f(A)$ for $A \in \{D, D'\}$.

(**3.14**) Example. For $D \in \hom^o_{T(\delta)}$ the propagating number $\#(D)$ is simply the number of components of D that meet both boundaries of the diagram. We have

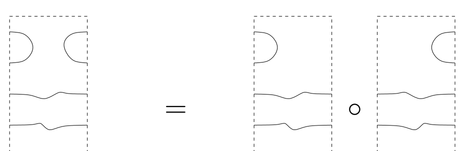
$$\#(D \circ D') \leq \min(\#(D), \#(D'))$$

so $\mathcal{C}_{T(\delta)}$ is filtered by $\#$ (with $\leq$ the natural order on $\mathbb{N}$). In a K-linear category with a given collection of bases we will adopt the convention that such a filter, if defined on the bases, takes the lowest value on linear combination X from the basis elements with finite support in X.

(**3.15**) The utility of such a filter is that it breaks each algebra $\mathrm{End}_C(n)$ into a nested sequence of ideals, the individual sections of which are generally easier to analyse. This raises the question of how to find such filters. A physical clue to this is given by the bottleneck picture in Section 2.3. There we see (at least heuristically) that it is not the homs that determine the filter but the transverse layers — which correspond to the *objects* in the category. Accordingly we are guided to make the following series of definitions.

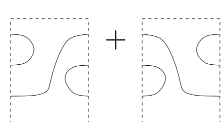
(**3.16**) Definition. A morphism D in a category $C = (S_C, \hom_C(-,-), \circ)$ *factors through* object $F \in S_C$ if $D = D' \circ 1_F \circ D''$ for some D', D''.
If C a K-linear category we say D factors K-linearly through F if it can be decomposed as a K-linear combination of morphisms each of which factors as above.

Thus D factors K-linearly through F if and only if $D \in \mathcal{I}_F$.
Example. (I) Diagram $D_1 \in \hom_{T(\delta)}(4,4)$ factors through 2 since



(15)

(II) In $\hom_{T(\delta)}(3,3)$ combination



factors K-linearly through 1 (since each diagram individually factors through 1), but does not itself factor through 1.

(**3.17**) Definition. For each category C define a relation on S_C by $F \geq_p F'$ if the map

$$\hom_C(F', F) \times \hom_C(F, F') \to \hom_C(F', F') \quad (16)$$
$$(A, B) \mapsto A \circ B \quad (17)$$

is surjective.

PROPOSITION. That is to say, $F \geq_p F'$ if $1_{F'}$ factors through F. □

(**3.18**) DEFINITION. For each K-linear category C define a relation on S_C by $F \geq_p^K F'$ if $1_{F'}$ factors K-linearly through F.

If the relations $\geq_p$ and $\geq_p^K$ agree on a K-linear category we call this a precious category.

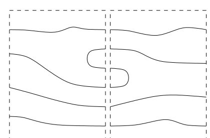
Such categories are not ubiquitous but they do exist, as the following pair of propositions shows.

PROPOSITION. *If C is the category of finite K-vector spaces then $F \geq_p F'$ if $dim(F) \geq dim(F')$; but $F \geq_p^K F'$ for all F, F'.*

Proof: For the first of these note that the rank of a composite map cannot exceed the rank of any factor. For the second note that any vector space map can be decomposed as a linear combination of projections onto one-dimensional subspaces. □

(**3.19**) PROPOSITION. *For $\mathcal{C}_{T(\delta)}$ (i) the relation $\geq_p$ is the usual natural order on the natural numbers, discarding pairs that are not congruent mod.2. (ii) the category is precious.*

Proof: To see that $m \geq_p m'$ when $m > m'$ (m, m' congruent mod.2) consider the following factorisation of 1_4:

$1_4 =$  (18)

(This also shows that $m \geq_p^K m'$ when $m > m'$.) When $m < m'$ note that since the total number of propagating lines in a diagram cannot increase in composition then $1_{m'}$ cannot factor through m. This proves (i). Any K-linear factorisation would be a combination of diagrams with the same problem. □

(**3.20**) PROPOSITION. *The relation $\geq_p$ is reflexive and transitive, for any category, but not in general antisymmetric.*
In particular, $\geq_p$ is a partial order only if C is a skeleton (a category in which $\hom(F, F')$ *contains an isomorphism only if $F = F'$).* □

If $\geq_p$ is antisymmetric we call the category C a *propagating category*, and poset $(S_C, \geq_p)$ the *propagating order* on C. We shall also assume that

$$S_C(F) := \{F' \in S_C \mid F' \leq_p F\}$$

is finite for every F. We say propagating category C is *terminal* if every $S_C(F)$ has a unique lowest element G_F, say, and every $\hom(G_F, G_F)$ is scalar (i.e. has a basis consisting only of 1_{G_F}).

EXAMPLE. In $\mathcal{C}_{T(\delta)}$, if n is even then $G_n = 0$; if n odd then $G_n = 1$.

We shall see later that in a K-linear category the order $\geq_p$ can play a significant role in the structure of the algebras $\mathrm{End}(F)$. Accordingly it behooves us to study it.

(**3.21**) Definition. (i) For each partial order $\preceq$ on object set S_C then $\#^{\preceq}(D)$ is the set of $\preceq$-lowest objects in S_C that D factors through.
(ii) We say category C is filtered (respectively weakly filtered) by $\preceq$ if $F \in \#^{\preceq}(D \circ D')$ implies $F \preceq F'$ (respectively $F' \not\prec F$) for all $F' \in \#^{\preceq}(D) \cup \#^{\preceq}(D')$.
(iii) If C is a propagating category then an element of $\#D := \#^{\leq_p}(D)$ is called a *propagating index* of D.
(iv) If $D \in \hom(F, F)$ has F as a propagating index it is said to be *loaded*. The subset of loaded homs is denoted $\hom^{-t}(F, F)$.

Proposition. In a propagating category (i) every isomorphism is loaded; but (ii) the converse need not be true; (iii) no $D \in \hom_C(F, F)$ has a propagating index $F' >_p F$.
Proof: (i): Every $D \in \hom(F, F)$ factors through F. Suppose D also factors through F'. Then 1_F also factors through F' (since D an isomorphism), but then $F' \geq_p F$. Thus $\#^{\leq_p}(D) = \{F\}$. (N.B. In particular, 1_F is loaded.)
(ii): But a loaded hom need not be an isomorphism. For example, if we consider the subcategory of TL excluding the object 2 then the diagram on the left in equation (15) has object 4 as a propagating index, but the diagram is not an isomorphism.
(iii): Evidently there exists $F'' \in \#D$ such that $F \geq_p F''$. But then $F' >_p F$ implies $F' >_p F''$ by transitivity, so no such F' can be a propagating index. □

(**3.22**) In light of the example in (ii), we shall call a category *object rich* if every loaded hom is an isomorphism.

Object richness implies a particularly simple structure in representation theory. However a K-linear category containing hom sets with more than one linearly independent isomorphism contains loaded non-invertible idempotents, so will not be object rich. In this case one can look for ways to add more objects to the category (a good idea if one has left some out, as in our TL example above, but a search in uncharted territory in general); or simply drop the category into its much larger 'categorical' module-category $(C, K-\mathrm{mod})$ by the Yoneda embedding (a well-defined procedure, but passing to an object which is, in general, very complex). Here we will follow a hybrid approach.

Proposition. A propagating category C is weakly filtered by $\geq_p$. A sufficient condition to be filtered by $\geq_p$ is if $\geq_p$ is a total order. □
Example. Category $\mathcal{C}_{T(\delta)}$ is filtered by $\geq_p$ (in the sense that the odd and even subcategories are so filtered, while the objects of the other parity are all zero-objects relative to each of these subcategories).

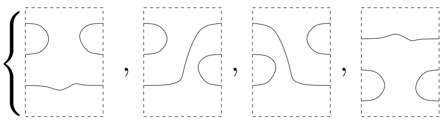
(**3.23**) Definition. A hom $D \in \hom_C(F, F')$ is *full* on F (respectively F') if there exists an element $D' \in \hom_C(F', F)$ such that $D \circ D'$ (respectively $D' \circ D$) is an isomorphism.
We write $\hom_{C,F}(F, F')$ (respectively $\hom_{C,F'}(F, F')$) for the subset full on F (respectively F').
Example. The left-hand diagram in equation (18) is full on 4.

3.3 On K-linear structure

(**3.24**) Definition. Let C be a K-linear category. Write $\hom_C^G(F, F')$ for

the subset of $\hom_C(F, F')$ of homs that factor through G, and the K-span thereof. (N.B. This is, by construction, the same as the subset of homs that factor K-linearly through G.)
EXAMPLE.

$$\hom^1_{T(\delta)}(3,3) = K \left\{ \quad \right\}$$



The following are obvious from the construction:
$\hom^G_C(F, F')$ has a basis of elements that factor through G;
$\hom^F_C(F, F') = \hom^{F'}_C(F, F') = \hom_C(F, F')$.

(**3.25**) PROPOSITION. *Recall that if C is a K-linear category then* $\hom_C(F, F')$ *is a left-*$\hom_C(F, F)$*-module and a right-*$\hom_C(F', F')$*-module, each by the action of composition in the category.*
We have that $\hom^G_C(F, F')$ *is a left-*$\hom_C(F, F)$*-submodule, and a right-*$\hom_C(F', F')$*-submodule.*

Proof: Let $a \in \hom_C(F, F)$ and $m \in \hom^G_C(F, F')$. We require to show that $a \circ m \in \hom^G_C(F, F')$, i.e. that $a \circ m$ factors K-linearly through G. But m so factors, so let $\sum_i \alpha_i m^l_i \circ m^r_i$ be a factorisation (i.e. $m^l_1 \in \hom_C(F, G)$ and so on). Then $\sum_i \alpha_i (a \circ m^l_i) \circ m^r_i$ is a factorisation of $a \circ m$. □

(**3.26**) PROPOSITION. *If $G \geq_p G'$ then* $\hom^G_C(F, F') \supseteq \hom^{G'}_C(F, F')$.

Proof: $m \in \hom^{G'}_C(F, F')$ implies that m is a linear combination of homs that factor through G'; thus $1_{G'}$ can be inserted in each of these factorisations. But if $G \geq_p G'$ then $1_{G'}$ factors through G. □
(**3.27**) DEFINITION. If C has a propagating order then

$$\hom^{=G}_C(F, F') \;:=\; \hom^G_C(F, F') / \sum_H \hom^H_C(F, F')$$

where the sum is over all H below G in the order.

Note that $\hom^{=G}_C(F, F')$ has a basis of elements that factor through G and nothing below G in the order; and that $\hom^{=G}_C(F, F') = 0$ if $G >_p F$ or F'.

(**3.28**) A propagating category C is *balanced* if for each pair of objects x, L, there is $M_x(L)$ a finite set, and S_L is a finite set (a basis of $\hom^{=L}_C(L, L)$; and in particular independent of x), and Γ a map such that

$$\Gamma : \cup_{L \leq x,y} M_x(L) \times S_L \times M_y(L) \to \hom^o_C(x, y)$$

is a bijection.

(Since this includes the case $x = y$, and $M_x(L)$ does not depend on y, we can potentially infer a lot about $M_x(L)$ from its role in the contruction of the regular module for the $\hom_C(x, x)$ K-algebra.)
EXAMPLE. Consider the category $\mathcal{C}_T$ whose homs are TL diagrams, with the usual TL composition except that closed loops are ignored. Then $M_x(l) = \hom_T(x, l) = \hom^o_{T(\delta)}(x, l)$ and $S_l = \{1_l\}$, and the map takes (a, b, c) to abc^t

where c^t is the image of $c \in \hom_T(y,l)$ under the map $-^t : \hom_T(y,l) \to \hom_T(l,y)$ which simply flips the diagrams.

In the next section we shall introduce several further examples of balanced propagating categories; and in section 6.1 we shall use the hom spaces introduced above to analyse the representation theory of these categories.

(**3.29**) A balanced propagating category can also be regarded as a version of a cellular or tabular category (an obvious generalisation of a cellular algebra in the sense of [16, 22], or tabular algebra in the sense of [18, 19]). We postpone details of the utility of this remark to a separate work.

We now turn to the construction of some concrete examples.

4 Graph categories

4.1 Set and Partition algebra notations

Recall that $\underline{n} = \{1, 2, ..., n\}$ and let $\underline{n}^i = \underline{n} \times \{i\}$. For S a set, $\mathbf{E}(S)$ is the set of partitions of S [25, 27]. Examples:

$$\mathbf{E}(\underline{2}) = \mathbf{E}(\{1,2\}) = \{\{\{1\},\{2\}\},\{\{1,2\}\}\}$$

$$\mathbf{E}(\{1,2,3\}) = \{\{\{1\},\{2\},\{3\}\},\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{1\},\{2,3\}\},\{\{1,2,3\}\}\}$$

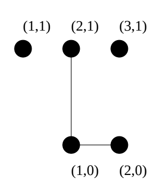
In this section element $p \in \mathbf{E}(S)$ may appear either as a partition or as an equivalence relation, depending on context (from which the form used will be clear), via the natural bijection. The next paragraphs have some examples.

Every function $f : S \to T$ defines an element $p_c(f) \in \mathbf{E}(S)$ by $s \sim_{p_c(f)} s'$ if $f(s) = f(s')$. A function f such that $p_c(f) = q \in \mathbf{E}(S)$ is called a *colour function* for q.

(**4.1**) We also use a diagrammatic realisation for partitions of S. Let $\Gamma(S)$ denote the set of loop free undirected graphs on vertex set S. Let

$$p : \Gamma(S) \to \mathbf{E}(S) \tag{19}$$

denote the map which takes graph g to the partition into connected components. In particular, to depict a partition in $\mathbf{E}(\underline{m}^1 \cup \underline{n}^0)$ we draw a row of m and a row of n vertices, and draw enough edges between them to indicate the partition. For example, in $\mathbf{E}(\underline{3}^1 \cup \underline{2}^0)$ the diagram



(20)

denotes the partition $\{\{(1,1)\},\{(2,1),(1,0),(2,0)\},\{(3,1)\}\}$.

Of course different graphs can have the same image under p. For example replacing any connected component by any spanning tree does not change the partition. Write $gen(g)$ for the maximum number of edges that can be removed from g without changing the partition. Evidently

$$gen(g) = |edge(g)| - (|g| - |p(g)|)$$

(**4.2**) Each reflexive relation ρ on set S to itelf has a symmetric, transitive closure. That is, a smallest element of $\mathbf{E}(S)$, regarded as a relation, which contains ρ as a subset. Define $\mathrm{T}C(\rho) \in \mathbf{E}(S)$ as this closure.

If $\rho \in \mathbf{E}(S)$, $\nu \in \mathbf{E}(T)$ are two equivalence relations, then $\rho \cup \nu$ is a reflexive relation on $S \cup T$ but is not transitively closed in general if S and T intersect. Define

$$\rho * \nu = \mathrm{T}C(\rho \cup \nu) \in \mathbf{E}(S \cup T).$$

If $p \in \mathbf{E}(S)$ and $T \subseteq S$ then $p|_T$ denotes the restriction of p to T. That is, the largest element of $\mathbf{E}(T)$ which is (as a relation) a subset of p.

If $p \in \mathbf{E}(S)$ and T a set, then $\#^T(p)$ denotes the number of parts of p which contain only elements of T.

For each bijection $f : S \to T$ there is a map

$$f : \mathbf{E}(S) \to \mathbf{E}(T)$$

by applying f to parts.

If S is a set then S^i is the image of S in $S \times \mathbb{Z}$ under $s \mapsto s^i = (s, i)$, and

$$\begin{array}{rcl} \sigma_1 : S \times \{0,1\} & \to & S \times \{1,2\} \\ (s,i) & \mapsto & (s, i+1) \end{array} \tag{21}$$

$$\begin{array}{rcl} \sigma_2 : S \times \{0,2\} & \to & S \times \{0,1\} \\ (s,2) & \mapsto & (s,1) \\ (s,0) & \mapsto & (s,0) \end{array} \tag{22}$$

Thus for $\rho \in \mathbf{E}(S^0 \cup T^1)$ and $\nu \in \mathbf{E}(\sigma_1(T^0 \cup U^1))$ we have

$$\rho * \nu \in \mathbf{E}(S^0 \cup T^1 \cup U^2).$$

4.2 The graph category

We now recall the definition of the *graph category* from [28] (one should also compare this construction with the tangle category [10]). Let g be a graph. In this section we will write $edge(g)$ for the edge set of g, and usually confuse g notationally with its vertex set.

(**4.3**) Let S be a set as before. By an S-graph we mean a finite graph g together with a 'structure' map

$$\lambda_g : S \to g$$

(this can be any map). Note that if $S' \subset S$ then each S-graph g restricts to an S'-graph $g|_{S'}$ by restricting the structure map.

(**4.4**) There is a map p_λ from S-graphs to $\mathbf{E}(S)$ which puts s, t in the same part if they label vertices in the same connected component. For example if g is the unique loop-free graph on one vertex then there is only one possible structure map and $p_\lambda(g)$ is the partition of S with only one part, for any S.

This should be contrasted with the map $p : \Gamma(S) \to \mathbf{E}(S)$, in which all graph vertices are labeled, so graphs are not regarded as equivalent under graph isomorphism. They coincide only when λ_g is the identity map.

There is an infinite fibre $p_\lambda^{-1}(q)$ of S-graphs over any $q \in \mathbf{E}(S)$. This fibre contains a graph $\omega(q)$ with no edges and $|q|$ vertices, in which the underlying labels on the vertices can be considered to be the parts of q, and the structure map assigns each $s \in S$ to the vertex whose label contains s.

(**4.5**) By an (S,T)-graph we mean an $(S^1 \cup T^0)$-graph. In this case we refer to the vertices labeled from S^1 as outputs and those labeled from T^0 as inputs. (This notation comes from the physical context described in Section 2.) By an $(n,m) - \mathit{graph}$ we mean an $(\underline{n}, \underline{m})$-graph.

For example, the graph shown in equation (20) is a $(3,2)$-graph with a bijective structure map.

Write $\hom_\Gamma(S,T)$ for the set of (S,T)-graphs, regarding any vertices not in the image of the structure map as unlabeled (so elements are strictly isomorphism classes with respect to the set of graph morphisms which commute with the structure map). That is, $\hom_\Gamma(S,T)$ is a certain set of partially labeled graphs (some vertices may have multiple labels). Let σ_1, σ_2 act on such a partially labeled graph by changing the labels in the obvious way.

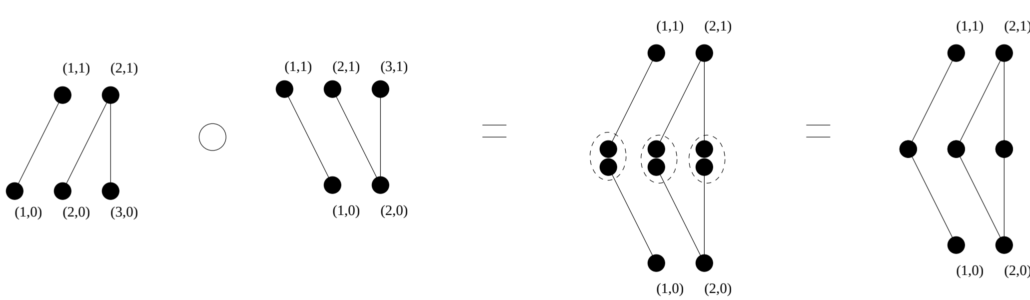
(**4.6**) For g, g' graphs, define the composite $g \cup_\Gamma g'$ to be the graph with vertex set $g \cup g'$ and edge set the (disjoint) union of the edge sets. Define a product

$$\circ : \hom_\Gamma(S,T) \times \hom_\Gamma(T,U) \to \hom_\Gamma(S,U)$$

by

$$g \circ g' = \sigma_2((\sigma_1(g) \cup_\Gamma g')|_{S^2 \cup U^0})$$

This amounts to drawing the two graphs one on top of the other, with the vertices whose T-label (as it were) coincide identified; and then stripped of their T-label.
Example:



Consider the element of $\hom_\Gamma(S,S)$ with *no* edges, and $|S|$ vertices, each vertex with two labels: s^0 and s^1 for some $s \in S$. It follows from the concatenation picture (or otherwise) that this is the identity element in $(\hom_\Gamma(S,S), \circ)$.

(**4.7**) Proposition. *The triple* $\Gamma = (\mathbb{S}_{Fin}, hom_\Gamma(-,-), \circ)$ *is a category.*

Proof: It remains to show associativity. Considering the concatenation picture we see that computation of $(g \circ g') \circ g''$ involves the same stack of diagrams as $g \circ (g' \circ g'')$; and that the order of 'internalisation' of the middle layers is unimportant. $\square$

Note that any graph in $\hom_\Gamma(S,T)$ with no edges for which the structure map restricts to a bijection on each of S^1 and T^0 is an isomorphism. This construction requires that $|S| = |T|$, whereupon there are $|S|!$ such graphs. The construction includes the identity if $S = T$. The composition then closes on this set of isomorphisms to form a submonoid that is isomorphic to the symmetric group S_n.

It follows that

(**4.8**) Proposition. *The subcategory* $\Gamma_\mathbb{N}$ *with object set* $\mathbb{N}$ *and hom sets* $hom_\Gamma(m,n)$ *is a skeleton in* Γ.

4.3 On graph invariants

One interesting way to proceed at this point is as follows. First extend $\Gamma_\mathbb{N}$ to a K-linear category. Then for example $\hom_{K\Gamma_\mathbb{N}}(0,0)$ is the free K-module with basis the set of all finite loop-free graphs (strictly speaking, isomorphism classes thereof). The aim is to find quotient relations in this category such that $\hom(0,0)$ is reduced to scalars — the scalar image of each graph thus being its invariant under this reduction. Both the invariant and the quotient *category* are potentially interesting.

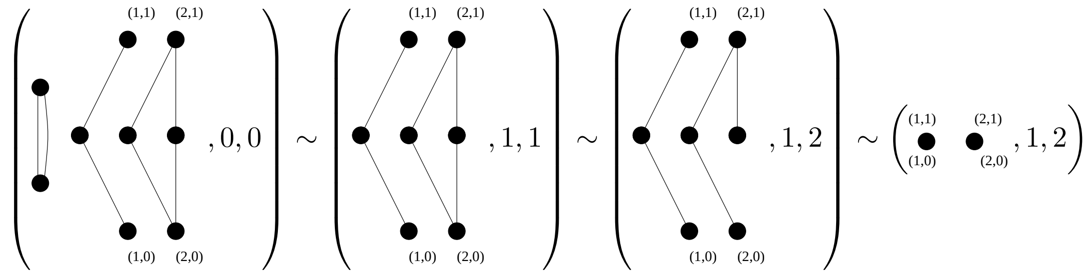
A set of quotient relations that gives rise to chromatic [8] and dichromatic polynomials as invariants is described in [28] (these relations are an extension to the categorical setting of relations used, for example, in [5] and references therein). For the category as a whole this quotient passes to the *partition category*. To introduce this we first adopt quotient relations that give rise to much more trivial invariants (but the same categorical structure).

(**4.9**) We may regard $(\mathbb{N}_0, +)$ as a category with one object. Thus $\Gamma \times \mathbb{N}_0$ is a category. We will consider it to have the *same* object set as Γ. We can consider $\Gamma \times \mathbb{N}_0 \times \mathbb{N}_0$ similarly. Now define $\hom_{\Gamma+}(S,T) = \hom_\Gamma(S,T) \times \mathbb{N}_0 \times \mathbb{N}_0$ (the second component can be called the *weight* and the third the *overflow*). Let $b(g)$ be the number of connected components of g having no labeled vertices. Define a relation on $\hom_{\Gamma+}(S,T)$ by

$$(g, m, n) \sim (g', m', n')$$

if $p_\lambda(g) = p_\lambda(g')$ and $b(g) + m = b(g') + m'$ and $gen(g) + n = gen(g') + n'$.

Example:



(**4.10**) Proposition. (i) *The relation* $\sim$ *is a congruence relation on the category* $\Gamma \times \mathbb{N}_0 \times \mathbb{N}_0$.
(ii) *The quotient category*

$$\mathbb{P}^+ := \Gamma \times \mathbb{N}_0 \times \mathbb{N}_0 / \sim$$

has homs that are equivalence classes of (g, m, n)*-triples. Each such equivalence class has a representative element whose first component* g *is of form* $\omega(q)$.

Proof: (i) Suppose $g \sim g'$ and $h \sim h'$. We RTS that $g \circ h \sim g' \circ h'$. The picture for $g \circ h$ produces connections between vertices in the same way as $g' \circ h'$, so $p_\lambda(g \circ h) = p_\lambda(g' \circ h')$. The second component works similarly. For the third component note that $gen(g \circ h) = gen(g) + gen(h) + X(g \circ h)$, where X is the number of occurences of pairs of vertices that are connected in both the g part and the h part of $g \circ h$. Since the connected components of (the labeled vertices of) g and g' (respectively h and h') agree we have $X(g' \circ h') = X(g \circ h)$. Thus

$$gen(g \circ h) = gen(g') + g_2' - g_2 + gen(h') + h_2' - h_2 + X(g' \circ h')$$

so

$$gen(g \circ h) + g_2 + h_2 = gen(g' \circ h') + g_2' + h_2'$$

(ii) The example above is sufficiently generic. □

(**4.11**) Proposition. *The restriction* $\mathbb{P}^x$ *of* $\mathbb{P}^+$ *to the subclass of finite cardinals is a skeleton.*

Let K be a ring and $\mathcal{C}$ a category. Recall that $K\mathcal{C}$ is the K-linear category extending $\mathcal{C}$.

(**4.12**) Suppose that $\delta, \kappa \in K$ and define a relation $\sim_\delta$ on $K \hom_{\mathbb{P}^x}(m, n)$ as follows. For $A, B \in \hom_{\mathbb{P}^x}(m, n)$ set $A \sim_\delta B$ if

$$\delta^{A_2} \kappa^{A_3} (A_1, 0, 0) = \delta^{B_2} \kappa^{B_3} (B_1, 0, 0)$$

and extend linearly.

(**4.13**) Proposition. *The relation* $\sim_\delta$ *is a congruence on* $K\mathbb{P}^x$, *so for each* δ, κ *we have a quotient*

$$\mathcal{C}_{\mathbb{P}(\delta,\kappa)} = (\mathbb{N}_0, \hom_{\mathbb{P}(\delta,\kappa)}(-,-), \circ)$$

a K-finite category. Each $\hom_{\mathbb{P}(\delta,\kappa)}(m,n)$ *has a basis of partitions* $\mathbf{E}(\underline{m}^1 \cup \underline{n}^0)$. *The specialisation $\kappa = 1$ is the partition category $\mathcal{C}_{\mathbb{P}(\delta)}$, and $P_n = End_{\mathbb{P}(\delta)}(n)$ is then the partition algebra. (We shall recall the definition of the partition category [25] shortly.)*

Note that $\hom_{\mathbb{P}(\delta)}(m,n)$ is a left-$\mathrm{End}_{\mathbb{P}(\delta)}(m)$-right-$\mathrm{End}_{\mathbb{P}(\delta)}(n)$-bimodule. Thus we have lots of functors between *module* categories. These functors were used to determine the structure of the partition algebra for $K = \mathbb{C}$ in [26]. (The structure over fields of finite characteristic is largely an open problem.)

5 The partition category

The *partition base* category $\mathbb{P}^\circ$ is constructed as follows. Set $O_{\mathbb{P}^\circ} = O_{\mathbb{S}}$. Then

$$\hom_{\mathbb{P}^\circ}(S,T) = \mathbf{E}(S^1 \cup T^0) \times \mathbb{N}_0$$

(so $\hom_{\mathbb{P}^\circ}(\emptyset,\emptyset) \cong \mathbb{N}$); and composition is partition algebra composition. That is:

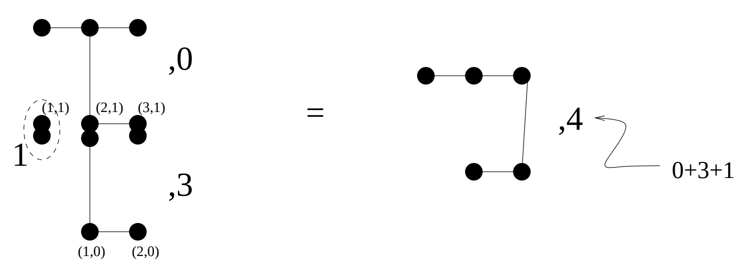
$$\begin{array}{rcl} \hom_{\mathbb{P}^\circ}(S,T) \times \hom_{\mathbb{P}^\circ}(T,U) & \to & \hom_{\mathbb{P}^\circ}(S,U) \\ (f,g) & \mapsto & f \circ g \end{array} \tag{23}$$

has $f \circ g$ given as follows.

$$(f \circ g)_1 = \sigma_2(\sigma_1(f_1) * g_1|_{S^2 \cup U^0}) \tag{24}$$

$$(f \circ g)_2 = f_2 + g_2 + \#^{T^1}(\sigma_1(f_1) * g_1) \tag{25}$$

(**5.1**) Example. Using the diagram realisation in (4.1), the first step is to concatenate the two diagrams in the product, as shown on the left here in case $S = \underline{3}$, $T = \underline{3}$, $U = \underline{2}$:



The second (integer) component of the new hom is the sum of the second components of the factors, plus the number of 'interior' components of the concatenated diagram.

Noting that every hom in $\mathbb{P}^\circ$ is a pair consisting of a partition and a number, then by convention, if the number is zero we may refer to the hom simply as a partition. The second (number) component is sometimes called the 'vacuum bubble' index, or the *weight*.

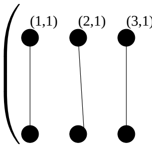
(**5.2**) Proposition. *The triple $\mathbb{P}^\circ = (O_{\mathbb{S}}, hom_{\mathbb{P}^\circ}, \circ)$ is a category.*

Proof: The unit in $\hom_{\mathbb{P}^\circ}(S,S)$ is the (weight 0) pair partition

$$1_S = (\{...,\{(s,0),(s,1)\},...\},0)$$

It is an exercise to check associativity. $\square$

Note that the set of homs of form



$\Big(\ldots , \mathbb{N}_0 \Big)$

is central in $\mathrm{End}_{\mathbb{P}^\circ}(\underline{3})$ (and similarly for any $\mathrm{End}_{\mathbb{P}^\circ}(S)$).

Note that if $f : S \to T$ is an isomorphism then

$$f' = (\{...,\{(s,0),(f(s),1)\},...\},0)$$

is an isomorphism in $\hom_{\mathbb{P}^\circ}(S,T)$. Every isomorphism can be constructed in this way. Thus

(**5.3**) Proposition. *The restriction of $\mathbb{P}^\circ$ to the subclass of finite cardinals is a skeleton for the restriction $\mathbb{P}^\circ_{Fin}$ (given by $O_{\mathbb{P}^\circ_{Fin}} = O_{\mathbb{S}_{Fin}}$).*

We denote this category as

$$\mathcal{C}_{\mathbb{P}} = (\mathbb{N}_0, \hom_{\mathbb{P}}(-,-), \circ)$$

where

$$\hom_{\mathbb{P}}(m,n) = \mathbf{E}(\underline{m}^1 \cup \underline{n}^0) \times \mathbb{N}_0$$

(**5.4**) Let K be a ring. Then $K\mathcal{C}_{\mathbb{P}}$ is the K-linear category extending $\mathcal{C}_{\mathbb{P}}$. Suppose that $\delta \in K$ and define a relation $\sim_\delta$ on $K \hom_{\mathbb{P}}(m,n)$ by $A \sim_\delta B$ if

$$\delta^{A_2}(A_1,0) = \delta^{B_2}(B_1,0)$$

if $A, B \in \hom_{\mathbb{P}}(m,n)$, and so on.

This is a congruence, so for each δ we have a quotient

$$\mathcal{C}_{\mathbb{P}(\delta)} = (\mathbb{N}_0, \hom_{\mathbb{P}(\delta)}(-,-), \circ)$$

a K-finite category. By construction $\hom_{\mathbb{P}(\delta)}(m,n)$ has basis $\mathbf{E}(\underline{m}^1 \cup \underline{n}^0)$. This is the partition category [25], and $P_n(\delta) = \mathrm{End}_{\mathbb{P}(\delta)}(n)$ is the partition algebra.

5.1 Subcategories

(**5.5**) The subcategory $\mathbb{S}^\circ$ of $\mathbb{S}$ in which only homs which are bijections are retained is a subcategory of $\mathbb{P}^\circ$ by identifying the object classes between $\mathbb{S}$ and $\mathbb{P}^\circ$ and taking $f \in \hom_{\mathbb{S}}(S,T)$ to the pair partition with pairs $\{s^0, f(s)^1\}$.

(**5.6**) The partition part of the partition product takes pair partitions to pair partitions, so that the subcategory $\mathbb{B}^\circ$ of $\mathbb{P}^\circ$ in which only homs which are pair partitions are retained is well defined, and is the Brauer base category. We have the factoring

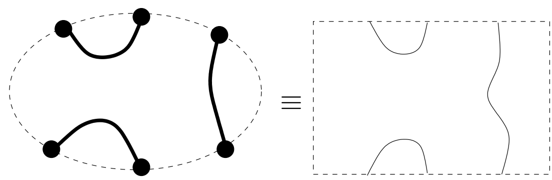
$$\mathbb{S}^\circ \hookrightarrow \mathbb{B}^\circ \hookrightarrow \mathbb{P}^\circ$$

The Brauer skeleton category is the skeleton of $\mathbb{B}^\circ$ with object set the finite cardinals. The hom set between cardinals n and m is denoted $\hom_{\mathbb{B}^\circ}(n, m)$ or $\hom_{\mathbb{B}}(n, m)$.

The congruence $\sim_\delta$ may be applied to define the Brauer category:

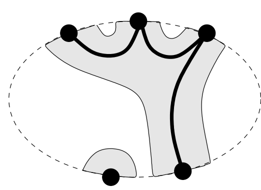
$$\mathcal{C}_{\mathbb{B}(\delta)} = (\mathbb{N}_0, \hom_{\mathbb{B}(\delta)}(-,-), \circ)$$

(**5.7**) Further, $\hom_{T(\delta)}(m, n) \subset \hom_{\mathbb{B}(\delta)}(m, n)$ is obtained by restricting to plane pair partitions. The corresponding category $\mathcal{C}_{T(\delta)}$ may be *identified* with the ordinary TL category discussed in Section 2 (Figure 3), in case $\delta = \sqrt{Q}$. (The diagrams are the same, although we have rotated through 90^o compared to the figures in Section 2.)



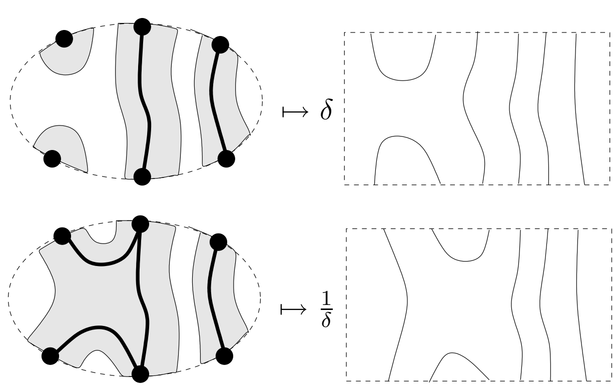
(26)

(**5.8**) Similarly we may define $\hom_{\mathbb{T}}(m, n) \subset \hom_{\mathbb{P}}(m, n)$ as the subset such that $A \in \hom_{\mathbb{T}}(m, n)$ implies that A_1 is a plane partition. Example:



As the figure illustrates, a plane partition may be thickened, leading to a kind of TL diagram (see also Section 2). Then the congruence $\sim_\delta$ defines the *even* TL category $\mathcal{C}_{\mathbb{T}(\delta)}$.

Note that there are some significant differences between these two constructions, $\hom_T(m, n)$ and $\hom_{\mathbb{T}}(m, n)$. For example we have a homomorphism from $\hom_{\mathbb{T}(\delta^2)}(n, m)$ to $\hom_{T(\delta)}(2n, 2m)$ illustrated by

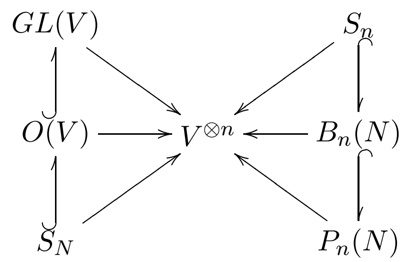


(27)

That is, the number of vertices is doubled, the parameter changes, and the diagrams must be rescaled.

6 Representations: Schur-Weyl duality

For $N \in \mathbb{N}$ let $V = K\{e_1, e_2, ..., e_N\}$. Then we have the following collection of pairs of commuting (indeed centralizing) actions:



(28)

Fix a field k. Then recall that Vect is the category of k-spaces. For G a group and V a G-module then $\mathrm{Vect}_{G,V}$ is the subcategory with objects

$$k, V, V^2, V^3, ...$$

and homs commuting with the diagonal action of G, i.e.

$$f : V^m \to V^n$$

such that

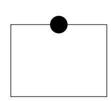
$$f\sigma v = \sigma f v \qquad \forall \sigma \in G$$

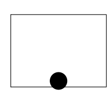
This inherits the tensor structure from Vect.

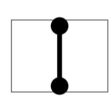
(**6.1**) The following functor

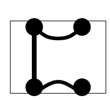
$$F_N : \mathcal{C}_{\mathbb{P}(N)} \to \mathrm{Vect}_{S_N,V}$$

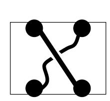
is a representation of $\mathcal{C}_{\mathbb{P}}$. We begin by giving the images of some elements (in case $N = 2$):

$\hom_{\mathbb{P}}(\underline{1}, \underline{0}) \ni$  $\mapsto \begin{pmatrix} 1 & 1 \end{pmatrix}$

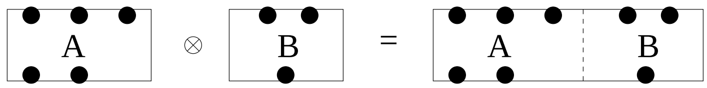
$\hom_{\mathbb{P}}(\underline{0}, \underline{1}) \ni$  $\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\hom_{\mathbb{P}}(\underline{1}, \underline{1}) \ni$  $\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hom_{\mathbb{P}}(\underline{2}, \underline{2}) \ni$  $\mapsto \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$

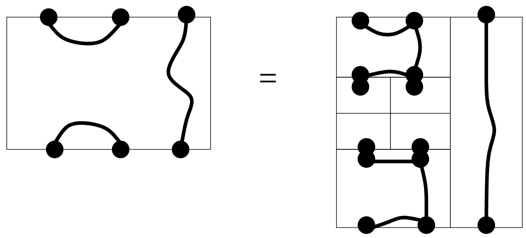
$\hom_{\mathbb{P}}(\underline{2}, \underline{2}) \ni$  $\mapsto \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$

Note that all the images are invariant under the appropriate S_2 action. We conclude by noting that $\mathcal{C}_{\mathbb{P}}$ is a tensor category with



and that the examples given above (respectively their direct generalisations to other N) generate.

(**6.2**) Example. Keeping with F_2:



$$\mapsto \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 1 \\ & 0 & 0 & \\ & 0 & 0 & \\ 1 & & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(omitted entries zero); while $N = 3$ gives

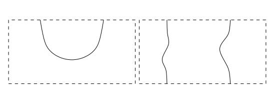
$$\ldots \quad = \begin{pmatrix} 1 & & & & 1 & & & & 1 \\ 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ 1 & & & & 1 & & & & 1 \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ 1 & & & & 1 & & & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(**6.3**) A *tensor representation* of a tensor category is a representation (a map to a tensor category) that commutes with the tensor operation.

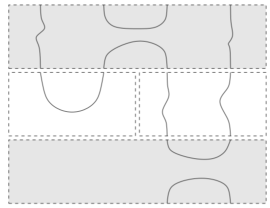
Suppose C a tensor category and X is an element of $\hom_C(l,m)$ then for each n we associate an ideal in $\hom_C(l \otimes n, m \otimes n)$ to X by

$$I_X^n = \mathrm{End}(l \otimes n)(X \otimes 1_n)\mathrm{End}(m \otimes n)$$

Example. In TL we have the case with $X \in \hom(2,0)$ given by

$X \otimes 1_2 =$ 

in which the ideal contains elements like



In this case it will be evident that $I_X^n = \hom_C(2+n, n)$.

Indeed the tensor structure on C defines an embedding of $\hom(l,m)$ in $\hom(l+n, m+n)$ by $X \mapsto X \otimes 1_n$ (along with other such embeddings). This gives us a way to interpret $\mathrm{End}(l+n)$ as a left $\mathrm{End}(l)$-module. For $l = m$ we call the associated functor from $\mathrm{End}(l+n) - \mathrm{mod}$ to $\mathrm{End}(l) - \mathrm{mod}$ restriction: $\mathrm{res}_1^{l+n} -$. If M_n is the representation of $\mathrm{End}(n)$ in a tensor representation then $\mathrm{res}_{n-1}^n M_n \cong M_1(1) \otimes M_{n-1}$.

By a similar token

$$\hom(n, n+2) \otimes_{\mathrm{End}(n+2)} M_{n+2} \cong M_n \tag{29}$$

(the key point being that $\mathrm{rank}(F_N(X)) = 1$).

(**6.4**) These F_N are called N-state Potts functors (because of their physical origin [24]). They restrict to representations of the Brauer category and Temperley–Lieb category. In the Brauer algebra case this is the representation associated to the action on tensor space in the Schur-Weyl duality diagram above.

In the Temperley–Lieb case we have described two possible restrictions. One is given by the example above (the particular example is the representation of D_1, via (26)). The other is (combining (27) with F_N), in case $N = 3$,

$$D_1 \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes 1_3 \otimes 1_3 \otimes \ldots$$

$$D_2 \mapsto \sqrt{3} \begin{pmatrix} 1 &&&&&&&& \\ 0 & 0 &&&&&&& \\ 0 & 0 & 0 &&&&&& \\ 0 & 0 & 0 & 0 &&&&& \\ &&&& 1 &&&& \\ &&&&& 0 &&& \\ &&&&&& 0 && \\ &&&&&&& 0 & \\ &&&&&&&& 1 \end{pmatrix} \otimes 1_3 \otimes \ldots$$

and so on. Note that this is a representation of $\mathrm{End}_{T(\sqrt{3})}(2n)$, whereas the other construction gives rise only to representations for integral δ values.

6.1 On module structure

The following analysis can be implemented for any of the algebras we have introduced, but here we use TL as an illustrative example (see [12] for the Brauer case; [26] for the partition algebra case; and many other references).

Fix δ and let A_n be the K-algebra $\mathrm{End}_{T(\delta)}(n)$. As noted in (3.7) above we may define a functor

$$\mathcal{F} : \mathrm{mod} - A_{2n} \to \mathrm{mod} - A_{2n-2}$$

by $M \mapsto \mathrm{Hom}(2n-2, 2n) \otimes_{A_{2n}} M$; and a functor

$$\mathcal{G} : \mathrm{mod} - A_{2n-2} \to \mathrm{mod} - A_{2n}$$

by $M \mapsto \mathrm{Hom}(2n, 2n-2) \otimes_{A_{2n-2}} M$. Unless $\delta = 0$ and $n = 1$ we have that

$$\mathrm{Hom}(2n-2, 2n) \otimes_{A_{2n}} \mathrm{Hom}(2n, 2n-2) \cong \mathrm{End}(2n-2) = A_{2n-2} \tag{30}$$

as an A_{2n-2}-bimodule. (By Proposition 3.19 the category composition gives a surjection from left to right. An inverse is defined (for example when $\delta \neq 0$) on diagrams as follows: equate the diagram to a suitably rescaled one with a loop added close to the right hand edge of the diagram; cut the diagram from side to side through its propagating lines and this loop.) We have further that $\mathrm{Hom}(2n, 2n-2)$ is a projective left $\mathrm{Hom}(2n, 2n)$-module; that

$$\mathrm{Hom}(2n, 2n-2) \otimes_{A_{2n-2}} \mathrm{Hom}(2n-2, 2n) \cong \mathrm{hom}^{2n-2}(2n, 2n) \tag{31}$$

and that

$$A_{2n} / \mathrm{hom}^{2n-2}(2n, 2n) \cong K \tag{32}$$

as a vector space.

One may use the functors $\mathcal{F}, \mathcal{G}$ to define a set of modules for each A_n that are a complete set of standard modules, in the sense that (i) they have a standard construction independent of δ and yet (ii) give rise to a basis for the Grothendieck group. That is, we may express the character of any module M as a combination of standard characters. For a given set of standard modules the collection of coefficients in this combination is called a Grothendieck vector, and here denoted $Gr(M)$.
Further (iii) Each standard module for A_n has simple head and is taken by $\mathcal{F}$ either to zero or else to a standard module for A_{n-2} (cf. [17, §6]). That is to say, the Grothendieck vector for the image $\mathcal{F}M$ of a module M is simply a localisation of $Gr(M)$ (i.e., a copy of $Gr(M)$ in which some of the coefficients have no role, since there is no corresponding standard module). Hereafter we assume that $Gr(M)$ is embedded in the global limit space (the space of the large n vector), thus

$$Gr(\mathcal{F}M) = Gr(M)$$

It follows from (30)-(32) (under the projective condition) that a labeling scheme for the standard modules of A_n is $\{\Delta(\lambda) \mid \lambda \in \mathbb{N};\ 0 \leq \lambda \leq n;\ \lambda \equiv n\ mod.2\}$ (when the projective condition fails we just have one too many labels). Thus $\mathbb{N}$ may be used as a labeling scheme for the entries in the Grothendieck vector $Gr(M)$ for any A_n (with some redundant entries).
It also follows that $\Delta(\lambda)$ is simple whenever A_n is semisimple. Thus in particular:

(**6.5**) Proposition. *Entries in $Gr(M)$ lie in $\mathbb{N}_0$ whenever A_n is semisimple.*

Consider the construction in (6.1) and Example (6.2) as providing a representation M_n of $\mathrm{End}_{T(\delta)}(n)$ (with $\delta = N$); and that in (6.4) as providing a representation $\overline{M}_{2n}$ of $\mathrm{End}_{T(\delta)}(2n)$ (with $\delta^2 = N$). Then both satisfy the conditions in (6.3) and we have, by (29),

(**6.6**) Proposition. *For $n > 1$*

$$\mathcal{F}M_n = M_{n-2} \qquad\qquad \mathcal{F}\overline{M}_{2n} = \overline{M}_{2n-2}$$

□

This implies that the Grothendieck vector for M_n can be considered as independent of n (or more precisely, as depending on it only though a localisation). It turns out that this is enough to determine the Grothendieck vector.

One proceeds as follows. First note that restriction provides another kind of functor between these categories. Its action on the Grothendieck vector is governed by the standard restriction rules. For the TL algebras these are $\mathrm{res}^{A_n}_{A_{n-1}}\Delta_0 \cong \Delta_1$ and otherwise

$$\mathrm{res}^{A_n}_{A_{n-1}}\Delta_\lambda \cong \Delta_{\lambda-1} + \Delta_{\lambda+1}$$

Note that

$$\mathrm{res}^{A_n}_{A_{n-1}} M_n \cong 1_N \otimes M_{n-1} \tag{33}$$

and

$$\mathrm{res}^{A_{2n}}_{A_{2n-2}} \overline{M}_{2n} \cong 1_N \otimes \overline{M}_{2n-2} \tag{34}$$

It follows that

$$\mathcal{X} Gr(M_-) = \ \delta \, Gr(M_-) \tag{35}$$

where

$$\mathcal{X} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \ddots \end{pmatrix}$$

That is the Grothendieck vector is an eigenvector of this infinite matrix. We obtain immediately

$$Gr(M_-) = (1, \delta, \delta^2 - 1, \delta(\delta^2 - 2),\ \delta^4 - 3\delta^2 + 1, \dots)^t \tag{36}$$

in either case (for $\overline{M}_-$ only the odd entries are relevant). This simple result is quite revealing. Reparameterising $\delta = q + q^{-1} = [2]$ we get

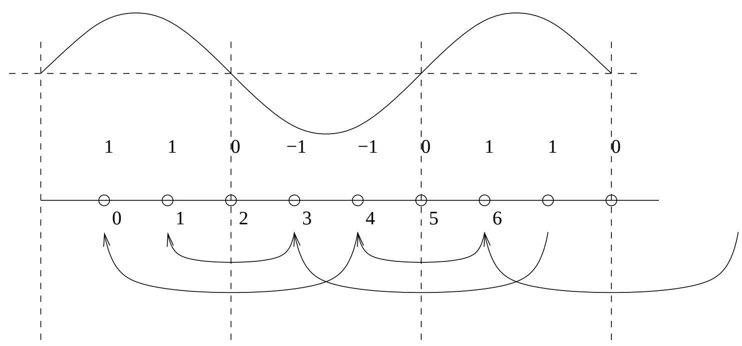
$$Gr(M_-) = (1, [2], [3], [4], [5], [6], \dots)^t$$

and although we have only constructed M_n for $\delta = N \in \mathbb{N}$ and $\overline{M}_{2n}$ for $\delta^2 = N \in \mathbb{N}$ this can be applied to determine representation theory of A_n in all specialisations. For example, when $[3] = 0$ ($\delta = N = 1$) we have $[4] = -1$:

$$Gr(M_-)|_{\delta=1} \ = \ (1, 1, 0, -1, -1, 0, \dots)^t$$

It follows from this (and Proposition 6.5 above) that A_n is not semisimple when $\delta = 1$ and $n > 2$. Similarly when $[4] = 0$ ($\delta^2 = N = 2$) we have $[5] = -1$; and when $[6] = 0$ ($\delta^2 = N = 3$) we have $[7] = -1$.

A negative λ-th entry in $Gr(M)$ implies that some standard module earlier in the labeling scheme with a positive entry ($\Delta(\mu)$ say) contains (at least the head of) $\Delta(\lambda)$; and that the module M contains the quotient of this standard by (the head of) $\Delta(\lambda)$. In other words, although the character for M formally contains the character for μ, this is an overcount, and the character for λ must be subtracted to correct this. In other words a negative λ-th entry is a signal of a homomorphism from $\Delta(\lambda)$ to $\Delta(\mu)$. By property (iii) of standards this says that λ and μ are in the same block. In our $\delta = 1$ example the first negative entry is at $\lambda = 3$, and the only possible homomorphism is to $\mu = 1$. These labels are in the same orbit of an A_1 affine reflection group action, and more generally these orbits describe the blocks of the algebra (when $K = \mathbb{C}$). We can visualise this with the following picture, which shows the labels λ embedded in the real line, with the affine reflection points represented by vertical dashed lines; and the reflections by curved arrows:

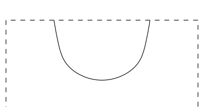


We also show in the figure that the Grothendieck vector comes from evaluating a sine curve (with origin set to the boundary of the 'dominant region') at the integral points. Thus our solution to the eigenvalue problem, which is essentially by fourier transform with a node at the origin (note that if $q = \exp(i\gamma)$ then $(1, [2], \ldots) = \frac{1}{\sin(\gamma)}(\sin(\gamma), \sin(2\gamma), \ldots)$), is a signal of an alcove geometric description of the block structure.

The representation theory of the Brauer algebra is much more complicated (see [11]), and the representation discussed above does not seem to provide sufficiently many constraints for its complete analysis. We now analyse a generalisation of the representation discussed above (due to Benkart [6, 7]) which turns out to be useful in this regard.

6.2 Generalisations

Returning to F_N, we could have implemented the TL part categorically as



$$\mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

This says that the two vertices at the ends of each edge must be the same colour — the Potts condition: $H_{ij} = \delta_{\sigma_i,\sigma_j}$ from Section 2. Organised as a matrix this is $H^i_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{ij}$ or for general N, $H^i_j = (1_N)_{ij}$. Instead we could have used the flip condition

$$H^i_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij}$$

(this is case $N = 2$ again), giving

$$D_1 \mapsto \begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 0 \end{pmatrix}$$

(**6.7**) Remark. This specific case coincides with the KS_n action in the top row of (28). The Temperley–Lieb algebra does not appear in this Schur-Weyl duality diagram, but the Temperley–Lieb algebra with $\delta = 2$ is a quotient of KS_n and we have

$$\sigma_1 \mapsto (1 - D_1) \mapsto \begin{pmatrix} 1 & & & \\ & 0 & -1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}$$

It is this action which may be q-deformed to give the dual action to that of the $U_q(sl_2)$ quantum group. (The deformation deforms $\delta = N = 2$ to $\delta = q + q^{-1}$ without changing N.)

(**6.8**) For $N > 2$ our flip construction no longer commutes with the S_N action on the bottom left side of (28). Consider now the case N even, and the $Sp(N)$ (instead of $O(N)$) action inside the $GL(N)$ action on the left side of (28). The $Sp(N)$ action does not contain the S_N action permuting basis elements (as $O(N)$ does). Instead it can be chosen to leave the form

$$H^i_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ij}$$

(this is case $N = 2$ again) fixed. Note that this gives $\delta = -2$ (and $\delta = -N$ in general).

More generally we could have used a mixed condition such as

$$H^i_j = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij}$$

This last does not make sense physically (where terms require a probablistic interpretation), but is fine in representation theory. Indeed the extension to the Brauer algebra is introduced in [7], where it is shown that this commutes with an $OSp(l, 2m)$ action (our example is $OSp(1, 2)$). In both Brauer and TL cases it gives

$$\delta = l - 2m$$

(so in our example $\delta = -1$).

Here we conclude by using the theory explained in the previous section to determine the Grothendieck vector for the TL action in the most general case. These actions have received renewed attention recently in the Brauer case, as a possible device for proving decomposition matrix conjectures (results in [11] show that the Brauer decomposition matrices are highly non-trivial, even in characteristic zero). We will return to the Brauer case elsewhere. The TL case is a useful paradigm since, as in the Brauer case, the Potts representation of $\delta = 1$ is rather trivial (it is 1-dimensional for all n). It is useful to have a suite of tensor space representations for $\delta = 1$ large enough to capture a much larger proportion (perhaps all) of the algebra structure.

In the TL case the algebra with parameter δ is isomorphic to that with parameter $-\delta$, so we have two sequences of representations informing the study of $\delta = 1$. That is $l - 2m = 1$ and $l - 2m = -1$. The former sequence is $(m, 2n) = (1, 0), (3, 2), (5, 4), ...$ and the latter $(1, 2), (3, 4), ...$

Following the notation of Section 6.1 let us denote by $M_n^{l,m}$ the representation of $\mathrm{End}_{T(\delta = l-2m)}(n)$ at hand. We again have

$$\mathcal{F}M_n^{l,m} = M_{n-2}^{l,m}$$

Note from (33) that the standard 'multiplicities' (in the sense of section 6.1) depend on $N = l + 2m$, rather than depending on l, m separately. That is, the Grothendieck vector is

$$Gr(M_-^{l,m}) = (1, N, N^2 - 1, N^3 - 2N, \ldots)^t \tag{37}$$

cf. (36). We have the following table of explicit Grothendieck vectors case by case:

$\lambda :$	0	1	2	3	4	...
$N = 2$ $l, m = 0, 1$ $\delta = -2$	1	2	3	4	5	...
$N = 3$ $l, m = 1, 1$ $\delta = -1$	1	3	8	21	55	...
$N = 4$ $l, m = 2, 1$ $\delta = 0$	1	4	15	56		...
$N = 5$ $l, m = 3, 1$ $\delta = 1$	1	5	24	115		...

(N.B. In the $\delta = 0$ case the $\lambda = 0$ and $\lambda = 2$ labelled 'standard' modules coincide, so if we retain this labeling then $(a, -, b, \ldots) \sim (a + b, -, 0, \ldots) \sim (0, -, a + b, \ldots)$ are equivalent Grothendieck vectors.)

(**6.9**) Proposition. *(i) In case $l = 2$, $m = 1$, the representation $M_n^{l,m}$ is non-semisimple for every even n; and (ii) in case $l, m > 1$, the representation $M_n^{l,m}$ contains at least one copy of every simple for every n.*

Proof: (i) A simple way to see this is first to note that $J = D_1 D_3 \ldots D_{2n-1}$ lies in the radical of the algebra (it generates a nilpotent double-sided ideal) when $\delta = 0$ as here. Then note that J is represented by a non-zero matrix in $M_{2n}^{2,1}$.

Alternatively, we can note that in particular, in case $l = 2$, $m = 1$, $n = 2$, the non-zero block of the representation of D_1 is

$$\tilde{U} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \overset{Jordan}{\mapsto} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(the representation is 16-dimensional overall, but all other entries are zero). Thus there is a non-split extension (of course this is implicit in the already noted fact that D_1 lies in the radical). Since $\mathcal{F}$ is exact there is an indecomposable summand of M_{2n} whose image under $\mathcal{F}$ is this non-split submodule. But if this indecomposable summand of M_{2n} were simple its Grothendieck vector would have non-zero entries of magnitude 1 and alternating sign (we invoke this standard TL result over $\mathbb{C}$ for the sake of brevity).

(ii) For the simple multiplicities note that the decomposition matrix for standard modules (into simple composition factors) is upper triangular. Thus if the Grothendieck vector is positive then so is the vector of simple composition multiplicities. □

This very simple result nicely illustrates the point. Firstly, there *is* no ordinary Potts $\delta = 0$ representation (since it would have $N = 0$). Secondly, the ordinary Potts representations are all semisimple, even in the cases when the algebra itself is not; whereas our $n = 2$ example is already manifestly non-semisimple (indeed it contains a copy of the two-dimensional indecomposable regular representation). Thirdly, the multiplicities in our table may here be interpreted as follows. In the $\delta = 0$ case the 'standard' modules Δ_0 and Δ_2 are isomorphic, so we can consider 16 copies to be distributed as 1+15 (with the 1 glued over one of the 15).

Actually a stronger result follows by combining the Temperley–Lieb structure theorem from [24] with a result on tensor ideals in [14], but we shall report on the non-semisimple structure of these representations in general elsewhere. As already noted a bigger (and open) question is the structure of the corresponding representations of the Brauer algebra. And for an even more thorough exercising of the techniques touched on here, see [2].

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