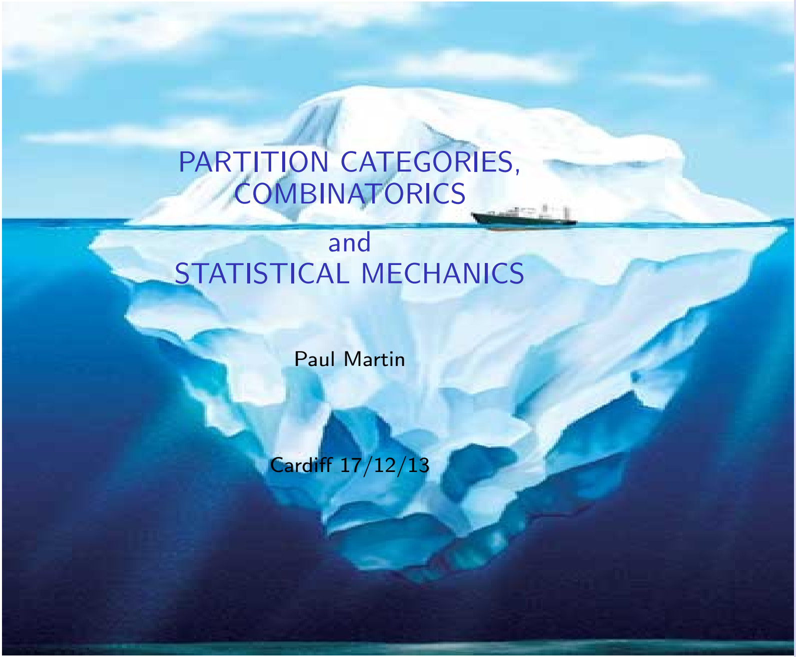

PARTITION CATEGORIES, COMBINATORICS

and STATISTICAL MECHANICS

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Cardiff 17/12/13

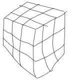
Overview

1. Potts Z-functor
2. Partition category
3. Geometric partition category
4. Combinatorics

Joint with Z Kadar and S Yu

$Q \in \mathbb{N}$, Potts partition function

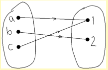
$$Z : \Gamma \longrightarrow \mathbb{Z}[x = e^{\beta}]$$



$\mapsto Z_G$ defined as follows.

- Potts configuration set $\underline{\Sigma}$ on graph G:

$$\underline{\Sigma} = \hom(V_G, \underline{Q})$$



(*in category* $\underline{\textit{Set}}$) ?

- Potts Hamiltonian:

$$H_G : \hom(V_G, \underline{Q}) \longrightarrow \mathbb{R}$$

$$f \mapsto H_G(f) = J \sum_{\langle ij \rangle \in E_G} \delta_{\sigma_i, \sigma_j} + h \sum_{i \in V_G} \delta_{\sigma_i, 1}$$

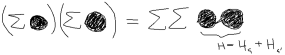
Constants $J = 1$, $h = 0$:

$$Z_G = \sum_{f \in \hom(V_G, \underline{Q})} \exp(\beta H_G(f))$$

Physics: look for stable behaviour in certain limits of large G.

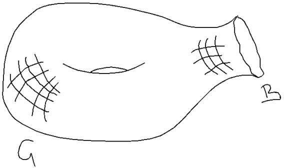
How compute?

$$Z_G Z_{G'} \;=\; Z_{G \sqcup G'}$$



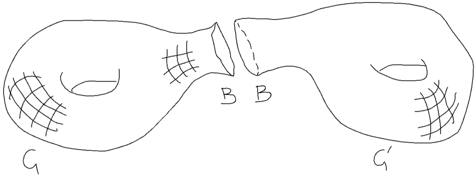
$B \subset V_G$ ‘boundary’, $b \in \hom(B, \underline{Q})$, define

$$Z_{G|_{B=b}} \;=\; \sum_{\substack{f\ s.t.\\ f|B\,=\,b}} \exp(\beta H_G(f))$$



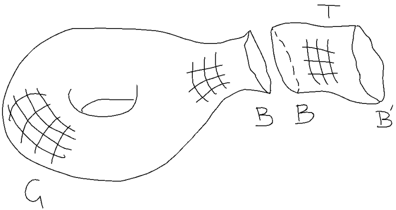
‘partition vector’

$$Z_{G|B} : \hom(B, \underline{Q}) \;\to\; \mathbb{Z}[x]$$
$$b \;\mapsto\; Z_{G|_{B=b}}$$



$$Z_{G|B=b|G'} \quad = \quad Z_{G|B=b}\, Z_{G'|B=b}$$

$$Z_{G.G'} \; = \sum_{b \in \hom(B,\underline{Q})} \text{'' \quad ''}$$



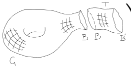
'partition matrix'

$$Z_{T|B,B'} : \hom(B,\underline{Q}) \times \hom(B',\underline{Q}) \longrightarrow \mathbb{Z}[x]$$

$$(b, b') \qquad \mapsto \quad Z_{T|B=b,B'=b'}$$

Potts Z-functor:

$$\underline{\Gamma} \;=\; \left(\begin{matrix} Fin \\ Sets \end{matrix} \;,\quad \begin{matrix} graphs \\ V_G \supset S, T \end{matrix} \;,\quad \bullet \right)$$



$Z{:}\underline{\Gamma} \to \underline{Mat}(\mathbb{Z}[x])$

$$Z(\underline{\Gamma}) \;=\; \left(\begin{matrix} Fin\ set\ S \\ \to \hom(S, \underline{Q}) \end{matrix} \;,\quad \begin{matrix} partition \\ matrix \end{matrix} \;,\quad . \right)$$

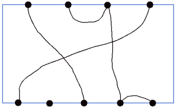
$$\underline{Set} \;=\; (Sets, \hom(-,-), \circ)$$

Underlying algebra:

k commutative ring, e.g. $\mathbb{C}$, $\delta \in k$

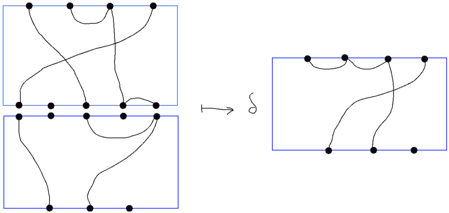
$$\mathcal{P} = (\mathbb{N}_0, k\mathcal{P}(n,m), *)$$

$\mathcal{P}(n,m)$ = set partitions of $\underline{n} \sqcup \underline{m}$.
Represent partition by picture of graph, using $\Pi : \Gamma \to \mathcal{P}$ (connected components).



$\in \mathcal{P}(4,5)$

$*$:



Theorem

$\mathcal{P}$ is a category.

Proof.

check well-definedness and axioms. □

Theorem

$\mathcal{J} = (\mathbb{N}_0, k\mathcal{J}(n,m), *)$ is a subcategory.

$\mathcal{J}(n,m)$ is subset of partitions into pairs.

Theorem

$\mathcal{T} = (\mathbb{N}_0, k\mathcal{T}(n,m), *)$ is a subcategory.

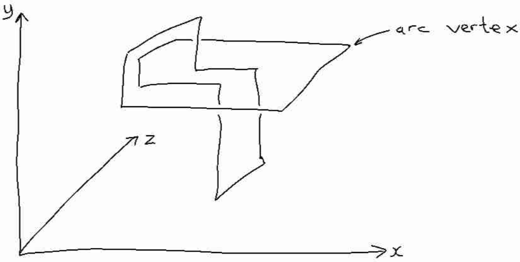
$\mathcal{T}(n,m)$ is subset of partitions with noncrossing pictures.

Aim: 'geometrical' extensions of $\mathcal{T} \subset \mathcal{J}$, interpolating between.

Definitions (geometry)

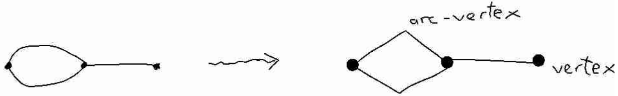
(91)

Polygonal knot K is embedding of $S^1 \xrightarrow{E} \mathbb{R}^3$



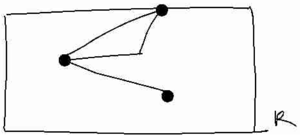
Regular knot is regular projection $p : K \longrightarrow \mathbb{R}^2$

(92) Polygonal graph : embedding of graph



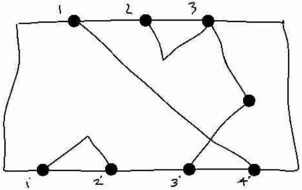
Regular graph (in $R \subset \mathbb{R}^2$) is regular projection

— locally like regular knot, except at vertex



(93) **Picture** $d = (V, \lambda, L, R)$

- V: vertex set
- $\lambda: V \hookrightarrow R \subset \mathbb{R}^2$
- L: image of some $g \in \Gamma(V)$ under regular projection



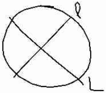
Convention: $\lambda(V) \cap \partial R$ labels this set as indicated.

— $\exists$ category of pictures and stacking.

(94) <u>Line</u> ℓ is regular projection of polygonal arc

<u>path</u> in d from x to y is line

$K \in (\ell) \cap L$ "regular" i.e.



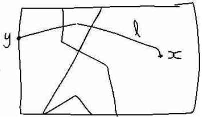
($\exists$ paths from x to y)

$D_d(\ell) = \#$ crossing pts of ℓ with L.

$D_d(x,y) = \min\{ D_d(\ell) : \partial\ell = \{x, y\}\}$

Suppose $L \cap LHE(R) = \emptyset$

$ht_d(x) = D_d(x,y)$ some y on LHE

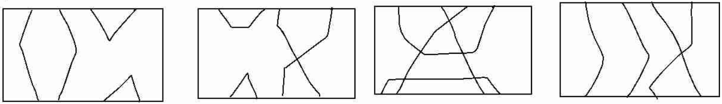


(95)

$$ht(d) = \begin{cases} \max\{ht_d(x) : x \text{ a crossing pt of } L\} \\ -1 \quad \text{otherwise} \end{cases}$$

$$ht(p) = \min\{ht(d) : \pi(d) = p\}$$

Examples



d	-1	0	1	2
p	-1	0	?	2

Definition

$\mathcal{P}^l(n,m) \subset \mathcal{P}(n,m)$ with $ht(p) \leq l$

Theorem

$\mathcal{P}^l = (\mathbb{N}_0, \mathcal{P}^l(n,m), *)$ *a subcategory.*

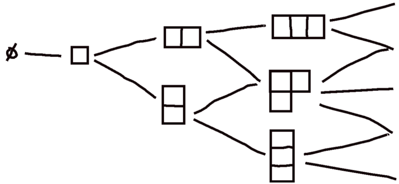
Proof.

Path in a stack of pictures is still a path... □

Theorem

$\mathcal{J}^l = (\mathbb{N}_0, \mathcal{J}^l(n,m), *)$ *a subcategory.*

Combinatorics.
Problem: Enumerate, say, $\mathcal{J}^l(n,n)$.

$\mathcal{Y} =$ 

Theorem (double-factorial)

$\mathcal{J}(n,n) \cong W^{\mathcal{Y}}_{\emptyset,\emptyset}(2n)$

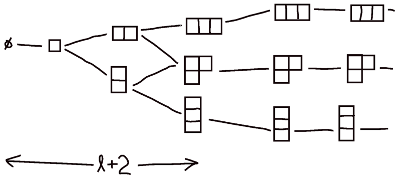
Cf. Robinson–Schented directed walks on $\mathcal{Y}$.

Theorem (Bell)

$\mathcal{P}(n,n) \cong W^{\mathcal{Y}_2}_{\emptyset,\emptyset}(2n)$

Theorem (Catalan)

$\mathcal{T}(n,n) \cong W^{A_\infty}_{0,0}(2n)$

$\mathcal{Y}^l =$ 

Theorem

$\mathcal{J}^l(n,n) \cong W^{\mathcal{Y}^l}_{\emptyset,\emptyset}(2n)$

Proof.

Category/Representation theory. □

Open problems: Loads!
How draw a low-height picture.
BMW / Knot invariants
Non-semisimple representation theory; Parabolic Kazhdan–Lusztig polynomials
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