Chapter 2

Computation

2.1 Transfer matrix formalism

2.1.1 Partition vectors

Suppose that we have fixed a graph Hamiltonian as in (1.2.3). Then for each graph G we have a partition function Z^G, associated to the Hamiltonian H_G.

(**2.1.1**) Let $Z^G_{V|_x}$ be Z^G but with vertex subset V *fixed* to x:

$$Z^G_{V|_x} = \sum_{s \text{ s.t. state } s|V=x} \exp(-\beta H)$$

Then the 'Partition vector' Z^G_V is a vector indexed by configurations of V, whose x-th entry, $(Z^G_V)_x$, is $Z^G_{V|_x}$.

(**2.1.2**) If $G = G' \cup G''$ where $V_{G'} \cap V_{G''} = V$, $E_{G'} \cap E_{G''} = \emptyset$, and H_G is 'local' in the sense that interactions are associated to pairs of vertices defined by edges, then the subgraph partition vector $Z^{G'}_V$ makes sense, and we have

$$Z^G = \sum_x (Z^{G'}_V)_x (Z^{G''}_V)_x \tag{2.1}$$

Typically G has topological properties (perhaps embedded in and representing some manifold), with respect to which V is a boundary, and the situation of equation(2.1) may be illustrated as in Figure 2.1 or 2.2.

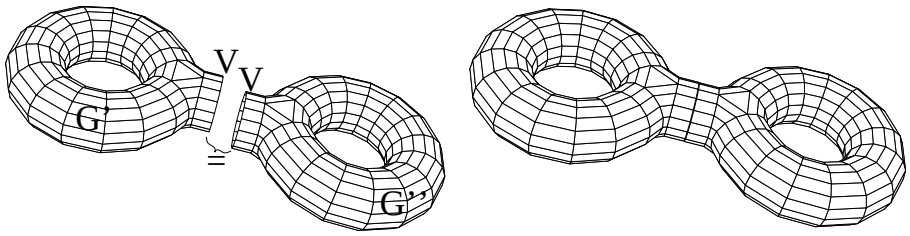
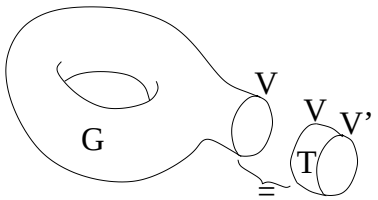


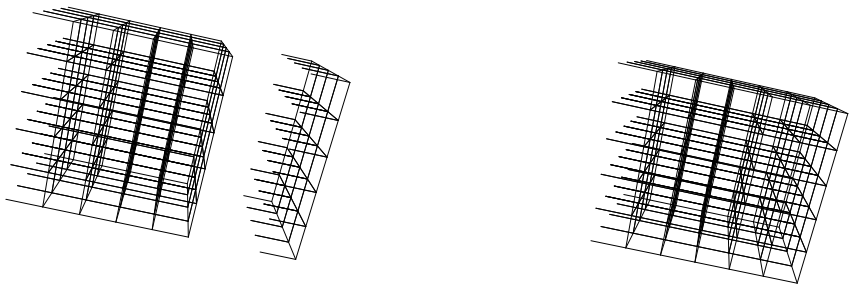
Figure 2.1: $\sum_x Z^{G'}_{V|_x} Z^{G''}_{V|_x} = Z^{G' \cup G''}$

□



$$\sum_x Z^G_{V|_x} Z^T_{V|_x \, V'|_y} = Z^{G \cup T}_{V'|_y}$$

Figure 2.2: Transfer Matrix $\mathcal{T}_{xy} = Z^T_{V|_x \, V'|_y}$.



1 $\rightsquigarrow$ 2

Figure 2.3: 1. Adding a lattice layer; 2. New larger lattice.

2.1.2 Transfer matrices

Figure 2.2 also server, formally, to define the *transfer matrix* $\mathcal{T} = Z^T_{V,V'}$, as a partition vector with two parts to the 'boundary' (note that this is simply an organisational arrangement). Suppose we iterate composition of a suitable $\mathcal{T}$, as illustrated in Figure 2.3. Then we get

$$Z^{big} = \langle \mathcal{T}^n \rangle$$

(for suitable initial and final boundary conditions $\langle - \rangle$). If $\{\lambda_i\}_i$ are the eigenvalues of $\mathcal{T}$ we have

$$Z^{big} = \langle \mathcal{T}^n \rangle = \sum_i k_i \lambda_i^n$$

where the k_is depend on the boundary conditions, but not n. For example with simple periodic b.c.s we have

$$\langle \mathcal{T}^n \rangle = \mathrm{Tr}(\mathcal{T}^n) = \sum_i \lambda_i^n$$

Note that here $\mathcal{T}$ is +ve symmetric, so the Perron–Frobenius Theorem implies

$$Z^{big} \sim k_0 \lambda_0^n \left(1 \; + \; \frac{k_1}{k_0} (\frac{\lambda_1}{\lambda_0})^n + \sum_{i>1} \frac{k_i}{k_0} (\frac{\lambda_i}{\lambda_0})^n \right) \sim k_0 \lambda_0^n$$

where λ_0 is the largest eigenvalue, unless $\lambda_1 \to \lambda_0$ as size$\to \infty$. So the Helmholtz free energy $\frac{1}{N} \ln(Z) \sim \ln(\lambda_0)$.

What about the physical role of other eigenvalues?

2.1.3 Correlation functions

Cold systems tend to be ordered, and hot systems disordered. Neither of these states exhibits long range *correlation* between local states. Thus only in the order/disorder transition region may there be such correlations. Experimentally, correlation of spins over long distance is indeed a signal of phase transition.

- Experimentally, at a fixed T away from T_c, an observation of the correlation of the state of two spins (say) as a function of their separation r, behaves like:
$$\langle \sigma_i \sigma_{i+r} \rangle \sim e^{-r/\rho}$$
(length scale $\rho(T)$ measured in terms of lattice spacing).
As $T \to T_c$, $\rho \to \infty$ (crucial in lattice Field Theory).

- In Stat Mech
$$\langle \sigma_i \sigma_{i+r} \rangle \sim \frac{(\mathcal{T}^{N_1} \hat{\sigma} \mathcal{T}^r \hat{\sigma} \mathcal{T}^{N_2})_.}{(\mathcal{T}^{N_1+r+N_2})_.} \sim \left(\frac{\lambda_\sigma}{\lambda_0}\right)^r$$
$$= \exp(-r \underbrace{(\ln(\lambda_0) - \ln(\lambda_\sigma))}_{\frac{1}{\rho}})$$

- So other eigenvalues besides λ_0 have physical significance. (NB labelled by operator content, not N, should not depend on N in limit.)

2.2 Practical calculation

2.2.1 Use the force: transfer matrix algebras

Next idea: We look for an algebra A and a representation R such that we can express
$$\mathcal{T} = R(X)$$
with $X \in A$; then organise the spectrum of $\mathcal{T}$ by simple components of R.

There is no simple recipe for finding A, R, X to make this work. We shall discuss a limited systematisation as we go.

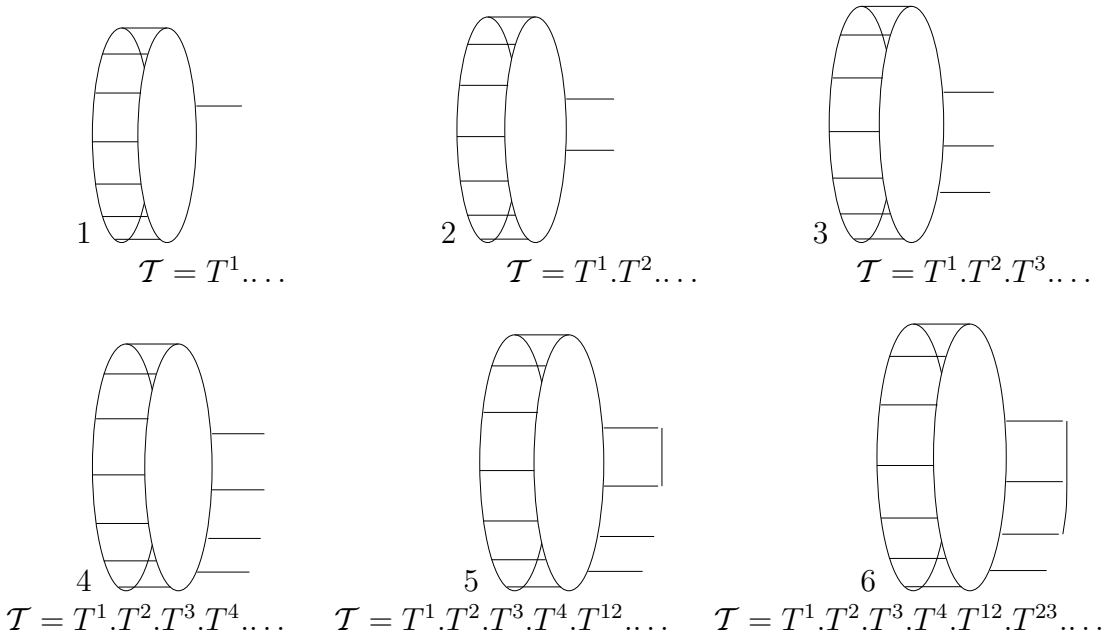


Figure 2.4: Growing a cylindrical lattice layer one interaction at a time

Local transfer matrix

The transfer matrix method (essentially requiring that the lattice can made up of a number of layers) grows the lattice a single layer at a time. Now we go further, and grow the lattice a single interaction at a time.

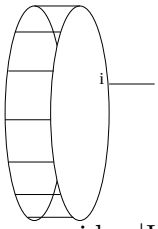
Let us picture the situation in which we have built some number of complete layers, and now proceed to start building a new layer. We start by adding a single new edge/interaction: see Figure 2.4(1). Proceeding as illustrated, here we get

$$\mathcal{T} = \prod_i T^i \prod_{\langle i,j \rangle} T^{ij} \tag{2.2}$$

What is T^i here?

Consider the following example. Take $H = -\beta \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$ (2-state case, say) on a graph made up of closed chain layers (hence a cylindrical lattice, as it were), as in our recent figures. Set $x = e^\beta$. Consider the partition vector Z^G_W for some assembly of complete layers of lattice G, relative to some collection of 'boundary' spins W (as in (2.1.1)). One natural arrangement is to take W to be the union of the states in some initial layer (on the left) and the states in the most recent layer

grown (on the right) — in which case the partition vector is the transfer matrix $\mathcal{T}^n$ for some n. Alternatively one might consider Z relative only to the states, V say, in the most recently grown layer — i.e. as $\langle \mathcal{T}^n$. But consider (for a moment) the partition vector relative to a *single* spin i in V, preparatory to adding a single new interaction involving that spin, as indicated:



$$Z_{\sigma_i} = \begin{pmatrix} Z_{\sigma_i=1} \\ Z_{\sigma_i=2} \end{pmatrix}$$

Now consider $|V| = m$ so Z_V is a Q^m-component vector

$$Z_V = \begin{pmatrix} Z_{V|\sigma_i=1} \\ Z_{V|\sigma_i=2} \end{pmatrix}$$

where each entry is a Q^{m-1}-component vector. The partition vector for the new system, over the new spin, after the new edge is added, is:

$$Z^+_{\sigma_i} = \begin{pmatrix} xZ_{\sigma_i=1} + Z_{\sigma_i=2} \\ Z_{\sigma_i=1} + xZ_{\sigma_i=2} \end{pmatrix} = \underbrace{\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}}_{T^i=(x-1)\mathbb{I}_Q+D_Q} Z_{\sigma_i}$$

Thus the prefactor matrix on the right is the local transfer matrix.

What is T^{ij}?

Similarly

$$Z^]_{\sigma_i\sigma_j} = \begin{pmatrix} xZ_{11} & & & \\ & Z_{12} & & \\ & & Z_{21} & \\ & & & xZ_{22} \end{pmatrix} = \underbrace{\begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & x \end{pmatrix}}_{T^{ij}=\mathbb{I}_{Q^2}+(x-1)C_Q} Z_{\sigma_i\sigma_j}$$

Let us define

$$u_i := \frac{1}{\sqrt{Q}}\mathbb{I}_Q \otimes .. \otimes \underbrace{D_Q}_{i-th} \otimes \mathbb{I}_Q \otimes .. \otimes \mathbb{I}_Q$$

$$u_{ij} := \sqrt{Q}\mathbb{I}_Q \otimes .. \otimes \underbrace{C_Q}_{i-th \text{ and } j-th} \otimes \mathbb{I}_Q \otimes .. \otimes \mathbb{I}_Q$$

NB, these obey

$$u_i^2 = \sqrt{Q}u_i \qquad u_{ij}^2 = \sqrt{Q}u_{ij} \qquad u_i u_{ij} u_i = u_i \qquad u_{ij} u_i u_{ij} = u_{ij} \tag{2.3}$$

$$[u_i, u_j] = [u_{ij}, u_{kl}] = [u_i, u_{kl}] = 0 \qquad \{i\} \cap \{k, l\} = \emptyset \tag{2.4}$$

Algebra

- As abstract relations (2.3-2.4) define *Graph TL algebra (GTLA) for the complete graph* K_m. A sort of TL version of a Coxeter–Artin group[1].
- GTLA for graph G is subalgebra with generators u_i and u_{ij} if $(i, j) \in G$.

  This is generally not finite rank[2] but $G = A_m$ case is $\cong$ ordinary TL algebra.
- Thus in 2d

$$\mathcal{T} = R\left(\prod_i (\frac{(x-1)}{\sqrt{Q}}1 + u_i) \prod_{ij} (1 + \frac{(x-1)}{\sqrt{Q}} u_{ij})\right)$$

  where R is a representation of OTLA.
- Thus spectrum of $\mathcal{T}$ decomposes by irreducible components of R. Thus correlation functions (particles) at least partially indexed by simples of algebra.
- This is the paradigm.

Global limit

- Even fixing the physical model, there is a $\mathcal{T}$, and hence a TMA, for each N.
- But physical observables, and hence spectrum components, defined essentially independently of N. For given N, spectrum components are (partly) indexed by simple module decomposition of

$$\mathcal{T} = R(X)$$

  , thus these can be indexed independently of N.
- Thus expect global limit to sequence of algebras, and localisation functors picking out fibres of "physically equivalent" modules.
- How change system size? Example:

  Freeze two spins together in transfer matrix layer.

  What does this look like at the level of algebra?

[1] (for Coxeter–Artin groups see (Ram's translation of) Brieskorn-Saito)

[2] quite interesting. See Martin-Saleur 93

The partition algebra and the Brauer algebra

For Q-state models the overarching algebra is the partition algebra. The partition algebra P_n has a basis of partitions of 2 rows of n vertices.

- P_n gives a representation of GTLA.
- consider also the Brauer subalgebra (pair partitions).

2.3 Analysis of results I: generalities and very low rank

We have seen quite generally that in the physical temperature region the limit free energy density is $\ln \lambda_0$ where λ_0 is the largest magnitude eigenvalue of the transfer matrix. What becomes of this when we look in the complex $x = \exp(\beta)$ plane, and in particular in our 'critical' neighbourhood of the physical region? (As defined in Section 1.1.3.)

To get a bit more out of the 1d Ising model here consider other boundary conditions. For example, for the A_N graph but with end spin states fixed

$$Z'(A_N) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}(\lambda_1^N + \lambda_2^N)$$

while

$$Z(\hat{A}_N) = \mathrm{Tr} \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}^N = \lambda_1^N + \lambda_2^N = (x+1)^N + (x-1)^N$$

There are a number of ways that we can recast this simple expression to help think about what might happen in general for large N.

Firstly, we can rewrite

$$(x+1)^N + (x-1)^N = (x-1)^N((\frac{x+1}{x-1})^N + 1)$$

Ignoring the first factor we have

$$Z \sim Y^N + 1 = Y^{N/2}(Y^{N/2} + Y^{-N/2})$$

(where $Y = \frac{x+1}{x-1}$), that is, the remaining zeros are distributed evenly around a circle. It is the same circle for any N, but the line density increases with N. Setting $Y = \exp(\beta')$

$$f = \frac{1}{N} \ln Z = \frac{1}{N}(\ln(2) + \frac{N}{2} \ln Y + \ln(\cosh(\frac{N}{2}\beta')))$$

so

$$U = -\frac{\partial \frac{1}{N} \ln Z}{\partial \beta'} = -1 - \tanh(\frac{N}{2}\beta')$$

That is, the internal energy changes fast (at $\beta' = 0$) for large N.

As we have already seen, the physics is dominated by the zeros close to the real line, so we can approximate

$$\lim_{N\to\infty} f \sim \beta'/2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} a(y)\ln(\beta' + iy)dy$$

where $a(y) = 1$ (in our case) is the line density of zeros. Thus

$$U \sim \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a(y)dy}{y - i\beta'}$$

The integrand has a simple pole at $y = i\beta$, so the integral changes by $2\pi a(0)$ as β' changes sign. In other words, if the limit line density $a(0) \neq 0$ the internal energy changes discontinuously at this point — a *first order phase transition*.

In practice, in more complicated systems, we can get

$$a(y) \sim |y|^{1-p} \qquad (0 \leq p \leq 1)$$

but we will return to this shortly.

Notice in our 2×2 transfer matrix example that the distribution of zeros corresponds to the locus of points where the largest eigenvalue is actually degenerate (with the other eigenvalue, regarded as an analytic function of β). In fact a large class of models have a transfer matrix reducible to a 2×2 polynomial matrix T'. As before

$$\lim_{N\to\infty} \frac{\ln Z}{N} = \lim_{N\to\infty} \ln(\lambda_+^N + \lambda_-^N) \stackrel{*}{=} \ln \lambda_+$$

where * means on the real axis. What happens to the zeros this time?

Consider the general identity

$$C^N + D^N = \prod_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} (C + \exp(\frac{2\pi i n}{N})D) = \prod_{n=1/2}^{\frac{N-1}{2}} (C^2 + D^2 + 2\cos(2\pi n/N)CD)$$

(the explicit limits are for the case N even — the reader will easily compute the odd case). Using this we can rewrite

$$\lim_{N\to\infty} \frac{\ln Z}{N} = \lim_{N\to\infty} \ln(\lambda_+^N + \lambda_-^N) = \frac{1}{2\pi}\int_0^{\pi} \ln(2(A^2 + B) + 2\cos y(A^2 - B))dy$$

where $A^2 - B = CD = \lambda_+\lambda_-$ and $2(A^2 + B) = C^2 + D^2 = \lambda_+^2 + \lambda_-^2$, that is

$$\lambda_\pm = A \pm \sqrt{B}$$

Since T' is polynomial, so are A and B, and hence the limit is (the log of) an infinite product of polynomials. One readily confirms that the zeros of this infinite product are the loci

$$|\lambda_+| = |\lambda_-|$$

and the endpoints of these loci (if any) are the points where $\lambda_+ = \lambda_-$, i.e. at roots of the polynomial B (if B nonvanishing).

(**2.3.1**) REMARKS: This analysis is essentially taken from [9, Ch.11].

2.4 The 2D Ising model: exact solution

Recall from §1.2.1 that a lattice is an embedding (i.e. a positional but not orientational fixing) of a set of spins in an underlying physical space (usually in a regular array). Then a lattice model is a model of the bulk behaviour of such a system of many interacting lattice spins, determined by a spin interaction Hamiltonian.

Recall the Potts model Hamiltonian (1.4). The Ising model is the two-state Potts model (up to some trivial Hamiltonian rescalings). It will be convenient to use the equivalent 'Ising form' of the Potts Hamiltonian here. Thus we have the collection of partition functions of form

$$Z = \sum_\sigma \exp(\beta \sum_{ij} (2\delta_{\sigma_i,\sigma_j} - 1))$$

where $\sum_{ij}$ is the sum over pairs of nearest neighbour sites in the lattice. In practice one focusses on a lattice or collection of lattices determined by the embedding space. In 2D this collection of lattices is (at least locally) the $n \times m$ square grids, with n, m large.

Our strategy in computing Z is to determine a transfer matrix $\mathcal{T}$ (acting on the space of states of an n-site layer of the lattice), such that $Z = \langle \mathcal{T}^m \rangle$, and then to compute by finding a basis for the state space in which $\mathcal{T}$ is diagonal.

In the Ising form the local transfer matrix T_i (from (2.2)) for an n-site wide lattice is

$$T_i = \begin{pmatrix} x & x^{-1} \\ x^{-1} & x \end{pmatrix}_i = x 1_{2^n} + x^{-1}\sigma_i^x = x(1_{2^n} + x^{-2}\sigma_i^x)$$

(this means T_i acts non-trivially on the ith factor in the layer configuration space, and acts trivially on all the other $n-1$ factors). Note that for any scalar θ

$$e^{\theta\sigma^x} = \cosh\theta 1 + \sinh\theta\sigma^x = \cosh\theta(1 + \tanh\theta\sigma^x)$$

so if we choose θ so that $\tanh\theta = x^{-2}$ we get

$$T_i = \frac{x}{\cosh(\theta)}\, e^{\theta\sigma_i^x} \;=\; (\cosh(\theta)\sinh(\theta))^{-1/2} e^{\theta\sigma_i^x} \;=\; \sqrt{2\sinh(2\beta)}\, e^{\theta\sigma_i^x} \qquad (2.5)$$

Meanwhile the local transfer matrix T_{ij} is

$$T_{ij} = \begin{pmatrix} x & & & \\ & x^{-1} & & \\ & & x^{-1} & \\ & & & x \end{pmatrix}_{ij}$$

(acting on the adjacent factors i, j), which can be written

$$T_{ij} = \exp\left(\beta \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\right) = \exp\left(\beta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)$$

Bruria Kaufman's (1949) idea is as follows. One notes that $V_1 = \prod T_i$ and $V_2 = \prod T_{ij}$ can both be equated to certain *spin representations* of rotations. (These are representations on tensor space of dimension 2^n.) We can then use an abstract relation to the eigenvalues of a smaller more manageable representation of the same rotations — the ordinary rotation matrices of dimension $2n$. Recall that these are generated by the simple plane rotations

$$w_{i\,i+1}(\theta) = 1_{i-1} \oplus \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \oplus 1_{2d-i-1} \qquad (2.6)$$

where the mixing occurs in the $i, i+1$-positions.

To understand the spin representations it is convenient to introduce *Clifford algebras*.

(**2.4.1**) A set $\{\Gamma_a\}_{a=1,\ldots,2n}$ of $2^n \times 2^n$ matrices such that $\Gamma_a^2 = 1$ and

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 0 \qquad (a \neq b)$$

are said to form a *Clifford algebra*.

(**2.4.2**) For example, with σ_i^x the usual Pauli matrix action on tensor space:

$$\Gamma^\bullet_{2i-1} := \left(\prod_{j=1}^{i-1} \sigma_j^x\right) \sigma_i^z \qquad\qquad \Gamma^\bullet_{2i} := \left(\prod_{j=1}^{i-1} \sigma_j^x\right) \sigma_i^y$$

obey these relations.

In this case note that

$$\Gamma^\bullet_{2i}\Gamma^\bullet_{2i-1} = \sigma^y_i\sigma^z_i = i\sigma^x_i \qquad \Gamma^\bullet_{2i+1}\Gamma^\bullet_{2i} = \sigma^x_i\sigma^z_{i+1}\sigma^y_i = i\sigma^z_i\sigma^z_{i+1}$$

Thus from (2.5) *et seq*

$$V_1 = \prod_i T_i \;=\; \kappa^n \prod_{i=1}^{n} e^{-i\theta\Gamma^\bullet_{2i}\Gamma^\bullet_{2i-1}} \qquad V_2 = \prod_i T_{i\,i+1} \;=\; e^{\beta\sigma^z_n\sigma^z_1}\prod_{i=1}^{n-1} e^{-i\beta\Gamma^\bullet_{2i+1}\Gamma^\bullet_{2i}}$$

where $\kappa = \sqrt{2\sinh(2\beta)}$, and at the last we have applied periodic boundary conditions.

[3]

(**2.4.4**) Fixing any Clifford algebra $\{\Gamma_a\}_a$ we define

$$S(w_{ab}(\theta)) = \cos\frac{\theta}{2}1 - \sin\frac{\theta}{2}\Gamma_a\Gamma_b = \exp(\frac{-1}{2}\theta\Gamma_a\Gamma_b)$$

Example: One can easily find a Clifford algebra $\{\Gamma^\circ_a\}_a$ such that

$$\exp(\frac{-1}{2}\theta\Gamma^\circ_1\Gamma^\circ_2) = \exp(\frac{-1}{2}\theta i\sigma^z_1) = \begin{pmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{pmatrix} \otimes 1_2 \otimes 1_2 ... \tag{2.7}$$

One can check that these matrices obey

$$S(w_{ab}(\theta))\Gamma_a S^{-1}(w_{ab}(\theta)) = \cos\theta\Gamma_a + \sin\theta\Gamma_b$$

[3] (**2.4.3**) Note: (i) If $\{\Gamma_a\}_a$ is a Clifford algebra, then so is $\{S\Gamma_a S^{-1}\}_a$ for any invertible matrix $S \in \mathrm{End}(\mathbb{C}^{2^n})$;
(ii) the matrices Γ'_a obtained from the $\Gamma^\bullet_a$s by swapping the roles of σ^x and σ^z are a Clifford algebra.
(iii) arbitrarily permute the numbering of the Γ_as in any Clifford algebra, and the Clifford relations will still be obeyed.
(iv) Also if $\{\Gamma_a\}$ is any Clifford algebra then $\Gamma' = r\Gamma_1 + s\Gamma_2$ obeys

$$\Gamma_a\Gamma' + \Gamma'\Gamma_a = \Gamma_a(r\Gamma_1 + s\Gamma_2) + (r\Gamma_1 + s\Gamma_2)\Gamma_a = r(\Gamma_a\Gamma_1 + \Gamma_1\Gamma_a) + s(\Gamma_a\Gamma_2 + \Gamma_2\Gamma_a) = 0$$

for all $a > 2$, for any r, s; and

$$(c\Gamma_1 + s\Gamma_2)(s\Gamma_1 - c\Gamma_2) + (s\Gamma_1 - c\Gamma_2)(c\Gamma_1 + s\Gamma_2) = 2(cs - sc) = 0$$

and so on. Following these calculations one eventually checks that Γ_1, Γ_2 can be replaced by the indicated linear combinations, so long as $c = \cos\theta$ and $s = \sin\theta$ for some θ. Evidently one can compose such transformations, so we have an action of the $2n$ dimensional rotation group transforming between realisations of the Clifford relations.

and so on. In other words conjugation by $S(w_{ab}(\theta))$ enacts the rotation $w_{ab}(\theta)$ on the space of Γ-matrices (not to be confused with the space on which the Γ-matrices act).

It follows that conjugation by $S(w_{ab}(\theta))S(w_{cd}(\theta'))$ realises the rotation $w_{ab}(\theta)w_{cd}(\theta')$. From this we have a kind of realisation of the group of rotations in $2n$-dimensions. (This is not quite a representation, since it is a double-cover, but this need not concern us.)

(**2.4.5**) Note from (2.6) that the spectrum of $w_{ab}(\theta)$ is $e^{i\theta}, e^{-i\theta}$ (and possibly some 1s). (Indeed any element in $SO(3)$ can be expressed as a simple rotation about some, not in general coordinate, axis; and hence has eigenvalues of the same form.) Meanwhile, noting (2.7), the spectrum of $S(w_{ab}(\theta))$ is $e^{\frac{i}{2}(\pm\theta)}$ (2^{n-1} copies of each).

Further, if $w = \prod_i w_{a_i b_i}(\theta_i)$ with all the $\{a_i, b_i\}$ distinct; and $S(w) = \prod_i S(w_{a_i b_i}(\theta_i))$, then the $2n$ eigenvalues of w are $\{e^{\pm i\theta_j}\}_j$; and, since the factors in $S(w)$ commute, the 2^n eigenvalue of $S(w)$ are

$$\text{Spectrum}(S(w)) \;=\; \{e^{\frac{i}{2}\sum_{j=1}^{n} \pm\theta_j}\}. \tag{2.8}$$

(Later we shall generalise this correspondence to elements of $SO(2n)$ that are products of commuting rotations about arbitrary sets of orthogonal axes, not just the nominal coordinate axes.)

(**2.4.6**) In these terms, writing $S^\bullet(w)$ for $S(w)$ in the $\Gamma^\bullet$ case, we have

$$V_1 = \kappa^n \prod_{i=1}^{n} S^\bullet(w_{2i\,2i-1}(2i\theta))$$

$$V_2 = \chi \prod_{i=1}^{n-1} S^\bullet(w_{2i+1\,2i}(2i\beta))$$

where χ is the periodic boundary operator. For a suitable treatment of the boundary (not quite simple periodic, but close enough) we have $\chi = S^\bullet(w_{1\,2n}(2i\beta))$.

(**2.4.7**) The idea now is to replace $\mathcal{T} = V_1 V_2$ with the corresponding product of rotation matrices $W = W_1 W_2$. We then find the eigenvalues of this product W. Since every rotation group element can be expressed as a product of commuting rotations (not necessarily respecting the initial axes) these eigenvalues will define a set of rotation angles $\{\theta_i\}_i$ as above. We claim that this set then give the spectrum of $\mathcal{T}$, as in (2.8).

We have (in a representative small example, $n = 4$), with $c = \cos(2i\theta)$, $s =$

$\sin(2i\theta)$, and so on,

$$W_1 = w_{12}(2i\theta)w_{34}(2i\theta)w_{56}(2i\theta)w_{78}(2i\theta) = \begin{pmatrix} c & s & & & & & & \\ -s & c & & & & & & \\ & & c & s & & & & \\ & & -s & c & & & & \\ & & & & c & s & & \\ & & & & -s & c & & \\ & & & & & & c & s \\ & & & & & & -s & c \end{pmatrix}$$

$$W_2 = \begin{pmatrix} c' & & & & & & & -s' \\ & c' & s' & & & & & \\ & -s' & c' & & & & & \\ & & & c' & s' & & & \\ & & & -s' & c' & & & \\ & & & & & c' & s' & \\ & & & & & -s' & c' & \\ s' & & & & & & & c' \end{pmatrix}$$

Note that W_1, W_2 are both fixed by the square of the matrix position shift operator (and that this is true for general n). Accordingly we can fourier transform to block diagonalise W. That is, we make a change of basis as follows. Define $f_z = (1, z, z^2, z^3, ..., z^{n-1})^t$, $v_z = (1,0)^t \otimes f_z$ and $v'_z = (0,1)^t \otimes f_z$. With $z^n = 1$ we have

$$W_1 \begin{pmatrix} v_z \\ v'_z \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} v_z \\ v'_z \end{pmatrix}$$

$$W_2 \begin{pmatrix} v_z \\ v'_z \end{pmatrix} = \begin{pmatrix} c' & zs' \\ -z^{-1}s' & c' \end{pmatrix} \begin{pmatrix} v_z \\ v'_z \end{pmatrix}$$

$$W \begin{pmatrix} v_z \\ v'_z \end{pmatrix} = \begin{pmatrix} cc' + z^{-1}ss' & zcs' - sc' \\ sc' - z^{-1}cs' & cc' + zss' \end{pmatrix} \begin{pmatrix} v_z \\ v'_z \end{pmatrix}$$

Noting that the determinant of the image matrix $W^{(z)}$ on the right is 1 we write the eigenvalues as $\lambda_\pm = e^{\pm l_z}$. Thus

$$e^{l_z} + e^{-l_z} = \text{Trace}(W^{(z)}) = 2\left(cc' + \frac{z + z^{-1}}{2} ss'\right)$$

That is,

$$\cosh(l_z) = \left(\cosh(2\theta)\cosh(2\beta) - \frac{z + z^{-1}}{2}\sinh(2\theta)\sinh(2\beta)\right)$$

Since z is any solution to $z^n = 1$, the complete set of l_zs is obtained from $z = e^{2\pi ik/n}$ with $k = 1, .., n$. One then finds that each l_z is positive for physical parameters. It follows that the largest among the eigenvalues for $\mathcal{T}$ that this gives:

$$\lambda = \exp\left(\frac{1}{2}\sum_{k=1}^{n} \pm l_{e^{2\pi ik/n}}\right)$$

is the case

$$\lambda_0 = \exp\left(\frac{1}{2}\sum_{k=1}^{n} l_{e^{2\pi ik/n}}\right)$$

Recall that this is for an n-site wide lattice. If the lattice is m sites long then $Z \sim \lambda_0^m$, or more usefully, $f \sim (1/n)\ln\lambda_0$, with this approximation getting better as m gets bigger and becoming an equality in the large m limit.

Now we follow [9, §4.1] for the rest of the analysis. Using that

$$\sinh(2\beta) = \sinh(2\theta)^{-1} \qquad \text{and} \qquad \coth(2\beta) = \cosh(2\theta)$$

we have

$$\cosh(l_z) = \left(\coth(2\beta)\cosh(2\beta) - \frac{z + z^{-1}}{2}\right)$$

which it is convenient to express via the integral representation

$$l_z = \frac{1}{\pi}\int_0^\pi dy \ln(2(\coth(2\beta)\cosh(2\beta) - \frac{z + z^{-1}}{2}) - 2\cos(y))$$

giving

$$\ln\lambda_0 = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{\pi}\int_0^\pi dy \ln(2\coth(2\beta)\cosh(2\beta) - 2\cos(2\pi k/n) - 2\cos(y))$$

Note how the difference in the way we have treated m and n is manifested here. We are looking at the free energy in the infinite-m limit (hence the integral), but with n still finite. We can rigorously bring the treatment of the two directions onto the same footing by taking the large n limit (which will covert the sum to a matching integral); or we can roughly discretise the integral to a sum over m terms, using the sum over n terms as a guide.

2.5 Anisotropic limit

Here we consider a 'continuous time' approximation to the Ising model. (Here 'time' is a misnomer for the layering direction in the transfer matrix formalism.)