Notes on the Brauer Algebra: characteristic free and characteristic zero representation theory

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Extract.
Rough draft.

Chapter 2

Brauer diagram category construction

Here K is a commutative ring, and we define the Brauer algebra [2] as an End-set in a suitable K-linear category. This Brauer diagram category is a natural subcategory of a partition category (see e.g. [8]). We recall the definition of the partition category and construct the Brauer category from this.

We then discuss the construction of specific elements of these algebras, such as idempotent elements of the centre. (This is related to Gram matrix problems discussed in Section **??** and 'discriminant' problems as discussed, for example, in [**?**, 5]. We mainly follow [11].)

We begin in §2.1 with some set theory notation.

2.1 Preliminaries

2.1.1 Set notation

For $n \in \mathbb{N}$ let $\underline{n} = \{1, 2, ..., n\}$, $\underline{n}' = \{1', 2', ..., n'\}$ and so on. Write

$$I' : \underline{n} \cup \underline{n}' \cup \underline{n}'' \cup \ldots \to \underline{n} \cup \underline{n}' \cup \underline{n}'' \cup \ldots$$

for the map which adds a (possibly further) prime to each element; and I^- for the map which removes a prime, or leaves a symbol unchanged if it is unprimed.

(**2.1.1**) For S a set, write $\mathcal{P}(S)$ for the power set of S. We regard this as a (hypercubical) lattice in the usual way. Thus $\mathcal{P}(S \times S)$ is the set of relations on S. If T is a set and $\rho \in \mathcal{P}(S \times S)$ then write $\rho|_T$ for the (possibly empty) restriction of ρ to a relation on $S \cap T$.

(**2.1.2**) If S is a set then $\mathbf{P}(S)$ is the set of partitions of S, and $\mathbb{E}(S)$ the set of equivalence relations. We will confuse these sets via their natural equivalence.

By convention, if we write $A \cup B$ for two partitions we shall intend the union of their images as (equivalence) relations. This will be a relation but not an equivalence relation in general (but see later).

(**2.1.3**) A relation on S may be representated as a directed graph on vertex set S (the details of the graph edge set from vertex to vertex are irrelevant except if the edge set is empty or not). The union $A \cup B$ above then corresponds to the union of edge sets and of vertex sets.

(**2.1.4**) We have

$$\mathbb{E}(S) \subset \mathcal{P}(S \times S)$$

Define

$$\mathrm{T}C : \mathcal{P}(S \times S) \to \mathcal{P}(S \times S)$$

by setting $\mathrm{T}C(R)$ to the smallest element of $\mathbb{E}(S)$ containing R.

The union of $A \in \mathbb{E}(S)$ and $B \in \mathbb{E}(T)$ is a relation (but not generally an equivalence) on $S \cup T$.

(**2.1.5**) For $A \in \mathbf{P}(S)$ define $\lambda = ||A||$ as the integer partition of $|S|$ such that λ_i is the degree of the i-th longest part of A. Thus $\lambda'_1 = |A|$.
For example,

$$||\{\{1,2\},\{3\}\}|| = (2,1).$$

We call $||-|| : \mathbf{P}(S) \to \Lambda_{|S|}$ the *shape* function.

(**2.1.6**) Every map $f : S \to T$ induces a map $f : \mathbb{E}(S) \to \mathbb{E}(T)$ and similarly on partitions. In particular the map

$$op : \underline{n} \cup \underline{n}' \to \underline{n} \cup \underline{n}'$$

is the one that toggles the prime ($i \leftrightarrow i'$).

(**2.1.7**) Note that if $f : S \to T$ is a bijection then $||f(A)|| = ||A||$.

2.1.2 Young diagrams

We confuse Young diagrams and integer partitions in the usual way. The set of all such is denoted Λ. We write Λ^* for Λ excluding the empty integer partition. Write Λ_n for the subset of partitions of n.

(**2.1.8**) A *multipartition* is an ordered list of integer partitions, i.e. an element of $hom(\mathbb{N}, \Lambda)$ (or of $hom(\underline{n}, \Lambda)$ for some n).
An *unordered multipartition* is an equivalence class of multipartitions under the action of reordering the list; i.e. a list of distinct partitions and multiplicities; i.e. a map from the set of partitions to the set of natural numbers — an element of $hom(\Lambda, \mathbb{N}_0)$.

(**2.1.9**) Let λ be an unordered multipartition. We say that a Young diagram μ is λ-*tilable* if it has a sequence of subdiagrams

$$\mu = \mu_0 \supset \mu_1 \supset ...\mu_l = \emptyset$$

such that each skew μ_i/μ_{i+1} is a diagram in λ and each such diagram occurs as many times in the filtration $(\mu_0/\mu_1, \mu_1/\mu_2, .., \mu_{l-1}/\mu_l)$ as in λ.
Write Λ_λ for the set of λ-tilable partitions.
For example:

$$\Lambda_{((2)^3)} = \{(2,2,2), (4,2), (6)\}$$

2.2 Aside: Representation theory generalities

Here are some useful reminders, from Brauer [3], Curtis and Reiner [4] and Benson [1].

A f.d. algebra A over field k is *Frobenius* if there is a linear map $L : A \to k$, such that $ker(L)$ contains no left or right ideal.
That is, A is Frobenius if there is $L \in A^*$ such that $L(ab) = 0$ for all $a \in A$ implies $b = 0$.

Note that each $L \in A^*$ defines an associative bilinear form $b_L : A \times A \to k$ via $b_L(a, b) = L(ab)$.
Here associativity means $b_L(ab, c) = b_L(a, bc)$.

Let R be the left regular representation of A. The bilinear form $b_R(a, b) = Trace(R(ab)) = Trace(R(a)R(b))$ is associative.

Theorem 1. *Let ideal H in ring A be nilpotent, and $e^2 = e \in A/N$. Then there is an idempotent $f \in A$ whose image in A/N is e.*

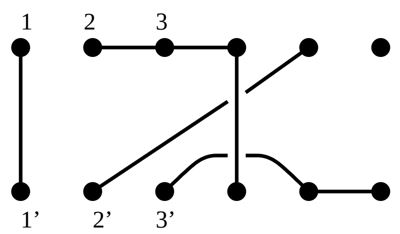
2.3 The partition algebra

We recall the definition of the partition algebra and category from [8] (see also, e.g., [10, 7]).

(**2.3.1**) Fix a ring K and $\delta \in K$. The partition algebra $P_n = P_n(\delta)$ has a basis of partitions of two rows of n objects: $\underline{n} \cup \underline{n}'$. We next describe the composition rule.

(**2.3.2**) We may represent partitions as graphs, with the object set as vertices. That is, we may represent a partition p by the graph of any relation whose RST closure gives p.

We adopt the usual convenience of confusing a graph with any *depiction* that encodes that graph. For example then:



represents the partition $\{\{1, 1'\}, \{2, 3, 4, 4'\}, \{5, 2'\}, \{6\}, \{3', 5', 6'\}\}$.

This realisation allows considerable freedom in the drawing of a typical partition. However we will adopt the arrangement of vertices into rows as drawn in the example as a rigid convention. Such a picture is then called a partition *diagram.*

More generally any digraph on a vertex set V together with a map from a set S to V (let us say an injective map, although even this can be relaxed) defines a relation and, by closure, a partition on S. In this case an element of V *not* in the image of S is called *internal.*

(**2.3.3**) Note that if we juxtapose two diagrams d, d' (each drawn as in our example) in a vertical stack, so that the meeting rows of vertices coincide pointwise, then we have a graph $d.d'$ on three rows of vertices. This then defines a partition on all three rows, or on any subset, and in particular defines a partition $p(d,d')$ on the subset consisting of the new top and bottom rows (relative to which, the middle row becomes internal).

The partition algebra product is defined on the basis of partitions:

$$q.q' = \delta^c p(d,d')$$

where d, d' are any representatives of q, q' and c is the number of connected components of $d.d'$ involving only internal vertices.

(**2.3.4**) The partition algebra P_n has identity element

$$1 = 1_n = \{\{1,1'\},\{2,2'\},...,\{n,n'\}\}$$

(**2.3.5**) The partition algebra P_n is generated by the elements (partitions)

$$\sigma_{ij} = \{\{1,1'\},\{2,2'\},...,\{i,j'\}.\{j,i'\},...,\{n,n'\}\}$$

$$A_i = \{\{1,1'\},\{2,2'\},...,\{i\},\{i'\},...,\{n,n'\}\}$$

$$A_{ij} = \{\{1,1'\},\{2,2'\},...,\{i,i',i+1,(i+1)'\},...,\{n,n'\}\}$$

Equivalent formulations of the multiplication rules are given, for example, in [9].

(**2.3.6**) We define a K-linear category

$$C_{\mathbf{P}} = (\mathbb{N}, \mathrm{Hom}_{\mathbf{P}}(-,-), \circ)$$

where $\mathrm{Hom}_{\mathbf{P}}(m,n) = K\mathbf{P}(\underline{m} \cup \underline{n}')$ and the composition is the obvious generalisation of the algebra composition.

(**2.3.7**) For d any partition appearing in the category $C_{\mathbf{P}}$ we write $\#(d)$ for the *propagating number* — the number of parts that contain both primed and unprimed elements. We write $\mathbf{P}(\underline{m} \cup \underline{n}')[l]$ for the subset of partitions d with $\#(d) = l$.

2.3.1 The Brauer algebra

(**2.3.8**) The Brauer algebra $B_n(\delta)$ is the subalgebra of $P_n(\delta)$ obtained by restricting the basis to the set $\mathbf{J}(\underline{n} \cup \underline{n}')$ of pair partitions (define $\mathbf{J}_n = \mathbf{J}(\underline{n} \cup \underline{n}')$ for short).

We write $\mathbf{J}_n[l]$ for the subset of $\mathbf{J}_n$ of diagrams with l propagating lines; and $\mathbf{J}_n(l)$ for the subset of $\mathbf{J}_n$ of diagrams with at most l propagating lines.

We define the special pair partitions

$$U_i = A_{i\,i+1} A_i A_{i+1} A_{i\,i+1}$$

Define $\mathbf{P}(m,n) = \mathbf{P}(\underline{m} \cup \underline{n}')$ and $\mathbf{J}(m,n) = \mathbf{J}(\underline{m} \cup \underline{n}')$. Keeping K fixed, the Brauer partition category is the subcategory of the partition category $C_{\mathbf{P}}$ given by

$$C_{\mathbf{J}} \;=\; (\mathbb{N}, K\mathbf{J}(-,-), \circ)$$

Automorphisms, arithmetic and idempotents

2.4 Spore function on partitions

The subset of generators σ_{ij} (from (2.3.5)) generate a copy of the symmetric group S_n in P_n (they are the pair permutations in S_n). Thus P_n is both a left and a right S_n-module by restriction. Indeed $\mathbf{P}(\underline{n} \cup \underline{n'})$ is a left and a right S_n-set.

(**2.4.1**) The invertible elements of $\mathbf{P}(\underline{n} \cup \underline{n'})$ in P_n are precisely the elements of S_n. From this we have, immediately, the inner automorphism group of P_n generated by these units.

(**2.4.2**) Write A^{S_n} for the orbit of $A \in \mathbf{P}(\underline{n} \cup \underline{n'})$ under conjugation by S_n. Define

$$A_\Sigma = A_\Sigma^{S_n} := \sum_{d \in A^{S_n}} d$$

Examples: note that if $A \in \mathbf{J}(\underline{n} \cup \underline{n'}) \subset \mathbf{P}(\underline{n} \cup \underline{n'})$ then $A^{S_n} \subset \mathbf{J}(\underline{n} \cup \underline{n'})$; and $A \in S_n$ implies $A^{S_n} \subset S_n$. In the latter case we have the usual observation that conjugacy classes are indexed by integer partitions of n.

Note that

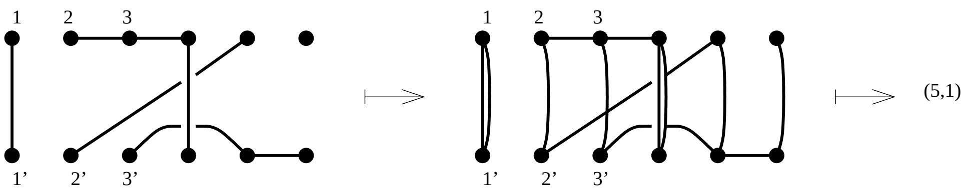
$$op(w) = w^{-1}$$

for $w \in S_n$.

(**2.4.3**) Define

$$Sp : \mathbf{P}(\underline{n} \cup \underline{n'}) \to \Lambda_n$$

by $Sp(A) = ||(\mathrm{T}C(A \cup 1_n))|_{\underline{n}}||$.
Example:



There are several more examples in §2.5.

Proposition 1. *For all $A \in \mathbf{P}(\underline{n} \cup \underline{n'})$ and $w \in S_n$:*

$$Sp(A) = Sp(wAw^{-1})$$

If $\#(A) = \#(B) = 0/1$ then $A^{S_n} = B^{S_n}$ if and only if $Sp(A) = Sp(B)$.

Proof. First part: Consider 2.1.7. Second part: Exercise. □

2.5 Primitive central idempotents

Our aim here is to compute the primitive central idempotents of the Brauer algebra over the field of rational polynomials in δ. This is for a number of reasons.

1. primitive central idempotents determine the blocks of an algebra. The Brauer algebra over the rational field is semisimple, but its idempotents are related (in a suitable sense [**?**]) to the primitive central idempotents in specific specialisations of δ over other fields, which have more complicated blocks.

2. we hope to gain information about the submodule structure of standard modules (modules that we shall describe later, that are simple over the field of rational polynomials in δ, but not in general).

3. we hope to get clues about analogues for the Brauer algebra of Young's orthogonal form (we have in mind the form of Leduc–Ram [6] and generalisations).

The difficulty of these problems tells us that constructing idempotents will also be hard. However both partition algebras and Brauer algebras are naturally filtered by certain ideals that are easy to construct. As a first step we can try to construct idempotents associated to these ideals.

2.5.1 Splitting idempotents

(**2.5.1**) Our approach follows [11]. There it is recalled firstly that if $J \subset A$ is an ideal in an algebra A, then the short exact sequence of A-bimodules

$$0 \to J \to A \to A/J \to 0$$

splits iff there is an idempotent $e_J \in A$ with the following properties.

1. $e_J \cong 1$ mod. J
2. $e_J J = J e_J = 0$

If e_J exists then note that $e_J \in Z(A)$, the centre of A; and e_J is unique with these properties.

(**2.5.2**) For $A' \subset A$ a subalgebra (or indeed any subset), then define $Z_{A'}(A)$ as the set of elements of A that commute with A'. Obviously $Z_{A'}(A) \supset Z(A)$. Thus we can start a search for elements of $Z(A)$ by looking for elements of $Z_{A'}(A)$.

2.5.2 The Brauer case

In our case $KS_n \subset B_n$, and KS_n has a nice action on B_n, so it is natural to consider $Z_{KS_n}(B_n)$. We are interested in elements of B_n that are invariant under conjugation by all elements of S_n. (The setup for the partition algebra is very similar.) Consider an element of form

$$x = \sum_{d \in \mathbf{J}(\underline{n} \cup \underline{n}')} c_d d \qquad \overset{x \in Z_{S_n}(B_n)}{=} \qquad wxw^{-1} = \sum_{d \in \mathbf{J}(\underline{n} \cup \underline{n}')} c_d w d w^{-1} = \sum_{d \in \mathbf{J}(\underline{n} \cup \underline{n}')} c_{w^{-1}dw} d$$

where we have used the fact that conjugation by $w \in S_n$ is a permutation on $\mathbf{J}(\underline{n} \cup \underline{n}')$. Thus $x \in Z_{S_n}(B_n)$ implies $c_d = c_{wdw^{-1}}$ for all w. Evidently for any d

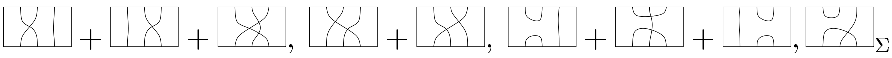
$$\sum_{w \in S_n} w d w^{-1} \in Z_{S_n}(B_n)$$

So (in characteristic 0, where the possible multiplicities in this sum are all units) $Z_{S_n}(B_n)$ has a basis of elements of this form.

Another basis of $Z_{S_n}(B_n)$ (in arbitrary characteristic) is the set of elements

$$\beta_n = \{d_\Sigma \mid d \in \mathbf{J}(\underline{n} \cup \underline{n}')\}.$$

Examples: $\beta_2 = \{1, \sigma_{12}, U_1\}$ (note that in this case B_2 itself is commutative);

$\beta_3 = \{1,$  $\}$.

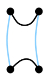
(**2.5.3**) The next question is how to construct this basis β_n in general. In other words, what are a set of representative elements of the orbits of $\mathbf{J}(\underline{n} \cup \underline{n}')$ under S_n-conjugation?

Firstly note that conjugation does not change the number l of propagating lines, so we can work separately in each section of the propagating line filtration. Accordingly let us decompose the basis:

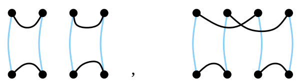
$$\beta_n = \bigsqcup_{l=n,n-2,\dots} \beta_n[l]$$

Within the l section, we see from Prop. 1 that the basis is partly indexed by possible images of the Sp map in this case.

Examples:

$Sp(\mathbf{J}_2[0]) = Sp($  $) = (2)$

$Sp(\mathbf{J}_3[1]) = Sp($  $) = \{(2,1),(3)\}$

$Sp(\mathbf{J}_4[0]) = Sp($  $) = \{(2,2),(4)\}$

(the faint lines are a reminder of the computation of Sp, and can otherwise be ignored);

$$Sp(\mathbf{J}_5(1)) = \{(2,2,1),(3,2),(4,1),(5)\}$$
$$Sp(\mathbf{J}_6(0)) = \{(2,2,2),(4,2),(6)\}$$

Exercise: compute $Sp(\mathbf{J}_6(2)) = Sp(\mathbf{J}_6[2]) \cup Sp(\mathbf{J}_6[0])$.

(**2.5.4**) More generally define

$$W_s^r = U_s U_{s+2} \dots U_{s+2(r-1)}, \qquad W = \prod_{i=1}^{\lfloor n/2 \rfloor} U_{2i-1} \overset{n\ even}{=} U_1 U_3 \dots U_{n-1}.$$

Then for $n = 2m$ we have

$$Sp(W) = (2^m). \tag{2.1}$$

For λ an integer partition of n with each λ_i even define

$$W_\lambda = \prod_{i=1} W^{\lambda_i/2-1}_{2+\sum_{j=1}^{i-1} \lambda_i}$$

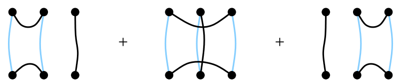
e.g. $W_{(6,4)} = W_2^2 W_8^1 = U_2 U_4 U_8$.

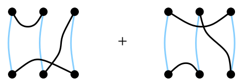
Proposition 2. *The image $Sp(\mathbf{J}_{2m+0/1}(0/1))$ includes only those partitions that, in the sense of (2.1.9), contain m distinct copies of the Young diagram (2). That is*

$$Sp(\mathbf{J}_{2m+0/1}(0/1)) = \Lambda_{((2)^m)} \qquad (resp.\ \Lambda_{((2)^m,(1))})$$

Proof. For convenience define $\Lambda_{0,n} = \Lambda_{((2)^m)}$ and $\Lambda_{1,n} = \Lambda_{((2)^m,(1))}$. The image of the spore map lies in $\Lambda_{l,n}$ because every non-propagating line in $d \in \mathbf{J}_{2m+0/1}(0/1)$ binds precisely two symbols into the same part in $Sp(d)$. The lines 'on the top' for example bind to a shape (2^m). And the lines on the bottom may then bind further. For example consider W in (2.1) above, with $Sp(W) = (2^m)$. To see surjectivity in case n even (case n odd is similar) then consider for example $Sp(WW_2^r)$, which gives $(2(r+1), 2^{m-r-1})$, and so on; so that $Sp(WW_\lambda) = \lambda$. □

(**2.5.5**) One sees immediately in these cases $l = 0, 1$ that they generate bases $\beta_n(l)$ for $Z_{S_n}(B_n) \cap J$ for the appropriate cases of $J = K\mathbf{J}_n(l)$. We write D_λ for the basis element labelled by partition λ, thus

$D_{(2,1)} =$ 

$D_{(3)} =$  $+\ 4\ others$

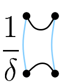
and $\beta_3(1) = \{D_\lambda | \lambda \in \Lambda_{((2),(1))}\} = \{D_{(2,1)}, D_{(3)}\}$.

(**2.5.6**) Let us write $J_n(l)$ for the ideal of B_n with basis $\mathbf{J}_n(l)$, and write $\psi_n(l)$ for the corresponding splitting idempotent in the sense of (2.5.1). We will see below (and it is well-known) that this idempotent exists in case K is the field of rational polynomials in δ. Assume we work in such a case. Define X_n by $\psi_n(0/1) = 1 + X_n$. Since X_n is central, and hence in $Z_{S_n}(B_n)$, we have

$$X_n = \sum_\lambda a_\lambda D_\lambda$$

where the scalars a_λ are to be determined. By (2.5.1) a necessary condition is given by $dX_n = -d$ for $d \in J_n(l)$ — in this case with $l = 0/1$. Thus in particular for $W = U_1 U_3 ... U_{n'}$ (n' the largest odd number below n) a necessary condition is $WX_n = -W$. Equating coefficients of W in this identity gives one linear condition on the unknowns. Our idea is to equate coefficients for a transversal of the orbits under S_n conjugation. Provided these give independent linear conditions then this is enough to determine the unknowns. (Since X_n exists for our K we do not need to check any of the other conditions.)

(**2.5.7**) Examples: For $n = 2$ we have $W = U_1$ and $X_2 = a_{(2)} U_1$, so $a_{(2)} \delta = -1$ and

$\psi_2(0) = 1 - \frac{1}{\delta}$  $= 1 - \frac{1}{\delta} D_{(2)}$

Remark: of course this gives a central idempotent decompositon of 1: $1 = \left(1 - \frac{1}{\delta} D_{(2)}\right) + \left(\frac{1}{\delta} D_{(2)}\right)$. This is not necessarily primitive. Indeed if 2 is a unit we can decompose further using the (central) idempotents in KS_2 — exercise.

(**2.5.8**) For $\psi_3(1)$ we have $\psi_3(1) = 1 + X_3$ with

$$X_3 = \sum_{\lambda \in \Lambda_{((2),(1))}} a_\lambda D_\lambda \;=\; a_{(2,1)} D_{(2,1)} + a_{(3)} D_{(3)}$$

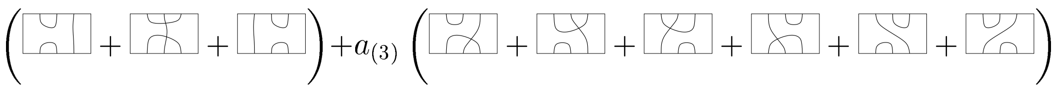
where $a_{(2,1)}, a_{(3)}$ are to be determined. Requiring $dX = -d$ for $d \in J$ is satisfied by requiring (say) $U_1 X = -U_1$, by the S_n-symmetry. Similarly we need only compute the coefficients of U_1 and of $U_1 U_2$. This gives

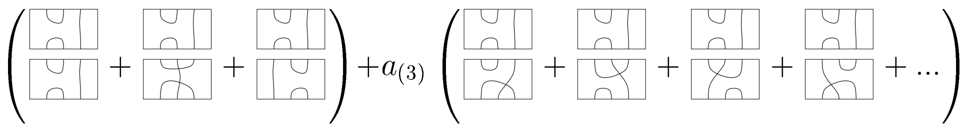
$$\begin{pmatrix} \delta & 2 \\ 1 & \delta+1 \end{pmatrix} \begin{pmatrix} a_{(2,1)} \\ a_{(3)} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

and hence

$$a_{(2,1)} = \frac{-(\delta+1)}{(\delta+2)(\delta-1)}, \qquad a_{(3)} = \frac{1}{(\delta+2)(\delta-1)}$$

Explicitly we have:

$X_3 = a_{(2,1)}$ 

$U_1 X_3 = a_{(2,1)}$ 

so the coefficient of U_1 in the condition $U_1 X_3 = -U_1$ is $\delta a_{(2,1)} + 2a_{(3)} = -1$, and so on.

(**2.5.9**) NOTES
1. the denominators of coefficients in our idempotents tell us a lot of representation theory! Over the rational polynomial field our idempotent is part of a complete decomposition of 1 into 'ordinary' primitive central idempotents. By the refinement theorem, the primitive central idempotents of any specialisation of the integral version of the algebra are the images of ordinary central idempotents, hence they are images of sums of ordinary primitive central idempotents — the failure of splitting down to the same level as the ordinary case being the signal of non-singleton blocks. This failure of splitting is signalled in the ordinary idempotents by the presence of denominators which prevent the specialisation. If a denominator vanishes like $(\delta - \delta_c)$ as $\delta \to \delta_c$ then we deduce that there is no such splitting idempotent at $\delta = \delta_c$, in other words a non-singleton block is formed.
2. cases $\mathbf{J}_n(l)$ with $l > 1$ require an extra layer of sophistication, which we shall address elsewhere.
3. we can read off the restriction rules for various B_n-modules restricted to S_n from our analysis. we observe that they agree with the known rules.
4. we can develop a version of this programme for the partition algebra. aspects of this have already been done, but the version of note 3 for the partition algebra is of current interest.

(**2.5.10**) For comparison we consider the 'natural' contravariant form on the 'Specht' module $\Delta_n(l)$ associated to this ideal []. This encodes a homomorphism from $\Delta_n(l)$

to its contravariant dual (with respect to the *op* involution). The form is computed via:

$$M_3(1) = \left(\begin{array}{c|ccc} \cdots & & & \\ \hline & \delta & 1 & 1 \\ & 1 & \delta & 1 \\ & 1 & 1 & \delta \end{array} \right)$$

— the gram matrix over the natural basis. We are interested in the rank of the form (the rank of the matrix) — and hence the rank of the homomorphism which, on general grounds, gives the dimension of the simple head when we pass to a ground ring that is a field.

The rank of the matrix is clearly full for generic δ. The non-full cases correspond to the zeros of the deteminant:

$$|M_3(1)| = (\delta - 1)^2(\delta + 2)$$

Ignoring the exponents for a moment, we see that our 'splitting idempotent' blows up at the correct values.

(**2.5.11**) In order to address note 1 further we now report some more specific cases. For $n = 5, l = 1$ we have $U_1U_3X_5 = -U_1U_3$. Equating coefficients of U_1U_3, $U_1U_3\sigma_4$, $U_1U_3U_2$ and $U_1U_3U_2\sigma_4\sigma_3$ we get

$$\begin{pmatrix} \delta^2 & 4\delta & 2\delta & 8 \\ \delta & \delta^2+\delta+2 & 2 & 4\delta+4 \\ \delta & 4 & \delta^2+\delta & 4\delta+4 \\ 1 & 2\delta+2 & \delta+1 & \delta^2+3\delta+4 \end{pmatrix} \begin{pmatrix} a_{(2,2,1)} \\ a_{(3,2)} \\ a_{(4,1)} \\ a_{(5)} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives (using sage [12]):

$$a_{(2,2,1)} = \frac{-(x-1)^2(x+2)^2(x^2+3x-2)}{\Delta_5}$$

$$a_{(3,2)} = a_{(4,1)} = \frac{(x-1)^2(x+2)^3}{\Delta_5}$$

$$a_{(5)} = \frac{-2(x-1)^2(x+2)^2}{\Delta_5}$$

where

$$\Delta_5 = (x-2)(x-1)^3(x+2)^3(x+4)$$

2.5.3 Exercises (and more cases)

(**2.5.12**) $n = 4$, $l = 0$. ... Here we need to determine $D_{(2,2)}$ and $D_{(4)}$, using $U_1U_3X_4 = -U_1U_3$. Equating coefficients of U_1U_3 and of $U_1U_3U_2$ we get:

$$\begin{pmatrix} \delta^2 & 2\delta \\ \delta & \delta^2+\delta \end{pmatrix} \begin{pmatrix} a_{(2,2)} \\ a_{(4)} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

This is almost the same as the $n = 3$ case — differing only by overall factors of δ. We get

$$a_{(2,1)} = \frac{-(\delta+1)}{\delta(\delta+2)(\delta-1)}, \qquad a_{(3)} = \frac{1}{\delta(\delta+2)(\delta-1)}$$

It is instructive to consider the difference with the $n = 3$ case. Here we have, nominally, for the 'natural' cv form on $\Delta_4(0)$:

$$|M_4(0)| = \delta^3(\delta - 1)^2(\delta + 2)$$

This (or rather the associated Smith form — exercise) tells us that the natural form is not well-defined when $\delta = 0$ (or rather it is the zero form, yielding the zero morphism, which is the only morphism/form in some cases, but is not the only morphism/form here). For other δ values the form is ok and the agreement is as before. For $\delta = 0$ we find that there is a renormalised form and it has full rank. *However*, in this case $J_4(0)$ lies in the radical, so there is no splitting idempotent, in agreement with our calculation.

Remark: How do we know a cv form is nonzero unique up to scalars? In our case the argument for this (essentially it is quasiheredity) does not hold integrally or in every specialisation. So the form is not necessarily natural integrally or in every specialisation. It is interesting to consider if/when the failures can be cast as degenerations and so, in this sense, naturality recovered. In the meanwhile our arguments must make reference to external facts (such as quasiheredity where applicable).

(**2.5.13**) $n = 6$, $l = 0$. ... Here we need $D_{(2,2,2)}$, $D_{(4,2)}$ and $D_{(6)}$, using $U_1U_3U_5X_6 = -U_1U_3U_5$. Equating coefficients of $U_1U_3U_5$, $U_1U_3U_5U_2$ and $U_1U_3U_5U_2U_4$ (say) we get

$$\begin{pmatrix} \delta^3 & 6\delta^2 & 8\delta \\ \dots & & \\ \dots & & \end{pmatrix} = \begin{pmatrix} D_{(2,2,2)} \\ D_{(4,2)} \\ D_{(6)} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Exercise: complete!

(**2.5.14**) $n = 7$, $l = 1$. ... Spores are $(2, 2, 2, 1), (3, 2, 2), (4, 2, 1), (5, 2), (6, 1), (7)$.

(**2.5.15**) $n = 8$, $l = 0$. ... Spores are $(2, 2, 2, 2), (4, 2, 2), (4, 4), (6, 2), (8)$.

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