Version: as in arXiv:0908.1500, but with several typos fixed.

The decomposition matrices of the Brauer algebra over the complex field

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1 Introduction

For each field k, natural number n and parameter $\delta \in k$, the Brauer algebra $B_n(\delta)$ is a finite dimensional algebra, with a basis of pair partitions of the set $\{1, 2, ..., 2n\}$ [3]. Indeed there is a $\mathbb{Z}[\delta]$-algebra $B_n^{\mathbb{Z}}$ (for δ indeterminate), free of finite rank as a $\mathbb{Z}[\delta]$-module, that passes to each Brauer algebra by the natural base change; and a collection of modules $\{\Delta^{\mathbb{Z}}(\lambda)\}_{\lambda \in \Lambda^n}$ for this algebra that are $\mathbb{Z}[\delta]$-free modules of known rank, so that

$$\Delta^k(\lambda) = k \otimes_{\mathbb{Z}[\delta]} \Delta^{\mathbb{Z}}(\lambda)$$

are $B_n(\delta)$-modules, and that there is a choice of field k extending $\mathbb{Z}[\delta]$ for which $\{\Delta^k(\lambda)\}_{\lambda \in \Lambda^n}$ is a complete set of simple modules. (The index set is $\Lambda^n = \Lambda_n \cup \Lambda_{n-2} \cup ... \cup \Lambda_{n_0}$ where Λ_n is the set of integer partitions of n, and $n_0 = 0$ for n even and $= 1$ for n odd [4].) Accordingly we are presented with the following tasks in studying the representation theory of $B_n(\delta)$:
(1) There are finitely many isomorphism classes of simple modules — index these.
(2) Describe the blocks (the reflexive-symmetric-transitive closure of the relation on the index set for simples given by $\lambda \sim \mu$ if simple modules $L(\lambda)$ and $L(\mu)$ are composition factors of the same indecomposable projective module).
(3) Describe the composition multiplicities of indecomposable projective modules (which follow from the composition multiplicities for the $\Delta^k(\lambda)$ (see for example [9, §16],[2, §1.9])).

Over the complex field, (1) was effectively solved in [4] (an index set is Λ^n, or $\Lambda^n \setminus \Lambda_0$ if $\delta = 0$ and n even), and (2) was solved in [6] (see references therein for other important contributions). Here we solve (3).

The layout of the paper is as follows. For each n, δ we wish to determine the Cartan decomposition matrix C given by $C_{\lambda\mu} = [P(\lambda) : L(\mu)]$, the composition multiplicity, where $\{P(\lambda)\}_{\lambda \in \Lambda^{n,\delta}}$ and $\{L(\lambda)\}_{\lambda \in \Lambda^{n,\delta}}$ are complete sets of imdecomposable projective and simple modules respectively.

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We firstly recall some organisational results to this end. We construct the modules $\Delta(\lambda)$, such that projective modules are filtered by these, with well-defined composition multiplicities denoted $(P(\lambda):\Delta(\mu))$; and that $C = DD^T$, where $D_{\lambda\mu} = (P(\lambda):\Delta(\mu)) = [\Delta(\mu):L(\lambda)]$ (so D is what might be called the Δ-decomposition matrix). Then we construct an inverse limit for the sets $\{\Lambda^{n,\delta}\}_n$ and show that the Cartan decomposition matrices (and the Ds) for all n can be obtained by projection from a corresponding limit. Next we give an explicit matrix D for each δ (this construction takes up the majority of the paper, and uses the block result [7, 8]). And finally we prove, in Section 7, by an induction on n, that this D is the limit Δ-decomposition matrix.

It is probably helpful to note that the original route to the solution of the problem was slightly different. It proceeded from a conjecture, following [19, §1.2], that D would consist of evaluations of parabolic Kazhdan–Lusztig polynomials for a certain reflection group given in, and parabolic determined by, our joint work in [7]. This is essentially correct, as it turns out, and without this idea we would not have had a candidate for D, the form of which then drives the proof of the Theorem. However the proof does not, in the end, lie entirely within the realms of Kazhdan-Lusztig theory and alcove geometry. Accordingly we do not use this framework, but instead a more general one within which the proof proceeds uniformly. With regard to the alcove geometry we restrict ourselves to incorporating some key ideas; and beyond that just a few remarks, where it seems helpful to explain strategy.

We return to discuss our parabolic Kazhdan-Lusztig polynomial solution in a second part to the paper: section 8 and thereafter.

As the derivation of our main result is somewhat involved, we end here with a brief preview of the result itself. For each fixed $\delta \in \mathbb{Z}$ (the cases $\delta \notin \mathbb{Z}$ are semisimple [22]), the rows and columns of the limit Δ-decomposition matrix D may be indexed by Λ, the set of all integer partitions. This matrix may be decomposed, of course, as a direct sum of matrices for the limit blocks. In this sense we may describe the blocks by a partition of Λ. As we shall see, there is a map for each block to the set $P_{even}(\mathbb{N})$ of subsets of $\mathbb{N}$ of even degree. Under these maps all the block summands of D (and for all δ) are identified with the same matrix. Thus we require only to give a closed form for the entries of this matrix. The closed form is given in Section 5, but an indication of its structure is given by a truncation to a suitable finite rank. Such a truncation is given in Figure 7 (the entries in this matrix encode polynomials that will be used later, and which must be evaluated at 1 to give the decomposition numbers; the *blank* entries evaluate to zero, and all other entries evaluate to 1).

This paper is a contribution toward a larger project, with Cox and De Visscher, aiming to compute the decomposition matrices of the Brauer algebras over fields of finite characteristic. This is a very much harder problem again (it includes the representation theory of the symmetric groups over the same fields as a sub-datum — see [7]), and so it is appropriate to present the characteristic zero case separately.

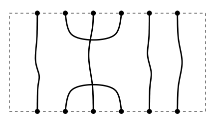
2 Brauer diagrams and Brauer algebras

We mainly base our exposition on the notations and terminology of [6], as well as key results from that paper. For self-containedness, however, we review the notation here. Our hypotheses are slightly more general than in [6], however many of the proofs in [6] go through essentially unchanged (as we shall indicate, where appropriate). We shall also make use of a categorical formulation of the Brauer algebra (a subcategory of the partition algebra category of [18, §7]).

(**2.1**) For $n \in \mathbb{N}$ we write S_n for the symmetric group, and $\underline{n} := \{1, 2, .., n\}$ and $\underline{n'} := \{1', 2', .., n'\}$ (and so on). For S a set we write $P(S)$ for the power set and J_S for the set of pair-partitions of S. We define $J_{n,m} = J_{\underline{n} \cup \underline{m'}}$. For example, in $J_{n,n}$ let us define

$$U_{ij} = \{\{1,1'\}, \{2,2'\}, ..., \{i,j\}, \{i',j'\}, ..., \{n,n'\}\} \tag{1}$$
$$(ij) = \{\{1,1'\}, \{2,2'\}, ..., \{i,j'\}, \{i',j\}, ..., \{n,n'\}\}$$

(**2.2**) An (n,m)-Brauer diagram is a representation of a pair partition of a row of n and a row of m vertices, arranged on the top and bottom edges (respectively) of a rectangular frame. Each part is drawn as a line, joining the corresponding pair of vertices, in the rectangular interval. We identify two diagrams if they represent the same partition. Write $\mathbf{Br}(n,m)$ for the set of (n,m)-Brauer diagrams (up to this identification). It will be evident that these diagrams can be used to describe elements of $J_{n,m}$. (In what follows it is usually safe to simply identify a diagram with its partition. If we need to emphasise the formal distinction we may write $d \mapsto [d]$ for the map $\mathbf{Br}(n,m) \xrightarrow{\sim} J_{n,m}$.) For example,



$\mapsto \;\; U_{24} \in J_{6,6}$

We then define a 'multiplication' $*$ as a composite map

$$J_{n,m} \times J_{m,l} \xrightarrow{\alpha} \mathbb{N}_0 \times J_{n,l} \xrightarrow{\beta} \mathbb{Z}[\delta]J_{n,l}, \qquad * = \beta \circ \alpha$$

as follows. Suppose d', d'' are diagrams representing the pair-partitions to be composed. Firstly vertically juxtapose the diagrams so that the two sets of m vertices meet (i.e. with d' over d''). This produces a diagram for an element d of $J_{n,l}$ (the pair partition of the vertices on the exterior of the combined frame); together with some number c of closed loops, which loops we discard. Thus we have a pair $(c,d) \in \mathbb{N}_0 \times J_{n,l}$. The final image is then $\delta^c d$ (i.e. we replace each closed loop formed in diagram composition by a factor δ).

For k a commutative ring and $\delta \in k$ we have a k-linear category with object set $\mathbb{N}_0$; and for each pair (n,m) of objects a hom set $kJ_{n,m}$ (or equivalently $k\mathbf{Br}(n,m)$); and composition k-linearly extending $*$. We denote this category by $\mathbf{Br}_\delta^k$, or just $\mathbf{Br}_\delta$ if k is fixed. (Here we allow $k = \mathbb{Z}[\delta]$ or any suitable base change.)

(**2.3**) The Brauer algebra $B_n(\delta)$ over k is the free k-module with basis $\mathbf{Br}(n,n)$ and the category composition. We write simply B_n for $B_n(\delta)$ where no ambiguity arises.

(**2.4**) Write $\mathbf{Br}^{\leq l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ consisting of diagrams with $\leq l$ propagating lines (lines from top to bottom); $\mathbf{Br}^l(m,n)$ for the subset with l propagating lines; and $\mathbf{Br}^{\underline{l}}(m,n)$

for the subset of these in which no pair of the l propagating lines cross each other. Write 1_r for the identity diagram in $\mathbf{Br}(r,r)$.

Note that $\mathbf{Br}^l(l,l)$ can be identified with the symmetric group S_l, so the category composition defines a bijection:

$$\mathbf{Br}^{\underline{l}}(m,l) \times \mathbf{Br}^l(l,l) \to \mathbf{Br}^l(m,l) \tag{2}$$

In particular $k\mathbf{Br}^l(m,l)$ is a free (right) kS_l-module of rank the number of elements in $\mathbf{Br}^{\underline{l}}(m,l)$.

(**2.5**) Define a product $\otimes : \mathbf{Br}(m,n) \times \mathbf{Br}(r,s) \to \mathbf{Br}(m+r, n+s)$ by placing diagrams side by side. For example the injection $i_{m+1,m+r} : \mathbf{Br}(m,n) \hookrightarrow \mathbf{Br}(m+r, n+r)$ defined by $d \mapsto d \otimes 1_r$ adds propagating lines $\{\{m+1, n+1'\}, ..., \{m+r, n+r'\}\}$.

(**2.6**) Example. The sets $\mathbf{Br}(1,1)$, $\mathbf{Br}(2,0)$ and $\mathbf{Br}(0,2)$ each have a single element, here denoted 1_1, u and u' respectively. The map $k\mathbf{Br}(1,3) \to k\mathbf{Br}(3,3)$ defined by $d \mapsto u \otimes d$ is an injection. As a right B_3-module we have

$$U_{12}U_{23}B_3 \cong k\mathbf{Br}(1,3) \tag{3}$$

(pictorially, the right action corresponds to acting with diagrams from $\mathbf{Br}(3,3)$ from *below*).

(**2.7**) Remark. A basic 'integral' version of the Brauer algebra is the case over the ring $k = \mathbb{Z}[\delta]$. Starting from this case, there are thus two aspects to the base change to a field: the choice of k and the choice of δ. More precisely this is the choice of k equipped with the structure of $\mathbb{Z}[\delta]$-algebra. Thus we have possible intermediate steps: base change to $k[\delta]$ (k a field); base change to $\mathbb{Z}$ (a $\mathbb{Z}[\delta]$-algebra by fixing $\delta = d \in \mathbb{Z}$). Each of these ground rings is a principal ideal domain and hence a Dedekind domain, and hence amenable to a P-modular treatment (see for example [9, §16],[2]).

3 Brauer-Specht modules

Here we construct the integral representations (in the sense of [2]) that we shall need. (These pass by base change, entirely transparently, to the *standard* modules of [6].)

(**3.1**) For $n \in \mathbb{N}$ we define $n_0 = 0, 1$ for n even, odd respectively. For any commutative ring k and $\delta \in k$, we have, as an elementary consequence of the composition rule, a sequence of $B_n(\delta)$-bimodules:

$$k\mathbf{Br}(n,n) = k\mathbf{Br}^{\leq n}(n,n) \supset k\mathbf{Br}^{\leq n-2}(n,n) \supset k\mathbf{Br}^{\leq n-4}(n,n) \supset ... \supset k\mathbf{Br}^{n_0}(n,n) \tag{4}$$

Note that the i-th section of the sequence (4) has basis $\mathbf{Br}^{n-2i}(n,n)$. For $n - 2i = l$ we write $k\mathbf{Br}^l(n,n)$ for this section. We have

$$k\mathbf{Br}^l(n,n) \cong \bigoplus_{w \in \mathbf{Br}^{\underline{l}}(l,n)} k\mathbf{Br}^l(n,l)\, w \tag{5}$$

as a left B_n-module; where all the summands are isomorphic to $k\mathbf{Br}^l(n,l)$ (a left B_n-module similarly, via the category composition, quotienting $k\mathbf{Br}(n,l)$ by $k\mathbf{Br}^{\leq l-2}(n,l)$).

Fixing a ring k, it will be evident that $\mathbf{Br}^l(m,l)$ is a basis for a free right kS_l-module by (2), and hence for a left-$B_m(\delta)$ right-kS_l bimodule, so long as $m \geq l$ and $m - l$ even.

(**3.2**) **Proposition**. *Fix any commutative ring k. The free k-module $k\mathbf{Br}^l(m,l)$ (which is a left $B_m(\delta)$-module by the action in (3.1)) is a projective right kS_l-module. Hence the functor*

$$k\mathbf{Br}^l(m,l) \otimes_{kS_l} - \;:\; kS_l{-}mod \;\;\rightarrow\;\; B_m(\delta){-}mod$$

between the categories of left-modules is exact.

Proof. As noted, $k\mathbf{Br}^l(m,l)$ is a direct sum of copies of the regular right kS_l-module. □

(**3.3**) Let $\Lambda_n = \{\lambda \vdash n\}$, the set of integer partitions of n. Let Λ be the set of all integer partitions; and define

$$\Lambda^n = \Lambda_n \cup \Lambda_{n-2} \cup \ldots \cup \Lambda_{n_0}, \qquad \text{and} \qquad \Lambda^{n,0} = \Lambda^n \setminus \Lambda_0$$

For $\lambda \vdash l$ let $\mathcal{S}(\lambda)$ denote the corresponding kS_l-Specht module (see e.g. [16]). For $m \geq l$ define

$$\Delta_m(\lambda) = k\mathbf{Br}^l(m,l) \otimes_{kS_l} \mathcal{S}(\lambda)$$

as the image of this Specht module under the functor in (3.2). Varying l, we have a set $\{\Delta_m(\lambda)\}_{\lambda \in \Lambda^m}$.

If we wish to emphasise the ring k notationally, we may write $\Delta^k_m(\lambda)$ for $\Delta_m(\lambda)$. Or if k is fixed as a field, we may write $\Delta^\delta_m(\lambda)$ ($\delta \in k$) for $\Delta_m(\lambda)$, so as to fix k as a $\mathbb{Z}[\delta]$-algebra. On the other hand, where unambiguous we may just write $\Delta(\lambda)$. We shall adopt analogous conventions for projective and simple modules.

(**3.4**) **Proposition**. *Fix n and suppose k is such that the left regular module ${}_{kS_l}kS_l$ is filtered by $\{\mathcal{S}(\lambda)\}_{\lambda \in \Lambda_l}$ for all $l \leq n$. Then the left regular module ${}_{B_n}B_n$ is filtered by $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$. In particular Brauer algebra projective modules over $\mathbb{C}$ (any δ) are filtered by $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$.*

Proof. Note first that if a module M is filtered by a set $\{N_i\}_i$, and these are all filtered by a set $\{N'_j\}_j$, then M is filtered by $\{N'_j\}_j$. By (3.1) the set $\{k\mathbf{Br}^l(n,l)\}_l$ gives (via the action therein) a left-B_n filtration of B_n. By Prop. 3.2 each factor itself has a filtration by Δs under the stated condition. For the last part, simply note that $\mathbb{C}S_l$ is semisimple, the modules $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ are indecomposable over $\mathbb{C}$ (for any δ — see e.g. [6], or cf. Prop.3.10 and [14, §6.2]), and each projective $P_n(\lambda)$ (say) a direct summand of ${}_{B_n}B_n$. □

(**3.5**) **Proposition**. [6, Lemma 2.4] *Let $b(\lambda)$ be a basis for $\mathcal{S}(\lambda)$. Then*

$$b_{\Delta_m(\lambda)} = \{a \otimes b \;:\; (a,b) \in \mathbf{Br}^{\underline{l}}(m,l) \times b(\lambda)\}$$

is a basis for $\Delta_m(\lambda)$.

Proof. This is a set of generators by (2). On the other hand this set passes to a basis (of the image) under the surjective multiplication map (using from [16] that $\mathcal{S}(\lambda)$ is a left ideal), so it is k-free. □

(**3.6**) We mention explicitly the following low rank cases, which form the bases for inductions later on. We have $B_0(\delta) \cong B_1(\delta) \cong k$. For $B_2(\delta)$ we have $\Delta_2(\emptyset)$, $\Delta_2(2)$, $\Delta_2(1^2)$, each of rank 1. These are inequivalent over $\mathbb{C}$ except when $\delta = 0$, where we have

$$\Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset)$$

Thus we may regard $\Delta_2(2)$, $\Delta_2(1^2)$ as the inequivalent simple $B_2(0)$-modules, and $P_2(2)$ is the self-extension of $\Delta_2(2)$, while $P_2(1^2) = \Delta_2(1^2)$.

3.1 Globalisation functors

Here we define certain functors that will allow us, in Section 3.2, to manipulate composition muliplicity data for all n simultaneously.

(**3.7**) For $n+m$ even the k-module $k\mathbf{Br}(n,m)$ is an algebra bimodule. Thus there is a functor between left-module categories

$$k\mathbf{Br}(n,m) \otimes_{B_m} - \;:\; B_m\!-\!\text{mod} \;\;\rightarrow\;\; B_n\!-\!\text{mod}$$

Let us write F for the functor $k\mathbf{Br}(n-2,n) \otimes_{B_n} -$; and G for the functor $k\mathbf{Br}(n,n-2) \otimes_{B_{n-2}} -$ for any n (if $\delta \in k$ a non-unit we shall exclude the case $n=2$ from this notation — cf. Prop.3.8).

(**3.8**) **Proposition**. *Suppose either $n>2$ or δ invertible in k. Then*
(I) the free k-module $k\mathbf{Br}(n-2,n)$ is projective as a right B_n-module; and indeed

$$k\mathbf{Br}(n-2,n) \;\cong\; \;\; e\;(k\mathbf{Br}(n,n))$$

as a right B_n-module, for a suitable idempotent $e \in k\mathbf{Br}(n,n)$ (for $n>2$ we may use $e = U_{12}U_{23}$; for $n=2$ use $e = \delta^{-1}U_{12}$).
(II) Functor $F : B_n\!-\!mod \rightarrow B_{n-2}\!-\!mod$ is exact; G is a right-exact right-inverse to F.

Proof. (I) Note that $U_{12}U_{23}$ is idempotent, so $k\mathbf{Br}(1,3)$ is projective by (3). This argument generalises without difficulty. (II) follows immediately (see e.g. [1]). □

(**3.9**) The first section in (4) obeys the algebra isomorphism

$$k\mathbf{Br}^{\leq n}(n,n)/k\mathbf{Br}^{\leq n-2}(n,n) \;\;\cong\;\; kS_n$$

Thus each kS_n-module induces an identical (as k-module) B_n-module, where the action of any diagram with fewer than n propagating lines is by 0. In particular, for $\lambda \vdash n$, $\mathcal{S}(\lambda) = \Delta_n(\lambda)$.

(**3.10**) **Proposition**. *For $\lambda \vdash l$ and regarding $\mathcal{S}(\lambda)$ as a B_l-module as in (3.9), we have*

$$\Delta_{2m+l}(\lambda) \cong\; G^{\circ m}\mathcal{S}(\lambda) \;=\; G^{\circ m}\Delta_l(\lambda)$$

for any m; unless $\delta \in k$ a non-unit, and $l=0$, in which case

$$\Delta_{2m+4}(\emptyset) \cong\; G^{\circ m}\Delta_4(\emptyset)$$

Proof. Via Proposition 3.5 and the various definitions. (The special case arises in the isomorphism of $k\mathbf{Br}(4,2) \otimes_{B_2} k\mathbf{Br}(2,0)$ with $k\mathbf{Br}(4,0)$, which is easy to show if δ has an inverse, and is false if not.) □

(**3.11**) Remark. If k is a field then in particular (unless $n=2$ and $\delta=0$) the category B_{n-2}–mod fully embeds in $B_n\!-\!$mod under G, and this embedding takes $\Delta_{n-2}(\lambda)$ to $\Delta_n(\lambda)$.

The embedding allows us to consider a formal limit module category (we take n odd and even together), from which all $B_n\!-\!$mod may be studied by 'localisation' (action of the functor F).

(**3.12**) **Proposition**. *The set* $\{head\ (\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ..., n_0\}$ *is a complete set of simple modules, up to isomorphism, for* $B_n(\delta)$ *over* $\mathbb{C}$ *for any* $\delta \in \mathbb{C}$.

Proof. To show that head$(\Delta_n(\lambda))$ is simple, first note that if $\delta \neq 0$ then every $\Delta_n(\lambda)$ is $G^{\circ m}\mathcal{S}(\lambda)$ for some m, and that $\mathcal{S}(\lambda)$ is simple over $\mathbb{C}$. From this, employing a mild generalisation of Prop. 3.8, we have that $F^{\circ m}(M) = 0$ for M any proper submodule of $\Delta_n(\lambda)$ (see e.g. [14, §6.2]), and hence that there is a unique maximal submodule Q_λ (say). Indeed the only case not covered by this argument (or indeed by [6]) is $\Delta_{2m}^{\delta=0}(\emptyset)$ $(m > 1)$. Here one can show directly that $G^{\circ(m-1)}\Delta_2(2)$ (with simple head) maps surjectively onto $\Delta_{2m}(\emptyset)$. Completeness follows from Prop. 3.4. □

Note that, regarded as a list $\{\text{head } (\Delta_n(\lambda))\}_{\lambda \in \Lambda^n}$ may give rise to multiple entries, depending on k and δ. Over the complex field there is no overcount with $\delta \neq 0$ (the heads of $\Delta_n(\lambda)$ and $\Delta_n(\mu)$ with $|\lambda| = |\mu|$ are non-isomorphic since their images under $F^{\circ m}$ are non-isomorphic; and if $|\lambda| > |\mu|$ then there is an idempotent that kills one but not the other), and with $\delta = 0$ just the element $\lambda = \emptyset$ should be excluded (as shown by the case treated above).
Note also that since $F^{\circ m}(Q_\lambda) = 0$, every composition factor below the head of $\Delta_n(\lambda)$ comes from $\Delta_n(\mu)$ with $|\mu| > |\lambda|$. Thus the set $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ (or, in case $\delta = 0$, the subset $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^{n,0}}$) is a basis for the Grothendieck group. Note also, e.g. from [4], that if k is a field extending $\mathbb{Z}[\delta]$ then $B_n(\delta)$ is semisimple, so in this case the Δ-modules are a complete set of simples.
This completes our summary of task (1) over $\mathbb{C}$.

(**3.13**) **Proposition**. [6, Lemma 2.6,Prop.2.7] *Let Ind– and Res– denote the induction and restriction functors associated to the injection* $B_n(\delta) \hookrightarrow B_{n+1}(\delta)$.
(i) We may identify the functors Res G– and Ind– (each from $B_n-mod \to B_{n+1}-mod$*).*
(ii) Over the complex field we have, for each $\lambda \in \Lambda^n$*, a short exact sequence*

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n+1}(\mu) \to Ind\, \Delta_n(\lambda) \to \bigoplus_{\mu \rhd \lambda} \Delta_{n+1}(\mu) \to 0$$

(recall $\mu \lhd \lambda$ *if* μ *is obtained from* λ *by removing one box from the Young diagram).*

Proof. (i) Unpack the definitions (see [6] for details). (ii) Note from (i) and Prop. 3.10 that it is enough to prove the equivalent result for restriction. Use the diagram notation above. Consider the restriction acting on the first n strings. We may separate the diagrams out into those for which the $n+1$-th string is propagating (which span a submodule, since action on the first n strings cannot change this property), and those for which it is not. The result follows by comparing with diagrams from the indicated terms in the sequence, using the induction and restriction rules for Specht modules. (See also [12].) □

3.2 Characters and Δ-filtration factors

For M a module, the notation $M = A_1//A_2//...$ shall indicate that M has a chain of submodules with sections $A_1, A_2, ...$ (up to isomorphism).

(**3.14**) Over the complex field the modules $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ have pairwise distinct characters except precisely in the case $n = 2, \delta = 0$ in (3.6). If $\delta \neq 0$ there is a unique expression for any character in terms of Δ-characters (see (3.12)). This means that the Δ-filtration multiplicities for a Δ-filtered module P, denoted $(P : \Delta_n(\lambda))$, are also uniquely defined (and, for P projective, coincide with the appropriate 'lifted' decomposition numbers [2, §1.9]). The set $\{P_n(\lambda)\}_\lambda$ of isomorphism classes of indecomposable projectives inherits its labelling scheme from the simples in the usual way.

For the case $\delta = 0$, when $n = 2$ the noted isomorphism means that these multiplicities are not uniquely defined. For all other n, however, the multiplicities of the $\Delta_n(\lambda)$'s with $\lambda \neq \emptyset$ are defined as before (consider the quotient algebra $B_n/k\mathbf{Br}^0(n,n)$ for example) and then the distinct character property of Δs precludes any remaining ambiguity. In particular, the sectioning of projectives in the block of $\Delta_n(\emptyset)$ up to $\lambda \vdash 4$ is indicated by

$$P_4(2) = \Delta_4(2)//\Delta_4(\emptyset) \qquad\qquad P_4(31) = \Delta_4(31)//\Delta_4(2)$$

(this is an easy direct calculation). In this sense we may treat $\delta = 0$ as a degeneration of the more general case, and treat the multiplicities $(P : \Delta_n(\lambda))$ as uniquely defined throughout. We do this hereafter.

(**3.15**) Recall from Proposition 3.10

$$G\Delta_n(\lambda) = \Delta_{n+2}(\lambda)$$

By Prop.3.10 and 3.12 the character of any B_n-module over $\mathbb{C}$ can be expressed as a not necessarily non-negative combination of Δ-characters:

$$\chi(M) = \sum_\lambda \alpha_\lambda(M)\, \chi(\Delta(\lambda)) \qquad (\alpha_\lambda(M) \in \mathbb{Z})$$

If in addition a module M has a Δ-filtration then this is a non-negative combination and (with the caveat mentioned in (3.14)) we have from (3.8) (cf. [11, Appendix], say) that

$$(GM : \Delta_{n+2}(\lambda)) = \begin{cases} (M : \Delta_n(\lambda)) & |\lambda| < n+2 \\ 0 & |\lambda| = n+2 \end{cases}$$

The functor G evidently takes projectives to projectives. It also preserves indecomposability, so

$$GP_n(\lambda) = P_{n+2}(\lambda) \tag{6}$$

Combining these results we see that the multiplicities $(P(\lambda) : \Delta(\mu))$ depend on n only through the range of possible values of λ. Thus for each δ (here with $k = \mathbb{C}$) there is a semiinfinite matrix D with rows and columns indexed by Λ such that

$$(P(\lambda) : \Delta(\mu)) = D_{\lambda,\mu}$$

for any n. Note that

$$(P(\lambda) : \Delta(\lambda)) = 1$$

and otherwise

$$(P(\lambda) : \Delta(\mu)) = 0 \text{ if } |\mu| \geq |\lambda| \tag{7}$$

In our case the 'standard' decomposition matrix D also determines the Cartan decomposition matrix C (see e.g. [2, §1.9]). That is $D_{\lambda,\mu} = (P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]$, so that $C = DD^T$. In particular there is an inverse limit of blocks that is a partition of Λ.

Equation(7) says that the matrix D is lower unitriangularisable. From this we have

(**3.16**) **Proposition**. *If P is a projective module containing $\Delta(\lambda)$ with multiplicity m and no $\Delta(\mu)$ with $|\mu| > |\lambda|$, then P contains $P(\lambda)$ as a direct summand with multiplicity m.* □

The induction functor takes projective modules to projective modules, and has a behaviour with regard to standard characters determined by Prop. (3.13). From this we see that

(**3.17**) **Proposition**. *For e_i a removable box of the Young diagram of λ,*

$$Ind\, P(\lambda - e_i) \cong P(\lambda) \bigoplus Q$$

where $Q = \oplus_\mu P(\mu)$ a possibly empty sum with no $\mu \geq \lambda$.

Proof: By Prop.3.16 a projective module is a sum of indecomposable projectives including all those with labels maximal in the dominance order of its standard factors. Now use (3.13). □

(**3.18**) Remark. From the definitions we have

$$F\Delta_n(\lambda) = \begin{cases} \Delta_{n-2}(\lambda) & |\lambda| < n \\ 0 & |\lambda| = n \end{cases}$$

(**3.19**) As we shall see shortly, the Young diagram labelling scheme we have for the various indecomposable modules, which is natural in light of (3.3), is the transpose of the labelling that it is convenient to work with in describing the blocks. For this reason it is convenient to define

$$\Delta_n(\lambda)' = \Delta_n(\lambda^T)$$

and similarly for simples and projectives.

4 Blocks

We now assemble the results we shall need on the blocks of the Brauer algebras. These include important results from [6], [7], [8] and extensions thereof. The Young diagram inclusion partial order $(\Lambda, \subset)$ restricts to a partial order on each block (any such construction evidently survives the inverse limit). By construction this order has a transitive reduction, that is, a directed graph that describes the limit of Hasse diagrams. This graph is key to our main result, and we describe it here. For example we endow the implicit definition of graph edges above (and in [6]) with an explicit contruction that we shall need.

4.1 The δ-balance condition

Recall that the content $c(b)$ of a box b in a Young diagram is $c(b)$ = column position - row position. In [6] we explain how it is that the block structure comes to depend on the relative content of the labelling Young diagrams (using in particular [12]). It will be convenient now to cast the appropriate content condition for blocks in various forms.

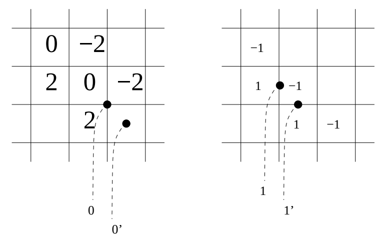


Figure 1: Possible π-rotation points.

(**4.1**) The δ-*charge* of a box in a Young diagram is

$$chg(b) \ := \ \delta - 1 - 2c(b)$$

(cf. the conjugate function $ch(b)$ used in [6]).

Just as for content, the lines of constant δ-charge run parallel to the main diagonal. The key difference from content is that the line of δ-charge 0 for given δ is no longer (unless $\delta = 1$) the main diagonal itself. That is, the δ-charge-0 main diagonal is shifted from the ordinary main diagonal of the Young diagram. (Indeed for δ even there are no boxes with charge 0, so the charge 0 line lies 'between' diagonal runs of charge +1 and charge -1 boxes.)

In the present setting, the point is as follows. Here we shall write $L(\mu) \sim^{\delta} L(\lambda)$ if simple modules $L(\mu)$, $L(\lambda)$ of $B_n(\delta)$ over $\mathbb{C}$ for given $\delta \in \mathbb{Z}$ are in the same block. A pair $\mu \subset \lambda$ gives modules *in the same block* only if the skew diagram λ^T/μ^T consists of $\pm$charge pairs of boxes [7]. (We give a precise statement shortly.)

For example, with $\delta = 2$ the skew $(2^2)/(1^2)$ contains ± 1, so potentially (and in fact) we have $L(2^2) \sim^{\delta=2} L(2)$.

(**4.2**) A Young diagram, or indeed any skew, can be considered as a planar graph all of whose faces are square. Its geometrical dual graph is obtained by drawing a vertex for each square face and drawing an edge between a pair of vertices whenever the corresponding pair of squares has a common edge. A skew is called a *chain* if its dual graph is a chain. A skew chain that is removable from a Young diagram is sometimes called a rim of that diagram. Here a *rim* is any skew that is a chain (i.e. not necessarily removable from a given Young diagram).

Two rims are δ-*opposite* if there is a rotation by π (hereafter called a π-rotation) of the plane about a point on the δ-charge-0 main diagonal that takes one into the other.
(Evidently this rotation is the same as reflection in the vertical line defined by the point of rotation; followed by reflection in the horizontal line defined by this point.)

Note that any such π-rotation is necessarily about a point positioned as shown in one of the cases in Figure 1.

Note further that such a rotation has the effect of exchanging boxes in specific pairs, that are $\pm$charge pairs. See Figure 2 for an example (in this case the position of the charge-0 diagonal corresponds to $\delta = 5$).

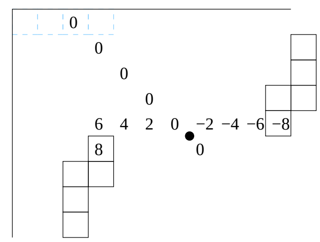


Figure 2: A π-rotation of rims (about the black dot shown) in case $\delta = 5$.

(**4.3**) A *minimal δ-balanced skew* (MiBS) is a skew that is a δ-opposite pair of rims such that no row of the skew is fixed by the associated π-rotation.

REMARK. Partition μ is a maximal δ-balanced subpartition of λ (as in [6]) if and only if λ^T/μ^T is a MiBS. (Proposition 4.7 below will serve to confirm this.) The explicit geometrical form of the construction of MiBS above (in contrast to the implicit construction in the definition of maximal δ-balanced subpartition given in [6]) will be crucial in what follows.

There are several examples of minimal δ-balanced skews shown in Figure 10.

(**4.4**) Define a relation $(\Lambda, \leftarrow^\delta)$ by $\mu \leftarrow^\delta \lambda$ if λ/μ is a minimal δ-balanced skew. Define $(\Lambda, <^\delta)$ as the partial order that is the transitive closure of this relation.

(**4.5**) **Lemma**. *Possible π-rotation points for a MiBS are of the forms shown in Figure 1. In case-$0'$ there can be no intersection of the skew with the row or column containing the point. In case-1 there can be no intersection of the skew with the row containing the point. Hence in either of these cases the skew is disconnected.* □

Define a partial order on the set of boxes occuring in Young diagrams by $b' > b$ if b' lies below *and* to the right of the top-left-hand corner of b (and $b' \neq b$).

(**4.6**) **Lemma**. *(Pinning Lemma) Let π_x be a rotation as above, and b, b' two boxes comparable in the above order, then*

$$b' > b \qquad \Rightarrow \qquad \pi_x(b) > \pi_x(b')$$

□

(**4.7**) **Proposition**. *(I) If $\mu \subset \lambda$ and λ/μ a MiBS, then there is no $\mu \subset \mu' \subset \lambda$ such that μ'/μ is a MiBS.*
(II) The relation $(\Lambda, \leftarrow^\delta)$ is the cover (transitive reduction) of the partial order $(\Lambda, <^\delta)$.

Proof. (I): Let π_0 be the rotation fixing λ/μ and suppose (for a contradiction) that π_γ fixes $\gamma = \mu'/\mu \subset \lambda/\mu$.

The positive charge part of λ/μ is connected, so there exists $b' \in \lambda/\mu'$ adjacent to $b \in \gamma$. Thus $\pi_0(b')$ lies in λ/μ adjacent to $\pi_0(b)$. Since λ/μ' is a skew over μ', we have $b' \not\leq b$ and hence (since adjacent) $b' > b$. Thus $\pi_0(b) > \pi_0(b')$ by Lemma 4.6.

Suppose for a moment that $\pi_0 = \pi_\gamma$ (i.e. they are rotations about the same point). Then $\pi_0(b') < \pi_\gamma(b)$, contradicting that γ is a skew over μ. Thus $\pi_0 \neq \pi_\gamma$.

Now, since $\pi_0 \neq \pi_\gamma$, π_0 fixes no pair $b, \pi_\gamma(b)$ in γ. Thus for example no charge appears more than once in γ, while all the charges appearing in γ appear twice in λ/μ. Thus in particular λ/μ is connected. Note that the rotation point of π_0 is necessarily half a box down and to the right of π_γ. It then follows from Lemma 4.5 that parts γ_+ and γ_- are disconnected from each other.

Let c be the lowest charge box in γ_+. The box $\pi_0(\pi_\gamma(c))$ is below and to the right of it. Thus there is a box of λ/μ to its immediate right. There cannot be a box of λ/μ above it (since γ is a skew over μ) so there is a box of λ/μ to the right of $\pi_0(\pi_\gamma(c))$. But the π_0 image of *this* is to the left of $\pi_\gamma(c) \in \gamma$, contradicting the γ skew over μ property.

Claim (II) follows from (I) since $\mu \subset \lambda$ is a necessary condition for $\mu <^\delta \lambda$ so any failure of the MiBS relation to be a transitive reduction implies the existence of a μ' contradicting (I). □

(**4.8**) **Theorem**. [6, Theorem 6.5] *If λ/μ is a minimal δ-balanced skew then*

$$Hom(\Delta_n^\delta(\lambda^T), \Delta_n^\delta(\mu^T)) \neq 0$$

Proof. Noting the formulation in [6, Theorem 6.5], it is enough to show that the definitions of MiBS and maximal δ-balanced subpartition are equivalent up to transposition. It is straightforward to show that every balanced skew in the sense of [6] contains a (transposed) MiBS. Equivalence then follows from (4.7). □

Write $\Lambda^{\sim\delta}$ for the reflexive-symmetric-transitive closure of the partial order $(\Lambda, <^\delta)$. Write $[\lambda]_\delta$ for the $\Lambda^{\sim\delta}$-class of $\lambda \in \Lambda$.

(**4.9**) **Proposition**. [6, Corollary 6.7] *The relation $\Lambda^{\sim\delta}$ gives the (transposed) block relation for $B_n(\delta)$ over the complex field.* □

(**4.10**) For any n, we write $\text{Proj}_\lambda-$ for the projection functor on the category $B_n(\delta) - \text{mod}$ onto the block associated to the class $[\lambda]_\delta$ (i.e. the block containing $\Delta_n^\delta(\lambda^T)$).

(**4.11**) Let $G_\delta(\lambda)$ be the λ-connected component of $(\Lambda, \leftarrow^\delta)$. This may thus be thought of as a directed acyclic graph. We call this the *block graph*.

4.2 The block graph

The structure of the graphs $G_\delta(\lambda)$ will be crucial for the statement and proof of the main Theorem. We can describe it as follows.

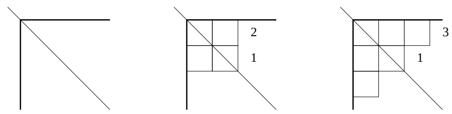
(**4.12**) Let $P_{even}(\mathbb{N}) \subset P(\mathbb{N})$ denote the set of subsets of $\mathbb{N}$ of even order. Define a directed graph, G_{even}, with vertex set $P_{even}(\mathbb{N})$; and labelled edges:

$$a \xrightarrow{\alpha} b \qquad \text{if} \qquad a \setminus b = \{\alpha\}, \;\; b \setminus a = \{\alpha+1\} \qquad (\alpha \in \mathbb{N})$$

$$a \xrightarrow{12} b \qquad \text{if} \qquad a \setminus b = \emptyset, \;\; b \setminus a = \{1,2\}$$

See Figure 4. (There is a corresponding graph G_{odd} with vertices given by subsets of $\mathbb{N}$ of odd order. The *toggle map* between the vertex sets given by toggling the presence of 1 so as to make an odd set even is readily seen to pass to a graph isomorphism (the edge labels 1 and 12 are interchanged).)

We shall shortly construct an isomorphism $G_\delta(\lambda) \cong G_{even}$ for each δ, λ. For now we note that the case $G_2(\emptyset)$ takes a relatively simple form. The vertex map, $o_2 : [\emptyset]_2 \to P_{even}(\mathbb{N})$, is as follows. First draw the main diagonal on the Young diagram, as in these three examples from $[\emptyset]_2$:



then count the number of boxes wholly or partly to the right of the diagonal in each row, and write down the subset of these numbers that are positive. Thus our examples become $\emptyset - \{2,1\} - \{3,1\} \cdots$. Comparing with (4.3) we readily see that o_2 passes to an isomorphism $G_2(\emptyset) \cong G_{even}$.

To generalise this it is useful to give an alternative statement which emphasises the geometrical nature of the block condition, following [7].

Suppose λ/μ a minimal δ-balanced skew. Note that if we suspend, for intermediate steps, the dominance requirement (the requirement to work with partitions rather than arbitrary compositions) then we can build λ from μ by a sequence of transformations on pairs of rows. Each transformation extends two rows: adding part of one row, and the corresponding opposite charges in the other row. The no-row-fixed condition of (4.3) ensures that it is always pairs of rows (as opposed to a single row) that are involved. For each row in question one takes the leading edge of the row in μ and performs the two reflections mentioned in (4.2). The vertical reflection (i.e., in a horizontal line) simply swaps the two rows. The other reflection takes this leading edge as far beyond the charge-0 diagonal as it was short of it beforehand. With these remarks in mind this transformation can be reformulated as we shall describe next (in (4.15) et seq.).

(**4.13**) Remark. Alternatively λ can be built by a sequence of transformations manipulating columns in pairs. The difference is firstly that, unless we transpose, the intermediate stages are neither partitions nor compositions (they are 'transpose compositions'); and secondly that it is possible in some cases to require a manipulation on a single column, rather than a pair; and thirdly that the no-row-fixed condition must still be imposed. In light of this we use here the rows-in-pairs version.

(**4.14**) Define a partial order $(\mathbb{R}^{\mathbb{N}}, \geq)$ by $v \geq w$ if $v_i \geq w_i$ for all i. (Write $v > w$ if $v \geq w$ and $v \neq w$.)

(**4.15**) For $\delta \in \mathbb{R}$ define

$$\rho_\delta \;=\; -\frac{\delta}{2}(1,1,...) - (0,1,2,...) \;\in \mathbb{R}^{\mathbb{N}}$$

For $\mathbb{Z}^f$ the subset of finitary elements of $\mathbb{Z}^{\mathbb{N}}$ define

$$e_\delta : \mathbb{Z}^f \;\hookrightarrow\; \mathbb{R}^{\mathbb{N}} \tag{8}$$

$$\lambda \;\mapsto\; \lambda + \rho_\delta \tag{9}$$

In other words, since $\Lambda \hookrightarrow \mathbb{Z}^f$, we have, for each δ, embedded our index set Λ into a Euclidean space. Thus our blocks $[\lambda]_\delta$ now correspond to collections of points in this space.

Examples: $e_2(\emptyset) = (0,0,0,0,...) - (1,1,1,1,...) - (0,1,2,3,...) \; = \; (-1,-2,-3,-4,...)$

$$e_{-1}(2) = (2,0,0,0,...) - \frac{-1}{2}(1,1,1,1,...) - (0,1,2,3,...) \; = \; (\frac{5}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, ...)$$

(**4.16**) Note that all the image points $e_\delta(\Lambda)$ are strictly descending sequences. We call such sequences *dominant*. Indeed all the image points $e_\delta(\Lambda)$ are *strongly* descending sequences, meaning that $v_i - v_{i+1} \geq 1$ for all i. We write A^+ for the set of strongly descending sequences.

Considering for a moment the magnitudes of terms in a sequence in A^+, we see that each magnitude occurs at most twice, i.e. in a sequence of form $(..., x, ..., -x, ...)$. We call such a $\pm x$ pairing a *doubleton*. Define a map

$$Reg : A^+ \to A^+$$

such that $Reg(v)$ is obtained from v by removing the doubletons.

For example

$$Reg(1,-1,-3,-4,-5,-6,...) = (-3,-4,-5,-6,...)$$

(note in this case that the input is $e_2((2,1))$ while the output is $e_6(\emptyset)$, that is, the Reg map can increase δ);

$$Reg(4,3,1,0,-1,-5,-6,...) = (4,3,0,-5,-6,...)$$

(**4.17**) For $\lambda \in \Lambda$ write $p_\delta(\lambda)$ for the set of pairs of rows $\{i,j\}$ such that $(\lambda+\rho_\delta)_j = -(\lambda+\rho_\delta)_i$ (i.e. $e_\delta(\lambda)_j = -e_\delta(\lambda)_i$). Write $s_\delta(\lambda)$ for the *singularity* of $e_\delta(\lambda)$:

$$s_\delta(\lambda) = |p_\delta(\lambda)|$$

(**4.18**) We say a sequence $v \in \mathbb{R}^\mathbb{N}$ is *regular* if no two terms have the same magnitude. Let $\mathbb{R}^{Reg}$ denote the set of regular sequences. Define a map

$$o : \mathbb{R}^{Reg} \cap A^+ \; \to \mathbb{Z}^\mathbb{N}$$

as follows. In the i-th term, $|o(v)_i|$ is the position of v_i in the magnitude ordering of the set of numbers appearing in v. The sign of $o(v)_i$ is the sign of v_i, unless $v_i = 0$ in which case the sign is chosen so as to make an even number of positive terms.
(Remark: this sign choice in case $v_i = 0$ is simply for definiteness. The definition of the function we eventually use (constructed next) will make it independent of this convention.)

(**4.19**) If v is a descending signed permutation of $(-1,-2,-3,...)$ then we define $v|_+ \in P(\mathbb{N})$ as follows. First take the subset of terms of v that are positive. Then, if this set is of odd order, toggle the presence of 1 in this set so as to make it even.

Define

$$\begin{array}{rcl} o_\delta : \Lambda & \to & P(\mathbb{N}) \\ \lambda & \mapsto & o(Reg(e_\delta(\lambda)))|_+ \end{array} \tag{10}$$

(**4.20**) Examples: $\emptyset \;\mapsto\; e_2(\emptyset) = (-1, -2, -3, ...) \;\mapsto\; \emptyset$

$$(3, 3) \mapsto (2, 1, -3, -4, ...) \mapsto \{1, 2\}$$

$$(3, 3, 3, 1) \mapsto (3, 2, 1, -2, -4, -5, ...) \mapsto \{1, 2\}$$

$$(4, 3, 3, 1) \mapsto (4, 2, 1, -2, -4, -5, ...) \mapsto \{1\} \overset{toggle}{\mapsto} \emptyset$$

(**4.21**) Given δ and λ we define

$$o^\lambda_\delta : P_{even}(\mathbb{N}) \to \Lambda$$

as follows (indeed we could extend the domain to $P(\mathbb{N})$ by applying the toggle map to $P_{odd}(\mathbb{N})$). First construct $e_\delta(\lambda)$. Note that this fixes the doubletons and (magnitudes of) singletons for its whole orbit, i.e. for every element of $o_\delta([\lambda]_\delta)$. We ignore the doubletons for a moment, and work out the magnitude order for the singletons. Note that the order in which the singletons can appear in a descending sequence is uniquely determined by their sign. Now for $v \in P_{even}(\mathbb{N})$ we give the positive sign to the corresponding singletons (in the magnitude order). Thus we have determined the singletons and their order in the sequence. The position of the doubletons is now forced, so the sequence $o_\delta(o^\lambda_\delta(v))$ is determined. But o_δ is readily invertible, so finally apply this inverse.

Example: $e^{(2)}_{-1}(\{1, 2, 4, 5\})$:
One easily sees that the doubletons of $e_{-1}(2)$ are just $\{5/2, -5/2\}$. The singletons have magnitudes $\{1/2, 3/2, 7/2, 9/2, 11/2, ...\}$ — where we have written them out in the magnitude order.
For $v = \{1, 2, 4, 5\}$ we give + signs to 1/2, 3/2, 9/2 and 11/2 and the remaining singletons are negative. Thus

$$e^{(2)}_{-1}(\{1, 2, 4, 5\}) = (\frac{11}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-5}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$$

(**4.22**) **Lemma**. *Fixing δ and λ, and hence $[\lambda]_\delta$, then o_δ and o^λ_δ are mutual inverses on $[\lambda]_\delta \leftrightarrow P_{even}(\mathbb{N})$.* $\square$

(**4.23**) **Theorem**. *For each δ, λ, the map o_δ passes to an isomorphism*

$$G_\delta(\lambda) \cong G_{even}$$

(via G_{odd} and the toggle map in case $o_\delta(\lambda)$ of odd order).

Lemma 4.22 shows that o_δ restricts to a bijection on vertex sets. The next few paragraphs build up to a proof (in (4.35)) of the graph isomorphism.

(**4.24**) **Proposition**. *Fix a block, i.e. a pair $(\delta, [\lambda]_\delta)$. If (v, w) is an edge in G_{even} with label α then the corresponding pair $(\mu, \lambda) = (o^\lambda_\delta(v), o^\lambda_\delta(w))$ gives λ/μ a minimal δ-balanced skew.*

This is just a useful restatement of part of Theorem 4.23.

4.3 Geometrical aspects of the block graph

(**4.25**) A Euclidean space together with a collection of hyperplanes defines a reflection group — the group generated by reflection in these hyperplanes. Note that

$$(ij) : (v_1, v_2, ..., v_i, ..., v_j, ...) \mapsto (v_1, v_2, ..., v_j, ..., v_i, ...)$$

$$(ij)_- : (v_1, v_2, ..., v_i, ..., v_j, ...) \mapsto (v_1, v_2, ..., -v_j, ..., -v_i, ...)$$

are reflection group actions on $\mathbb{R}^{\mathbb{N}}$. Write $\mathcal{D}$ for the group generated by these (all $i < j$). Write $\mathcal{D}v$ for the orbit of a point $v \in \mathbb{R}^{\mathbb{N}}$ under the action of $\mathcal{D}$. Write $\mathcal{D}_+$ for the subgroup $\langle (ij) \rangle_{ij}$.

(**4.26**) Note that $\mathcal{D}$ does not preserve the image $e_\delta(\Lambda)$, for any δ. Indeed the closure of the dominant region (in the sense of (4.16)) is a fundamental region for the $\mathcal{D}_+$ action on $\mathbb{R}^{\mathbb{N}}$. This region is bounded by the reflection hyperplanes $\{(i\ i+1)\}_{i \in \mathbb{N}}$ (as is the region of ascending sequences). Although the blocks are not precicely $\mathcal{D}$-orbits (we will see that in a suitable sense)

$$\text{orbit} \cap \text{dominant} = \text{block}$$

Comparing the definitions of minimal δ-balanced skew (4.3), e_δ and $(ij)_-$ we see that

(**4.27**) **Lemma**. *If λ/μ is a minimal δ-balanced skew then $e_\delta(\lambda)$ can be obtained from $e_\delta(\mu)$ by a sequence of one or more transformations by $(ij)_-$s, extending rows in pairs of δ-balanced part-rows. Specifically*

$$e_\delta(\lambda) = \left(\prod_{ij} (ij)_- \right) e_\delta(\mu)$$

where the product is over pairs of rows in the skew, from the outer pair to the inner pair. □

Note also that no subset of this product, applied to $e_\delta(\mu)$, results in a dominant weight.

It follows that the $\mathcal{D}$ action on λ, via this construction, at least traverses the block $[\lambda]_\delta$. In [7] it is shown that it intersects no other block.

(**4.28**) For $v \in \mathbb{R}^{\mathbb{N}}$ define

$$V(v) = \mathcal{D}v \cap A^+$$

The partial order $(\mathbb{R}^{\mathbb{N}}, \leq)$ restricts to a partial order $(V(v), \leq)$. The latter (unlike the former) has a unique transitive reduction. This reduction thus defines a directed acyclic graph, denoted $\mathbf{G}(v)$.

(**4.29**) **Proposition**. [8, Prop.7.1] *For $\lambda \in \Lambda$ the map e_δ restricts to a bijection $[\lambda]_\delta \to V(\lambda + \rho_\delta)$; and this bijection extends to a graph isomorphism*

$$G_\delta(\lambda) \cong \mathbf{G}(\lambda + \rho_\delta).$$

Proof: By [7, Th.5.2] we have that e_δ defines a bijection between $[\lambda]_\delta$ and $V(\lambda + \rho_\delta)$. Note that $\mu \subset \nu \in \Lambda$ if and only if $e_\delta(\mu) < e_\delta(\nu)$. Thus, restricting this to $[\lambda]_\delta$, the graphs are covers (transitive reductions) of isomorphic partial orders. These covers thus agree on arbitrarily large finite sub-orders, and hence agree. □

Note that v is regular if and only if every sequence in $\mathcal{D}v$ is regular.

(**4.30**) **Proposition**. [8, Prop.7.2] *For $v \in A^+$ the map Reg restricts to a bijection $V(v) \to V(Reg(v))$; and this bijection extends to a graph isomorphism*

$$\mathbf{G}(v) \cong \mathbf{G}(Reg(v))$$

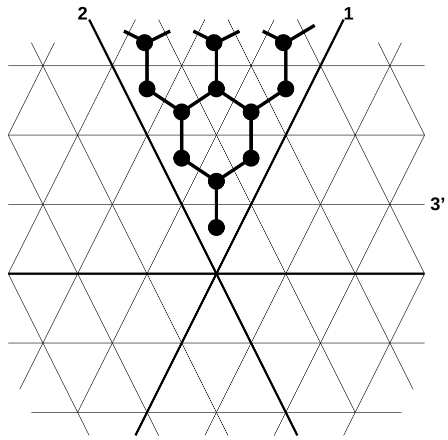


Figure 3: Example of a dominant dual graph: case affine-A_2/A_2

Proof: The set of doubletons is an invariant of the elements of $V(v)$, and there is a unique way of adding these into an element of $V(Reg(v))$ that keeps the sequence decreasing. Thus the restriction of *Reg* here has an inverse, i.e. the set map is a bijection. Now suppose $t, u \in A^+$ and $a \in \mathbb{R}$ such that

$$s = (t_1, t_2, ..., t_i, a, t_{i+1}, ...) \qquad s' = (u_1, u_2, ..., u_j, a, u_{j+1}, ...)$$

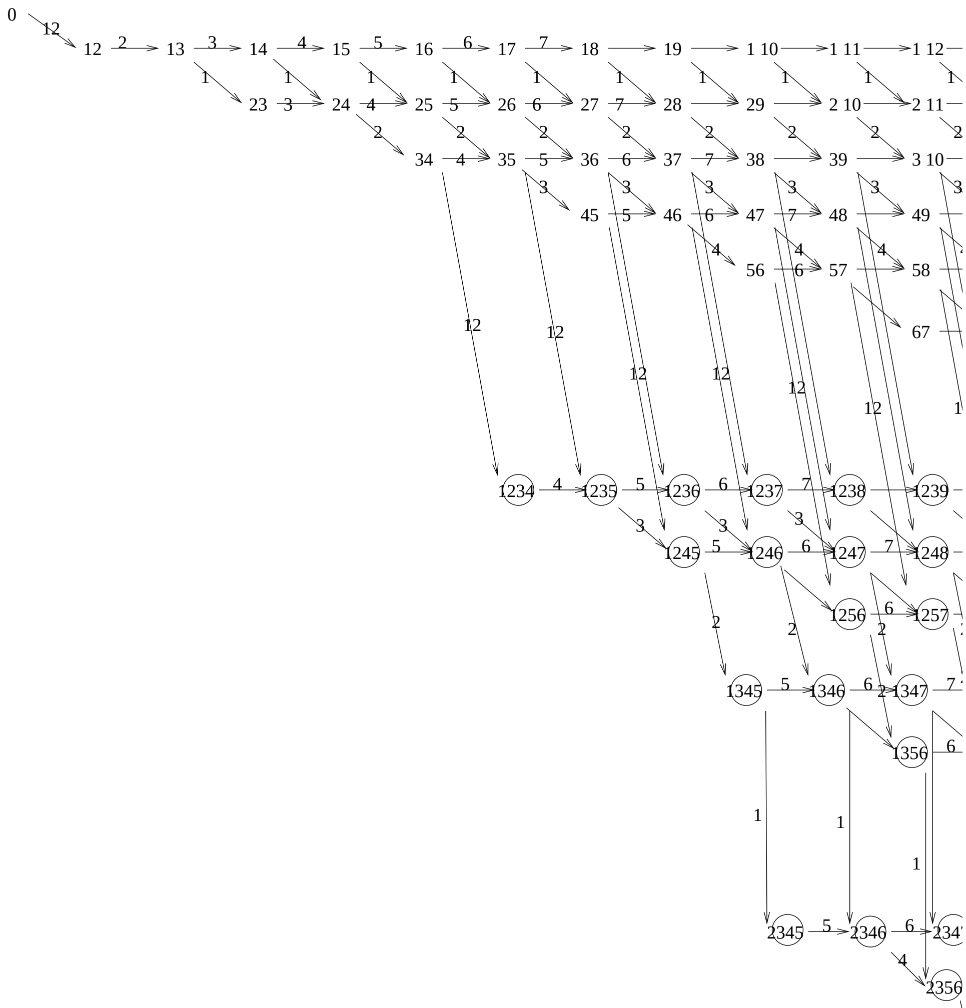
are in A^+. Then $t < u$ if and only if $s < s'$. The *Reg* map can be built from pairs of such moves, so $t < u$ if and only if $Reg(t) < Reg(u)$, which establishes the graph isomorphism. □

(**4.31**) To any Euclidean space V and set of hyperplanes $\mathbb{H}$ we may associate a *dual graph* $D(\mathbb{H})$. This has a vertex for each connected component of the space with the hyperplanes removed (called an 'alcove'); and an edge whenever the closures of two alcoves intersect in a defining subset of a hyperplane (called a 'wall').

If the set of hyperplanes is closed (under the reflections they define) it may be generated by a minimal set defined by the walls bounding a single alcove [15] (or see Section 8.1). This minimal set of hyperplanes is thus in bijection with the edges out of the dual graph vertex for the chosen 'fundamental' alcove. We have then two different enhancements of $D(\mathbb{H})$ to include *edge labels*: *left edge labelling* associates to each edge (a, b) the hyperplane defined by (a, b); *right edge labelling* requires the choice of a prefered alcove C' and associates to (a, b) the wall of C' in the same reflection group orbit as the wall $\overline{a} \cap \overline{b}$ defined by (a, b).

Given a pair of a closed set of reflection hyperplanes and a closed subset $\mathbb{H}_+$ (a parabolic), a *dominant dual graph* is the intersection of the dual graph with a fundamental chamber (a connected component of the space with just the subset removed). For example Figure 3 shows the dominant dual graph for affine-A_2 (generated by the hyperplanes 1,2 and 3' shown) over the subset corresponding to A_2 (generated by the hyperplanes 1 and 2). If (as in the example) $\mathbb{H}_+$ is maximal [15] then only one alcove in each chamber has a subset of walls defining $\mathbb{H}_+$, and then by default one chooses the fundamental alcove to be the one such in the fundamental chamber.

Figure 4: The beginning of the graph G_{even}, with edge labels. (Vertex labels have been written in an obvious shorthand.)



We write G_{alc} for the dominant dual graph of our reflection group action $\mathcal{D}$ above (with parabolic $\mathcal{D}_+$) corresponding to the choice of $S_{\mathcal{D}_+} = \{(i\ i{+}1) : i \in \mathbb{N}\}$ as reflection hyperplanes bounding the fundamental chamber, and (to make contact with the given notion of dominance) such that descending sequences lie in the fundamental chamber; and of $\{(12)_-\} \cup S_{\mathcal{D}_+}$ as reflection hyperplanes bounding the fundamental alcove.
(Figure 4 shows a graph isomorphic to G_{alc}, using an isomorphism we shall explain next.)

(**4.32**) **Lemma**. [8, Cor.7.3] *If $v \in \mathbb{R}^{\mathbb{N}}$ is regular then it lies within an alcove; and $V(v)$ consists of a point within each dominant alcove. Thus $\mathbf{G}(v) \cong G_{alc}$.* □

A convenient example of a regular v is $e_2(\emptyset)$. In light of the lemma we may use the orbit of $e_2(\emptyset)$ to label dominant alcoves. In particular $e_2(\emptyset)$ itself lies in the fundamental alcove. By considering the effect of simple reflections in this case, such as

$$(15)_-(4, 3, -1, 2, -5, ...) = (5, 3, -1, -2, -4, ...)$$

we see:

(**4.33**) **Lemma**. *The map from $V(e_2(\emptyset))$ to subsets of $\mathbb{N}$ of even order which discards all negative entries coincides with the final step in $o_2 : [\emptyset]_2 \to P(\mathbb{N})$ and extends to a graph isomorphism $G_{alc} \cong G_{even}$.*

(**4.34**) Remark. The relationship between the $\mathcal{D}$ action between adjacent vertices in G_{alc} and the edge labels in G_{even} is not, perhaps, transparent in this isomorphism, and we shall not need it explicitly for the computation of decomposition matrices. It is useful in the discussion of parabolic Kazhdan–Lusztig polynomials, however. We shall resturn to describe it in the second part of the paper.

(**4.35**) **Theorem**. *For all δ, λ the map o_δ passes (via $o_2^{\emptyset}$) to an isomorphism*

$$G_\delta(\lambda) \cong G_{alc}$$

Proof. By (4.29), (4.30), (4.32) and (4.33). □

This is a remarkable result, since the right hand side does not depend on λ or even δ.

5 Decomposition data

In this section we prepare the structures needed in the statement of the main result. The idea comes from solving for parabolic Kazhdan–Lusztig polynomials for the $\mathcal{D}/\mathcal{D}_+$ system (a highly non-trivial exercise). However the *proof* of the main result requires a more general approach, so we do not emphasise the Kazhdan–Lusztig theory aspect at this stage. (See later.)

5.1 Hypercubical decomposition graphs

(**5.1**) Let $\mathbf{b} : P(\mathbb{N}) \to \{0, 1\}^{\mathbb{N}}$ denote the natural bijection. For example:

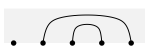
$$\mathbf{b} : \{1, 3, 5, 6\} \mapsto 101011$$

(if set a is finite we omit the open string of 0s on the right).
Define $\mathbf{b}_\delta : \Lambda \to \{0,1\}^{\mathbb{N}}$ by $\mathbf{b}_\delta(\lambda) = \mathbf{b}(o_\delta(\lambda))$.

(**5.2**) A generalisation of Brauer diagrams is to allow singleton vertices. A vertex pairing in such a diagram *covers* a vertex if the pair lie either side of it. A *TL-diagram* (TL as in Temperley–Lieb) is here a diagram drawn in the positive quadrant of the plane, consisting of a collection of vertices drawn on the horizontal part of the boundary (countable by the natural numbering from left to right); together with a collection of *non-crossing* arcs drawn in the positive quadrant, each terminating in two of the vertices, such that no vertex terminates more than one arc, and no arc covers a singleton vertex. An example is: 
It will be convenient to label each arc by the associated pair of numbered vertices.

Remark. As with a Brauer diagram, it is the vertex pairings (and here singletons) rather than the precise routes of the arcs that are important.

(**5.3**) Each binary sequence b has a TL-diagram $d(b)$ constructed as follows.
1. Draw a row of vertices, one for each entry in b (up to the last non-zero entry).
2. For each binary subsequence 01 draw an arc connecting the corresponding vertices.
3. Consider the sequence obtained by ignoring the vertices paired in 2. For each subsequence 01 draw an arc connecting these vertices (it will be evident that this can be done without crossing).
4. Iterate this process until termination (it will be evident that it terminates, since the sequence is getting shorter).
5. Note that this process terminates either in the empty sequence or in a sequence of 1s then 0s (either run possibly empty). Finally connect the run of vertices binary-labelled 1 in adjacent pairs (if any) from the left. Leave the remaining vertices as singletons.

Example: $d(10011) =$  A number of examples are shown in Figure 5.

(**5.4**) For $a \in P(\mathbb{N})$ we write Γ_a for the list of arcs (i.e. pairs) corresponding to 01 subsequences, and an initial 11 subsequence (i.e. if there is one in the 12-position); and Γ^a for the list of all arcs.

In particular, for example,

$$\Gamma_{1356} = \{\{2,3\},\{4,5\}\} \qquad\qquad \Gamma^{1356} = \{\{2,3\},\{4,5\},\{1,6\}\}$$

See Figure 5 for more examples. We may write $\Gamma_{\delta,\lambda}$ for $\Gamma_{o_\delta(\lambda)}$, and Γ^λ_δ for $\Gamma^{o_\delta(\lambda)}$.

(**5.5**) A *hypercubical directed graph* is a rooted directed graph isomorphic to the cover of the subset partial order on some set S. There is a notion of *parallel* edges (edges corresponding to deleting the same element of S). The edges coming out of the top vertex are called *shoulder* edges, and every edge is parallel to one of these.
There is an obvious association with the notion of the (geometrical) hypercube or hypercuboid, i.e. the $\{0,1\}$-span of any linearly independent collection of vectors in a space. The notion of parallel edges comes from this.

(**5.6**) Each $a \in P(\mathbb{N})$ defines a hypercubical directed graph h^a, as follows. The vertices are binary sequences (these should be considered as identified with elements of $P(\mathbb{N})$ by the bijection, but it is convenient to treat them as binary sequences for the construction). Firstly a defines a binary

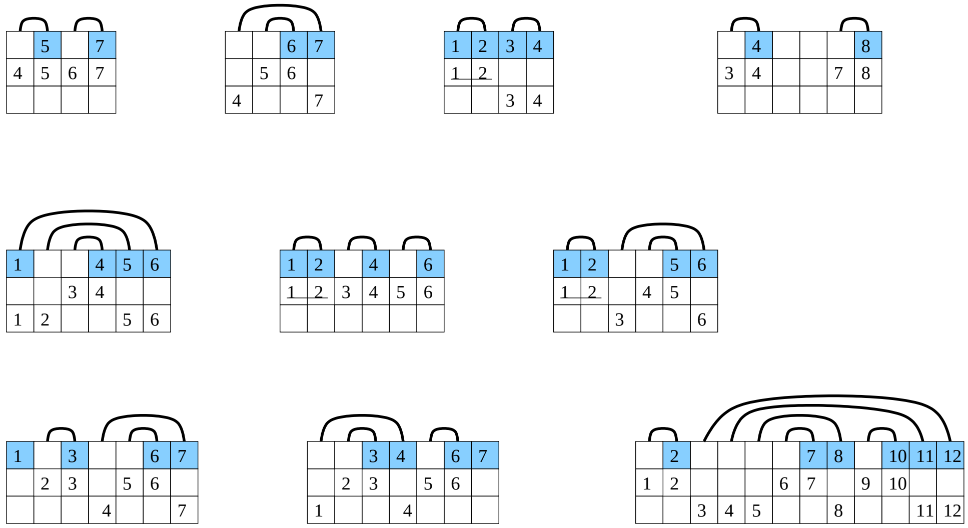


Figure 5: Examples for the map from sequences to TL-diagrams, and to sets of pairs. In each case the sequence for a set $a \in P(\mathbb{N})$ is indicated in the first (shaded) row of boxes. The second row shows the set of pairs of numbers Γ_a extracted from the TL construction. The third row shows the further pairs added to obtain the set Γ^a.

sequence $\mathbf{b}(a)$ and hence a TL-diagram $d(\mathbf{b}(a))$. The top sequence in h^a is the defining sequence $\mathbf{b}(a)$. There is an edge out of this corresponding to each completed arc in the TL-diagram $d(\mathbf{b}(a))$. The sequence at the other end of a given edge is obtained from the original by replacing $01 \to 10$ (or $11 \to 00$) at the ends of this arc. Indeed every parallel edge in the hypercube follows this transformation rule.

There is an example in Figure 6 (and an example starting from given δ and λ in Section 5.2).

(**5.7**) Note from the construction that these hypercubes are multiplicity-free. That is, no two vertices have the same label.

Since fixing a block $[\lambda]_\delta$ establishes a bijection between $P_{even}(\mathbb{N})$ and $[\lambda]_\delta$ the construction for h^a also defines a hypercubical directed graph $h_\delta(\mu)$ for each pair $(\delta, \mu) \in \mathbb{Z} \times \Lambda$, obtained by applying o^μ_δ to the vertices. That is, abusing notation slightly,

$$o_\delta(h_\delta(\mu)) = h^{o_\delta(\mu)}$$

(**5.8**) We label each edge of the hypercube (i.e. each direction) by $\{\alpha, \alpha'\}$, where α, α' are the positions of the ends of the arc associated to this edge.

If label $\alpha' = \alpha + 1$ for an 01-arc, we may just label the edge by α. If $\{\alpha, \alpha'\} = \{1, 2\}$ for a 11-arc we may just label the edge by 12. Note that these α-edges and 12-edges in particular then coincide with edges of G_{even}, although other edges do not.

(**5.9**) It follows from the construction and Theorem 4.23 that if a vertex of some hypercube $h_\delta(\tau)$

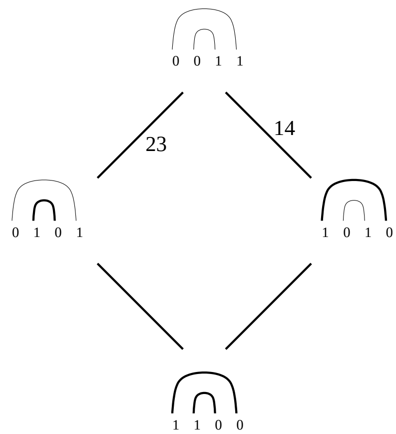


Figure 6: Hypercube $h^{3\,4}$ (showing the TL arcs used in the construction).

is $\mathbf{b}_\delta(\lambda)$ for some λ, then a vertex beneath it down an α or 12-edge is $\mathbf{b}_\delta(\mu)$ for some μ a maximal δ-balanced subpartition of λ.

(**5.10**) Note that we have assigned a hypercube to each appropriate binary sequence and hence to each vertex of G_{even}. Thus for any given block $[\lambda]_\delta$ we have asigned a hypercube to each partition in the block. The vertices in this hypercube then correspond to partitions in the same block (the defining one, together with one of each of some collection below the defining one). In this way we can use the hypercubes to determine, for each δ, a matrix (of almost all 0s, and some 1s), with rows and columns labelled by partitions. The 1's in any given row are given by the vertices of the hypercube associated to the partition labelling that row.
In light of this interpretation we shall write $h_\delta(\mu)_\nu = 1$ if ν appears in $h_\delta(\mu)$, and $= 0$ otherwise. We will see in Theorem 7.1 that the resultant matrix gives our block decomposition matrix.

It will also be useful to consider an intermediate encoding, between the hypercube and the constant matrix row, in which we record the *depth* i of each entry in the hypercube, by writing v^i (v a formal parameter) instead of 1 in the appropriate position. (Thus this polynomial version evaluates to the decomposition matrix at $v = 1$.) The first few vertices of this form are shown in Figure 7, using the $P(\mathbb{N})$ labelling scheme.

5.2 Hypercubical decomposition graphs: tools and examples

(**5.11**) Here is a concrete example of $h_\delta(\lambda)$ with $\delta = 2$. We take $\lambda = (7, 7, 6, 5, 3, 2)$ so

$$\lambda + \rho_2 = (6, 5, 3, 1, -2, -4, -7, -8, ...)$$

giving $o_2(\lambda) = \{1, 3, 5, 6\}$ and hence $\Gamma^\lambda_\delta = \{\{2, 3\}, \{4, 5\}, \{1, 6\}\}$. The specific hypercube (with integer partitions at the vertices) is thus (a) in Figure 8. In the figure we have recorded both the

	0	12	13	14	23	15	24	16	25	34	17	26	35	18	27	36	45	1234	19	28	37	46	1235	1X	29	38	47	56	1236	1245	1 11	2 10	39	48	57	1237	1246	1345	1 12
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1345									2				1																	1								0	
1 12																															1								0

Figure 7: Table encoding array of polynomials in the G_{even} labelling scheme (every non-zero polynomial is of form v^i, and the entry shown is i).

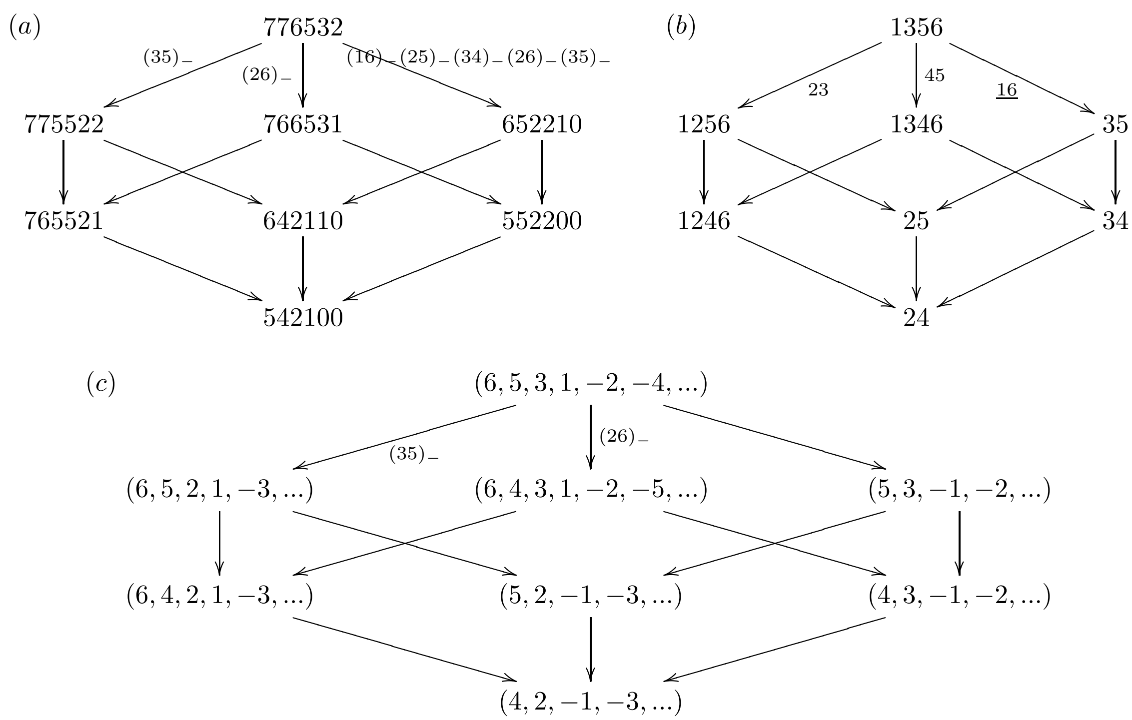


Figure 8: Three labellings of the same hypercube in case $\delta = 2$: (a) partition labelling; (b) $P(\mathbb{N})$ labelling; (c) descending sequence labelling.

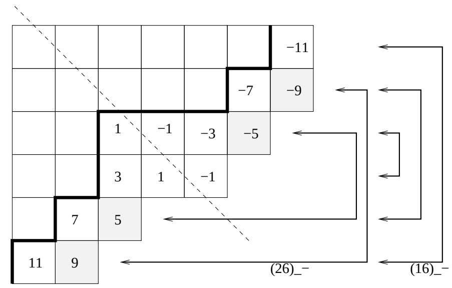


Figure 9: Explicit reflections on $\lambda = 776532$ in case $\delta = 2$.

α-action and the specific reflection group action required to achieve it on each edge (for the shoulder layer). The version in (b) shows the G_{even} vertex labels. The version in (c) shows the ρ_δ-shifted vertex labels. Figure 9 shows the explicit reflections and composite reflection in the shoulder. Note that the composite can be built as five dominance preserving but not all commuting reflections.

(**5.12**) Keeping the same δ, λ, now consider $\lambda - e_i$ in case $i = 4$.

This gives $(7,7,6,4,3,2) \rightsquigarrow (6,5,3,0,-2,-4,-7,...) \overset{o_2}{\rightsquigarrow} \{1,3,5,6\}$ (by the toggle rule). This means that the hypercube $h_\delta(\lambda - e_i)$ is isomorphic to that for λ above, so in particular the α-actions (the formal edge labels) are the same. Note also that the specific reflections (realising these α-actions) in the shoulder of $h_\delta(\lambda - e_i)$ are the same as for λ.

Remark. We show in Section 6 that so long as e_i does not 'separate' a MiBS (in the sense of (6.5)) this holds true in general. That is the hypercubes are isomorphic and the reflections needed to move through the hypercube are the same.

A more complicated example is given in Figure 12. We conclude this Section with some tools for manipulating these hypercubes, that we shall need later.

(**5.13**) Let $b = (b_1, b_2, ...)$ be a binary sequence, and α a natural number. Then $\hat{\alpha}b$ is the sequence obtained from b by inserting 01 in the $\alpha, \alpha+1$ positions (i.e. so that this pair become the elements in the α and $\alpha+1$ positions in the sequence, with any terms at or above these positions in b bumped two places further up in $\hat{\alpha}b$).
Similarly $\check{\alpha}b$ is the sequence obtained from b by inserting 10 in the $\alpha, \alpha+1$ positions.
Examples: $\hat{2}01 = 0011$, $\check{2}01 = 0101$.

(**5.14**) Let h be a hypercube (i.e. the $\{0,1\}$-span of any linearly independent collection of vectors, as before), and α a vector outside the span of h (or an operator that can otherwise be considered to shift all the vertices of h by the same amount in a new direction). Then by αh we mean the translate of h determined by α, and by $(1,\alpha)h$ we mean the new hypercube which contains h and a translate of h by α together with the edges in the α direction.

More specifically, if h is a hypercube whose vertices are binary sequences, all of which have 01 (or all 11) in the $\alpha, \alpha+1$ positions, then αh is the hypercube defined from h by modifying this

01 → 10 (respectively 11 → 00). In this case $(1, \alpha)h$ is the hypercubical union of h and αh.

If the bumped sequence $\hat{\alpha}\mathbf{b}_\delta(\lambda)$ makes sense, then by $\hat{\alpha}h_\delta(\lambda)$ we understand the corresponding vertex-modified hypercube (insert 01 at the same position in every vertex binary sequence, and modify any edge labels affected by this accordingly). Note that this is not a hypercube of form $h_\delta(\mu)$, but a subgraph of somesuch. Similarly define $\check{\alpha}h_\delta(\lambda)$ (and note that $\check{\alpha}h_\delta(\lambda) = \alpha\hat{\alpha}h_\delta(\lambda)$). Note that $\check{\alpha}h_\delta(\lambda)$ is another hypercube not of form $h_\delta(\mu)$. However

$$(1, \alpha)\hat{\alpha}h_\delta(\lambda) = h_\delta(\mu) \qquad \text{where } \mathbf{b}_\delta(\mu) = \hat{\alpha}\mathbf{b}_\delta(\lambda) \tag{11}$$

This is simply a restatement of part of the definition (5.6), that will be useful later.

6 Embedding properties of δ-blocks in Λ

In this section we consider how the block graphs embed in $\mathbb{R}^{\mathbb{N}}$ and hence how the embeddings of the different block graphs relate to each other. The result (3.13) means, loosely speaking, that the usual *metrical* structure on $\mathbb{R}^{\mathbb{N}}$ has relevance in our representation theory. This, together with the embedding results we develop here, will allow us to pass information between blocks.

(**6.1**) Suppose $w \in \mathcal{D}$ such that $we_\delta(\lambda) = e_\delta(\mu)$. When δ is fixed we may write $w.\lambda$ for μ. Also if λ is a vertex of G_{even} or $G_\delta(\mu)$ and α is the label on an edge out of λ we write $\alpha\lambda$ for the vertex at the other end.

(**6.2**) The isomorphism implicit in Theorem 4.35 between any pair of block graphs $G_\delta(\lambda)$ and $G_\delta(\lambda')$ defines a pairing of each vertex in $G_\delta(\lambda)$ with the corresponding vertex in $G_\delta(\lambda')$. A pair of block graphs is *adjacent* if they have the same singularity, and every such pair of vertices is adjacent as a pair of partitions.

(**6.3**) Remark. If λ, λ' are adjacent partitions in the same $\mathcal{D}$-facet (in the alcove geometric sense) then the corresponding pair of graphs are adjacent, since the same reflection group elements serve to traverse these graphs [7], and reflection group elements preserve adjacency of partitions. We shall need to show adjacency of a more general pairing of graphs.

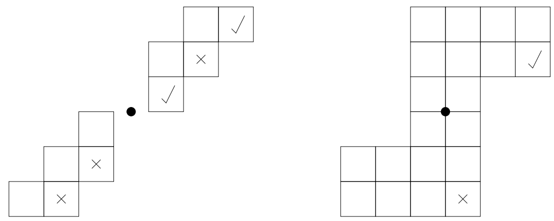
(**6.4**) For given λ, if $\lambda' = \lambda - e_i$ in (6.2) above we write

$$f_i : [\lambda]_\delta \ \rightarrow \ [\lambda - e_i]_\delta$$

for the restriction of the graph isomorphism to vertices. (Strictly speaking f_i depends on λ too, but we suppress this for brevity.)

(**6.5**) Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i. Suppose that $\lambda/\alpha\lambda$ is a MiBS containing e_i. Write π_α for the π-reflection fixing this MiBS. Then note that $\pi_\alpha(e_i)$ is an addable box of $\alpha\lambda$. If $\lambda/\alpha\lambda \setminus \{e_i, \pi_\alpha(e_i)\}$ is not a MiBS (of $\lambda - e_i$) we say that e_i *separates* $\lambda/\alpha\lambda$.

(**6.6**) Examples: crosses show boxes that separate; ticks show boxes that do not:



(**6.7**) **Lemma**. *(Charge-row lemma) Fix any δ. If a row i of partition λ ends in a box with charge c we have*

$$(\lambda + \rho_\delta)_i = -\frac{c}{2} + \frac{1}{2}$$

(**6.8**) **Lemma**. *Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i such that singularity $s_\delta(\lambda) = s_\delta(\lambda - e_i)$. Then*
(I) $o_\delta(\lambda) = o_\delta(\lambda - e_i)$;
(II) There does not exist a weight $\lambda - e_i - e_{i'}$ δ-balanced with λ.
(III) There does not exist a weight $(\lambda - e_i) + e_i + e_{i'}$ δ-balanced with $\lambda - e_i$.

Proof. Write x for $(\lambda + \rho_\delta)_i$. That is

$$\lambda + \rho_\delta \sim (..., w, \underbrace{x}_{i}, y, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., w, \underbrace{x-1}_{i}, y, ...) \tag{12}$$

with $w > x$ and $y < x - 1$. From this we see that $x - 1$ cannot occur in $\lambda + \rho_\delta$ (else it would occur twice in $\lambda + \rho_\delta - e_i$, contradicting the descending property of the latter); and similarly x cannot appear in $\lambda + \rho_\delta - e_i$.
Note also that for $\lambda - e_i - e_{i'}$ to be δ-balanced with λ we would have to have (for $x \geq 1$)

$$\lambda + \rho_\delta - e_i - e_{i'} \sim (..., \underbrace{x-1}_{i}, ..., \underbrace{-x}_{i'}, ...) \tag{13}$$

We now split into two cases, depending on whether the set of pairs $p_\delta(\lambda) = p_\delta(\lambda - e_i)$.
(A) If $p_\delta(\lambda) = p_\delta(\lambda - e_i)$:

(I) The argument depends on the value of x. We split into subcases (i-v).
(i) If $x - 1 > 0$: then $-(x-1) < 0$ cannot appear in either sequence (suppose it appears in the j-th position, then $\{i, j\} \in p_\delta(\lambda + \rho_\delta - e_i)$ contradicting hypothesis (A)); and similarly $-x$ cannot appear in either (else again p_δ changes between them). It follows that x appears in $Reg(\lambda + \rho_\delta)$ and $x - 1$ in the corresponding position in $Reg(\lambda + \rho_\delta - e_i)$; and that these sequences otherwise agree.

Suppose then that x is, say, the l-th smallest magnitude entry in $Reg(\lambda+\rho_\delta)$. If there is a smaller magnitude entry it's magnitude is smaller than $x - 1$, by the argument following Equations(12) and the argument above. Since all these other entries are the same for the other sequence, $x - 1$ is the l-th smallest magnitude entry in $Reg(\lambda + \rho_\delta - e_i)$. Thus o_δ is unchanged.

(ii) If $x = 1$: then we have $\lambda + \rho_\delta \sim (..., w > 1, \underbrace{x = 1}_{i}, y < 0, ...)$. We note that $-x = -1$ still cannot appear in either sequence (else p_δ changes). Thus 1 in $Reg(\lambda + \rho_\delta)$, respectively 0 in $Reg(\lambda + \rho_\delta - e_i)$, is the smallest magnitude entry. If there are an even number of other positive entries then this entry does not contribute to o_δ in either case (in the former by the toggle rule, and in the latter by the definition of the o-map). If there are an odd number of other positive entries then this entry contributes to o_δ in both cases (similarly). Thus o_δ is unchanged.

(iii) If $x = 0$: then we have $\lambda + \rho_\delta \sim (..., w > 0, \underbrace{x = 0}_{i}, y < -1, ...)$ and this time the hypothesis determines that $-(x-1) = 1$ cannot appear in either sequence. Thus 0 in $Reg(\lambda+\rho_\delta)$, respectively -1 in $Reg(\lambda + \rho_\delta - e_i)$, is the smallest magnitude entry. If there are an even number of strictly positive entries then this entry does not contribute to o_δ in either case. If there are an odd number of positive entries then this entry contributes an element 1 to o_δ in former cases (by the definition of the o-map); the entry -1 does not contribute in the latter case, but there is an element 1 by the toggle rule. Thus o_δ is unchanged.

(iv) If $x = 1/2$: then we have $\lambda + \rho_\delta \sim (..., w > 1, \underbrace{x = 1/2}_{i}, y < -1, ...)$. Evidently there is no -1/2 in the former or 1/2 in the latter, so the terms in the i-th position are the smallest magnitude terms in their respective sequence, with all else equal. Again by the toggle rule o_δ is unchanged.

(v) If $x < 0$: then neither $-x$ nor $-(x-1)$ can appear in either sequence (else hypothesis (A) is violated much as before). The argument is then much as in (i).

(II) For $x \geq 1$, by equation (13) δ-balance here would require $-x+1$ in the i'-position in $\lambda+\rho_\delta$, and this is already disallowed under hypothesis (A). The case $x = 1/2$ does not arise; and the cases $x \leq 0$ are similar to the above, with the order of i, i' reversed.

(III) By the rules of balance $e_{i'}$ cannot be in the same row as e_i, so $(\lambda+\rho_\delta+e_{i'})_i = (\lambda+\rho_\delta)_i = x$. This would require that in the balance partner $(\lambda + \rho_\delta - e_i)_{i'} = -x$, but this is already disallowed under hypothesis (A).

(B) If $p_\delta(\lambda) \neq p_\delta(\lambda - e_i)$:

(I) Write x for $(\lambda + \rho_\delta)_i$ as before. Then from Equation(12) we see firstly that $-x$ occurs in $\lambda + \rho_\delta$ and $1 - x$ occurs in $\lambda + \rho_\delta - e_i$ (if neither occurs then p_δ does not change between them; if only one occurs then s_δ changes); of course it follows immediately that $1 - x, -x$ occur (and are adjacent) in both; secondly, by the same argument as above $x - 1$ does not occur in $\lambda + \rho_\delta$.

In computing o_δ we discount the $\pm x$ pair in $\lambda + \rho_\delta$ and the $\pm(x-1)$ pair in $\lambda + \rho_\delta - e_i$. The discrepancy is thus now a $1 - x$ in $\lambda + \rho_\delta$ compared to a $-x$ in $\lambda + \rho_\delta - e_i$. But if $1 - x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta$ then $-x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta - e_i$, with all else equal, so o_δ is unchanged.

(II) By equation (13) δ-balance would require $-x + 1$ in the i'-position in $\lambda + \rho_\delta$ as before. Although this is not disallowed here, it forces the $-x$ to lie in the next (that is, the $i' + 1$) position. This would force a second $-x$ in the same position in $\lambda + \rho_\delta - e_i - e_{i'}$, which would thus not be descending — a contradiction.

(III) Since $(\lambda + \rho_\delta - e_i)_i = x - 1$ we would require $(\lambda + \rho_\delta + e_{i'})_{i'} = 1 - x$ for balance. Thus $(\lambda + \rho_\delta)_{i'} = -x$. But we have already seen that $\lambda + \rho_\delta$ contains both $1 - x, -x$, so this would require $\lambda + \rho_\delta + e_{i'}$ containing $1 - x$ in two positions — a contradiction.

□

(6.9) Lemma. *Fix δ and suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before. Suppose λ has an edge down labelled α, i.e. $\lambda/\alpha\lambda$ is a MiBS; and let w be the product of commuting reflections such that $we_\delta(\lambda) = e_\delta(\alpha\lambda)$, as in Lemma (4.27). Then*
(I) $we_\delta(\lambda - e_i)$ is dominant;
(II) $we_\delta(\lambda - e_i) = e_\delta(\alpha(\lambda - e_i))$;
(III) $\alpha(\lambda - e_i) \overset{\triangleleft}{\underset{\triangleright}{}} \alpha\lambda$ (i.e. they are adjacent).

Proof. (I) We split into two cases:

If e_i does not intersect $\lambda/\alpha\lambda$ then $we_\delta(\lambda - e_i)$ is the same as $we_\delta(\lambda)$ everywhere except in row i: $we_\delta(\lambda - e_i) = we_\delta(\lambda) - e_i$. Since $\lambda - e_i$ is dominant, $\lambda_i > \lambda_{i+1}$, but $(\alpha\lambda)_i = \lambda_i$ in this case, and $(\alpha\lambda)_{i+1} \leq \lambda_{i+1}$, so $(\alpha\lambda)_i > (\alpha\lambda)_{i+1}$, so $\alpha\lambda - e_i$ is dominant, so $e_\delta(\alpha\lambda - e_i) = we_\delta(\lambda - e_i)$ is dominant.
If e_i intersects $\lambda/\alpha\lambda$ then $\pi_\alpha(e_i)$ is addable to $\alpha\lambda$ as noted in (6.5). That is $e_\delta(\alpha\lambda + \pi_\alpha(e_i)) = we_\delta(\lambda - e_i)$ is dominant.

(II) Firstly note that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ by Lemma 6.8, so $\alpha(\lambda - e_i)$ makes sense. Similarly we have $o_\delta(\alpha(\lambda - e_i)) = o_\delta(\alpha\lambda)$ (since both are equal to the formal set $\alpha o_\delta(\lambda)$).

Since $we_\delta(\lambda - e_i)$ is dominant (by (I)) in the $\mathcal{D}$-orbit of $\lambda - e_i$ there is some $\mu \in [\lambda - e_i]_\delta$ such that $we_\delta(\lambda - e_i) = e_\delta(\mu)$. Since it is adjacent to $e_\delta(\alpha\lambda)$ and has the same singularity, then by Lemma (6.8) (applied appropriately) $o_\delta(\mu) = o_\delta(\alpha\lambda)$. That is, $\mu = \alpha(\lambda - e_i)$.

(III) Follows immediately from (II).

□

(6.10) Lemma. *Fix δ. Suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before, and $\alpha\lambda/\lambda$ is MiBS (i.e. α is an edge up from λ). Then there is a reflection group element w such that $w.\lambda = \alpha\lambda$ (so $w.\alpha\lambda = \lambda$) and $w.(\lambda - e_i)$ is dominant; whereupon $w.(\lambda - e_i) = \alpha(\lambda - e_i)$.*

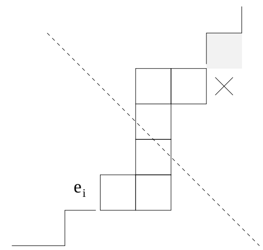
Proof. Suppose $w.(\lambda - e_i)$ is dominant. Then it is $\mu \in [\lambda - e_i]_\delta$ adjacent to $w.\lambda = \alpha\lambda$ with the same singularity, hence the same o_δ by Lemma (6.8). Thus it is enough to show that $w.(\lambda - e_i)$ is dominant.

Given that $w.\lambda$ is dominant, any *failure* of dominance of $w.(\lambda - e_i)$ must involve the i-th row itself being shorter than row-$i+1$ in $w.(\lambda - e_i)$ (i.e. row-$i+1$ intersects the MiBS); or a row with which row-i is paired in w (j, say) being longer than row-$j-1$ in $w.(\lambda - e_i)$. We must consider the cases: (A) e_i lies 'behind' the skew (i.e. it's image under the π-rotation π_α that fixes $\alpha\lambda/\lambda$ extends some row of the skew); or (B) not.

(A) In this case the failure would have to be that the image of e_i under the π-rotation broke dominance, i.e. extended beyond the row above it.

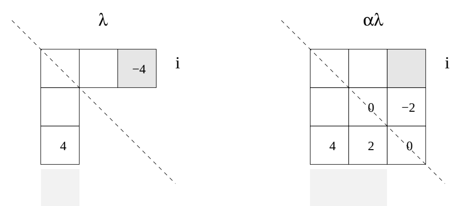
Suppose e_i is behind other than the last row of the skew. Then there is a box of the skew immediately to its right and one immediately below it. The π-rotation images of these are behind and above the image of e_i, so $w.(\lambda - e_i)$ is dominant.

On the other hand, suppose e_i is behind the last row of the skew. For example:



(the box $\pi_\alpha(e_i)$ is marked $\times$). Here $w.(\lambda - e_i)$ is dominant unless the box above $\pi_\alpha(e_i)$ is missing from λ. But if this is missing then this row and the i-row are a singular pair in $\lambda - e_i$. Neither row can be in a singular pair in λ so this contradicts the hypothesis.

(B) If the i-th row is not moved by w then the failure would have to be that the skew $\alpha\lambda/\lambda$ includes a box directly under e_i. But in that case a δ-balanced box to e_i given by $\pi_\alpha(e_i)$ is directly to the left of the skew, and we have a setup something like the following:



(the δ-balanced box is the box marked 4). If there is no box below the $\pi_\alpha(e_i)$ in λ then row-i is not in a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$, thus $s_\delta(\lambda) \neq s_\delta(\lambda - e_i)$ so we can exclude this. If there is a box below the $\pi_\alpha(e_i)$ in λ then this row and row-i are a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$. In this case, a w which also has a factor acting on the i-th and undrawn row has the same effect on λ as one which does not. Its effect on $\lambda - e_i$ is to restore the box e_i and to add a box in the undrawn row. This $w.(\lambda - e_i)$ is dominant since the added box is under a box added in the original skew.

□

Since the block graph is connected we may use Lemmas 6.9 and 6.10 to show:

(**6.11**) **Theorem**. *(Embedding Theorem)* *If* $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ *then* $G_\delta(\lambda)$ *is adjacent to* $G_\delta(\lambda - e_i)$. □

(**6.12**) Lemma (6.8)(I) says that if the partitions $\lambda, \lambda - e_i$ have the same singularity then they pass to the same point on the block graph G_{even}. That is

$$f_i(\lambda) = \lambda - e_i$$

and so on. Thus for $\mu \in [\lambda]_\delta$

$$h_\delta(\lambda)_\mu = h_\delta(\lambda - e_i)_{f_i(\mu)}$$

(6.13) Lemma. *Fix δ. No pair of weights of form λ and $\lambda - e_i + e_j$ are in the same block (unless $i = j$).*
That is, no pair of weights of form $\lambda + e_i$ and $\lambda + e_j$ are in the same block (unless $i = j$).

Proof. Such a pair cannot meet the charge-pair form of the balance condition [6], since each of the skews involved has rank 1. □

(6.14) Lemma. *If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then for all pairs $(\mu, f_i(\mu)) \in [\lambda]_\delta \times [\lambda - e_i]_\delta$*

$$Proj_\lambda Ind\, \Delta_n(f_i(\mu))' = \Delta_{n+1}(\mu)'$$

$$Proj_{f_i(\lambda)} Ind\, \Delta_n(\mu)' = \Delta_{n+1}(f_i(\mu))' \tag{14}$$

Proof. Note that the pair $(\mu, f_i(\mu))$ are adjacent by Theorem 6.11. For any ν

$$\text{Ind}\ \Delta(\nu)' = \Big(+_j \Delta(\nu + e_j)'\Big) + \Big(+_k \Delta(\nu - e_k)'\Big)$$

For $\nu = f_i(\mu)$ adjacent to μ, one of these summands is $\Delta(\mu)'$. Specifically either (i) $\mu = \nu + e_l$ (some l); or (ii) $\mu = \nu - e_l$ (some l).

In case (i) other summands are of form $\mu - e_l + e_j$, $\mu - e_l - e_k$. By Lemma (6.13) the former are not in $[\mu]_\delta$, and since $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ we may use Lemma (6.8)(II) to exclude the latter. The other case is similar. □

7 The Decomposition Theorem

(7.1) Theorem. *For each $\delta \in \mathbb{Z}$ and $\lambda \in \Lambda$, the hypercube $h_\delta(\lambda)$ gives the λ-th row of the $[\lambda]_\delta$-block of the global Brauer algebra Δ-decomposition matrix D over $\mathbb{C}$. That is*

$$(P_n^\delta(\lambda)' : \Delta_n^\delta(\mu)') = h_\delta(\lambda)_\mu$$

for all $n \geq |\lambda|$; or equivalently

$$P_n^\delta(\lambda)' = +_{\mu \in h_\delta(\lambda)}\ \Delta_n^\delta(\mu)'$$

(Recall we omit $\lambda = \emptyset$ in case $\delta = 0$.)
This data determines the Cartan decomposition matrix C for any finite n by (3.15).

Proof. We prove for a fixed but arbitrary δ, working by induction on n. The base cases are $n = 0, 1$, which are trivial (and $n = 2$ for $\delta = 0$, which is straightforward). We assume the theorem holds up to level $n - 1$, and consider $\lambda \vdash n$. (For $|\lambda| < n$ the result holds by (3.15) and the inductive assumption.)

The λ-th row of D encodes the standard content of projective module $P(\lambda)'$. We apply the induction functor to a suitable $P(\lambda - e_i)'$ in level $n - 1$ (known by the inductive assumption), and use Prop.(3.17):

$$\text{Proj}_\lambda \text{Ind}\ P(\lambda - e_i)' \cong P(\lambda)' \bigoplus Q$$

(some Q). Thus the challenge is to determine the Δ-content of $\mathrm{Proj}_\lambda \mathrm{Ind}\ P(\lambda - e_i)'$ and Q. In general determining Q can be complicated, but we will show that there is always a choice of $\lambda - e_i$ which makes it tractable.

Note that if λ is at the bottom of its block then the claim is trivially true by (3.15). If λ is not at the bottom of its block then the binary sequence $\mathbf{b}_\delta(\lambda)$ has at least one 01 (or initial 11) subsequence. Thus we can choose e_i to be a removable box from the skew associated to the corresponding edge α of $h_\delta(\lambda)$. (We sometimes write $\mu = \alpha\lambda$ for the partition at the other end of this edge, so the skew is $\lambda/\mu = \lambda/\alpha\lambda$.) Note that this skew is a minimal δ-balanced skew, by (5.9).

The next step depends on whether the skew $\lambda/\alpha\lambda$ is of form (1)+(1), or otherwise.

7.1 Properties of minimal δ-balanced skews

(**7.2**) We will say that a skew $\lambda/\alpha\lambda$ is *boxy* if every box in it lies within a (2^2)-shape that also lies within the skew. In our case, these are the skews in which the pair of rims fully overlap (i.e. run side-by-side). Thus in our case boxy skews have a terminal (2^2)-shape at each end, in which the largest magnitude charges reside. Note that since no (2^2)-shape has a removable box of largest magnitude charge, neither does a boxy skew (on the other hand every such shape has a removable box of next-largest magnitude, and one can see that the largest of these is removable at one end of the boxy skew or the other). An example is given in Figure 10(iii).

If a minimal skew is neither of form (1)+(1) nor boxy we shall say that it is generic.

(**7.3**) **Lemma**. *Let $\lambda/\mu = \lambda/\alpha\lambda$ be a minimal δ-balanced skew. Then there are a pair of boxes in the skew of greatest magnitude charge. In case the skew is of shape (1)+(1) both of these are removable; in the boxy cases (such as (2^2)) neither are removable (but precisely one of the next-largest is removable); and otherwise precisely one of them is removable.*

Proof. All statements are (by now) clear except the last. For this note that if both were removable this would contradict that $\alpha\lambda$ is a maximal δ-balanced subpartition, since removing just this pair from λ would give a larger δ-balanced subpartition; while if neither were removable then again this would contradict the maximal δ-balanced subpartition property, since removing the complement (i.e. the boxes in $\lambda/\alpha\lambda$ *not* in this pair) would give a larger δ-balanced subpartition. □

(**7.4**) We call a removable box of largest magnitude charge (among those removable in the given skew) a *rim-end removable box.* (Since the skew is a (possibly touching) pair of rims, and this box lies at one of the outer ends.)

(**7.5**) Examples of minimal skews are shown in Figure 10. The rim-end removable boxes (as labelled by charge) in the figure are (i) 22; (ii) -16; (iii) 8.
(For $\delta = 1$ example (i) is, in greater detail,

$$\lambda + \rho_1 = (25/2, 23/2, 21/2, 19/2, 17/2, 11/2, 9/2, -3/2, -9/2, -11/2, -17/2, -19/2, -21/2, ...)$$

which is five-fold singular (in the sense of (4.17)), giving $o_1(\lambda) = \{2, 3\}$ for its 'valley' set.)

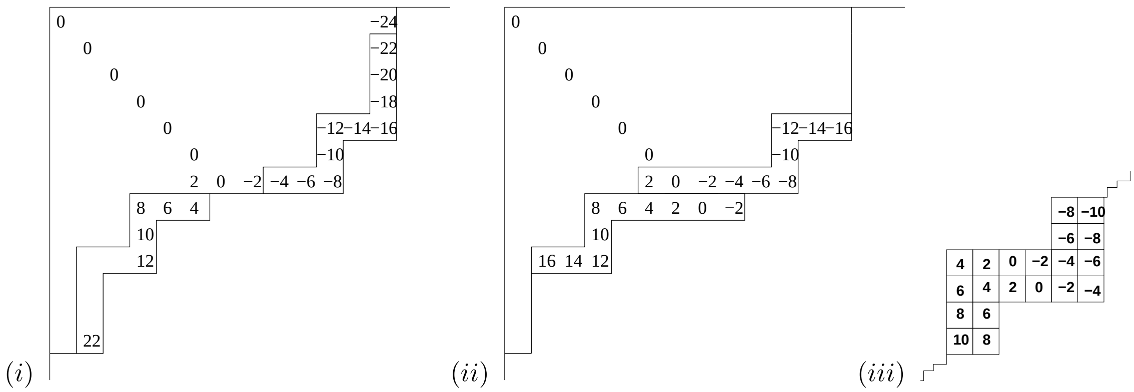


Figure 10: Examples of minimal δ-balanced skews.

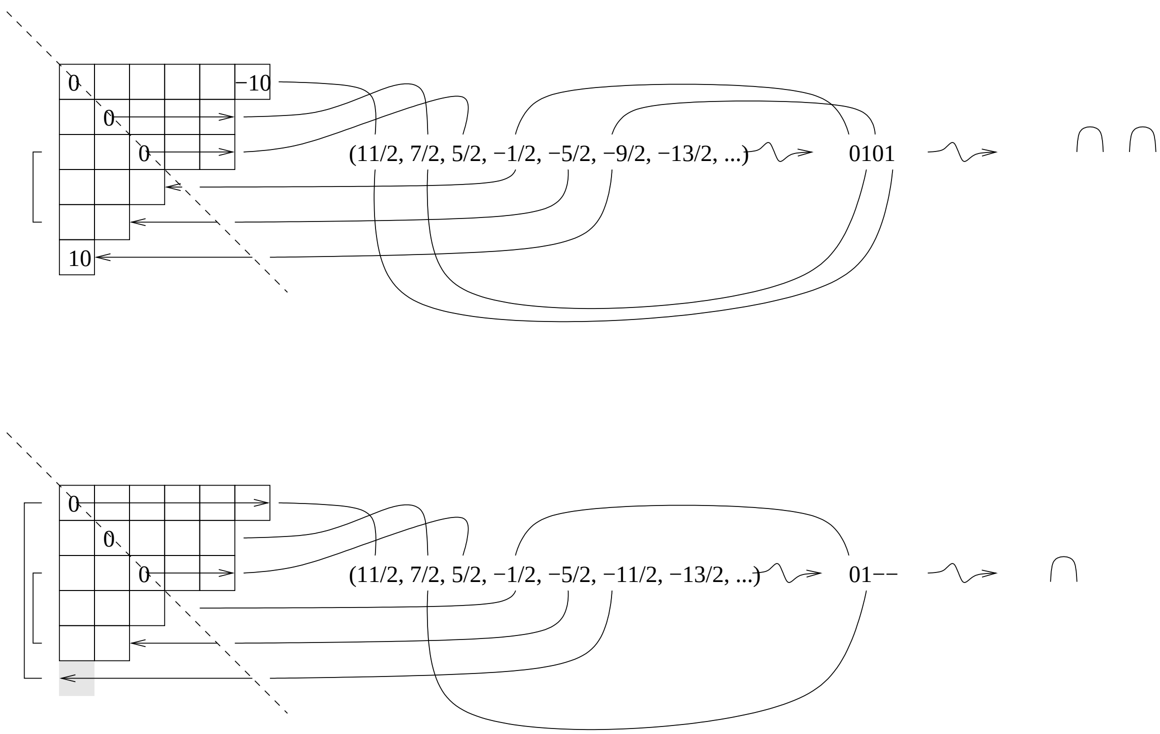


Figure 11: Two examples showing the passage from Young diagram λ, via corresponding (case $\delta = 1$) descending sequence $\lambda + \rho_1$, to binary sequence and TL diagram. The connecting lines indicate the precise passage of data through the process. The two cases are related in the form λ, $\lambda - e_i$, illustrating a step up in singularity (the singular pairs of rows in each case are marked on the left).

7.2 Cases in the inductive step

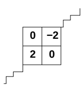
(**7.6**) **Proposition**. *Fix δ, and hence an identification between descending sequences and partitions. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. Then the singularities obey*

$$s_\delta(\lambda - e_i) = \begin{cases} s_\delta(\lambda) + 1 & \textit{if } |\lambda/\alpha\lambda| = 2 \\ s_\delta(\lambda) & \textit{otherwise} \end{cases}$$

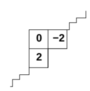
Proof: If $|\lambda/\alpha\lambda| = 2$ we are in the $(1) + (1)$ or (1^2) case, and the charges in the boxes are (say) x and $-x$. Removing x (from row i) we get a row ending in charge $x + 2$, giving $(\lambda + \rho_\delta)_i = -\frac{x+2}{2} + \frac{1}{2} = -\frac{x+1}{2}$ (by Lemma 6.7). The row ending in $-x$ has $(\lambda + \rho_\delta)_j = -\frac{-x}{2} + \frac{1}{2} = \frac{x+1}{2}$ thus these two rows are now a singular pair.
Figure 11 gives an example.

Suitable examples of the generic situation are given in Example 7.5. If the upper end of a rim ends in a row (of length greater than 1), such as the upper rim in Example 7.5(ii), which ends in -16, then the end box of this row is removable, but its balance partner is not. It follows that singularity is unchanged on removing the end-box e_i, since this row becoming part of a singular pair would imply a removable balance partner. (Thus $o_\delta(\lambda - e_i) = o_\delta(\lambda)$, indeed we remain in the same facet.)
If the lower end of a rim ends in a column (of length greater than 1), such as the lower rim in (i), which ends in 22, then the end-box of this column is removable. This time $\lambda - e_i$ lies on different hyperplanes to λ, but *overall* singularity is unchanged.
(In the particular example $-21/2 \to -23/2$.)

In the case (2^2) we have



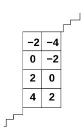
$\mapsto (..., 3/2, 1/2, \leq -5/2, ...)$ $\rightsquigarrow$



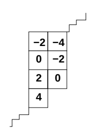
$\mapsto (..., 3/2, -1/2, \leq -5/2, ...)$

which shows that the singularity does not change.

For the remaining (boxy) cases there are a couple of analogous variations to the generic 'ends in row/column' cases treated above. Here we merely illustrate with a couple of examples. In the case (2^4) we have

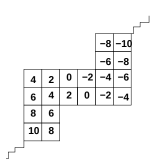


$\mapsto (..., 5/2, 3/2, 1/2, -1/2, \leq -7/2, ...)$ $\rightsquigarrow$



$\mapsto (..., 5/2, 3/2, 1/2, -3/2, \leq -7/2, ...)$

which shows that the singularity does not change, although the wall does. In the case (3^2) we have (similarly embedded, in general) $(3^2) \mapsto (..., 2, 1, \leq -3, ...) \rightsquigarrow (32) \mapsto (..., 2, 0, \leq -3, ...)$ which has the same singularity (and wall set). A more typical boxy skew is



$\mapsto (..., 11/2, 9/2, 7/2, 5/2, -5/2, -7/2, \leq -13/2, ...)$

Removing the removable 8 here changes $-7/2 \to -9/2$, giving the same singularity (different wall). □

(**7.7**) **Proposition**. *Fix δ. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. In the cases in which the skew is neither of form $(1)+(1)$ nor (1^2)*
(i) the Δ-decomposition data for $P(\lambda)'$ is the 'translate' *of that for $P(\lambda-e_i)'$:*

$$(P(\lambda)' : \Delta(\mu)') = (P(\lambda-e_i)' : \Delta(f_i(\mu))') \qquad \forall \mu \in [\lambda]_\delta$$

(ii) This verifies the inductive step for the main theorem in such cases. That is, $h_\delta(\lambda) \cong h_\delta(\lambda - e_i)$.

Proof: Consider the 'translation' $\mathrm{Proj}_\lambda \mathrm{Ind}\, P(\lambda-e_i)'$ of $P(\lambda-e_i)'$. By Proposition 3.17

$$\mathrm{Proj}_\lambda \mathrm{Ind}\, P(\lambda-e_i)' \;=\; P(\lambda)' \oplus Q$$

with $Q = \mathrm{Proj}_\lambda Q$ some projective, possibly zero. In the cases under consideration (skew neither $(1)+(1)$ nor (1^2)) each standard module occuring in $P(\lambda-e_i)'$ induces precisely one standard module after projection onto the block of λ, by Lemma 6.14 (noting Proposition 7.6). More specifically, writing

$$P(\lambda-e_i)' = \;+\!\!\!\!+_\mu\, c_\mu\, \Delta(f_i(\mu))' \tag{15}$$

(for some multiplicities c_μ), using (6.4); then

$$P(\lambda)' \oplus Q = \mathrm{Proj}_\lambda \mathrm{Ind}\, P(\lambda-e_i)' \;=\; +\!\!\!\!+_\mu\, c_\mu\, \mathrm{Proj}_\lambda \mathrm{Ind}\, \Delta(f_i(\mu))' = \;+\!\!\!\!+_\mu\, c_\mu\, \Delta(\mu)'$$

On inducing again and projecting back to the block of $\lambda - e_i$, by (14) we have

$$\mathrm{Proj}_{\lambda-e_i} \mathrm{Ind}\, (P(\lambda)' \oplus Q) \;=\; +\!\!\!\!+_\mu\, c_\mu\, \Delta(f_i(\mu))'$$

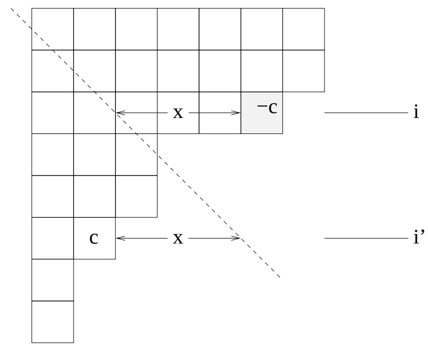
That is, each standard module occuring in $(P(\lambda)' \oplus Q)$ induces precisely one standard module after projection onto the block of $\lambda - e_i$. Comparing with (15), it follows that this second 'translation' may be identified with $P(\lambda-e_i)'$ again. Since this is indecomposable, the *first* translation cannot be split, and hence is precisely $P(\lambda)'$ — with the same decomposition pattern.

For the last part use (6.12). □

The remaining cases needed to move between level n and $n-1$ are skews of form (1)+(1).

(**7.8**) **Proposition**. *Fix δ. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. Then in the cases in which the skew is of form $(1)+(1)$ or (1^2)*
(I) the sequence $\mathbf{b}_\delta(\lambda) = \hat{\alpha}\mathbf{b}_\delta(\lambda-e_i)$. Thus, hypercube $h_\delta(\lambda) = (1,\alpha)\hat{\alpha}h_\delta(\lambda-e_i)$ (i.e. has increased 'dimension' by +1 compared to $h_\delta(\lambda-e_i)$). The sequence $\mathbf{b}_\delta(\alpha\lambda) = \check{\alpha}\mathbf{b}_\delta(\lambda-e_i)$ (i.e. differs from $\mathbf{b}_\delta(\lambda-e_i)$ by insertion of subsequence 10 in the α position).
(II) The Δ-decomposition data for $P(\lambda)'$ is in agreement with the above, in the sense of the equality in the main theorem: if the inductive assumption $(P(\lambda-e_i)' : \Delta(\nu)') = h_\delta(\lambda-e_i)_\nu$ (all ν) holds, then $(P(\lambda)' : \Delta(\mu)') = h_\delta(\lambda)_\mu$ (all μ).

Proof: (I) As shown in the proof of Prop. 7.6 (or see below), removing e_i from λ makes that row part of a singular pair with the row containing the box with opposite charge. Thus $\mathbf{b}_\delta(\lambda-e_i)$ differs from $\mathbf{b}_\delta(\lambda)$ in that a pair which contributed an 01 sequence in the latter does not contribute to the descending sequence in the former — i.e. $\mathbf{b}_\delta(\lambda - e_i)$ differs by the removal of this 01 sequence. (Figure 11 serves as an example here.) It remains to confirm the *position* of the removal. The situation $\lambda/\alpha\lambda \sim (1) + (1)$ is well illustrated by the following typical example:



In general we have

$$\lambda + \rho_\delta \sim (..., \underbrace{\underbrace{x+1}_{i}, ..., -x}_{01}, ...)$$

Altogether the bracketed pair contribute an 01 in binary as indicated. The $x+1$ lies at some position l, say, in the *magnitude* order, depending on the rest of λ. Confer

$$\lambda - e_i + \rho_\delta \sim (..., \underbrace{x}_{i}, ..., -x, ...)$$

Here the $x, -x$ are a singular pair, so do not appear in the magnitude order — to obtain its binary representation from that of λ one deletes the binary pair 01 in the $l-1, l$ position. That is, $\mathbf{b}_\delta(\lambda) = \widehat{l-1}\ \mathbf{b}_\delta(\lambda - e_i)$. Finally

$$\alpha\lambda + \rho_\delta = \lambda - e_i - e_{i'} + \rho_\delta \sim (..., \underbrace{\underbrace{x}_{i}, ..., -x-1}_{10}, ...)$$

Since the α action on λ manifests (by definition) as $10 \leftrightarrow 01$ in the $\alpha, \alpha+1$ position of $\mathbf{b}_\delta(\lambda)$ we see that position $l - 1 = \alpha$ as claimed. The other assertions follow immediately.

(II) Applying $\mathrm{Proj}_\lambda -$ to Proposition 3.13(ii) here we get a short exact sequence

$$0 \to \Delta(\lambda - e_i - e_{i'})' \to \mathrm{Proj}_\lambda \mathrm{Ind}\ \Delta(\lambda - e_i)' \to \Delta(\lambda)' \to 0$$

(non-split, by [6, Lemma 4.10]). That is

$$\mathrm{Proj}_\lambda \mathrm{Ind}\ \Delta(\lambda - e_i)' \ = \Delta(\lambda)' + \Delta(\lambda - e_i - e_{i'})' = \Delta(\lambda)' + \Delta(\alpha\lambda)' \tag{16}$$

(non-split). Translating $P_{\lambda - e_i} := P(\lambda - e_i)'$ away from and then back to $\lambda - e_i$ therefore produces a projective whose dominating content is two copies of $\Delta(\lambda - e_i)'$ (one from each of the summands on the right of (16)). Indeed every Δ-filtration factor of $P_{\lambda-e_i}$ engenders at most two factors in $\text{Proj}_{\lambda-e_i}\text{Ind}\ (\text{Proj}_\lambda\text{Ind}\ P_{\lambda-e_i})$ (we shall be able to make a precise statement shortly). Hence, by (3.16), $\text{Proj}_{\lambda-e_i}\text{Ind}\ (\text{Proj}_\lambda\text{Ind}\ P_{\lambda-e_i}) = P_{\lambda-e_i} \oplus P_{\lambda-e_i}$. It follows that

$$\text{Proj}_\lambda\text{Ind}\ P_{\lambda-e_i} = P_\lambda$$

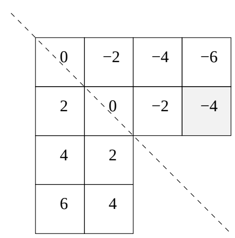
It remains to show that $(\text{Proj}_\lambda\text{Ind}P(\lambda-e_i)' : \Delta(-)') = h_\delta(\lambda)$ (given $(P(\lambda-e_i)' : \Delta(-)') = h_\delta(\lambda-e_i)$). We do this next (the reader may also find Example 7.9 helpful).

For each Δ_μ occuring in the $P(\lambda-e_i)'$ decomposition we will see that the translation is $\Delta_\mu \rightsquigarrow \Delta_{\mu+} + \Delta_{\mu-}$ for some pair $\mu+, \mu-$ in the λ-orbit. For $\lambda - e_i$ itself we have seen in the proof of (I) that $b_\delta(\lambda-e_i)$ gives $b_\delta(\lambda)$ and $b_\delta(\alpha\lambda)$ by inserting 01 (respectively 10) in the α position. For other $\mu \in h_\delta(\lambda-e_i)$, note that the relevant singular pair of rows in $\lambda-e_i$, while not contributing to the magnitude order (since they are singular) are formally permuted (in the $\mathcal{D}$-action sense) along with the rest of the rows, in the collection of reflection group actions that traverse $h_\delta(\lambda-e_i)$. Thus they (jointly) maintain a formal position in the magnitude order, between two terms that are properly consecutive in this order. The difference with μ_+, μ_- is that in these one of the pair is extended by 1, or contracted by one. Thus the singularity is broken, and the pair appear properly in the order, between the given two terms, and hence bumping up the larger of the two. Since $\mu + \rho_\delta$ is just a signed permutation of $\lambda - e_i + \rho_\delta$ (and hence just a permutation, as far as the magnitudes are concerned), the position of the pair in the magnitude order, and hence the position of the bump in the binary representation, is at α, the same as for $\lambda - e_i$. That the collection thus engendered overall is $h_\delta(\lambda)$ now follows directly from Equation(11). Indeed, for $\mu \in h_\delta(\lambda-e_i)$, and $\check{\alpha}\mu, \hat{\alpha}\mu$ the two partitions associated to μ by the doubling $h_\delta(\lambda) = (1,\alpha)\hat{\alpha}h_\delta(\lambda-e_i)$, we have (non-split [6, Lemma 4.10])

$$0 \to \Delta(\check{\alpha}\mu)' \to \text{Proj}_\lambda\text{Ind}\ \Delta(\mu)' \to \Delta(\hat{\alpha}\mu)' \to 0$$

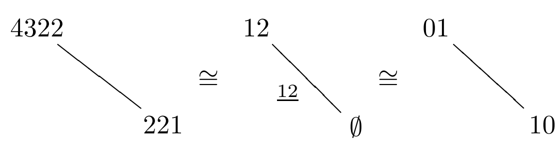
(From an alcove geometric perspective one may view this argument as follows: Since $[\lambda-e_i]_\delta$ is a strictly more singular orbit than $[\lambda]_\delta$ the reflection group elements moving through $h_\delta(\lambda-e_i)$ will also serve to move the pair $\lambda, \alpha\lambda$ through these pairs $\mu+, \mu-$, thus they remain adjacent above and below μ.)
□

(**7.9**) Example for Proposition 7.8: $\delta = 1$, computing for $\lambda = 4422$ via $\lambda - e_2 = 4322$. We have



In particular $e_1(4322) = (7/2, 3/2, -1/2, -3/2, ...)$ so $o_1(4322) = \text{toggle}(\{2\}) = \{1,2\}$. By the inductive hypothesis we have

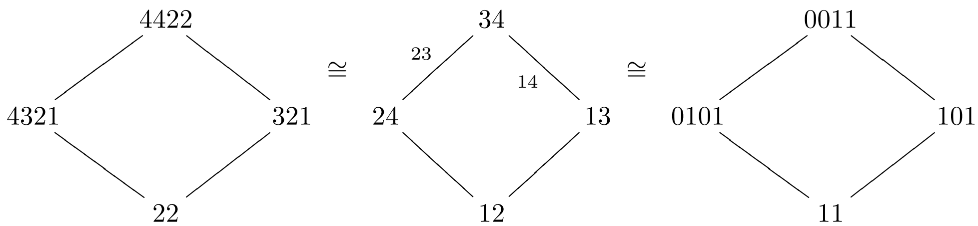
$(P(4322) : \Delta(-)) = h_1(4322)_- =$



Here the first form of the hypercube is in partition labelling; the second form is in $P(\mathbb{N})$ labelling (having applied the toggle); and the last is the untoggled binary representation. Note that we have reverted to the untoggled form at the last since we will be inserting an 01 subsequence (removing the need for the toggle) at the next step. Translating off the wall we get $4322 + 221 \rightarrow (4422 + 4321) + (321 + 22)$. In binary this corresponds to $01 \rightarrow 0**1 \rightarrow 0101 + 0011$ and $10 \rightarrow 1**0 \rightarrow 1100 + 1010$. These four sequences therefore encode the content of P_{4422}.

Meanwhile

$h_1(4422) =$



confirming the assertion of the Theorem in this case.
Note how the insertion of a binary pair in the α position, and action of α on that pair, transforms $h_1(4322)$ to produce $h_1(4422)$. The effect is (i) to extend the hypercube by a new generating direction (labelled by α); (ii) the generating edge inherited from $h_1(4322)$ changes label from 12 to 14 due to the bump (which illustrates how such non-G_{even} edge labels arise in this construction).

Proposition 7.8 completes the last case for the main inductive step for the Theorem. □

8 Background: parabolic Kazhdan–Lusztig polynomials

In the remainder of the paper we explain where the idea for hypercubical decomposition graphs comes from.

Associated to each Coxeter system C and parabolic A, acting as reflection groups on a suitable space, is an alcove geometry on that space. For each such pair C/A there is, therefore, an array $P = P(C/A)$ of Kazhdan–Lusztig polynomials — one for each ordered pair of alcoves. (Deodhar's recursive formula [10] computes these polynomials in principle. However it generally tells us very little about them in practice.) These polynomials are of interest from a number of points of view. For example they are often important in representation theory (see [21, 19] and references therein). So, with the reflection group pair $\mathcal{D}/\mathcal{D}_+$ manifesting itself in Brauer algebra block theory (as we have seen), one is motivated to compute them in this case.

8.1 Chamber geometry

We first need to review the notion of chamber geometry. In this we follow Humphreys [15]. (Alcove geometry is a mild generalisation associated to the group/parabolic pair. Humphreys introduces this in the context of affine extensions, but it serves equally well in general.)

Let V be a Euclidean space, and (W, S) a Coxeter system with an action generated by reflections on V. Let H_s be the reflection hyperplane of $s \in S$, or indeed of any reflection $s \in W$ generated by these. For T any subset of S let $[T]$ be the set of reflections generated by T. Set

$$\mathbb{H}_T = \bigcup_{t \in [S] \setminus [T]} H_t$$

A chamber is a maximal connected component of $V \setminus \mathbb{H}_\emptyset$. Write $\mathcal{C}_W$ for the set of chambers.

The set $H'_t = H_t \setminus \mathbb{H}_{\{t\}}$ (the subset of hyperplane H_t that intersects no other hyperplane) may similarly be broken up into connected components. At most one of these components intersects any given chamber closure $\overline{C}$. If H'_t intersects $\overline{C}$ in this way it is called a wall of C.

For any given C, the set $\{t \;:\; H'_t \cap \overline{C} \neq \emptyset\}$ of ts that make up its walls functions as a choice of S in W (i.e. they are an equivalent choice of Coxeter generators to the original set S). On the other hand S may or may not determine such a C uniquely.

The choice of a prefered chamber C_0 corresponds to the choice of a simple system in V, and the associated reflections are simple reflections. (Given a non-commuting pair of these, the conjugate of one by the other is also a reflection, but not 'simple' in this choice.)

A reflection s in W is simple for chamber B if its hyperplane H_s makes a wall of B (NB simple for B is not the same as simple, unless $B = C_0$). For our purposes it will be convenient to think specifically of the intersection of the hyperplane with the chamber closure (i.e. this facet) as the wall (thus we distinguish the walls of distinct chambers in general, even if they come from the same hyperplane).

(**8.1**) The reflection action of W acts to permute $\mathcal{C}_W$. This action is transitive and indeed regular (simply transitive). See for example [15, §1.12].
Note that W does *not* act transitively on V, or specifically, on the set of walls. The walls of C_0 are representatives for the W orbits of the set of all walls.

Regularity says that we may identify $\mathcal{C}_W$ with W, and the action of W with the left-action on itself. In particular write

$$A = w_A C_0 \tag{17}$$

(so we may indentify C_0 with 1).

Note that it follows from this identification that there is another commuting action of W on $\mathcal{C}_W$, corresponding to the right-action of W on itself.

Noting the choice of C_0, define a length function on $\mathcal{C}_W$: $l_W(A)$ is the number of hyperplanes separating A from C_0. (If W is clear from context we shall write simply $l = l_W$.)

(**8.2**) We define a digraph $G(W,S)$ with vertex set $\mathcal{C}_W$ by (A,B) an edge if $B = tA$ with t simple for A and $l(B) = l(A) + 1$.

We call t the left-action label of edge (A, tA).

By (17) the edge (A, tA) may also be written $(w_A C_0, t w_A C_0)$. The image under w_A of a particular 'initial' edge (C_0, sC_0) $(s \in S)$ is

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A s w_A^{-1} w_A C_0) = (A, w_A s w_A^{-1} A)$$

Using the right-action this can be expressed as

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A C_0 s) = (A, As)$$

We call this s the right-action label of the edge. (With this label the graph is essentially the right Cayley graph $\Gamma(W,S)$, and s is the 'colour' label.)

Evidently $G(W,S)$ is a rooted acyclic digraph, with root C_0.

(**8.3**) Let $v \in C_0$, and let Wv be the W-orbit of v in V. In the same way as above we may associate a graph to this orbit. It will be evident that this graph is isomorphic to $G(W,S)$, for any such v.

8.2 Alcove geometry

Let (W',S') be a system containing (W,S) as a parabolic subsystem, with both acting on V. The chambers of W' are then called alcoves. Thus the alcoves are a further subdivision of the chambers of W. Write $\mathcal{A} = \mathcal{C}_{W'}$ for the set of alcoves, and X^+ for the set of alcoves lying in C_0. Thus X^+ is a representative set for the W-orbits of $\mathcal{A}$. (In this setting we will call any $v \in C_0$ *dominant*.)

Choose C' a prefered alcove in C_0. As before, the hyperplanes bounding C' determine S' (a superset of S, by the inclusion in C_0).

The digraph $G(W',S')$ has vertex set $\mathcal{A}$, and (A,B) an edge if $B = sA$ with s simple for A and $l_{W'}(B) = l_{W'}(A)+1$. This is evidently a rooted acyclic digraph, with root C'. The edges are in correspondence with the set of walls, and may thus be partitioned into W'-orbits, labelled by the walls of C'.

(**8.4**) Let $G_a = G_a(W',W)$ denote the full subgraph of $G(W',S')$ with vertex set X^+. This is still rooted. Thus any alcove $A \in X^+$ may be reached from C' by a sequence of simple reflections, always remaining in X^+.
We shall denote the poset defined by the acyclic digraph G_a as $(X^+,<)$.

The array $P = P(W'/W)$ is a (generally semiinfinite) lower unitriangular matrix, with row and column positions indexed by X^+. It is natural to organise this data into rows (although it is also of interest to organise it into columns). These rows are thus 'finite' (i.e. of finite support), while the columns are not in general.

8.3 The recursion for $P(W'/W)$

The recursion for rows of P above the root in the poset (acyclic digraph) order may be given as follows (see [21] for equivalent constructions). Write $P = (p_{AB})_{A,B\in X^+}$. To compute the row p_A for alcove A we first compute another polynomial for each alcove D, p'_{AD}, also denoted $p'_A(D)$ as follows. (Actually $p'_A(D)$ can depend on the choice made next in the computation, but p_A does not and we supress this dependence in notation.)
Pick an edge (B,A) in G_a ending at A (so p_B is known). For each alcove D let $\Gamma^{\pm}_D$ be the set of alcoves D' of G_a such that (D',D) (resp. (D,D')) is an edge in the orbit of the edge (B,A). (By (8.2) we can express $(B,A) = (B,Bs)$, $s \in S'$, whereupon any such D' must obey $(D',D) = (D',D's) = (Ds,D)$ (respectively $(D,D') = (D,Ds)$).) Then

$$p'_A(D) = \sum_{D'\in\Gamma^+_D} (vp_B(D) + p_B(D')) + \sum_{D'\in\Gamma^-_D} (v^{-1}p_B(D) + p_B(D')) \tag{18}$$

(As noted there is at most one edge in the orbit of (B, A) involving any alcove D. Thus at most one of these sums is non-trivial, and that contains only one entry. In particular (B, A) is in its own orbit, so $p'_A(A) = v^{-1}p_B(A) + p_B(B) = 1$.)

To obtain the row of P that we want from p'_A it is then necessary to perform a subtraction in case the evaluation $p'_A(D)(v = 0)$ is non-zero for any $D < A$:

$$p_A = p'_A - \sum_{D<A} p'_A(D)(v = 0)\, p_D$$

(But we shall see that the sum always vanishes in the case we are interested in. So in our case $p_A = p'_A$.)

In order to work with this rule in any given alcove geometry it is necessary to be able to manipulate the graph G_a and its edge orbits efficiently. In Section 9 we set up the requisite machinery for the case $\mathcal{D}/\mathcal{D}_+$.

9 The reflection group action $\mathcal{D}$ on $\mathbb{R}^{\mathbb{N}}$

Define $v_- = o_2(\emptyset) = -(1, 2, 3, ...) \in \mathbb{R}^{\mathbb{N}}$. In Section 4.3 we chose the alcove containing v_- as C', for the reflection group $\mathcal{D}$. (We shall refer to $\mathcal{D}v_-$ as the *fully-regular* orbit.)

Thus our choice of C' corresponds to choosing $S_{\mathcal{D}} = \{(12)_-,\ (i\ i{+}1)\}_{i\in\mathbb{N}}$ for the Coxeter generating set of $\mathcal{D}$.

The orbit $\mathcal{D}v_-$ consists in the set of *co-even permutations* (signed permutations of v_- with an even number of positive terms). By (8.1) this orbit (and hence each of the others) is isomorphic, via the left action of $\mathcal{D}$ upon it, to the (limit) regular representation. It is easy to check that the action we are using is the left-regular action. By (8.2) it is the associated right action that we need to determine in order to compute (18). This commuting right action corresponds to signed permutations of the *entries* in the sequence, rather than signed permutations of the *positions*. For example

$$(4, 3, -1, -2, -5, ...)(45) = (5, 3, -1, -2, -4, ...)$$

Via the isomorphism between $V(v_-)$ and $P_{even}(\mathbb{N})$ we understand left- and right-actions of $w \in \mathcal{D}$ on any $a \subset \mathbb{N}$ (noting that wa, respectively aw, is not necessarily expressible in $P(\mathbb{N})$, since it is not necessarily dominant). When aw *is* dominant we shall see now that the right-action transformation $a \to aw$ is expressible in a simple form in $P(\mathbb{N})$ which facilitates computation of the pKLps. Let G_e denote the simple relabelling of G_{alc} from $P(\mathbb{N})$ using the above isomorphism. (We shall shortly be able to identify G_e with G_{even}.) The following crucial result is routine to show.

(9.1) Theorem. *Let $a \subset \mathbb{N}$ of even degree. Then there exists an edge $(a, a(\alpha, \alpha + 1))$ in G_e iff $a \cap \{\alpha, \alpha+1\} = \{\alpha\}$, whereupon $a(\alpha, \alpha+1) \cap \{\alpha, \alpha+1\} = \{\alpha+1\}$; and an edge $(a, a(12)_-)$ in G_e iff $a \cap \{1, 2\} = \emptyset$, whereupon $a(12)_- \cap \{1, 2\} = \{1, 2\}$. Every edge is one of these types.* □

That is, we may associate edge labels corresponding to the right-action in G_e, taken from the Coxeter generating set $S_{\mathcal{D}}$ (as required by (8.1)). To streamline still further we may write simply α

as 'right-action' label for edges of form $(\lambda, \lambda(\alpha, \alpha+1))$ and 12 for $(\lambda, \lambda(12)_-)$. This makes explicit the identification with G_{even}. See Figure 4.

(**9.2**) Remark. The left-action labels are of course different in this regard. Only elements of form $(ij)_-$ preserve dominance.

A convenient summary of the above is as follows (when we speak of an edge orbit on G_{even} we shall mean the orbit induced by the graph isomorphism with G_{alc} from the edge orbit thereon):

(**9.3**) **Theorem**. *Two edges in G_{even} pass to G_{alc} edges in the same $\mathcal{D}$-orbit (up to direction) if and only if they have the same label.* $\square$

10 Solving the polynomial recursion

To give an indication of the nature of the data set, note that a table of the first few parabolic Kazhdan–Lusztig polynomials is encoded in Figure 7 (these first few may even be computed by brute force if desired). Now we solve the recursion in closed form.

10.1 Hypercubes revisited

As we have noted in Theorem 9.1, the right-action of $\mathcal{D}$ takes a particularly simple form when between 'dominant' elements, i.e. between elements expressable as $a \subset \mathbb{N}$. We define $\langle\alpha\rangle a = a(\alpha, \alpha+1)$ to be this action between dominant elements. *I.e. only for the appropriate domain.* (Because the underlying descending sequences consist first of positive terms of descending magnitude, and then negative terms of ascending magnitude, we call $a \subset \mathbb{N}$ a valley set, and $\langle\alpha\rangle$ a valley edge operator.)

(**10.1**) We generalise the set of valley edge operators $\langle\alpha\rangle$ as follows.

Operator $\langle ij\rangle$ has action defined in case one of i, j is in a, and swaps it for the other (i.e. swaps the side of the valley that each of i, j are on).
Example

$$\langle 36\rangle 56 = 35$$

(Thus $\langle\alpha\rangle = \langle\alpha \quad \alpha{+}1\rangle$. NB, Throughout this section we shall continue to write simply αa for $\langle\alpha \;\; \alpha{+}1\rangle a$ where no ambiguity arises.)
Where defined, each such operator acts involutively; and, where defined, takes a to $\langle ij\rangle a$ comparable to a in the G_{even} order.
Each such operator has the same effect on the given a as some (strictly descending (or ascending)) sequence of $\langle\alpha\rangle$ edge operators. In our example

$$56 \xrightarrow{4} 46 \xrightarrow{3} 36 \xrightarrow{5} 35$$

Operator $\langle \underline{ij}\rangle$ has action defined in case both or neither of i, j are in a, and toggles this state.
Example

$$\langle \underline{16}\rangle 1456 = 45$$

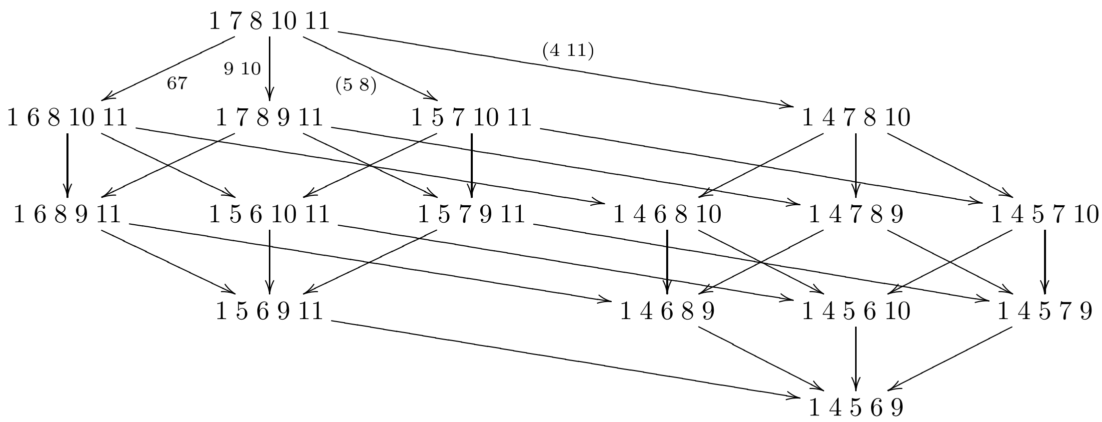


Figure 12: The hypercube $h^{1\ 7\ 8\ 10\ 11}$.

which expands, for example, as

$$1456 \xrightarrow{3} 1356 \xrightarrow{2} 1256 \xrightarrow{4} 1246 \xrightarrow{5} 1245 \xrightarrow{12} 45$$

(**10.2**) Remark. Let v be the fully-regular (FR) image of $a \in P(\mathbb{N})$ such that $\langle ij \rangle a$ is defined. Unless $j = i+1$ it does *not* follow that the fully-regular image of $\langle ij \rangle a$ is given by the right-action of (i, j) on v. Note, for example, that the underlying descending sequence of $\langle ij \rangle a$ is *not* in general a pair permutation of that of a.

(**10.3**) Let S be a set of generalised valley edge labels, and $a \in P(\mathbb{N})$. If for each subset $S' \subseteq S$ the elements of S' may be applied to a in any order to obtain the same set, and this set lies below a in G_{even}, then the *dominant hypercube* $hh(a, S)$ is the digraph consisting of this collection of sets (vertices) and edges.

(**10.4**) In Section 5.1 we defined a map $\mathbf{b} : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ and a map d from binary sequences to TL-diagrams. It will be convenient to write $\mathcal{T}(a) = d(\mathbf{b}(a))$. We also defined Γ_a and Γ^a (for $a \in P(\mathbb{N})$, note). By construction we have

$$h^a = hh(a, \Gamma^a)$$

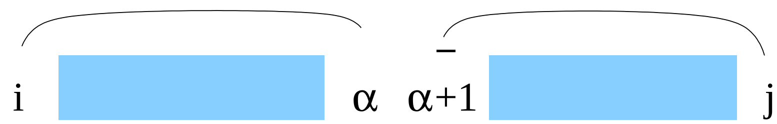
(with the understanding that if $\{i, j\}$ appears in Γ^a and is a subset of a then the edge operator is $\langle \underline{ij} \rangle$).

See Figure 12 for an example.

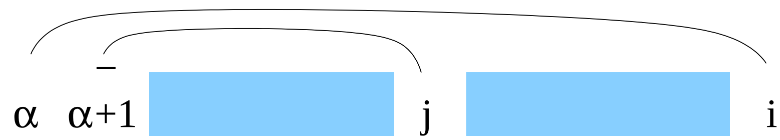
(**10.5**) **Lemma**. *Suppose* $\{\alpha, \alpha+1\} \in \Gamma_a$ *(so* $\langle \alpha \rangle a < a$*). Let* $\{\alpha\} \cup X$, $\{\alpha+1\} \cup Y$ *be parts in* $\mathcal{T}(\langle \alpha \rangle a)$ *(*X, Y *could contain a vertex or be empty). Then* $\mathcal{T}(a)$ *differs from* $\mathcal{T}(\langle \alpha \rangle a)$ *in that these parts are replaced by* $\{\alpha, \alpha+1\}$, $X \cup Y$ *(*$X \cup Y$ *may be empty).*

Proof: It is clear that $\{\alpha, \alpha+1\}$ is in $\mathcal{T}(a)$, so it remains to consider X, Y; and to show that all

other pairs agree between $\mathcal{T}(a)$ and $\mathcal{T}(\langle\alpha\rangle a)$.
If $X \cup Y = \emptyset$ then $\alpha, \alpha+1$ singletons in $\langle\alpha\rangle a$ and there are no pairs bridging over them, so no other pair is changed between $\langle\alpha\rangle a$ and a.
If $X = \{i\}$, $Y = \{j\}$ say, then $j \in \langle\alpha\rangle a$ (since $\alpha+1 \notin \langle\alpha\rangle a$ by construction). Suppose $j > \alpha+1$ and $i < \alpha$. Then we are in a situation like



By construction there are no 11 pairs in the i, α or $\alpha+1, j$ intervals. The algorithm for extracting the sequences in the shaded regions will thus operate in the same way for each sequence. In a the algorithm generates a pair at $\alpha, \alpha+1$ as already noted, so we may pass to an iteration where these and both shaded parts have been dealt with. Vertex i is not involved in a pair from below (else it would be in $\langle\alpha\rangle a$), and $j \in a$, so we get a pair $\{i, j\}$ as required.
Suppose $j > \alpha+1$ and $i > j$. Then we are in a situation like



The same argument goes through until noting that $\alpha, i \in \langle\alpha\rangle a$, so that there is an even number of 1s in the remainder sequence (algorithm stage 5) left of α. This even property still holds for a, so j is not involved in a pair from below. Again we have the required outcome.
The other cases are similar. □

10.2 Kazhdan–Lusztig polynomial Theorem

We continue to use labels $a \subset \mathbb{N}$ for alcoves. Thus the rows (and columns) of the parabolic Kazhdan–Lusztig polynomial array $P(\mathcal{D}/\mathcal{D}^+)$ may be indexed by these labels. That is, there is a polynomial $p_a(b) = p_{a,b}$, in the formal variable v, for each pair $a, b \in P_{even}(\mathbb{N})$. We write $p_a = \{p_{a,b}\}_{b \in P_{even}(\mathbb{N})}$ for the complete row of the array labelled by a.

Following on from (5.10) we define polynomial h^a_b by $h^a_b = v^i$ if b appears in hypercube h^a at depth i; and $h^a_b = 0$ if b does not appear in h^a.

(**10.6**) **Theorem**. *Let $a, b \subset \mathbb{N}$ label alcoves. The hypercube h^a gives the parabolic Kazhdan–Lusztig polynomials in the row p_a as follows:*

$$p_{a,b} = h^a_b$$

Proof: We work by induction on the graph order. We can then get the polynomials for a by looking at the polynomials for $\langle\alpha\rangle a$, where α labels one of the edges in the 'shoulder' of the hypercube associated to Γ_a. Specifically, by the definition of P in Section 8.3, Theorem 9.3, and the inductive assumption we need to determine all the dominant α images of vertices in $h^{\langle\alpha\rangle a}$.

For any α and $b \in P(\mathbb{N})$ let

$$\langle\alpha\rangle h^b := \{\langle\alpha\rangle c \mid c \in h^b;\ \langle\alpha\rangle c \text{ defined}\}$$

For example $\langle\alpha\rangle h^{\langle\alpha\rangle a} \ni a$ since $\langle\alpha\rangle\langle\alpha\rangle a = a$. Similarly let $\langle\alpha\rangle^2 h^b = \{c \mid c \in h^b;\ \langle\alpha\rangle c \text{ defined}\}$.

Note that there is a map $\langle\alpha\rangle h^{\langle\alpha\rangle a} \to \langle\alpha\rangle^2 h^{\langle\alpha\rangle a}$ given by $c \mapsto \langle\alpha\rangle c$, and that this is a bijection between disjoint sets.

By Section 8.3 (equation(18)) and Theorem 9.3 an alcove label c appears in p_a (i.e. polynomial $p_{a,c} \neq 0$) if there is a c' in $p_{\langle\alpha\rangle a}$ that, as a vertex of G_{even}, has an edge labelled α attached to it, and either $c = c'$ or $c = \langle\alpha\rangle c'$ (strictly speaking there is a subtraction to perform after equation(18), but we shall see that all such are null). The vertices of p_a will thus be those occuring in $\langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$, i.e. as a vertex set:

$$p_a \ \sim\ \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$$

Note that by the bijection and the inductive hypothesis every alcove label appears in at most one way, and hence that every polynomial will be of form v^i.

We need to check that this set of vertices p_a agrees with those of h^a, and that they aquire the right powers via this identification.

For any b define

$$\Gamma^b \setminus \alpha = \Gamma^b \setminus \{\alpha, \alpha{+}1\}$$

$$\Gamma^b(\alpha) = \{\{i,j\} \in \Gamma^b \mid \{i,j\} \cap \{\alpha, \alpha{+}1\} = \emptyset\}$$

Consider the 'ideal' $I_{\langle\alpha\rangle a}$ with vertices $c \leq \langle\alpha\rangle a$ in hypercube h^a. Note that this sub-hypercube has shoulder $\Gamma^a \setminus \alpha$; that is

$$I_{\langle\alpha\rangle a} = hh(\langle\alpha\rangle a, \Gamma^a \setminus \alpha) \tag{19}$$

and that the quotient of h^a by this ideal has the same shoulder set. Note also that this quotient $h^a/I_{\langle\alpha\rangle a}$ consists of the images under α of the vertices in $I_{\langle\alpha\rangle a}$, as exemplified in Figure 13.

It follows from Lemma 10.5 that $\Gamma^a \setminus \alpha$ agrees with the set $\Gamma^{\langle\alpha\rangle a}(\alpha)$ of pairs in $\Gamma^{\langle\alpha\rangle a}$ that do not intersect α or $\alpha + 1$, *except* that if there are pairs α, i and $\alpha{+}1, j$ in $\Gamma^{\langle\alpha\rangle a}$ then there will be a pair i, j in $\Gamma^a \setminus \alpha$ (that obviously does not appear in $\Gamma^{\langle\alpha\rangle a}$):

$$\Gamma^{\langle\alpha\rangle a}(\alpha) \ \subseteq\ \Gamma^a \setminus \alpha \tag{20}$$

From (19) and (20) we have that $hh(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ is a subgraph of $I_{\langle\alpha\rangle a}$ and hence of h^a (albeit one layer down from the 'head'), and also of $h^{\langle\alpha\rangle a}$.

As noted, all the vertices in the subgraph $hh(\langle\alpha\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ of $h^{\langle\alpha\rangle a}$ have α-images (and these images are above in the graph order). Thus all these vertices and images appear in p_a (by the inductive assumption $p_{\langle\alpha\rangle a} \equiv h^{\langle\alpha\rangle a}$ and the constructive definition of p_a from $p_{\langle\alpha\rangle a}$). The power of v for each image vertex is inherited from the original vertex (for example $p_a(a) = p_a(\langle\alpha\rangle\langle\alpha\rangle a) = p_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = v^0$), while the power of v for the original vertex is raised by 1 (example: $p_a(\langle\alpha\rangle a) = v p_{\langle\alpha\rangle a}(\langle\alpha\rangle a) = v v^0 = v^1$). We see, therefore, that *all these vertices have the correct exponent.*
The other vertices in the shoulder of $h^{\langle\alpha\rangle a}$ (the ones, if any, at the end of edges of form α, i and

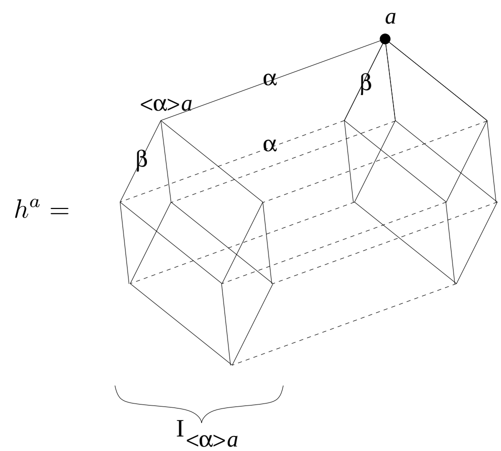


Figure 13:

$\alpha{+}1, j)$ do not have α-images. Thus we have agreement between h^a and $p_a \sim \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$ except for the ideal generated by $\langle ij\rangle a$ as above (if any) in h^a on the one hand; and the possible descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ that *do* have α-images on the other.

It remains to show that these contributions match up (with the correct powers).

If there is no such $\langle ij\rangle a$ then one can show that there are not descendents of $\langle\alpha, i\rangle\langle\alpha\rangle a$ and $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$ in $h^{\langle\alpha\rangle a}$ with α-images and we are done. So let us suppose there is $\langle ij\rangle a$ in h^a. Note that for our a we have

$$\langle ij\rangle a = \langle\alpha, i\rangle\langle\alpha{+}1, j\rangle\langle\alpha\rangle a \tag{21}$$

See Figure 14 for a representative example of this. We have there

$$a = \ 1\ 5\ 8\ 10\ 11\ 12 \stackrel{\langle 45\rangle}{\to} 1\ 4\ 8\ 10\ 11\ 12 \stackrel{\langle 5\ 12\rangle}{\to} 1\ 4\ 5\ 8\ 10\ 11 \stackrel{\langle 3\ 4\rangle}{\to} 1\ 3\ 5\ 8\ 10\ 11 \ = \langle 3\ 12\rangle a$$

A similar version works for $\langle \underline{ij}\rangle$ operators.

The $\langle ij\rangle a$ in $h^{\langle\alpha\rangle a}$ is in level $i = 2$ by (21), and has a hypercube $hh(\langle ij\rangle a, \Gamma^{\langle\alpha\rangle a}(\alpha))$ below it. All the elements of this hypercube have α-images, since $\langle\alpha\rangle, \langle ij\rangle$ commute. Note for example that $\langle ij\rangle a$ itself has an α-image (although $\langle ij\rangle a$ is below $\langle\alpha{+}1, j\rangle\langle\alpha\rangle a$, which does not have an α-image, in the *graph* order), and that its α-image $\langle\alpha\rangle\langle ij\rangle a$ is *below* it in the graph order. The other labels in the ideal behave similarly. Thus the polynomials asigned by Equation(18) to the relevant part of $p_a \sim \langle\alpha\rangle^2 h^{\langle\alpha\rangle a} \cup \langle\alpha\rangle h^{\langle\alpha\rangle a}$ are, for v^i the relevant polynomial from $p_{\langle\alpha\rangle a}$, v^i (for the α-image) and v^{i-1} (the vertex 'left behind') respectively. The -1 compensates for the fact that the vertex appears in $h^{\langle\alpha\rangle a}$ one layer lower than in h^a (where it appears in the shoulder in the case of $\langle ij\rangle a$ itself for example), so subject to the working assumptions we verify $p_a \equiv h^a$.

Note finally that this -1 increment only occurs for the vertex $\langle ij\rangle a$ and those below it, and thus for polynomials v^i with exponent $i \geq 2$. Thus we never have an increment of form $v^1 \to v^{1-1} = v^0$ (which would incur a subtraction in the polynomial construction). The only remaining working assumption is the inductive assumption.

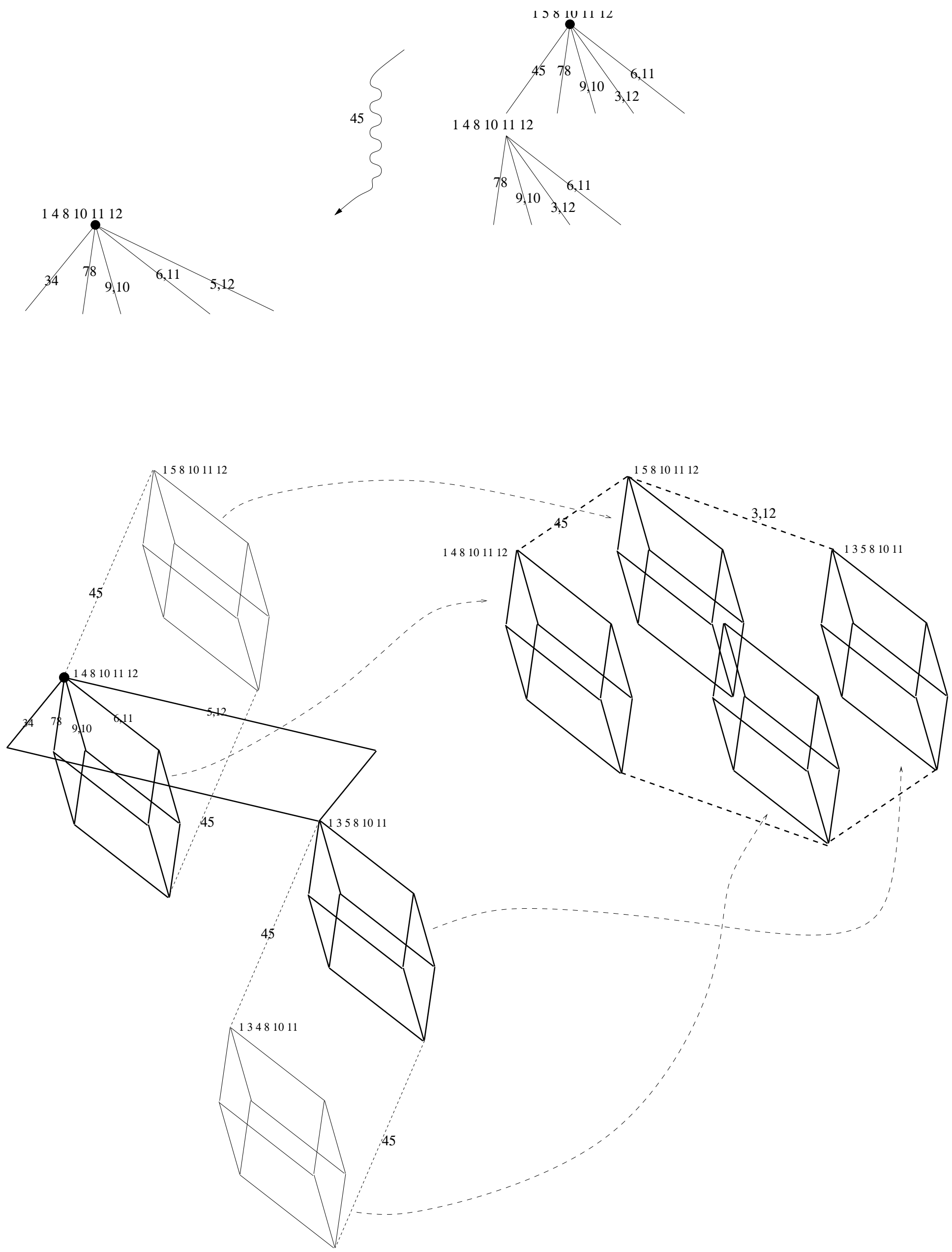


Figure 14: A representative example. The top part of the figure shows the shoulder of h^a for $a = \{1, 5, 8, 10, 11, 12\}$; and the shoulder of the ideal below $\langle 45 \rangle a = \{1, 4, 8, 10, 11, 12\}$ within h^a. Immediately below-left of this is the shoulder of $h^{\langle 45 \rangle a}$ itself (showing that this hypercube is bigger than the corresponding ideal within h^a). The bottom-left part of the figure shows all of the vertices in $h^{\langle 45 \rangle a}$ that have $\langle 45 \rangle$ images (and a couple which do not, that are relevant for the construction); together with a representation of those images. The bottom-right part shows how all these vertices may be collected together to constitute the vertices of h^a.

□

Concluding remarks. As already noted, a significant mathematical application of this work is hoped to be as a base for corresponding calculations over fields of finite characteristic (cf. [7, §6]). A *physically* motivated application is in computing eigenvectors of the Young matrix (the adjacency matrix of the Young graph [17]), which are involved in certain quantum spin chain computations (see e.g. [5]). We note that formal connections between parabolic Kazhdan–Lusztig polynomials and Brauer algebra decomposition matrices can be constructed in principle by other approaches, such as in [20]. However such formal approaches do not give access to the specific decomposition numbers that we compute here (and which are required for the applications mentioned). Finally we note that [13] includes formulations of Kazhdan–Lusztig polynomials related to the $\mathcal{D}/\mathcal{D}_+$ case, considered from an entirely different perspective.

Acknowledgements. I thank Anton Cox and Maud De Visscher for many useful discussions and suggestions; Robert Marsh for useful conversations; and Catharina Stroppel for useful comments on the first version of the paper. I thank Paula Martin for support.

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