SYMBOLIC AND ALGEBRAIC DYNAMICAL SYSTEMS

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Part 1. Introduction

This article gives a brief survey of a class of dynamical systems which contains shifts of finite type (both one- and multi-dimensional), as well as actions of one or more commuting automorphisms of a compact abelian group. Here we use the term 'dimension' to refer to the rank of the parameter group for the action, not to the space itself. The connection between classical (one-dimensional) shifts of finite type and automorphisms of compact abelian groups via Markov partitions is classical and well understood (cf. [2], [5], [61], and [40, Chap. 6]). A perhaps less classical connection between the two was pointed out in [33] and [57], where expansive automorphisms of compact abelian groups (and, more generally, expansive $\mathbb{Z}^d$-actions by such automorphisms) are written as shifts of finite type whose alphabet is a finite-dimensional torus. This point of view, combined with tools from commutative algebra, allows a systematic treatment of $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups and leads to a fairly comprehensive theory of such actions. The wealth of interesting and unexpected properties exhibited by such 'algebraic' $\mathbb{Z}^d$-actions shows that multi-parameter ergodic theory (i.e. the study of $\mathbb{Z}^d$- and $\mathbb{R}^d$-actions) is much more than an elementary extension of the classical theory of $\mathbb{Z}$-actions and flows.

Although this connection between algebraic $\mathbb{Z}^d$-actions and d-dimensional shifts of finite type with possibly infinite alphabet has benefited the study of algebraic $\mathbb{Z}^d$-actions, its impact on the theory of higher-dimensional shifts of finite type has been restricted to a few intriguing examples, suggestions and conjectures. Nevertheless the insight into $\mathbb{Z}^d$-actions gained in the algebraic context has helped to stimulate a number of recent publications on higher-dimensional shifts of finite type (cf. [45]–[47], [48]–[49], [64]).

1.1. Notation

Throughout this article the term *compact space* will mean a compact metrizable space. Any *metric* on a such compact space will be assumed to be compatible with its topology. If X is a compact space we write $\mathcal{B}_X$ for the Borel sigma-algebra of X. A probability measure μ on $\mathcal{B}_X$ will simply be called a probability measure *on* X.

A Borel space $(Y, \mathcal{T})$ is *standard* if it is Borel isomorphic to $(X, \mathcal{B}_X)$ for some compact space X. A measure space $(Y, \mathcal{T}, \mu)$ is *standard* if $(Y, \mathcal{T})$ is a standard Borel space and μ is a sigma-finite measure on $\mathcal{T}$.

As usual we write $\mathbb{Z}$ for the integers, $\mathbb{N} = \{0, 1, \dots\}$ for the natural numbers, $\mathbb{Q}$ for the rationals, $\mathbb{R}$ for the reals, $\mathbb{C}$ for the complex numbers and set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Although the multiplicative group

$$\mathbb{S} = \{c \in \mathbb{C} : |c| = 1\}$$

of complex numbers of absolute value 1 is isomorphic to $\mathbb{T}$ it will be convenient to distinguish notationally between these groups.

For any set S we write $|S|$ for the cardinality of S. If $S' \subset S$ is a subset, then $1_{S'} \colon S \to \mathbb{R}$ denotes the indicator function of S'.

1.2. Dynamical properties of $\mathbb{Z}^d$-actions

Let X be a compact space, δ a metric on X, $d \geq 1$ and $T\colon \mathbf{n} \mapsto T^{\mathbf{n}}$ a continuous action of $\mathbb{Z}^d$ on X. A probability measure μ on X is (T-)*invariant* if $\mu T^{\mathbf{n}} = \mu$ for every $\mathbf{n} \in \mathbb{Z}^d$.

If T' is a second continuous $\mathbb{Z}^d$-action on a compact space X' then T and T' are *topologically conjugate* if there exists a homeomorphism $\phi\colon X \to X'$ with

$$\phi \cdot T^{\mathbf{n}} = T'^{\mathbf{n}} \cdot \phi \tag{1.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. If the map $\phi\colon X \to X'$ in (1.1) is only surjective then T' is a *topological factor* of T.

Other notions of conjugacy will appear later in this article.

The following dynamical properties of a continuous $\mathbb{Z}^d$-action T on a compact space X are obviously preserved under topological conjugacy.

Definition 1.1. (1) A point $x \in X$ is *periodic* if its orbit $\{T^{\mathbf{n}}x : \mathbf{n} \in \mathbb{Z}^d\}$ is finite. For every subgroup of finite index $\Lambda \subset \mathbb{Z}^d$ we denote by

$$\mathrm{Fix}_\Lambda(T) = \{x \in X : T^{\mathbf{n}}x = x \ \text{ for every } \ \mathbf{n} \in \Lambda\} \tag{1.2}$$

the set of points in X with *period* Λ. The action T has *dense periodic points* if

$$\mathrm{Per}(T) = \bigcup_{\substack{\Lambda \subset \mathbb{Z}^d \\ |\mathbb{Z}^d/\Lambda| < \infty}} \mathrm{Fix}_\Lambda(T)$$

is dense in X. The action T has *finitely many periodic points of any period* if $|\mathrm{Fix}_\Lambda(T)| < \infty$ for every subgroup of finite index $\Lambda \subset \mathbb{Z}^d$.

(2) A point x is *transitive* if $\{T^{\mathbf{n}}x : \mathbf{n} \in \mathbb{Z}^d\}$ is dense in X, and *totally transitive* if $\bigcap_{K \geq 0} \{T^{k\mathbf{n}}x : k \geq K\}$ is dense in X for every nonzero $\mathbf{n} \in \mathbb{Z}^d$.

The action T is (*totally*) *transitive* if it has a (totally) transitive point. The action T is *topologically mixing* if, for any pair $\mathcal{O}_1, \mathcal{O}_2$ of nonempty open sets in X, $\mathcal{O}_1 \cap T^{\mathbf{n}}(\mathcal{O}_2) \neq \varnothing$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$. An elementary argument shows at, for any topologically mixing action T on X, the set of totally transitive points is a dense G_δ in X.

Definition 1.2. The action T is *expansive* if there exists a constant $c > 0$ such that

$$\sup_{\mathbf{m} \in \mathbb{Z}^d} \delta(T^{\mathbf{m}}x, T^{\mathbf{m}}y) > c \tag{1.3}$$

for all pairs of points $x \neq y$ in X. Any $c > 0$ satisfying (1.3) is called an *expansive constant* of T. If T is expansive it obviously has finitely many periodic points of any given period.

Definition 1.3. (1) The action T has *weak specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \geq 1$ with the following property: for every finite collection $Q_1, \dots, Q_t$ of rectangles $Q_j = \prod_{i=1}^d \{a_i, \dots, b_i\} \subset \mathbb{Z}^d$ with

$$\mathrm{dist}(Q_j, Q_k) = \min_{\mathbf{m} \in Q_j,\, \mathbf{n} \in Q_k} \|\mathbf{m} - \mathbf{n}\| \geq p(\varepsilon) \quad \text{for } 1 \leq j < k \leq t, \tag{1.4}$$

and for every collection of points $x^{(1)}, \dots, x^{(t)}$ in X, there exists a point $y \in X$ with

$$\delta(T^{\mathbf{n}} y, T^{\mathbf{n}} x^{(j)}) < \varepsilon \quad \text{for all } \mathbf{n} \in Q_j,\ 1 \le j \le t. \tag{1.5}$$

(2) The $\mathbb{Z}^d$-action T has *strong specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \ge 1$ with the following property: for every collection of rectangles $Q_1, \dots, Q_t$ in $\mathbb{Z}^d$ satisfying (1.4) and every subgroup $\Lambda \subset \mathbb{Z}^d$ with

$$\operatorname{dist}(Q_j + \mathbf{q}, Q_k) = \min_{\mathbf{m} \in Q_j + \mathbf{q},\, \mathbf{n} \in Q_k} \|\mathbf{m} - \mathbf{n}\| \ge p(\varepsilon) \tag{1.6}$$

whenever $1 \le j, k \le t$ and $\mathbf{q} \in \Lambda \smallsetminus \{\mathbf{0}\}$, and for every collection of points $x^{(1)}, \dots, x^{(t)}$ in X, there exists a point $y \in X$ satisfying (1.5) and with

$$T^{\mathbf{m}} y = y$$

for every $\mathbf{m} \in \Lambda$.

Our next task is the definition of topological entropy. If $E \subset \mathbb{Z}^d$ is a nonempty finite set and $\varepsilon > 0$ we call a set $Y \subset X$ (E, ε)-*spanning* if there exists, for every $x \in X$, an element $y \in Y$ with $\delta(T^{\mathbf{n}} x, T^{\mathbf{n}} y) < \varepsilon$ for every $\mathbf{n} \in E$. The set $Y \subset X$ is (E, ε)-*separated* if $\sup_{\mathbf{n} \in E} \delta(T^{\mathbf{n}} y, T^{\mathbf{n}} y') > \varepsilon$ for all pairs of points $y \ne y'$ in Y. We set

$$r_E(\varepsilon) = \min\{|Y| : Y \subset X \text{ is } (E, \varepsilon)\text{-spanning}\},$$
$$s_E(\varepsilon) = \max\{|Y| : Y \subset X \text{ is } (E, \varepsilon)\text{-separated}\},$$

and observe that $r_E(\varepsilon) \le s_E(\varepsilon) < \infty$ by compactness. Finally, if $\mathcal{U}, \mathcal{V}$ are covers of X by open sets we put

$$N(\mathcal{U}) = \min\{|\mathcal{U}'| : \mathcal{U}' \subset \mathcal{U} \text{ is a cover of } X\},$$
$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U},\, V \in \mathcal{V}\}.$$

Definition 1.4. For every $M \ge 0$ we set

$$C_M = \{0, \dots, M-1\}^d \subset \mathbb{Z}^d. \tag{1.7}$$

Let

$$h_{\text{span}}(T) = \sup_{\varepsilon > 0} \limsup_{M \to \infty} \frac{1}{|C_M|} \log r_{C_M}(\varepsilon),$$
$$h_{\text{sep}}(\sigma_X) = \sup_{\varepsilon > 0} \limsup_{M \to \infty} \frac{1}{|C_M|} \log s_{C_M}(\varepsilon),$$
$$h_{\text{top}}(T) = \sup_{\mathcal{U}} \limsup_{M \to \infty} \frac{1}{|C_M|} \log N\Bigl(\bigvee_{\mathbf{n} \in C_M} T^{-\mathbf{n}} \mathcal{U}\Bigr).$$

Then $h_{\text{span}}(T) = h_{\text{sep}}(T) = h_{\text{top}}(T)$, and this common value is the *topological entropy* of T, denoted by $h(T)$.

Definition 1.5. A T-invariant probability measure μ on X is *ergodic* under T (or T is ergodic on $(X, \mathcal{B}_X, \mu)$) if whenever $E \in \mathcal{B}_X$ satisfies $T^{\mathbf{n}} E = E$ for all $\mathbf{n} \in \mathbb{Z}^d$ then $\mu(E) = 0$ or 1. If $r \ge 2$ is an integer, we say that μ is *mixing of order* r if

$$\lim_{k \to \infty} \mu(B_1 \cap T^{\mathbf{n}_k^{(2)}}(B_2) \cap \dots \cap T^{\mathbf{n}_k^{(r)}}(B_r)) = \mu(B_1) \cdots \mu(B_r) \tag{1.8}$$

for every choice of Borel sets $B_1, \dots, B_r$ in X and every sequence $((\mathbf{0} = \mathbf{n}_k^{(1)}, \dots, \mathbf{n}_k^{(r)}), k \geq 1)$ in $(\mathbb{Z}^d)^r$ with $\lim_{k\to\infty} \mathbf{n}_k^{(j)} - \mathbf{n}_k^{(i)} = \infty$ for $1 \leq i < j \leq r$. For $r = 2$ we obtain the usual definition of mixing.

Definition 1.6. Let μ be a T-invariant probability measure on X. The *metric entropy* $h_\mu(T)$ is defined by

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}) = \sup_{\mathcal{P}} \limsup_{M\to\infty} \frac{1}{|C_M|} H_\mu\Big(\bigvee_{\mathbf{n}\in C_M} T^{-\mathbf{n}}\mathcal{P}\Big), \tag{1.9}$$

where the supremum is taken over all finite Borel partitions $\mathcal{P}$ of X, and where $H_\mu(\mathcal{Q}) = -\sum_{Q\in\mathcal{Q}} \mu(Q) \log \mu(Q)$ is the μ-entropy of a finite partition $\mathcal{Q} \subset \mathcal{B}_X$. The measure μ has *completely positive entropy* if $h_\mu(T, \mathcal{P}) > 0$ for every nontrivial finite partition $\mathcal{P}$ of X (*nontrivial* means that $\mu(P) < 1$ for every $P \in \mathcal{P}$).

The *variational principle* states that

$$h(T) = \max_\mu h_\mu(T), \tag{1.10}$$

where the maximum is taken over all T-invariant probability measures μ on X (the existence of this maximum is part of the variational principle — cf. e.g. [44]). A T-invariant probability measure μ on X has *maximal entropy* if $h_\mu(T) = h(T)$.

Definition 1.7. Two measure-preserving $\mathbb{Z}^d$-actions T and T' on standard probability spaces $(X, \mathcal{S}, \mu)$ and $(X', \mathcal{S}', \mu')$ are measurably conjugate if there exists a measure space isomorphism $\phi\colon (X, \mathcal{S}, \mu) \to (X', \mathcal{S}', \mu')$ satisfying (1.1) μ-a.e. for every $\mathbf{n} \in \mathbb{Z}^d$. A measure-preserving $\mathbb{Z}^d$-action T on a probability spaces $(X, \mathcal{S}, \mu)$ is *Bernoulli* if it is measurably conjugate to a $\mathbb{Z}^d$-action T' on a probability space $(X', \mathcal{S}', \mu')$ of the following form: there exists a standard probability space $(Y, \mathcal{T}, \nu)$ such that $X' = Y^{\mathbb{Z}^d}$, $\mathcal{S}' = \mathcal{T}^{\mathbb{Z}^d}$ is the product Borel field on X', $\nu = \mu^{\mathbb{Z}^d}$ is the product measure, and T' is the shift-action on $\mathbb{Z}^d$ on X' given by

$$({T'}^{\mathbf{n}} x)_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{m}}) \in X'$ (cf. (2.1)).

1.3. Expansive subdynamics

Recently a framework for understanding expansive $\mathbb{Z}^d$-actions T has been developed called "expansive subdynamics" [10]. Roughly speaking, a given subspace V of $\mathbb{R}^d$ the action T may remain "expansive" when restricting to iterates in $\mathbb{Z}^d$ that lie within a bounded distance of V. The set of such expansive subspaces with a fixed dimension forms an open set. Within a connected component of this set the dynamical properties of T remain fixed or vary nicely, while they generally change rather abruptly when passing from one component to anther, analogous to a "phase transition." Hence one approach to understanding expansive $\mathbb{Z}^d$-actions is to identify the expansive components, and then describe the dynamics within each component.

To make these notions precise, let $\mathbb{G}_k$ denote the compact Grassman manifold of k-dimensional subspaces (or *k-spaces*) of $\mathbb{R}^d$. Say that $V \in \mathbb{G}_k$ is *expansive for T* if there are $c > 0$ and $r > 0$ such that if $x \neq y$ then

$\sup\{\delta(T^{\mathbf{n}}x, T^{\mathbf{n}}y) : \mathrm{dist}(\mathbf{n}, V) \le r\} > c$. Let $E_k(T)$ denote the set of expansive k-spaces for T, and $N_k(T) = \mathbb{G}_k \smallsetminus E_k(T)$.

Examples 1.8.

(1) Let

$$X = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{m,n} + x_{m+1,n} + x_{m,n+1} = 0 \text{ for all } m, n \in \mathbb{Z}\}$$

and T be the $\mathbb{Z}^2$ action on X generated by the horizontal and vertical shifts (see also Example 2.4(1)). Then $N_1(T)$ consists of the three lines parallel to the faces of the unit simplex in $\mathbb{R}^2$ [10, Example 2.8].

(2) Consider the 3-dimensional version of (1). Let $\mathbf{e}_j$ denote the standard unit vectors in $\mathbb{R}^3$, put

$$X = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^3} : x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} + x_{\mathbf{n}+\mathbf{e}_3} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^3\},$$

and let T be the $\mathbb{Z}^3$-action generated by the shifts in each of the coordinate directions. Then $N_2(T)$ is topologically the union of six arcs [10, Example 2.9].

(3) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$

Then A and B are commuting automorphisms of $X = \mathbb{T}^4$ inducing a $\mathbb{Z}^2$-action T. Here $N_1(T)$ consists of exactly two lines, both irrational [10, Example 2.10]. This example emphasises the necessity of considering irrational subspaces of $\mathbb{R}^d$, a point made explicitly by Katok and Spatzier [26] in their work on Weyl chambers and invariant measures for smooth hyperbolic $\mathbb{Z}^d$-actions.

Except for the trivial case when X is finite (where all subspaces are expansive), the expansive subdynamics of any action is nontrivial [10, Thm. 3.6].

Theorem 1.9. *Let X be an infinite compact metric space and T be a $\mathbb{Z}^d$-action on X. Then $N_{d-1}(T)$ is a nonempty compact subset of $\mathbb{G}_{d-1}$. A k-space is in $N_k(T)$ if and only if it is a subspace of some $(d-1)$-space in $N_{d-1}(T)$.*

Thus the entire set of nonexpansive subspaces is completely determined by those of co-dimension one.

Which nonempty compact sets of $\mathbb{G}_{d-1}$ can arise as some $N_{d-1}(T)$? Although the answer for general d is not known, when $d = 2$ there is almost a complete answer [10, Prop. 4.1].

Proposition 1.10. *Let $d = 2$ and K be a nonempty compact subset of $\mathbb{G}_1$ that is not a singleton set consisting of one irrational line. Then there is a $\mathbb{Z}^2$-action T for which $N_1(T) = K$.*

The following question remains open.

Problem 1.11. Let L be an irrational line in $\mathbb{R}^2$. Does there exist a $\mathbb{Z}^2$-action T such that $N_1(T) = \{L\}$?

A *k-frame* is an ordered k-tuple of linearly independent vectors in $\mathbb{R}^d$. Let $\mathbb{F}_k$ denote the set of all k-frames, equipped with the topology inherited from $(\mathbb{R}^d)^k$. Define $s_k \colon \mathbb{F}_k \to \mathbb{G}_k$ by sending a k-frame to the k-space it spans. An *expansive component* for T is a connected component of the open set $s_k^{-1}(E_k(T))$. Within an expansive component dynamical properties of T such an entropy (both topological and measure-theoretic), zeta function, and being Markov are constant or vary nicely. For example, measure-theoretic entropy is given by a k-form within an expansive component of k-frames. However, abrupt changes in such properties typically occur at the boundaries of expansive components. For further examples and problems see [10]. If T is a $\mathbb{Z}^d$-action defined by commuting automorphisms of a finite-dimensional torus, the connected components of $s_{d-1}^{-1}(E_{d-1}(T))$ are precisely the *Weyl chambers* of T considered in [26].

Part 2. Multi-dimensional Markov systems

In this part we introduce a general class of $\mathbb{Z}^d$-actions which contains the majority of examples of such actions in the literature.

2.1. Markov systems and shifts of finite type

Let $\mathcal{A}$ be a compact set, the *alphabet*, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the space of all maps $x\colon \mathbb{Z}^d \to \mathcal{A}$. For every $E \subset \mathbb{Z}^d$ we denote by $\pi_E\colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^E$ the projection $x \mapsto \pi_E(x) = x|_E$, where $x|_E$ is the restriction to E of the map $x\colon \mathbb{Z}^d \to \mathcal{A}$. If $x \in \mathcal{A}^{\mathbb{Z}^d}$ then $x_{\mathbf{n}}$ is the $\mathbf{n}$-th coordinate of x, i.e. the value of x at $\mathbf{n}$. The space $\mathcal{A}^{\mathbb{Z}^d}$ is compact in the product topology and we define, for every $\mathbf{m} \in \mathbb{Z}^d$, a homeomorphism $\sigma^{\mathbf{m}}$ of $\mathcal{A}^{\mathbb{Z}^d}$ by setting

$$(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}} \tag{2.1}$$

for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-action $\mathbf{m} \mapsto \sigma^{\mathbf{m}}$ is the *shift-action* on $\mathcal{A}^{\mathbb{Z}^d}$. A subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is *shift-invariant* if $\sigma^{\mathbf{m}}(X) = X$ for every $\mathbf{m} \in \mathbb{Z}^d$. A closed, shift-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is called a *subshift* of $\mathcal{A}^{\mathbb{Z}^d}$. The restriction of σ to a subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ will be denoted by σ_X.

If $\mathcal{A}$ and $\mathcal{A}'$ are compact sets and $X \subset \mathcal{A}^{\mathbb{Z}^d}$, $X' \subset \mathcal{A}'^{\mathbb{Z}^d}$ are subshifts then X and X' are *topologically conjugate* (or X' is a *factor* of X) if the shift-actions σ_X and $\sigma_{X'}$ are topologically conjugate (or $\sigma_{X'}$ is a topological factor of σ_X). Topologically conjugate subshifts will be regarded as essentially the same throughout this article.

Definition 2.1. A subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is *Markov* if there exist a nonempty finite subset $F \subset \mathbb{Z}^d$ and a closed subset $P \subset \mathcal{A}^F$ such that

$$X = X_{(F,P)} = \{x \in \mathcal{A}^{\mathbb{Z}^d} : \pi_F \cdot \sigma^{\mathbf{m}} x \in P \text{ for every } \mathbf{m} \in \mathbb{Z}^d\}. \tag{2.2}$$

The set P in (2.2) is the collection of *permissible* (or *allowed*) words. More generally, we denote by

$$\Pi_{(F,P)}(E) = \{x \in \mathcal{A}^E : \pi_{(E-\mathbf{m})\cap F}(x) \in \pi_{(E-\mathbf{m})\cap F}(P) \text{ for every } \mathbf{m} \in \mathbb{Z}^d\} \tag{2.3}$$

the set of *allowed configurations* on an arbitrary subset $E \subset \mathbb{Z}^d$. In this notation we have that $X_{(F,P)} = \Pi_{(F,P)}(\mathbb{Z}^d)$.

Note that a closed shift-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is Markov if and only if there exists a finite set $F \subset \mathbb{Z}^d$ such that

$$X = X^{(F)} = \{x \in \mathcal{A}^{\mathbb{Z}^d} : \pi_F(x) \in \pi_F(X) \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}. \tag{2.4}$$

In order to verify the equivalence of the definitions (2.2) and (2.4) it suffices to set $P = \pi_F(X)$ and to observe that $X^{(F)} = X_{(F,P)}$ in the sense of (2.2).

If the alphabet $\mathcal{A}$ is finite then a Markov shift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is called a *shift of finite type.*

Finally, if $\mathcal{A}$ is finite and $X \subset \mathcal{A}^{\mathbb{Z}^d}$ a subshift, then X is a *sofic shift* if it is a factor of a shift of finite type.

By using topological conjugacy we can simplify the definition of a Markov system slightly.

Lemma 2.2. *Let $\mathcal{A}$ be a finite alphabet, $F \subset \mathbb{Z}^d$ a finite set, $P \subset \mathcal{A}^F$, $X = X_{(F,P)}$ the shift of finite type defined by* (2.2), *and let*

$$F' = \{0,1\}^d \subset \mathbb{Z}^d. \tag{2.5}$$

Then there exist a finite alphabet $\mathcal{A}'$ and a subset $P' \subset \mathcal{A}'^{F'}$ such that the shifts of finite type $X_{(F,P)} \subset \mathcal{A}^{\mathbb{Z}^d}$ and $X_{(F',P')} \subset \mathcal{A}'^{\mathbb{Z}^d}$ are topologically conjugate.

Proof. There exist an integer $M \geq 1$ and an element $\mathbf{n} \in \mathbb{Z}^d$ with $F + \mathbf{n} \subset \{0, \dots, M\}^d$. Put $\mathcal{A}' = \pi_{\{0,\dots,M-1\}^d}(X)$ and define a continuous injective map $\phi' \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}'^{\mathbb{Z}^d}$ by setting

$$\phi'(x)_{\mathbf{n}} = \pi_{\{0,\dots,M-1\}^d}(x)$$

for every $\mathbf{n} \in \mathbb{Z}^d$. Then ϕ' satisfies (1.1), and

$$X' = \phi'(X) = X'^{(F')} \subset \mathcal{A}'^{\mathbb{Z}^d}$$

with F' given by (2.5). □

Note that every continuous $\mathbb{Z}^d$-action T on a compact space X is trivially conjugate to the shift-action of $\mathbb{Z}^d$ on the Markov system $X_{(F',P)}$, where $\mathcal{A} = X$, $F' \subset \mathbb{Z}^d$ is defined by (2.5) and $P = (T^{\mathbf{n}}x,\, x \in X,\, \mathbf{n} \in F)$. However, if we impose certain restrictions on the alphabet $\mathcal{A}$, such as finiteness or finite-dimensionality, then Definition 2.1 becomes much more useful: one important class of examples — the shifts of finite type — arises when $\mathcal{A}$ is finite. In Part 5 we shall be interested in the case where $\mathcal{A}$ is a finite-dimensional torus and $P \subset \mathcal{A}^F$ a closed subgroup.

The general philosophy behind the definition of Markov systems is the following: whereas there are many examples of smooth or 'geometric' $\mathbb{Z}$-actions on 'small' spaces (such as finite-dimensional manifolds), every such action of $\mathbb{Z}^d$, $d > 1$, on a finite-dimensional manifold must have zero entropy, since the individual elements of $\mathbb{Z}^d$ all have finite entropy. It follows that all sufficiently 'chaotic' actions of $\mathbb{Z}^d$ with $d > 1$ must either live on very large spaces or lack smoothness.

The definition of a Markov system can help to overcome this problem to some extent: if $\mathcal{A}$ and $P \subset \mathcal{A}^F$ are, for example, finite-dimensional manifolds, then the Markov system $X_{(F,P)}$ is in a sense given by smooth data, but the shift-action of $\mathbb{Z}^d$ on $X_{(F,P)}$ can have positive entropy.

However, there is a price to pay: it is usually quite difficult (and, in general, undecidable) to determine even the most elementary properties of $X_{(F,P)}$ from these initial data, even in the case where the alphabet $\mathcal{A}$ is finite. The most notorious of these difficulties is the following: there is no algorithm which determines, given a finite alphabet $\mathcal{A}$, a nonempty finite set $F \subset \mathbb{Z}^d$ and a nonempty set $P \subset \mathcal{A}^F$, whether the space $X_{(F,P)}$ in (2.2) is nonempty or not ([4], [50], [67]).

This undecidability problem is closely related to the existence of nonempty shifts of finite type $X_{(F,P)}$ without periodic points (cf. Definition 1.1 and Example 2.6 (2)). Suppose for simplicity that $d = 2$, $F = \{0,1\}^2 \subset \mathbb{Z}^2$ and $P \subset \mathcal{A}^F$. If every nonempty shift of finite type contained a periodic point, we would have the following algorithm for deciding whether $X_{(F,P)}$ is nonempty:

for every $n \geq 1$, consider the set of allowed configurations $\Pi_{(F,P)}(Q^{(n)}) \subset \mathcal{A}^{Q^{(n)}}$ (cf. (1.7)). Then we can find an $n \geq 1$ for which exactly one of the following possibilities holds: either there exists an allowed configuration $y \in \Pi_{(F,P)}(Q^{(n)})$ which is periodic (in the sense that the restrictions of y to the left and right (resp. top and bottom) edges of $Q^{(n)}$ match), or $\Pi_{(F,P)}(Q^{(n)}) = \varnothing$. In the former case $X_{(F,P)} \neq \varnothing$, and in the latter case $X_{(F,P)} = \varnothing$.

For concrete examples of higher-dimensional shifts of finite type (e.g. for those arising in statistical mechanics) it is usually easy to check that the space is nonempty. However, the undecidability problem mentioned above is an indication of the difficulty of obtaining general mathematical statements about higher-dimensional Markov systems and shifts of finite type.

We conclude this section with a list of examples which will serve both as an illustration of the concept of a Markov system and as an introduction to the topics of the following sections.

2.2. Examples

We begin with some examples of Markov systems.

Examples 2.3 (*One-dimensional Markov systems*)**.**

(1) Let $\mathcal{A}$ be a compact set, $f\colon \mathcal{A} \to \mathcal{A}$ a continuous map, $F = \{0,1\}$ and $P = \{(a, f(a)) : x \in \mathcal{A}\} \subset \mathcal{A} \times \mathcal{A} = \mathcal{A}^F$ the graph of f. The shift σ on $X_{(F,P)}$ is topologically conjugate to the restriction of f to $Y = \bigcap_{n\geq 1} f^n(\mathcal{A})$. For example, let $\mathcal{A} = \mathbb{T}$, $F = \{0,1\}$ and $P = \{(3t, 2t) : t \in \mathbb{T}\}$. Then

$$X_{(F,P)} = \{x \in \mathbb{T}^{\mathbb{Z}} : 3x_n = 2x_{n+1} \text{ for every } n \in \mathbb{Z}\},$$

and the shift σ on $X_{(F,P)}$ corresponds to 'multiplication by $\frac{3}{2}$' on $\mathbb{T}$, made into a group automorphism. In Part 5 we shall see that σ on $X_{(F,P)}$ is expansive, mixing, and that $h(\sigma) = h_\lambda(\sigma) \log 3$, where λ is the normalized Haar measure of $X_{(F,P)}$ (cf. Definitions 1.4 and 1.6). Further examples of this kind and their dynamical properties will be investigated in Part 5.

(2) Let $\mathcal{A}$ be a compact manifold, $F = \{0,1\}$, $P \subset \mathcal{A}^2$ a submanifold, and let $\pi_i\colon \mathcal{A}^2 \to \mathcal{A}$, $i = 1,2$, be the projection onto the i-th coordinate. Even under the additional hypothesis that the maps $\pi_i\colon P \to \mathcal{A}$ are surjective and bounded-to-one there appears to be no systematic analysis of the shift σ on the space $X_{(F,P)}$ in the literature.

Examples 2.4 (*Two-dimensional Markov systems*)**.**

(1) Let $\mathcal{A} = \mathbb{T}$, $F = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ and

$$P = \{(t_{(0,0)}, t_{(1,0)}, t_{(0,1)}) \in \mathbb{T}^F : t_{(0,0)} + t_{(1,0)} + t_{(0,1)} = 0\}.$$

Then

$$\begin{aligned} X_{(F,P)} = \{x \in \mathbb{T}^{\mathbb{Z}^2} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = 0 \\ \text{for every } (m_1,m_2) \in \mathbb{Z}^2\}. \end{aligned} \tag{2.6}$$

The shift-action σ of $\mathbb{Z}^2$ on $X_{(F,P)}$ will be discussed in Part 5.

If one replaces $\mathcal{A} = \mathbb{T}$ in (2.6) by the group $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ one obtains *Ledrappier's example* of a mixing shift of finite type which is not 3-mixing (cf. [37]). Again we refer to Part 5 for details.

(2) One can generalize Example (1) further and set $\mathcal{A} = \mathrm{SU}(2)$, the group of unitary 2×2-matrices, $F = \{(0,0),(1,0),(0,1)\}$ and

$$P = \{(g_{(0,0)}, g_{(1,0)}, g_{(0,1)}) \in \mathrm{SU}(2)^F : g_{(0,0)} \cdot g_{(1,0)} \cdot g_{(0,1)} = 1\},$$

where 1 denotes the identity element in SU(2). Very little appears to be known about the shift-action on $X_{(F,P)}$ (cf. [57], p. xi).

More generally, let $\mathcal{A}$ be a compact manifold, $F = \{(0,0),(1,0),(0,1)\}$ and $P \subset \mathcal{A}^F$ a submanifold such that the restrictions to P of the three coordinate projections $\pi_{\mathbf{n}} \colon \mathcal{A}^F \to \mathcal{A}$, $\mathbf{n} \in F$, are surjective and bounded-to-one. What can be said about the resulting shift-action σ of $\mathbb{Z}^2$ on $X_{(F,P)}$?

We next turn to some examples of shifts of finite type.

Example 2.5 (*One-dimensional shifts of finite type*).

Let $d = 1$ and $\mathcal{A} = \{1,\dots,r\}$ for some $r \geq 2$. If $X = X_{(F,P)} \subset \mathcal{A}^{\mathbb{Z}}$ is a shift of finite type then Lemma 2.2 allows us to assume without loss in generality that $F = \{0,1\}$. In this case $P \subset \mathcal{A}^F$ may be written as an $r \times r$ matrix $P = (P(i,j)$ of 0's and 1's, with $P(i,j) = 1$ if and only if $(i,j) \in P$. For reasons of economy we assume that $\mathcal{A}$ is as small as possible, i.e. that P has no zero rows or columns.

An elementary classical argument shows that X is transitive if and only if there exists, for every $(i,j) \in \mathcal{A}^2$, a nonzero integer $m = m(i,j)$ with $P^m(i,j) > 0$, where P^m is the m-th matrix power of P. The shift of finite type X is totally transitive if we can choose $m(i,j) > 0$ for every $(i,j) \in \mathcal{A}^2$. The shift of finite type X is mixing if there exists an integer $m > 0$ with $P^m(i,j) > 0$ for every $(i,j) \in \mathcal{A}^2$.

For $k = 2$ the matrix $P = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ determines a transitive shift of finite type, $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ determines a totally transitive, but not mixing, shift of finite type, and $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ yields a mixing shift of finite type.

For a detailed discussion of one-dimensional shifts of finite type we refer to Part 3.

Examples 2.6 (*Higher-dimensional shifts of finite type*).

(1) (Wang tilings) Let $\mathcal{T}$ be a finite nonempty set of distinct, closed 1×1 squares (tiles) with coloured edges such that no horizontal edge has the same colour as a vertical edge: such a set $\mathcal{T}$ is called a collection of *Wang tiles* (cf. [50], [67]). For each $\tau \in \mathcal{T}$ we denote by $\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau)$ the colours of the right, top, left and bottom edges of τ, and we write $\mathcal{C}(\mathcal{T}) = \{\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau) : \tau \in \mathcal{T}\}$ for the set of colours occurring on the tiles in $\mathcal{T}$. A Wang tiling w by $\mathcal{T}$ is a covering of $\mathbb{R}^2$ by such that

(i) every corner of every tile in w lies in $\mathbb{Z}^2 \subset \mathbb{R}^2$,

(ii) two tiles of w are only allowed to touch along edges of the same colour, i.e. $\mathsf{r}(\tau) = \mathsf{l}(\tau')$ whenever τ, τ' are horizontally adjacent tiles with τ to the left of τ', and $\mathsf{t}(\tau) = \mathsf{b}(\tau')$ if τ, τ' are vertically adjacent with τ' above τ.

We identify each such tiling w with the point

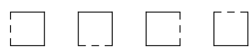
$$w = (w_{\mathbf{n}}) \in \mathcal{T}^{\mathbb{Z}^2},$$

where $w_{\mathbf{n}}$ is the unique element of $\mathcal{T}$ whose translate covers the square $\mathbf{n} + [0,1]^2 \subset \mathbb{R}^2$, $\mathbf{n} \in \mathbb{Z}^2$. The set $W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^2}$ of all Wang tilings by $\mathcal{T}$

is obviously a shift of finite type of the form (2.2)–(2.5), and is called the *Wang shift* of $\mathcal{T}$.

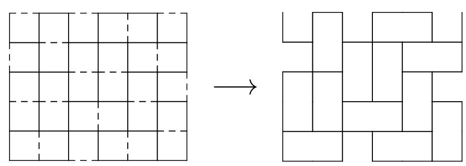
This definition has obvious analogues for any $d \geq 1$: a collection of *d-dimensional Wang tiles* is a finite set $\mathcal{T}$ of d-dimensional cubes with coloured faces, and the corresponding *Wang shift* $W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^d}$ consists of all coverings of $\mathbb{R}^d$ by translates of copies of elements of $\mathcal{T}$ with non-overlapping interiors satisfying (i)–(ii) above (mutatis mutandis).

Here is an explicit example of a two-dimensional Wang shift: let $\mathcal{T}_D$ be the set of Wang tiles



(2.7)

with the colours $\mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v}$ on the solid horizontal, broken horizontal, solid vertical and broken vertical edges. The following picture shows a partial Wang tiling of $\mathbb{R}^2$ by $\mathcal{T}_D$ and explains the name 'domino tiling' for such a tiling (cf. [25], [65], [12], [55]): two tiles meeting along an edge coloured h or v form a single vertical or horizontal 'domino'.



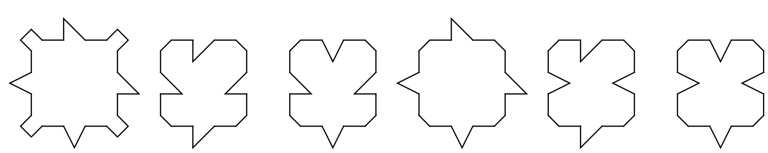
(2.8)

The Wang shift $W_D \subset \mathcal{T}_D^{\mathbb{Z}^2}$ of $\mathcal{T}_D$ is called the *Domino* (or *Dimer*) *Shift*, and is one of the few higher dimensional shifts of finite type for which the dynamics is understood to some extent (cf. [12], [16], [20], [21], [64] and [56]). The shift-action σ_{W_D} of $\mathbb{Z}^2$ on W_D is topologically mixing, and its topological entropy $h(\sigma_{W_D})$ was determined in [25] (cf. also [12] and [64]):

$$h(\sigma_{W_D}) = \frac{1}{4}\int_0^1 \int_0^1 (4 - 2\cos 2\pi s - 2\cos 2\pi t)\, ds\, dt. \tag{2.9}$$

For further discussion of Wang tilings we refer to Section 4.1.

(2) (Shifts of finite type without periodic points) Consider the following set $\mathcal{T}'$ of six polygonal tiles, introduced in [50], each of which which should be thought of as a 1×1 square with various bumps and dents.



(2.10)

We denote by $\mathcal{T}$ the set of all tiles which are obtained from (2.10) by allowing horizontal and vertical reflections as well as rotations by multiples of $\frac{\pi}{2}$ of elements in $\mathcal{T}'$. As in the preceding Example (1) we consider the set $W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^2}$ consisting of all tilings of $\mathbb{R}^2$ by translates of elements of $\mathcal{T}$ aligned to the integer lattice (as much as their bumps and dents allow). The set $W_{\mathcal{T}}$ is obviously a two-dimensional shift of finite type, and in [50] it was proved that $W_{\mathcal{T}}$ is uncountable and has no periodic points. If we allow each (or even only one) of these tiles to occur in two different colours with no restriction on adjacency of colours then we obtain a two-dimensional shift of finite type with positive entropy, but still without periodic points.

We remark in passing that the paper [50] also contains an explicit set $\mathcal{T}$ of Wang tiles for which the following *completion* (or *extension*) *problem* is undecidable: for a given subset $E \subset \mathbb{Z}^2$ and a given allowed configuration $y \in \mathcal{T}^E$, does there exist a point $x \in W_{\mathcal{T}}$ with $\pi_E(x) = y$?

An earlier construction of a tiling system without periodic points appeared in [4], but the number of tiles used there was considerably greater. For a detailed discussion of tilings and shifts of finite type without periodic points we refer to [47] and [49].

Part 3. One-dimensional shifts of finite type

The classical theory of shifts of finite type with parameter group $\mathbb{Z}$ has reached a mature stage in which most questions have good answers. This is mainly due to our ability to represent one-dimensional shifts of finite type using matrices whose entries are nonnegative integers, and then invoke tools from linear algebra. In this Part we will briefly survey the most important results and techniques, relying on references to [40] for more detailed motivation, proofs, many examples, and a comprehensive bibliography.

3.1. Graph shifts

We consider one-dimensional shifts of finite type with alphabet $\mathcal{A} = \{1, 2, \dots, r\}$. As noted in Lemma 2.2, by possibly modifying the alphabet we may assume that $F = \{0, 1\} \subset \mathbb{Z}$, i.e., the permitted patterns have length two. Hence they can be described by an $r \times r$ matrix P of 0's and 1's, where $P(i, j) = 1$ if and only if (i, j) is permitted. The *vertex shift* $X_{(F,P)}$ is defined as

$$X_{(F,P)} = \{x = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}} : P(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}.$$

The terminology arises from considering the directed graph G_P with vertex set $\mathcal{A}$ and one edge from vertex i to vertex j if and only if $P(i, j) = 1$. Then points in $X_{(F,P)}$ are described as the sequence of vertices traversed by bi-infinite paths on G_P.

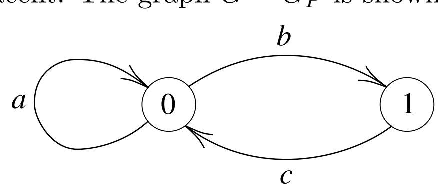
It turns out to be more useful, as well as mathematically convenient, to use sequences of edges rather than sequences of vertices. For example, this allows graphs with multiple edges between a given pair of vertices. This is the graphical representation of shifts of finite type that we will use here.

Definition 3.1. Let G be a directed graph with edge set $\mathcal{E}$. The *graph shift* determined by G is

$$X_G = \{\, x = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{E}^{\mathbb{Z}} : \text{ edge } x_{i+1} \text{ follows edge } x_i \text{ in } G\}.$$

If A is the adjacency matrix of G, we will also use X_A for the same graph shift, and also σ_G or σ_A for the corresponding shift-maps.

Example 3.2 (*Golden Mean shift*)**.** Let $\mathcal{A} = \{0, 1\}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $F = \{0, 1\}$. Then $X_{(F,P)}$ is the collection of all bi-infinite strings of 0's and 1's with no two 1's adjacent. The graph $G = G_P$ is shown below.



The graph shift X_G has alphabet $\mathcal{E} = \{a, b, c\}$ and permitted set of blocks $P = \{aa, ab, bc, ca, cb\}$.

We will be working with matrices whose entries are contained in the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of nonnegative integers. For brevity, we will refer to such matrices as simply $\mathbb{N}$-*matrices*.

The properties of topological transitivity and topological mixing for shifts of finite type can be characterized in terms of their defining $\mathbb{N}$-matrices.

Definition 3.3. An $\mathbb{N}$-matrix A is *irreducible* if for every i and j there is an $n \geq 1$ such that $(A^n)_{ij} > 0$. An $\mathbb{N}$-matrix A is *primitive* if there is an $n \geq 1$ such that $A^n > 0$.

Lemma 3.4. *Let A be an $\mathbb{N}$-matrix and σ_A be the corresponding shift of finite type.*

(1) *σ_A is topologically transitive if and only if A is irreducible.*
(2) *σ_A is topologically mixing if and only if A is primitive.*

To a large extent the theory of one-dimensional shifts of finite type can be reduced to that of topologically transitive, or even mixing, shifts of finite type. To describe this reduction, let A be a general $r \times r$ $\mathbb{N}$-matrix. Say that i and j *communicate* if there are paths from i to j and from j to i in the directed graph with adjacency matrix A. This is an equivalence relation, and the resulting directed graph of equivalence classes cannot contain any loops. This allow the renumbering of states so that A acquires the block triangular form

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ * & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & A_k \end{bmatrix}. \tag{3.1}$$

Here the A_i are irreducible square $\mathbb{N}$-matrices and the $*$'s represent possibly nonzero rectangular matrices. Then the dynamics of σ_A is largely determined by those of the σ_{A_i}.

Suppose now that A is irreducible. The greatest common divisor p of all of the periods of periodic points of σ_A is called the *period* of A. It is elementary to show that A^p has the block diagonal form

$$A^p = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{bmatrix},$$

where each of the A_j is a square, primitive matrix. This allows an analysis of irreducible matrices in terms of primitive matrices (see [40, Sec. 4.4] for further details).

3.2. Codes and conjugacy

One reason for the exceptionally explicit and combinatorial character of dynamics on shift spaces is the observation that continuous shift-commuting maps have finite descriptions.

Definition 3.5. Let X and Y be subshifts over finite alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. A *sliding block code* ϕ from X to Y is a mapping $\phi\colon X \to Y$ of the form

$$\phi(x)_i = \Phi(x_{i-m}, x_{i-m+1}, \dots, x_{i+n}),$$

where $\Phi\colon \mathcal{A}^{m+n+1} \to \mathcal{B}$ is a fixed function.

Clearly a sliding block code $\phi\colon X \to Y$ is continuous, and also shift-commuting in the sense that $\phi \cdot \sigma_X = \sigma_Y \cdot \phi$. The converse is also true (cf. Theorem 6.2.9 of [40]).

Theorem 3.6. *Let (X, σ_X) and (Y, σ_Y) be subshifts over finite alphabets, and $\phi\colon X \to Y$ be continuous with $\phi \cdot \sigma_X = \sigma_Y \cdot \phi$. Then ϕ is given by a sliding block code.*

Thus we may use the combinatorially defined sliding block codes when describing conjugacy between shifts of finite type.

Example 3.7. Consider the two shifts of finite type $X_{(F,P)}$ and X_G from Example 3.2, with alphabets $\mathcal{A} = \{0, 1\}$ and $\mathcal{B} = \{a, b, c\}$, respectively. Let $\Phi\colon \mathcal{A}^2 \to \mathcal{B}$ be defined by $\Phi(0, 0) = a$, $\Phi(0, 1) = b$, and $\Phi(1, 0) = c$, yielding a sliding block code $\phi\colon X_{(F,P)} \to X_G$ (here $m = 0$ and $n = 1$). Also, let $\Psi\colon \mathcal{B} \to \mathcal{A}$ by $\Psi(a) = 0$, $\Psi(b) = 0$, and $\Psi(c) = 1$, yielding a sliding block code $\psi\colon X_G \to X_{(F,P)}$ (here $m = n = 0$). Note that both $\psi \cdot \phi$ and $\phi \cdot \psi$ are the identity maps, so that the shifts on $X_{(F,P)}$ and X_G are topologically conjugate.

A fundamental question is when two shifts of finite type are topologically conjugate.

Problem 3.8 (Conjugacy Problem). Given two $\mathbb{N}$-matrices A and B, determine whether σ_A is topologically conjugate to σ_B.

R. Williams found one answer to the Conjugacy Problem, which we now describe (see [40, Chap. 7] for a complete account).

Definition 3.9. Let A and B be $\mathbb{N}$-matrices. An *elementary equivalence from A to B* is a pair (R, S) of rectangular $\mathbb{N}$-matrices satisfying $A = RS$ and $B = SR$. We say that A is *strong shift equivalent to B* if there is a finite chain of elementary equivalences from A to B of the form $A = A_0 = R_1S_1$, $S_1R_1 = A_1 = R_2S_2$, $S_2R_2 = A_2 = R_3S_3$, $\dots$, $S_{\ell-2}R_{\ell-2} = A_{\ell-1} = R_{\ell-1}S_{\ell-1}$, $S_{\ell-1}R_{\ell-1} = A_\ell = B$.

Theorem 3.10. *Let A and B be $\mathbb{N}$-matrices. Then σ_A is topologically conjugate to σ_B if and only if A is strong shift equivalent to B.*

Examples 3.11. (1) In Example 3.7 we showed that if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

then σ_A and σ_B are topologically conjugate. Here the pair (R, S), where

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

gives an elementary equivalence from A to B.

(2) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_1 S_1, \quad S_1 R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = A_1$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = R_2 S_2, \quad S_2 R_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A_2$$

$$A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = R_3 S_3, \quad S_3 R_3 = \begin{bmatrix} 2 \end{bmatrix} = B.$$

Hence σ_A and σ_B are topologically conjugate.

(3) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 1 & 1 \end{bmatrix}$. It was not known for a long time whether A and B are strong shift equivalent. K. Baker used a computer search to find a chain of length seven of elementary equivalences from A to B, with some intermediary matrices of size 4×4 (cf. Example 7.3.12 of [40]).

Example 3.11 points up two serious shortcomings of Theorem 3.10: there is no *a priori* bound on the length of the chain of elementary equivalences, nor on the sizes of the intermediate matrices. There are still some simple examples of pairs of 2×2 matrices where strong shift equivalence is not yet decided.

We turn to a weaker form of equivalence for $\mathbb{N}$-matrices.

Definition 3.12. Let A and B be $\mathbb{N}$-matrices and $\ell \geq 1$. A *shift equivalence of lag* ℓ from A to B is a pair (R, S) of rectangular $\mathbb{N}$-matrices such that (i) $AR = RB$, (ii) $SA = BS$, (iii) $RS = A^\ell$, and (iv) $SR = B^\ell$. We say A and B are shift equivalent if there is a shift equivalence of some lag from one to the other.

You can visualize these conditions (i)–(iv) for, say, $\ell = 2$ in the following commutative diagram.

$$\begin{tikzcd}
\bullet \arrow[r, "A"] \arrow[d, "S"] & \bullet \arrow[r, "A"] \arrow[d, "S"] & \bullet \arrow[r, "A"] \arrow[d, "S"] & \bullet \arrow[r, "A"] \arrow[d, "S"] & \bullet \arrow[r, "A"] \arrow[d, "S"] & \bullet \arrow[r] \arrow[d, "S"] & \cdots \\
\bullet \arrow[r, "B"'] \arrow[urr, "R"] & \bullet \arrow[r, "B"'] \arrow[urr, "R"] & \bullet \arrow[r, "B"'] \arrow[urr, "R"] & \bullet \arrow[r, "B"'] \arrow[urr, "R"] & \bullet \arrow[r, "B"'] & \bullet \arrow[r] & \cdots
\end{tikzcd}$$

Remarks 3.13. (1) If A is strong shift equivalent to B with a chain of ℓ elementary equivalences, then there is a shift equivalence from A to B of lag ℓ.

(2) There is a general algorithm that will decide whether pair of $\mathbb{N}$-matrices is shift equivalent or not.

For many years it was conjectured that shift equivalence implies strong shift equivalence. However, using some newly discovered invariants of subtle power, Kim and Roush [31], building on joint work with Wagoner [32] have recently shown this conjecture to be false.

An *automorphism* of a shift of finite type σ_A is a conjugacy of σ_A to itself, i.e., a shift commuting homeomorphism $\phi\colon X_A \to X_A$. The set of all automorphisms of σ_A forms a group under composition, called the *automorphism group* of σ_A and denoted by $\mathrm{aut}(\sigma_A)$. Since sliding block codes are given in finite terms, $\mathrm{aut}(\sigma_A)$ is at most countable. However, if σ_A is mixing then marker constructions can be used to embed every finite group into $\mathrm{aut}(\sigma_A)$. For more information about $\mathrm{aut}(\sigma_A)$ see [11]. There are still some open problems about automorphism groups. For example, are the automorphism groups of the full 2-shift and full 3-shift isomorphic as groups?

Let $Q_n(\sigma_A)$ denote the set of points in X_A with *least* period n under σ_A. Each automorphism of σ_A restricts to a bijection of $Q_n(\sigma_A)$ that commutes with the shift. Does every such bijection of $Q_n(\sigma_A)$ come from some automorphism of σ_A? Surprisingly, the answer is no! For example, Kim, Roush, and Wagoner [32] found a mixing shift of finite type with exactly two fixed points that they proved cannot be switched by an automorphism. Further work on the action of automorphisms on periodic points, initiated by Boyle and Krieger [9], eventually led Kim and Roush to their counterexample to the Shift Equivalence Conjecture referred to above.

3.3. Invariants

Suppose we can attach to each shift of finite type an "object," which could be a real number, or function, or group, or other mathematical object, so that topologically conjugate shifts are assigned the same value of the object. Such an assignment is called a *conjugacy invariant*, or simply *invariant*. If two shifts are assigned different values of the invariant, then we know for sure that they are not conjugate (although of course the converse may fail). For example, the number of fixed points is an invariant that distinguishes between the full 2-shift and the Golden Mean shift.

The topological entropy $h(\sigma_A)$ of a shift of finite type σ_A is the growth rate as $n \to \infty$ of the number of allowed blocks of length n with appear in points in X. We may assume without loss that A has no zero rows or zero columns. Then the number of n-blocks in X_A is $\sum_{i,j}(A^n)_{ij}$. The Perron-Frobenius theory of nonnegative matrices implies that the spectral radius λ_A of A is an eigenvalue of A, that $\lambda_A \ge |\mu|$ for all other eigenvalues μ of A, and that the growth rate of terms in A^n is $\log \lambda_A$.

Theorem 3.14. $h(\sigma_A) = \log \lambda_A$.

Example 3.15. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $\lambda_A = (1+\sqrt{5})/2$, the Golden Mean. This explains the terminology in Example 3.2.

Let $p_n(\sigma_A)$ denote the number of points in X_A of period n (i.e., fixed by σ_A^n). Clearly this number is a conjugacy invariant for each $n = 1, 2, \dots$. Note that $p_n(\sigma_A) = \operatorname{tr} A^n$. Artin and Masur assembled all of these invariants into a single function, which is analogous to the Riemann zeta function from classical number theory.

Definition 3.16. The Artin-Masur zeta function of a shift of finite type σ_A is defined as

$$\zeta_{\sigma_A}(t) = \exp\left[\sum_{n=1}^{\infty} \frac{p_n(\sigma_A)}{n}\, t^n\right].$$

The reason for using this form rather than a more traditional generating function is that it has a product formula over all "prime" orbits analogous to the Euler product formula for the classical zeta function.

Example 3.17. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, with eigenvalues $\lambda = (1+\sqrt{5})/2$ and $\mu = (1-\sqrt{5})/2$. Then $p_n(\sigma_A) = \operatorname{tr} A^n = \lambda^n + \mu^n$. Hence

$$\begin{aligned}\zeta_{\sigma_A}(t) &= \exp\left[\sum_{n=1}^{\infty} \frac{\lambda^n+\mu^n}{n}\, t^n\right] = \exp\left[\sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n} + \sum_{n=1}^{\infty} \frac{(\mu t)^n}{n}\right] \\ &= \exp\bigl[-\log(1-\lambda t) - \log(1-\mu t)\bigr] = \frac{1}{(1-\lambda t)(1-\mu t)} = \frac{1}{1-t-t^2}.\end{aligned}$$

This calculation generalizes to shifts of finite type.

Theorem 3.18. *Let A be an $\mathbb{N}$-matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$. Then*

$$\zeta_{\sigma_A}(t) = \frac{1}{\prod_j (1-\lambda_j t)} = \frac{1}{\det(1-tA)}.$$

Which numbers can arise as entropies of shifts of finite type, and which functions arise as zeta functions? To answer the first question define a *Perron number* to be an algebraic integer greater than or equal to 1 that is strictly larger than the absolute values of its algebraic conjugates (the other roots of its minimal polynomial).

Theorem 3.19. *A real number $\lambda \ge 1$ is the spectral radius of a primitive $\mathbb{N}$-matrix if and only if it is a Perron number. Furthermore, λ is the spectral radius of a (not necessarily primitive) $\mathbb{N}$-matrix if and only if some power λ^n is a Perron number.*

The characterization of zeta functions is much more difficult and has just recently been achieved. We will describe the result for primitive matrices only, but the irreducible and general cases can be deduced from it. First, note that $\zeta_{\sigma_A}(t)^{-1} = \prod_{j=1}^{r}(1-\lambda_j t)$ depends only on the nonzero eigenvalues of A. Let $\Lambda = \{\lambda_1, \dots, \lambda_k\}$ be a candidate list of k nonzero complex numbers, with repetitions possible. If Λ is the list of nonzero eigenvalues of a primitive matrix, then it must satisfy three conditions. First, the polynomial $\prod_{j=1}^{k}(1-\lambda_j t)$ must have integer coefficients. Second, the Perron-Frobenius theory says that there must be one entry $\lambda_i \ge 1$ such that $\lambda_i > |\lambda_j|$ for all $j \ne i$. To describe the third condition, let μ denote the Möbius function defined to be $(-1)^r$ for a product of r distinct primes and 0 otherwise. The number of orbits of *least* period n for σ_A is then given by $\sum_{d|n} \mu(n/d) p_d(\sigma_A)$, where the sum is over all divisors d of n. Thus if $\operatorname{tr}(\Lambda^n) = \sum_{j=1}^{k} \lambda_j^n$, then we must have the "net trace" condition that $\sum_{d|n} \mu(n/d) \operatorname{tr}(\Lambda^d) \ge 0$ for all $n \ge 1$. Boyle and Handelman [8] conjectured that these three conditions were also sufficient, and proved an analogue of this for nonnegative real matrices using techniques from symbolic dynamics. Very recently Kim, Ormes, and Roush [29] have established this conjecture, using techniques from formal power series.

Theorem 3.20. *Let $\Lambda = \{\lambda_1, \dots, \lambda_k\}$ be a list of nonzero complex numbers. Then a necessary and sufficient condition for Λ to be the list of nonzero eigenvalues of a primitive $\mathbb{N}$-matrix are that Λ satisfy the following:*

(1) $\prod_{j=1}^{k}(1-\lambda_j t) \in \mathbb{Z}[t]$.
(2) *There is a $\lambda_i \geq 1$ such that $\lambda_i > |\lambda_j|$ for all $j \neq i$.*
(3) $\sum_{d|n} \mu(n/d)\,\mathrm{tr}(\Lambda^d) \geq 0$ *for all $n \geq 1$.*

This characterizes the possible zeta functions of mixing shifts of finite type, and there are consequent characterizations of the zeta functions of irreducible and general shifts of finite type as well (cf. [29]). Note that because of the dominant entry in Λ, the last condition in the theorem can be checked computationally very efficiently.

One of the most sophisticated invariants for shifts of finite type is the dimension group, which was originally motivated from the theory of C^*-algebras.

Definition 3.21. Let A be an $r \times r$ $\mathbb{N}$-matrix. The *eventual range* $\mathcal{R}_A$ of A is $A^r(\mathbb{Q}^r)$. The *dimension group* Δ_A of A is

$$\Delta_A = \{\, \mathbf{v} \in \mathcal{R}_A : A^k \mathbf{v} \in \mathbb{Z}^r \quad \text{for some } k \geq 0 \,\}.$$

The *dimension semigroup* Δ_A^+ of A is

$$\Delta_A^+ = \{\, \mathbf{v} \in \mathcal{R}_A : A^k \mathbf{v} \in \mathbb{N}^r \quad \text{for some } k \geq 0 \,\}.$$

The *dimension group automorphism* δ_A of A is the restriction of A to Δ_A. The *dimension triple* of A is $(\Delta_A, \Delta_A^+, \delta_A)$.

Note that Δ_A^+ gives Δ_A an ordering that is preserved by δ_A. There is an intrinsic topological description of the dimension triple which shows that it is a conjugacy invariant [40, Thm. 7.5.13]. In fact, this invariant turns out to be shift equivalence (see [40, Thm. 7.5.8]).

Theorem 3.22. *Two $\mathbb{N}$-matrices are shift equivalent if and only if they have isomorphic dimension triples.*

3.4. Embeddings and factors

Consider continuous, shift-commuting maps $\phi\colon X_A \to X_B$. If ϕ is one-to-one it is called an *embedding* of X_A into X_B, while if it is onto it is called a *factor map* of X_A onto X_B. This section describes conditions under which such maps exist.

First note that if there is an embedding of X_A into X_B, then the growth rate of words in X_A cannot be greater than that for X_B, so that $h(\sigma_A) \leq h(\sigma_B)$. In addition, $p_n(\sigma_A) \leq p_n(\sigma_B)$ since an embedding must in particular be one-to-one on periodic points. When the entropy inequality is strict, Krieger proved these necessary conditions are also sufficient.

Theorem 3.23. *Let A and B be irreducible $\mathbb{N}$-matrices such that $h(\sigma_A) < h(\sigma_B)$ and $p_n(\sigma_A) \le p_n(\sigma_B)$ for all $n \ge 1$. There there is an embedding of X_A into X_B.*

Note that since $p_n(\sigma_A) = \operatorname{tr} A^n$, the entropy inequality shows that for all large enough n we have that $p_n(\sigma_A) \le p_n(\sigma_B)$. Hence the periodic point condition is really only in doubt for a finite number of low periods, which can be easily checked.

The proof of this theorem makes use of the fundamental notion of a *marker set*, which is a finite union M of cylinder sets such that M moves disjointly from itself under "moderate" powers of the shift, and a point whose orbit misses M for a "long" amount of time looks very much like a periodic point of "low" period. The embedding is constructed between markers that are not too far apart using the entropy condition, while for markers very far apart the periodic point condition is used. These very different coding schemes can be stitched together into a seamless sliding block code (see Section 10.1 of [40] for complete details).

There is an analogous result for factor maps $\phi\colon X_A \to X_B$ due to Boyle. Here the words appearing in X_B must come from words in X_A, so that $h(\sigma_A) \ge h(\sigma_B)$. Furthermore, any point of period n in X_A must map to a point whose period divides n, and it must have a place to go. Again, assuming strict inequality in entropy, the periodic point condition is also sufficient.

Theorem 3.24. *Let A and B be irreducible $\mathbb{N}$-matrices with $h(\sigma_A) > h(\sigma_B)$ and such that if $p_n(\sigma_A) > 0$ then there is a divisor d of n such that $p_d(\sigma_B) > 0$. Then there is a factor map from X_A onto X_B.*

The equal entropy situation is much more delicate, and is not completely solved. See Chapter 12 of [40] for an account of what is known.

Although conjugacy between shifts of finite type of equal entropy is delicate, there is a weaker notion of equivalence for which there is a complete answer.

Definition 3.25. Two subshifts (X, σ_X) and (Y, σ_Y) are *finitely equivalent* is there is a shift of finite type W together with finite-to-one factor codes $\phi_X\colon W \to X$ and $\phi_Y\colon W \to Y$.

Transitivity of this relation depends on the fiber product construction (see [40, Def. 8.3.2]. Since finite-to-one factor codes preserve entropy, finitely equivalent shifts of finite type much have equal entropy. For irreducible shifts of finite type the converse is also true (see [40, Thm. 8.3.7]).

Theorem 3.26. *Two irreducible shifts of finite type are finitely equivalent if and only if they have the same entropy.*

The image of a shift of finite type under a factor code need not have finite type, as the following example shows.

Example 3.27. Let $\mathcal{A} = \{0, 1\}$ and Y be the collection of all bi-infinite sequences of 0's and 1's such that between successive 1's there are an even number of 0's. We call Y the *even shift*.

Let X_G be the graph shift from Example 3.2. Define $\Phi(a) = 1$, $\Phi(b) = 0$, and $\Phi(c) = 0$, and let $\phi\colon X_G \to Y$ be the corresponding sliding block code.

It is easy to check that ϕ is onto, so provides a factor code with domain a shift of finite type. However, Y is *not* a shift of finite type. For if it were, it would be determined by a finite number of permitted blocks. Since both $10^{2n+1} = 1000\dots000$ and $0^{2n+1}1$ are permitted, for sufficiently large n so would $10^{2n+1}1$, contradicting the definition of the even shift.

This example points out the difference in defining subshifts using memory of finite length versus memory with a finite amount (e.g., keeping track of whether one has seen an even or an odd number of 0's). The latter notion leads to the idea of sofic shifts.

Definition 3.28. A *labeled graph* $\mathcal{G} = (G, \mathcal{L})$ consists of a directed graph G with edge set $\mathcal{E}$, and a labeling $\mathcal{L} \colon \mathcal{E} \to \mathcal{A}$ of the edges with labels from some finite alphabet $\mathcal{A}$. The *sofic shift* determined by $\mathcal{G}$ is

$$X_{\mathcal{G}} = \{ \dots \mathcal{L}(x_{-1}x_0x_1 \dots \in \mathcal{A}^{\mathbb{Z}} : \dots x_{-1}x_0x_1 \dots \in X_G \}.$$

For instance, the even shift in Example 3.27 is sofic by suing the labeling $\mathcal{L}(a) = 1$, $\mathcal{L}(b) = 0$, and $\mathcal{L}(c) = 0$ of the Golden Mean shift. The term "sofic" is adapted from the Hebrew works for finite.

The importance of the class of sofic shifts is that it is the "closure" of the collection of shifts of finite type under the operation of factor codes.

Theorem 3.29. *A subshift with finite alphabet is conjugate to a sofic shift if and only if it is the image of a shift of finite type under a factor code.*

Part 4. Multi-dimensional shifts of finite type

As mentioned in Part 2, a general theory of multi-dimensional shifts of finite type is in some sense impossible. Even the less ambitious project of finding a reasonably general class of multi-dimensional shifts of finite type which is amenable to systematic analysis has not met with much success. Even the simplest and most natural examples (such as some of the those arising from lattice systems in statistical mechanics) lead to problems of remarkable and unexpected difficulty. Out of necessity this section will therefore be restricted to a somewhat idiosyncratic collection of examples, partial results and problems.

We begin our discussion with an easy, but perhaps not completely trivial observation. Let $\mathcal{A}$ and $\mathcal{A}'$ be finite alphabets, and let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a shift of finite type (i.e. $X = X_{(F,P)}$ for some finite subsets $F \subset \mathbb{Z}^d$ and $P \subset \mathcal{A}^P$). If $X' \subset \mathcal{A}'^{\mathbb{Z}^d}$ is a subshift which is topologically conjugate to X, then X' is again a shift of finite type (this follows from the characterization (2.4) of Markov systems, taking into account the finiteness of the alphabets). This shows that the definition of a shift of finite type is *intrinsic* and not tied to a particular representation of the shift.

4.1. Wang tilings and cohomology of shifts of finite type

For the definition of a set $\mathcal{T}$ of d-dimensional Wang tiles we refer to Example 2.6 (1). There we saw that the Wang shift $W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^d}$ is a shift of finite type. The following elementary proposition establishes the converse of this observation.

Proposition 4.1. *Let $\mathcal{A}$, $F \subset \mathbb{Z}^d$ and $P \subset \mathcal{A}^F$ be nonempty finite sets, and let $X = X_{(F,P)} \subset \mathcal{A}^{\mathbb{Z}^d}$ be the corresponding shift of finite type defined in* (2.2)*. Then there exists a collection $\mathcal{T}$ of Wang tiles such that the subshifts $X_{(F,P)}$ and $W_{\mathcal{T}}$ are conjugate.*

Proof. For simplicity we restrict ourselves to dimension $d = 2$. Lemma 2.2 allows us to assume that $F = \{0,1\}^2 \subset \mathbb{Z}^2$. We set $\mathcal{T} = \pi_F(X_{(F,P)}) \subset \mathcal{A}^F$ and consider each

$$\tau = \boxed{\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \\ x_{(0,0)} & x_{(1,0)} \end{smallmatrix}} \in \mathcal{T} \tag{4.1}$$

as a unit square with the 'colours' $\left[\begin{smallmatrix} x_{(0,0)} & x_{(1,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \end{smallmatrix}\right]$ along its bottom and top horizontal edges, and $\left[\begin{smallmatrix} x_{(0,1)} \\ x_{(0,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(1,1)} \\ x_{(1,0)} \end{smallmatrix}\right]$ along its left and right vertical edges. With this interpretation we obtain a one-to-one correspondence between the points $x = (x_{\mathbf{n}}) \in X$ and the Wang tilings $w = (w_{\mathbf{n}}) = (\pi_F \cdot \sigma^{\mathbf{n}}(x)) \in \mathcal{T}^{\mathbb{Z}^2}$. □

This correspondence allows us to regard each shift of finite type as a Wang shift and vice versa. However, the correspondence is a bijection only up to topological conjugacy: if we start with a shift of finite type $X \subset \mathcal{A}^{\mathbb{Z}^2}$ of the form (2.2)–(2.5), view it as the Wang $W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^2}$ with $\mathcal{T} = \pi_F(X)$, and then interpret $W_{\mathcal{T}}$ as a shift of finite type as above, we do not end up with X, but with the 2-block representation of X. We simplify terminology by introducing the following definition.

Definition 4.2. Let $\mathcal{A}$ be a finite set and $X \subset \mathcal{A}^{\mathbb{Z}^2}$ a shift of finite type, $\mathcal{T}$ a set of Wang tiles and $W_{\mathcal{T}}$ the associated Wang shift. We say that $W_{\mathcal{T}}$ *represents* X if $W_{\mathcal{T}}$ is topologically conjugate to X. Two Wang shifts $W_{\mathcal{T}}$ and $W_{\mathcal{T}'}$ are *equivalent* if they are topologically conjugate as shift of finite type.

Since any given infinite shift of finite type X has many different — and quite dissimilar — representations by Wang shifts one may ask whether these different representations of X have anything in common. The answer to this question turns out to be related to a measure of the 'complexity' of the shift of finite type X. In order to continue we need the notion of a continuous 1-cocycle on a shift of finite type $X \subset \mathcal{A}^{\mathbb{Z}^d}$, where $\mathcal{A}$ is a finite alphabet.

Definition 4.3. Let G be a discrete group with identity element 1_G. A map $c\colon \mathbb{Z}^d \times X \to G$ is a *continuous cocycle* for the shift-action σ of $\mathbb{Z}^d$ on X defined in (2.1) if $c(\mathbf{n}, \cdot)\colon X \to G$ is continuous for every $\mathbf{n} \in \mathbb{Z}^d$ and

$$c(\mathbf{m}+\mathbf{n}, x) = c(\mathbf{m}, \sigma^{\mathbf{n}}(x))c(\mathbf{n}, x) \tag{4.2}$$

for all $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. The cocycle c is a *homomorphism* if $c(\mathbf{n}, \cdot)$ is constant for every $\mathbf{n} \in \mathbb{Z}^d$, and c is a *coboundary* if there exists a Borel map $b\colon X \to G$ such that

$$c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}(x))^{-1}b(x) \tag{4.3}$$

for all $x \in X$ and $\mathbf{n} \in \mathbb{Z}^d$. The map b in (4.3) is the *cobounding function* of c. Two continuous cocycles $c, c'\colon \mathbb{Z}^d \times X \to G$ are *continuously cohomologous*, with *transfer function* $b\colon X \to G$, if

$$c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}(x))^{-1}c'(\mathbf{n}, x)b(x)$$

for all $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. A continuous cocycle $c\colon \mathbb{Z}^d \times X \to G$ is *trivial* if it is continuously cohomologous to a homomorphism.

For the remainder of this section we again restrict ourselves to the case $d = 2$.

Let $\mathcal{T}$ be a collection of (2-dimensional) Wang tiles, and let $W_{\mathcal{T}} \subset \mathcal{T}$ be the Wang shift of $\mathcal{T}$. Following [66] we write

$$\Gamma(\mathcal{T}) = \langle \mathcal{C}(\mathcal{T}) | \mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau),\ \tau \in \mathcal{T}\rangle \tag{4.4}$$

for the free group generated by the colours occurring on the edges of elements in $\mathcal{T}$, together with the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in \mathcal{T}$. The countable, discrete group $\Gamma(\mathcal{T})$ is called the *tiling group* of $\mathcal{T}$ (or of the Wang shift $W_{\mathcal{T}}$). From the definition of $\Gamma(\mathcal{T})$ it is clear that the map $\theta\colon \Gamma(\mathcal{T}) \to \mathbb{Z}^2$, given by

$$\theta(\mathsf{b}(\tau)) = \theta(\mathsf{t}(\tau)) = (1,0), \qquad \theta(\mathsf{l}(\tau)) = \theta(\mathsf{r}(\tau)) = (0,1), \tag{4.5}$$

for every $\tau \in \mathcal{T}$, is a group homomorphism whose kernel is denoted by

$$\Gamma_0(\mathcal{T}) = \ker(\theta). \tag{4.6}$$

If $W_{\mathcal{T}}$ is a Wang shift we obtain not only the corresponding tiling group $\Gamma_0(\mathcal{T}) \subset \Gamma(\mathcal{T})$, but also a *tiling cocycle* $c_{\mathcal{T}}\colon \mathbb{Z}^2 \times W_{\mathcal{T}} \to \Gamma(\mathcal{T})$ of $W_{\mathcal{T}}$ by setting

$$c_{\mathcal{T}}((1,0), w) = \mathsf{b}(w_{\mathbf{0}}), \ \ c_{\mathcal{T}}((0,1), w) = \mathsf{l}(w_{\mathbf{0}}) \tag{4.7}$$

for every $w \in W_{\mathcal{T}} \subset \mathcal{T}^{\mathbb{Z}^2}$, and by using (4.2) to extend $c_{\mathcal{T}}$ to a map from $\mathbb{Z}^2 \times W_{\mathcal{T}}$ to $\Gamma(\mathcal{T})$ (the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in \mathcal{T}$, in (4.4) are precisely what is needed to allow such an extension).

Any representation of a shift of finite type X by a Wang shift $W_{\mathcal{T}}$ clearly induces a continuous cocycle $c\colon \mathbb{Z}^2 \times X \to \Gamma(\mathcal{T})$: if $\phi\colon X \to W_{\mathcal{T}}$ is a topological conjugacy, then we set

$$c(\mathbf{n}, x) = c_{\mathcal{T}}(\mathbf{n}, \phi(x)) \tag{4.8}$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$. The next result shows that every continuous cocycle on X with values in a discrete group G arises essentially in this manner.

Theorem 4.4. *Let $\mathcal{A}$ be a finite set, $X \subset \mathcal{A}^{\mathbb{Z}^2}$ a shift of finite type, G a discrete group and $c\colon \mathbb{Z}^2 \times X \to G$ a continuous cocycle. Then there exist a Wang shift $W_{\mathcal{T}}$ representing X and a group homomorphism ϕ from the tiling group $\Gamma(\mathcal{T})$ of $\mathcal{T}$ to G such that*

$$c(\mathbf{m}, \psi(w)) = \phi \cdot c_{\mathcal{T}}(\mathbf{m}, w) \tag{4.9}$$

for every $w \in W_{\mathcal{T}}$ and $\mathbf{m} \in \mathbb{Z}^2$.

Different Wang shifts representing the same shift of finite type X may lead to wildly different tiling cocycles. For certain shifts of finite type X, however, there exists a single Wang shift $W_{\mathcal{T}}$ representing X whose tiling cocycle determines *every* continuous cocycle $c\colon \mathbb{Z}^2 \times X \to G$ with values in a discrete Group G.

Definition 4.5. Let $\mathcal{A}$ be a finite alphabet, $d \geq 1$ and $X \subset \mathcal{A}^{\mathbb{Z}^d}$ a shift of finite type. A continuous cocycle $c^*\colon \mathbb{Z}^d \times X \to G^*$ is *fundamental* if there exists, for every countable group G and every continuous cocycle $c\colon \mathbb{Z}^d \times X \to G$, a group homomorphism $\eta\colon G^* \to G$ such that the cocycle $\eta \cdot c^*\colon \mathbb{Z}^d \times X \to G$ is continuously cohomologous to c.

If X has such a fundamental cocycle then Theorem 4.4 allows us to assume without loss in generality that $c^* = c_{\mathcal{T}}$ for some Wang shift $W_{\mathcal{T}}$ representing X. What is surprising is that some shifts of finite type have fundamental cocycles; the extent of this rigidity phenomenon is not understood at this stage.

We conclude this section with a few examples.

Example 4.6. Let $\mathcal{A}$ be a finite alphabet and $X \subset \mathcal{A}^{\mathbb{Z}}$ a mixing shift of finite type (cf. Example 2.5). Then X does not possess a fundamental cocycle (cf. [56]).

Example 4.7 (*Full shifts*)**.** Let $d \geq 2$ and $X = \mathcal{A}^{\mathbb{Z}^d}$ the full shift with a finite alphabet $\mathcal{A}$. Then the cocycle $c^*\colon \mathbb{Z}^d \times X \to \mathbb{Z}^d$ given by

$$c^*(\mathbf{n}, x) = \mathbf{n}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$ is fundamental. This statement is equivalent to saying that every continuous cocycle on X with values in a discrete group G is trivial.

The full shifts are not the only ones with this property. For example, the d-dimensional *hard sphere model*

$$X = \Big\{ x \in \{0,1\}^{\mathbb{Z}^d} : \sum_{i=1}^{d} x_{\mathbf{n}} x_{\mathbf{n}+\mathbf{e}^{(i)}} = 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^d \Big\} \tag{4.10}$$

has no nontrivial continuous cocycles (here $\mathbf{e}^{(i)}$ is the i-th unit vector in $\mathbb{Z}^d$, $i = 1, \dots, d$). The shift of finite type X in (4.10) can be described more informally as the set of points in the d-dimensional 2-shifts in which every 1 is surrounded by 0's.

A more general sufficient condition for the absence of nontrivial continuous cocycles can be expressed in terms of a specification property (cf. [56]).

Example 4.8 (*Domino tilings*)**.** Let k, k', l, l' be positive integers and let $R = [0, k+1) \times [0, l+1) \subset \mathbb{R}^2$, $R' = [0, k'+1) \times [0, l'+1) \subset \mathbb{R}^2$. We interpret R and R' as rectangular 'dominoes' and write $X = X_{(k,l),(k',l')}$ for the set of all tilings of $\mathbb{R}^2$ by integer translates of R and R'. In order to see that X is (topologically conjugate to) a Wang system we introduce a set

$$\mathcal{T} = \{\tau_{i,j} : 0 \le i < k, 0 \le j < l\} \cup \{\tau'_{i,j} : 0 \le i < k', 0 \le j < l'\}$$

of Wang tiles with the colours $\mathsf{b}(\cdot), \mathsf{r}(\cdot), \mathsf{t}(\cdot), \mathsf{l}(\cdot)$ on the bottom, right, top and left edges. These colours are chosen as follows:

(i) $\mathsf{l}(\tau_{i,0}) = \mathsf{l}(\tau'_{i',0}) = \mathsf{r}(\tau_{i,l-1}) = \mathsf{r}(\tau'_{i',l'-1}) = \mathsf{V}$ for $0 \le i < k$ and $0 \le i' < k'$,
(ii) $\mathsf{r}(\tau_{i,j}) = \mathsf{l}(\tau_{i,j+1}) = \mathsf{v}_{j+1}$ and $\mathsf{r}(\tau'_{i',j'}) = \mathsf{l}(\tau'_{i',j'+1}) = \mathsf{v}_{j'+1}$ for $0 \le i < k$, $0 \le i' < k'$, $0 \le j < l-1$ and $0 \le j' < l'-1$,
(iii) $\mathsf{b}(\tau_{i,j}) = \mathsf{t}(\tau_{i,j}) = \mathsf{b}(\tau'_{i',j'}) = \mathsf{t}(\tau'_{i',j'}) = \mathsf{H}$ for $0 \le i \le k-1$, $0 \le i' \le k'-1$, $0 \le j < l-1$ and $0 \le j' < l'-1$,
(iv) $\mathsf{b}(\tau_{0,l-1}) = \mathsf{b}(\tau_{0,l'-1}) = \mathsf{t}(\tau_{k-1,l-1}) = \mathsf{t}(\tau_{k'-1,l'-1}) = \mathsf{H}$,
(v) $\mathsf{t}(\tau_{i,l-1}) = \mathsf{b}(\tau_{i+1,l-1}) = \mathsf{a}_j$ for $0 \le i < k-1$, and $\mathsf{t}(\tau_{i',l'-1}) = \mathsf{b}(\tau_{i'+1,l'-1}) = \mathsf{b}_{j'}$ for $0 \le i' < k'-1$.

Any tile $\tau \in \mathcal{T}$ occurring in a Wang tiling of $\mathbb{R}^2$ must be contained in an integer translate of one the rectangles R or R' with the colours H and V on the horizontal and vertical edges, respectively. This shows that the Wang shift $W_{\mathcal{T}}$ is topologically conjugate to X and that X is therefore a shift of finite type. For the remainder of this discussion we identify $X = X_{(k,l),(k',l')}$ with $W_{\mathcal{T}}$.

In [18] the following result was proved: $X = W_{\mathcal{T}}$ is topologically mixing if and only $\gcd(k, k') = \gcd(l, l') = 1$; furthermore, if X is mixing then the tiling cocycle $c^* = c_{\mathcal{T}} \colon \mathbb{Z}^2 \times X \to \Gamma(\mathcal{T})$ is fundamental.

In order to describe this cocycle a little more explicitly we write $\mathbb{Z}_{/m} = \mathbb{Z}/m\mathbb{Z}$ for the cyclic group of order $m \ge 1$ and denote by $\mathbb{Z}_{/m} * \mathbb{Z}_{/n}$ the free product of two such groups. For every $m, n \ge 1$ there exists a group homomorphism $\kappa_{m,n} \colon \mathbb{Z}_{/m} * \mathbb{Z}_{/n} \to \mathbb{Z}_{/m} \times \mathbb{Z}_{/n}$ which consists of making the elements of $\mathbb{Z}_{/m}$ and $\mathbb{Z}_{/n}$ commute with each other. Put

$$\Gamma = \Gamma_{(k,l),(k',l')} = \mathbb{Z}^2 \times (\mathbb{Z}_{/k} * \mathbb{Z}_{/l'}) \times (\mathbb{Z}_{/k'} * \mathbb{Z}_{/l}),$$

and let $\Delta = \Delta_{(k,l),(k',l')} \subset \Gamma$ be the subgroup

$$\begin{aligned}\Delta = \{((m,n),\gamma,\gamma') \in \Gamma : \kappa_{k,l'}(\gamma) &= (m \pmod{k}, n \pmod{l'}),\\ \kappa_{k',l}(\gamma') &= (m \pmod{k'}, n \pmod{l})\}.\end{aligned} \tag{4.11}$$

Then there exists a unique group isomorphism $\eta \colon \Gamma(\mathfrak{T}) \to \Delta$ with

$$\eta(\mathsf{H}) = ((1,0), 1*0, 1*0), \qquad \eta(\mathsf{V}) = ((0,1), 0*1, 0*1),$$

where $1*0$ and $0*1$ are the images of 1 under the obvious embeddings $\mathbb{Z}_{/k}, \mathbb{Z}_{/l'} \hookrightarrow \mathbb{Z}_{/k} * \mathbb{Z}_{/l'}$ and $\mathbb{Z}_{/k'}, \mathbb{Z}_{/l} \hookrightarrow \mathbb{Z}_{/k'} * \mathbb{Z}_{/l}$, respectively (the uniqueness of η follows from the fact that $\Gamma(\mathfrak{T})$ is generated by H and V, since all the other colours v_j, a_i and $\mathsf{b}_{i'}$ can be expressed recursively in terms of H and V).

In this notation the fundamental cocycle $c^* = \eta \cdot c_{\mathfrak{T}} \colon \mathbb{Z}^2 \times X \to \Delta$ of X is given by

$$c^*((1,0),x) = \eta(\mathsf{b}(x_{\mathbf{0}})), \qquad c^*((0,1),x) = \eta(\mathsf{l}(x_{\mathbf{0}})), \tag{4.12}$$

for every $x \in X = W_{\mathfrak{T}} \subset \mathfrak{T}^{\mathbb{Z}^2}$.

The domino shift $W_{\mathfrak{T}_D}$ appearing in (2.7) corresponds to the choice $k = 2, l = 1, k' = 1, l' = 2$. In this case $\Gamma(\mathfrak{T}_D)$ is isomorphic to the subgroup

$$\Delta_{(2,1),(1,2)} \subset \mathbb{Z}^2 \times C_2 * C_2$$

consisting of all $((m,n),\gamma) \in \mathbb{Z}^2 \times C_2 * C_2$ with

$$\kappa_{2,2}(\gamma) = (m \pmod{2}, n \pmod{2})$$

(cf. (4.11)). From this one concludes easily that every continuous cocycle $c \colon \mathbb{Z}^2 \times W_{\mathfrak{T}_D} \to A$ with values in a discrete abelian group A must be trivial (cf. [56]).

Further examples of shifts of finite type possessing a fundamental cocycle can be found in [18] and [55]–[56]. We note in passing that it is not known whether Ledrappier's example has a fundamental cocycle (cf. [56] and Example 2.4 (1)).

Fundamental cocycles are not only an expression of cohomological rigidity, but also have a geometrical interpretation. Let $W_{\mathfrak{T}}$ be a two-dimensional Wang shift, and let $E \subset \mathbb{Z}^2$ be a finite set. Suppose that we have an allowed element $y \in \mathfrak{T}^{\mathbb{Z}^2 \smallsetminus E}$; how can one tell whether y has an allowed extension to all of $\mathbb{Z}^2$, i.e. whether there exists an element $x \in W_{\mathfrak{T}}$ with $\pi_{\mathbb{Z}^2 \smallsetminus E}(x) = y$? Since this is a finite problem it is obviously decidable. However, if we ask whether there exists a finite set $E' \supset E$ and an element $x \in W_{\mathfrak{T}}$ with $\pi_{\mathbb{Z}^2 \smallsetminus E'}(x) = \pi_{\mathbb{Z}^2 \smallsetminus E'}(y)$, i.e. whether y can be extended after possibly 'correcting' finitely many coordinates, then the problem becomes much more complicated. Let us call y *weakly extensible* if it admits such an extension after a finite correction.

If we are interested in weak extensibility we may assume for simplicity the the bounded set $E \subset \mathbb{Z}^2$ is a rectangle of the form $\{0,\dots,m-1\} \times \{0,\dots,n-1\}$. For any allowed element y of $\mathfrak{T}^{\mathbb{Z}^2 \smallsetminus E}$ we consider the word $w(E,y)$ in the tiling group formed by the product

$$\begin{aligned}&\mathsf{r}(y_{(-1,0)})^{-1} \cdots \mathsf{r}(y_{(-1,n-1)})^{-1} \cdot \mathsf{b}(y_{(0,n)})^{-1} \cdots \mathsf{b}(y_{(m-1,n)})^{-1}\\ &\quad \cdot \mathsf{l}(y_{(m,n-1)}) \cdots \mathsf{l}(y_{(m,0)}) \cdot \mathsf{t}(y_{(m-1,-1)}) \cdots \mathsf{t}(y_{(0,-1)})\end{aligned} \tag{4.13}$$

of the colours on the edges of the tiles touching E in anti-clockwise direction, starting from the bottom left hand corner of E (written from right to left, and with the colours along the top and left edges of E inverted). From the relations in the tiling group it is clear that, if $E' \supset E$ is another rectangle, then $w(E', y) = w(E, y)$. Similarly, if $E'' \subset E$ is a rectangle and $y' \in \mathcal{T}^{\mathbb{Z}^2 \smallsetminus E'}$ is an allowed element with $\pi_{\mathbb{Z}^2 \smallsetminus E}(y') = \pi_{\mathbb{Z}^2 \smallsetminus E}(y)$, then $w(E, y) = w(E, y') = w(E'', y')$. It follows that $w(E, y) = 1$ (the identity element in $\Gamma(\mathcal{T})$) if y is weakly extensible. On the other hand, if $w(E, y) \neq 1$, then y is obviously not weakly extensible. This obstruction to weak extensibility was the motivation for introducing the tiling group $\Gamma(\mathcal{T})$ in [17] and [66].

If $\mathcal{A}$ is a finite alphabet, $X \subset \mathcal{A}^{\mathbb{Z}^2}$ a shift of finite type and $W_{\mathcal{T}}$ is a Wang shift representing X, then this observation yields a potential obstruction to weak extensibility of allowed elements $y \in \mathcal{A}^{\mathbb{Z}^2 \smallsetminus E}$, where $E \subset \mathbb{Z}^2$ is a finite set. However, as different representations of X by Wang shifts may lead to different answers (cf. [56]), the search for obstructions to weak extensibility might conceivably involve infinitely many different representation of X by Wang shifts. It can be shown that any fundamental cocycle c^* of X expresses the total information about the patching of holes that can be extracted from *all* Wang shifts representing a shift of finite type X: if $W_{\mathcal{T}}$ is a Wang shift representing X such that c^* is a homomorphic image of the tiling cocycle $c_{\mathcal{T}}$ in the sense of Theorem 4.4 then *every* tiling cocycle of a Wang shift representing X is continuously cohomologous to a homomorphic image of $c_{\mathcal{T}}$, and no representation of X by a Wang system can yield obstructions to weak extensibility beyond those arising from $W_{\mathcal{T}}$. However, even if there exists such a fundamental cocycle then the resulting information may may not provide *complete* information about weak extensibility (cf. [18] and [56]). For example, the tiling cocycle of the domino shift $W_{\mathcal{T}_D}$ arising from the tiles (2.7) does not detect all allowed points $w \in \mathcal{T}_D^{\mathbb{Z}^2 \smallsetminus Q_3}$ which are not weakly extensible (cf. (1.7)).

For a connection between fundamental cocycles and the *fundamental group* of a two-dimensional shift of finite type introduced in [23] we refer to [56].

4.2. Miscellaneous results and problems

4.2.1. Topological conjugacy, embeddings and factor maps. While there exists a very detailed theory of topological conjugacy and factor maps of one-dimensional shifts of finite type (cf. Part 3), almost nothing is known about this topic for higher-dimensional shifts of finite type. For example, if $X = \{0, 1\}^{\mathbb{Z}^2}$, there exist many uncountable-to-one factor maps $\phi\colon X \to X$ (such as the map given by $\phi(x)_{\mathbf{n}} = x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} \pmod 2$ for every $x \in X$ and $\mathbf{n} \in \mathbb{Z}^2$). The triviality of the fundamental cocycle of X mentioned in 4.7 can be used to show that there exist no bounded-to-one factor maps $\phi\colon X \to X$ which are open; are there bounded-to-one factor maps $\phi\colon X \to X$ which are not open? This question was raised, for example, in [23].

Another open question is when a d-dimensional shift of finite type X can be embedded in another d-dimensional shift of finite type Y. Partial results concerning an extension of Krieger's embedding theorem 3.23 to d-dimensional shifts of finite type can be found in [38].

4.2.2. Measures of maximal entropy. Let M be a positive integer, $\mathcal{A} = \{-M, \dots, -1, 1, \dots, M\}$, and let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be the set of all points $x = (x_{\mathbf{n}}) \in \mathcal{A}^{\mathbb{Z}^2}$ such that $x_{\mathbf{n}} x_{\mathbf{n}\pm(1,0)} > -1$ and $x_{\mathbf{n}} x_{\mathbf{n}\pm(0,1)} > -1$ for all $\mathbf{n} \in \mathbb{Z}^2$ (i.e. no positive coordinate of x can sit next to a negative one unless they are both equal to ± 1). The shift X was introduced in [13] under the name *Iceberg Model*. It is clear that X is a mixing shift of finite type with positive topological entropy. In [13] it was also shown that, if $M = 1$, then X has a unique measure of maximal entropy which is Bernoulli by [15]. For M sufficiently large, X has exactly two distinct ergodic shift-invariant measures of maximal entropy, both of which are Bernoulli (cf. [14]). A variation of the definition of the iceberg model in [14] yields, for any $k \geq 1$ and $d \geq 2$, a mixing d-dimensional shift of finite type with exactly k ergodic measures of maximal entropy. The paper [14] also contains examples of 3-dimensional shifts of finite type with uncountably many distinct ergodic shift-invariant measures of maximal entropy.

4.2.3. Gibbs relations and equilibrium states. Let $\mathcal{A}$ be a finite alphabet and $X \subset \mathcal{A}^{\mathbb{Z}^d}$ a d-dimensional shift of finite type. The *Gibbs* (or *homoclinic*) equivalence relation is defined by

$$\Delta_X = \{(x, y) \in X \times X : x_{\mathbf{n}} \neq y_{\mathbf{n}} \text{ for only finitely many } \mathbf{n} \in \mathbb{Z}^d\}. \tag{4.14}$$

For every $x \in X$, the equivalence class $\Delta_X(x) = \{y \in X : (x, y) \in \Delta_X\}$ is countable. From [22] it follows that the *saturation*

$$\Delta_X(B) = \bigcup_{x \in B} \Delta_X(x)$$

of every Borel set $B \subset X$ is again Borel.

For example, if $X = \mathcal{A}^{\mathbb{Z}^d}$ is the full shift with alphabet $\mathcal{A}$, the Gibbs relation Δ_X is *minimal*, i.e. the equivalence class $\Delta_X(x)$ of every $x \in X$ is dense in X. The same is obviously true for the Iceberg Models in Subsection 4.2.2. For the Domino Shift W_D in Example 2.6 (1), the Gibbs relation Δ_{W_D} is not minimal, but still *topologically transitive*: there exists an $x \in W_D$ whose equivalence class $\Delta_{W_D}(x)$ is dense in W_D. For Ledrappier's example $X \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ described in Example 2.4 (1) the Gibbs relation Δ_X is *trivial*, i.e. $\Delta_X(x) = \{x\}$ for every $x \in X$.

A probability measure μ on a shift of finite type $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is *quasi-invariant* under Δ_X if $\mu(\Delta_X)(B) = 0$ for every Borel set $B \subset X$ with $\mu(B) = 0$. If μ is quasi-invariant under Δ_X we say that two Borel maps $\psi_1, \psi_2 \colon \Delta_X \to \mathbb{R}$ *coincide modulo* μ if there exists a μ-null Borel set $N \subset X$ with $\psi_1 = \psi_2$ on $\Delta_X \smallsetminus ((X \times N) \cup (N \times X))$.

Any probability measure μ on X which is quasi-invariant under Δ_X has a well-defined *Radon-Nikodym derivative* on Δ_X: there exists a Borel map $\rho_\mu \colon \Delta_X \to \mathbb{R}$ such that

$$\log \frac{d\mu\phi}{d\mu}(x) = \rho_\mu(\phi(x), x) \ \ \mu\text{a.e.}, \tag{4.15}$$

for every homeomorphism $\phi \colon X \to X$ with $(\phi(x), x) \in \Delta_X$ for every $x \in X$ (such a homeomorphism of X is called a *finite coordinate change*, and the quasi-invariance of μ under Δ_X guarantees that μ is also quasi-invariant under every such ϕ). This Radon-Nikodym derivative is well-defined in the sense that any two Borel maps $\rho_\mu, \rho'_\mu \colon \Delta_X \to \mathbb{R}$ satisfying (4.15) for every finite coordinate change ϕ coincide modulo μ.

Conversely, let $f \colon X \to \mathbb{R}$ be a continuous map which decays sufficiently fast; for the following discussion we assume for simplicity that f depends only on finitely many coordinates, i.e. that there exists a finite set $E \subset \mathbb{Z}^d$ with $f(x) = f(y)$ for all $x, y \in X$ which coincide on E. Then the map $c_f \colon \Delta_X \to \mathbb{R}$ with

$$c_f(x,y) = \sum_{\mathbf{n} \in \mathbb{Z}^d} (f(\sigma^{\mathbf{n}}x) f(\sigma^{\mathbf{n}}y)) \tag{4.16}$$

for every $(x,y) \in \Delta_X$ is well-defined and satisfies that

$$c_f(x,y) + c_f(y,z) = c_f(x,z)$$

for all $(x,y), (x,z) \in \Delta_X$. A probability measure μ on X is a *Gibbs measure* of f if μ is quasi-invariant unter Δ_X and $\rho_\mu = c_f \pmod{\mu}$. For background we refer to [52]–[53].

From the definition of a measure of maximal entropy it is clear that any such measure must be a Gibbs state of the function 0 on Δ_X. However, even if a shift of finite type X has a unique measure of maximal entropy, the function 0 may have many different equilibrium states. In [12] it was shown that the *Domino Shift* has a unique shift-invariant measure of maximal entropy; however, the function 0 has uncountably many shift-invariant and ergodic equilibrium states with different entropies (cf. [16] and [20]–[21]).

Part 5. Algebraic $\mathbb{Z}^d$-actions

In this part we discuss dynamical properties of $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups. Although this class of actions is obviously very special, it is diverse enough to exhibit many of the new phenomena encountered in the transition from $\mathbb{Z}$ to $\mathbb{Z}^d$ while nevertheless lending itself to systematic study. The key to the study of these actions is their connection with commutative algebra and arithmetical algebraic geometry. By exploiting this algebraic setting one can construct a rich supply of examples with specified dynamical properties, and by combining algebraic and dynamical tools one obtains a sufficiently detailed understanding of this class of $\mathbb{Z}^d$-actions to glimpse at least the beginnings of a general theory of $\mathbb{Z}^d$-actions.

5.1. The structure of algebraic $\mathbb{Z}^d$-actions

An *algebraic* $\mathbb{Z}^d$*-action* is an action $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ of $\mathbb{Z}^d$ by continuous automorphisms of a compact abelian group X. If $d = 1$ we will not distinguish notationally between a single automorphism α of a compact abelian group X and the $\mathbb{Z}$-action $n \mapsto \alpha^n$ defined by α.

As was shown in [33], [54] and [57], commutative algebra provides a natural and effective formalism for algebraic $\mathbb{Z}^d$-actions. In particular, any such action is Markov under a mild finiteness condition (such as expansiveness).

Let us begin with a few elementary dynamical facts about algebraic $\mathbb{Z}^d$-actions. If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the normalized Haar measure λ_X of X is obviously invariant under α. Two algebraic $\mathbb{Z}^d$-actions α and α' on compact abelian groups X and X' are *algebraically conjugate* if there exists a continuous group isomorphism $\phi\colon X \to X'$ with

$$\phi \cdot \alpha^{\mathbf{n}} = \alpha'^{\mathbf{n}} \cdot \phi \tag{5.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$. Topological conjugacy of α and α' was defined in Section 1.2. Finally we call α and α' *measurably conjugate* if there exists a measure space isomorphism $\phi\colon (X, \mathcal{B}_X, \lambda_X) \to (X', \mathcal{B}_{X'}, \lambda_{X'})$ satisfying (5.1) λ_X-*a.e.* for every $\mathbf{n} \in \mathbb{Z}^d$.

Properties of α like *ergodicity*, (r-)*mixing*, *completely positive entropy* or *Bernoullicity* are always understood with reference to the measure λ_X.

Let $d \geq 1$, and let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. We write a typical element $f \in R_d$ as

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \tag{5.2}$$

with $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $c_f(\mathbf{m}) = 0$ for all but finitely many $\mathbf{m}$.

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. The additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \widehat{\alpha^{\mathbf{m}}}(a) \tag{5.3}$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{m}}}$ is the automorphism of $M = \widehat{X}$ dual to $\alpha^{\mathbf{m}}$. In particular,

$$u^{\mathbf{m}} \cdot a = \widehat{\alpha^{\mathbf{m}}}(a) \tag{5.4}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in M$. The module M is called the *dual module* of the $\mathbb{Z}^d$-action α.

Conversely, any R_d-module M determines an algebraic $\mathbb{Z}^d$-action α_M on the compact abelian group $X_M = \widehat{M}$ with $\alpha_M^{\mathbf{m}}$ dual to multiplication by $u^{\mathbf{m}}$ on M for every $\mathbf{m} \in \mathbb{Z}^d$ (cf. (5.4)). Note that X_M is metrizable if and only if M is countable.

The simplest examples of R_d-modules are those of the form $M = R_d/I$, where $I \subset R_d$ is an ideal (such modules are called *cyclic*). A module M is *Noetherian* if every strictly increasing sequence of submodules

$$M \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$$

is finite. Since the ring R_d itself is Noetherian (as a module over itself), the module M is Noetherian if and only if it is finitely generated, i.e. if and only if there exist elements $a_1, \dots, a_m \in M$ with $M = R_d \cdot a_1 + \dots + R_d \cdot a_m$. In particular, every cyclic R_d-module is Noetherian.

Examples 5.1. Let $d \geq 1$.

(1) Let $M = R_d$. Since R_d is isomorphic to the direct sum $\sum_{\mathbb{Z}^d} \mathbb{Z}$ of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}^d$, the dual group $X = \widehat{R_d}$ is isomorphic to the Cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}}) = (x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d)$ with $x_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and choose the following identification of $X^{R_d} = \widehat{R_d}$ and $\mathbb{T}^{\mathbb{Z}^d}$: for every $x \in \mathbb{T}^{\mathbb{Z}^d}$ and $f \in R_d$,

$$\langle x, f \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}}, \tag{5.5}$$

where f is given by (5.2). Under this identification the $\mathbb{Z}^d$-action α^{R_d} on $X_{R_d} = \mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action (2.1).

(2) Let $I \subset R_d$ be an ideal, and let $M = R_d/I$. Since M is a quotient of the additive group R_d by an $\widehat{\alpha_{R_d}}$-invariant subgroup (i.e. by a submodule), the dual group $X_M = \widehat{M}$ is the α_{R_d}-invariant subgroup

$$\begin{aligned} X_{R_d/I} &= \{x \in X_{R_d} = \mathbb{T}^{\mathbb{Z}^d} : \langle x, f \rangle = 1 \text{ for every } f \in I\} \\ &= \Bigl\{ x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = 0 \pmod 1 \\ &\qquad\qquad \text{for every } f \in I \text{ and } \mathbf{m} \in \mathbb{Z}^d \Bigr\}, \end{aligned} \tag{5.6}$$

and $\alpha_{R_d/I}$ is the restriction of the shift-action α_{R_d} to the shift-invariant subgroup $X_M \subset \mathbb{T}^{\mathbb{Z}^d}$.

Conversely, let $X \subset \mathbb{T}^{\mathbb{Z}^d} = \widehat{R_d}$ be a closed subgroup, and let

$$X^{\perp} = \{f \in R_d : \langle x, f \rangle = 1 \text{ for every } x \in X\}$$

be the annihilator of X in $\widehat{R_d}$. Then X is shift-invariant if and only if $X^{\perp}$ is an ideal in R_d.

It is sometimes useful to write Equation (5.6) is a slightly different form. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$ given by (5.3)–(5.4). For every $f \in R_d$ we denote by

$$f(\alpha) = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \alpha^{\mathbf{m}} \colon X \to X \tag{5.7}$$

the group homomorphism dual to multiplication by f on M (cf. (5.3)). In this notation

$$X_{R/I} = \bigcap_{f \in I} \ker(f(\alpha_{R_d})). \tag{5.8}$$

Conversely, if $S \subset R_d$ is any subset, then

$$X_{R/(S)} = \bigcap_{f \in S} \ker(f(\alpha_{R_d})), \tag{5.9}$$

where $(S) = S \cdot R_d$ is the ideal in R_d generated by S.

(4) Let M be a Noetherian R_d-module, and let $\{a_1, \dots, a_k\}$ be a set of generators for M, i.e. $M = R_d \cdot a_1 + \dots + R_d \cdot a_k$. The surjective homomorphism $(f_1, \dots, f_k) \mapsto f_1 \cdot a_1 + \dots + f_k \cdot a_k$ from R_d^k to M induces a dual injective homomorphism $\phi\colon X_M \to X_{R_d^k} \cong (\mathbb{T}^k)^{\mathbb{Z}^d} = Y$ such that $\phi \cdot \alpha_M^{\mathbf{n}} = \alpha_{R_d^k}^{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$. In particular, ϕ embeds X_M as a closed, shift-invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$. Conversely, if $X \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup, then $\hat{X} = R_d^k / X^{\perp}$, and $X^{\perp}$ is a submodule of R_d^k.

We identify $(\mathbb{T}^k)^{\mathbb{Z}^d}$ with $(\mathbb{T}^{\mathbb{Z}^d})^k$ in the obvious manner and set, for every $a = (a^{(1)}, \dots, a^{(k)}) \in R_d^k$ and $y = (y^{(1)}, \dots, y^{(k)}) \in (\mathbb{T}^{\mathbb{Z}^d})^k$,

$$\langle y, a \rangle = \prod_{j=1}^{k} \langle y^{(j)}, a^{(j)} \rangle \tag{5.10}$$

(cf. (5.5)). Equation (5.10) allows us to view R_d^k as the dual group of $(\mathbb{T}^k)^{\mathbb{Z}^d}$. As R_d^k is a Noetherian module over R_d, it has a finite generator $a_1, \dots, a_l$, and

$$\phi(X_M) = \{y \in (\mathbb{T}^k)^{\mathbb{Z}^d} : \langle u^{\mathbf{m}} \cdot a_i, y \rangle = 1 \text{ for every } i = 1, \dots, l \text{ and } \mathbf{m} \in \mathbb{Z}^d\}.$$

In particular, $\phi(X_M) \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ is Markov.

For further examples, background and details we refer to [33], [54] and [57].

5.2. Dynamical properties of algebraic $\mathbb{Z}^d$-actions

A prime ideal $\mathfrak{p} \subset R_d$ is *associated with* an R_d-module M if

$$\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$$

for some $a \in M$. The set of (distinct) prime ideals associated with a Noetherian R_d-module M is finite. For details we refer to [36].

Since an algebraic $\mathbb{Z}^d$-action α is completely determined by its dual module M, one can in principle express all dynamical properties of α by properties of M. The following results establish a partial correspondence between algebraic properties of M and dynamical properties of α_M and show that the prime ideals associated with an R_d-module M contain much of the dynamical information about the $\mathbb{Z}^d$-action α determined by this module. For convenience we summarise these results in Figure 1 on page 41.

If $I \subset R_d$ is an ideal we denote by

$$V_{\mathbb{C}}(I) = \{c = (c_1, \dots, c_d) \in (\mathbb{C}^{\times})^d : f(c) = 0 \text{ for every } f \in I\} \tag{5.11}$$

the *variety* of I, where $\mathbb{C}^{\times} = \mathbb{C} \smallsetminus \{0\}$.

Theorem 5.2. ([54] and Theorem 6.5 in [57]). *Let $\mathfrak{p} \subset R_d$ be a prime ideal and $\alpha = \alpha_{R_d/\mathfrak{p}}$ the algebraic $\mathbb{Z}^d$-action on $X = X_{R_d/\mathfrak{p}} = \widehat{R_d/\mathfrak{p}}$ defined by* (5.3)–(5.4).

(1) *For every $\mathbf{n} \in \mathbb{Z}^d$ the following conditions are equivalent.*
   (a) *$\alpha^{\mathbf{n}}$ is ergodic,*
   (b) *$\mathfrak{p} \cap \{u^{l\mathbf{n}} - 1 : l \geq 1\} = \varnothing$.*
(2) *The following conditions are equivalent.*
   (a) *α is ergodic,*
   (b) *$\alpha^{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$,*
   (c) *$\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \Gamma\} \not\subset \mathfrak{p}$ for every subgroup of finite index $\Gamma \subset \mathbb{Z}^d$.*
(3) *The following conditions are equivalent.*
   (a) *α is mixing* (*either topologically or w.r.t. λ_X*),
   (b) *for every nonzero $\mathbf{n} \in \mathbb{Z}^d$, $\alpha^{\mathbf{n}}$ is ergodic,*
   (c) *for every nonzero $\mathbf{n} \in \mathbb{Z}^d$, $\alpha^{\mathbf{n}}$ is mixing,*
   (d) *$\mathfrak{p} \cap \{u^{\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} = \{0\}$.*
(4) *Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup of finite index. Then the following conditions are equivalent.*
   (a) *$|\mathrm{Fix}_{\Lambda}(\alpha)| < \infty$,*
   (b) *$V_{\mathbb{C}}(\mathfrak{p}) \cap \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1 \text{ for every } \mathbf{n} \in \Lambda\} = \varnothing$, where $c = (c_1, \dots, c_d)$, $\mathbf{n} = (n_1, \dots, n_d)$ and $c^{\mathbf{n}} = c_1^{n_1} \cdots c_d^{n_d}$.*
(5) *The following conditions are equivalent.*
   (a) *α is expansive,*
   (b) *$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$.*

An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X satisfies the *descending chain condition* if every strictly decreasing sequence

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$$

of closed, α-invariant subgroups is finite. Elementary duality considerations show that α satisfies the descending chain condition if and only if the dual module $M = \widehat{X}$ is Noetherian (cf. Proposition 5.4 in [57]). We can now state our next result.

Theorem 5.3. ([54] and Theorem 6.5, Proposition 6.6 in [57]). *Let M be a countable R_d-module and $\alpha = \alpha_M$ the algebraic $\mathbb{Z}^d$-action on $X = X_M = \widehat{M}$ determined by M.*

(1) *For every $\mathbf{n} \in \mathbb{Z}^d$ the following conditions are equivalent.*
   (a) *$\alpha^{\mathbf{n}}$ is ergodic,*
   (b) *$\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated with M.*
(2) *The following conditions are equivalent.*
   (a) *α is ergodic,*
   (b) *$\alpha_{R_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated with M.*

(3) *The following conditions are equivalent.*
   (a) *α is mixing,*
   (b) *$\alpha_{R_d/\mathfrak{p}}$ is mixing for every prime ideal $\mathfrak{p}$ associated with M.*

(4) *The following conditions are equivalent.*
   (a) *α is expansive,*
   (b) *M is Noetherian and $\alpha_{R_d/\mathfrak{p}}$ is expansive for every prime ideal $\mathfrak{p} \subset R_d$ associated with M.*

(5) *If M is Noetherian (or, equivalently, if α satisfies the descending chain condition) the following conditions are equivalent for every subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index.*
   (a) $|\mathrm{Fix}_\Lambda(\alpha)| < \infty$,
   (b) $|\mathrm{Fix}_\Lambda(\alpha_{R_d/\mathfrak{p}})| < \infty$ *for every prime ideal $\mathfrak{p}$ associated with M.*

(6) *If M is Noetherian then α has dense periodic points.*

Corollary 5.4. ([35], Corollaries 3.10–3.11 in [54], Corollaries 6.14–6.15 in [57]). *Let X be a compact group (not necessarily abelian) and α an expansive $\mathbb{Z}^d$-action by automorphisms of X. If $Y \subset X$ is a closed normal α-invariant subgroup then the $\mathbb{Z}^d$-actions α_Y and $\alpha_{X/Y}$ induced by α on Y and X/Y are both expansive. Furthermore, if X is connected, then it is abelian.*

Examples 5.5. Although for $d = 1$ there is no real need for all the algebraic machinery appearing above, it is still instructive to understand single automorphisms of compact abelian groups in this setting.

(1) Let α be the automorphism of $\mathbb{T}^2$ determined by the matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{Z})$ with irreducible characteristic polynomial $f(u) = u^2 - u - 1$. Example 5.1 (2) shows that

$$X_{R_1/(f)} = \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : x_n + x_{n+1} - x_{n+2} = 0 \pmod 1 \text{ for every } n \in \mathbb{Z}\},$$

where $(f) = fR_1 \subset R_1$ is the principal prime ideal generated by f, and $\alpha_{R_1/(f)}$ is the shift (2.1) on X. The coordinate map $\pi_{\{0,1\}} : X \to \mathbb{T}^2$ establishes an algebraic conjugacy between α and $\alpha_{R_1/(f)}$ (cf. (5.1)). Theorem 5.2 implies that α is mixing and expansive.

If we start with a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ which is not conjugate to its companion matrix with $\mathrm{GL}(2, \mathbb{Z})$, such as $A = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, the R_1-module $M = \widehat{\mathbb{T}^2} \cong \mathbb{Z}^2$ associated with this automorphism is slightly more complicated. The characteristic polynomial of A is given by $f(u) = u^2 - 4u - 1$ and is again irreducible, but $M \ncong R_1/(f)$: in this case M has a submodule $N \subset M$ with $|M/N| = 2$ and $N \cong R_1/(f)$. However, (f) is the only prime ideal associated with M, so that the dynamical properties of A are again determined by f (cf. Examples 5.3 (1)–(2) in [57]).

(2) Let α be the automorphism of $\mathbb{T}^n$, $n \geq 2$, determined by a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$, and let $f \in R_1$ be the characteristic polynomial of A. Then the associated prime ideals of the module $\widehat{\mathbb{T}^n}$ are precisely the principal prime ideals arising from the irreducible divisors of f. In conjunction with the Theorems 5.2 and 5.3 this yields the classical assertions that A is ergodic (and hence mixing) if and only if f has no roots which are roots of unity, and that A is expansive if and only if f has no roots of absolute value 1.

If A is equal to the companion matrix of f, i.e. if

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{n-2} & -c_{n-1} \end{bmatrix},$$

where $f = c_0 + \dots + c_{n-1}u^{n-1} + u^n$, then the map $\phi\colon X_{R_1/(f)} \to \mathbb{T}^n$, defined by

$$\phi(x) = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

for every $x \in X_{R_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$, is an algebraic conjugacy between $\alpha_{R_1/(f)}$ and α. In general the matrix A is conjugate to the companion matrix of f over $\mathbb{Q}$, but not over $\mathbb{Z}$, and the dual module $M = \widehat{\mathbb{T}^n}$ of α has a submodule of finite index $N \subset M$ which is isomorphic (as a module) to $R_1/(f)$. From this one obtains easily that there exist continuous surjective finite-to-one group homomorphisms $\phi_1\colon \mathbb{T}^n \to X_{R_1/(f)}$ and $\phi_2\colon X_{R_1/(f)} \to \mathbb{T}^n$ with $\phi_1 \cdot \alpha = \alpha_{R_1/(f)} \cdot \phi_1$ and $\alpha \cdot \phi_2 = \phi_2 \cdot \alpha_{R_1/(f)}$ (cf. Section 9 in [57]).

(3) Consider the irreducible polynomials

$$f_1 = 2, \qquad f_2 = u - 2, \qquad f_3 = 3 - 2u,$$

in R_1. Then $\alpha_i = \alpha_{R_1/(f_i)}$ is the shift on $X_i = X_{R_1/(f_i)}$ with

$$\begin{aligned} X_1 &= \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 2x_n = 0 \pmod 1 \text{ for every } n \in \mathbb{Z}\}, \\ X_2 &= \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 2x_n = x_{n+1} \pmod 1 \text{ for every } n \in \mathbb{Z}\}, \\ X_3 &= \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 3x_n = 2x_{n+2} \pmod 1 \text{ for every } n \in \mathbb{Z}\}, \end{aligned}$$

respectively. From Theorem 5.2 it is clear that α_i is ergodic and expansive in each case. Clearly, α_1 is the full two-shift. We define surjective group homomorphisms $\phi_i\colon X_i \to \mathbb{T}$, $i = 2, 3$, by $\phi_i(x) = x_0$ for every $x = (x_n) \in X_i$. Then $\phi_2 \cdot \alpha = M_2 \cdot \phi_2$, where $M_2 x = 2x$ for every $x \in \mathbb{T}$. In other words, α_2 is multiplication by 2 on $\mathbb{T}$, made invertible. Similarly we see that α_3 corresponds to 'multiplication by 3/2' on $\mathbb{T}$.

Examples 5.6. We use results in this section to determine the dynamics of the systems in Example 2.4.

(1) Let $f = 1 + u_1 + u_2 \in R_2$. As f is irreducible, the principal ideal $(f) = fR_2$ is prime, and Theorem 5.2 shows that $\alpha = \alpha_{R_2/(f)}$ is mixing. If ω is a primitive third root of unity, then $(\omega, \omega^2) \in V_{\mathbb{C}}(f) \cap \mathbb{S}^2$. According to Theorem 5.2 this implies that α is nonexpansive, and that there are uncountably many periodic points with period $\Lambda = 3\mathbb{Z}^2$.

(2) Let $\mathfrak{p} = (2, 1 + u_1 + u_2) \subset R_2$ be the prime ideal generated by 2 and $1 + u_1 + u_2$. As $V_{\mathbb{C}}(f) = \varnothing$, $\alpha = \alpha_{R_2/\mathfrak{p}}$ is expansive, and α is mixing (cf. Theorem 5.2 (3) and (5)).

5.3. Mahler measure and entropy of algebraic $\mathbb{Z}^d$-actions

Although the calculation of entropy of $\mathbb{Z}^d$-actions with $d > 1$ tends to be quite difficult, there exists a precise entropy formula for algebraic $\mathbb{Z}^d$-actions.

The *Mahler measure* of a polynomial $f \in R_d$ is defined as

$$\mathsf{M}(f) = \begin{cases} \exp\bigl(\int_{\mathbb{S}^d} \log |f(\mathbf{s})| \, d\mathbf{s}\bigr) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0, \end{cases} \tag{5.12}$$

where $d\mathbf{s}$ denotes integration with respect to the normalized Haar measure on the multiplicative group $\mathbb{S}^d \subset \mathbb{C}^d$, and where f is regarded as a function on $\mathbb{C}^d$. The Mahler measure of a polynomial is its geometric mean over the multiplicative torus $\mathbb{S}^d$. Note that $\mathsf{M}(f) = \mathsf{M}(u^{\mathbf{n}} f)$ for every $f \in R_d$ and $\mathbf{n} \in \mathbb{Z}^d$.

Lemma 5.7. (Theorem 3.1 in [42] and Proposition 16.1 in [57]). *Let* $f = c_0 + c_1 u + \cdots + c_s u^s \in R_1$ *with* $c_0 c_s \neq 0$. *Then*

$$\log \mathsf{M}(f) = \log |c_s| + \sum_{j=1}^{s} \log^+ |\xi_j|,$$

where $\xi_1, \dots, \xi_s$ *are the roots of* f *and* $\log^+ t = \max\{0, \log t\}$ *for every* $t > 0$.

An element $g \in R_d$ is a *generalized cyclotomic polynomial* if it is of the form $g(u_1, \dots, u_d) = u^{\mathbf{m}} c(u^{\mathbf{n}})$, where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, $\mathbf{n} \neq \mathbf{0}$, and c is a cyclotomic polynomial in a single variable. From Lemma 5.7 it is clear that, for any $f \in R_1$, $\mathsf{M}(f) = 1$ if and only if f is a finite product of generalized cyclotomic polynomials (cf. Lemma 19.1 in [57]). The proof of the corresponding assertion for $f \in R_d$ is nontrivial:

Lemma 5.8. ([6], [62] and Theorem 19.5 in [57]). *Let* $f \in R_d$. *Then* $\mathsf{M}(f) = 1$ *if and only if* f *is a product of generalized cyclotomic polynomials.*

The following theorem shows that, for any prime ideal $\mathfrak{p} \subset R_d$, the entropy of the $\mathbb{Z}^d$-action $\alpha_{R_d/\mathfrak{p}}$ can be expressed as a logarithmic Mahler measure.

If M is an arbitrary R_d-module we call a sequence of submodules

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$$

a *prime filtration* if there exists, for every $j \geq 0$, a prime ideal $\mathfrak{p}_j \subset R_d$ with $M_j/M_{j+1} \cong R_d/\mathfrak{p}$. If M is Noetherian it has a finite prime filtration (cf. [36] or [57], Corollary 6.2).

Theorem 5.9. (Theorems 4.1, Lemma 4.3, Theorem 4.4 in [42] and Theorem 18.1 and Proposition 18.6 in [57]).

(1) *Let* $\mathfrak{p} \subset R_d$ *be a prime ideal. Then*

$$h(\alpha_{R_d/\mathfrak{p}}) = \begin{cases} |\log \mathsf{M}(f)| & \text{if } \mathfrak{p} = (f) \text{ is principal,} \\ 0 & \text{if } \mathfrak{p} \text{ is not principal.} \end{cases}$$

(2) *Let* M *be a Noetherian* R_d*-module,* $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = \{0\}$ *a prime filtration of* M*, and let* $\mathfrak{p}_0, \dots, \mathfrak{p}_{r-1}$ *be a sequence of prime ideals in* R_d *with* $M_j/M_{j+1} \cong R_d/\mathfrak{p}_j$ *for every* $j = 0, \dots, r-1$. *Then*

$$h(\alpha_M) = \sum_{j=0}^{r-1} h(\alpha_{R_d/\mathfrak{p}_j}).$$

(3) *Let M be a countable R_d-module, and let $(N_j,\ j \geq 1)$ be an increasing sequence of Noetherian submodules of M with $M = \bigcup_{j\geq 1} N_j$. Then*

$$h(\alpha_M) = \lim_{j\to\infty} h(\alpha_{N_j}).$$

Note that, in the last statement of Theorem 5.9, α_{N_j} is algebraically conjugate to the $\mathbb{Z}^d$-action induced by α_M on $X_M/N_j^{\perp}$ for every $j \geq 1$.

The next result shows that zero entropy, positive entropy and completely positive entropy are again characterized by the prime ideals associated with the dual module of α.

Theorem 5.10. (Theorem 6.5 in [42] and Proposition 19.4 in [57]). *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $M = \widehat{X}$ be its dual module.*

(1) *$h(\alpha) = 0$ if and only if $h(\alpha_{R_d/\mathfrak{p}}) = 0$ for every prime ideal $\mathfrak{p} \subset R_d$ associated with M;*
(2) *$h(\alpha) > 0$ if and only if $h(\alpha_{R_d/\mathfrak{p}}) > 0$ for some prime ideal $\mathfrak{p} \subset R_d$ associated with M;*
(3) *α has completely positive entropy if and only if $h(\alpha_{R_d/\mathfrak{p}}) > 0$ for every prime ideal $\mathfrak{p} \subset R_d$ associated with M.*

In general, there exists a closed α-invariant subgroup $Y \subset X$ such that $h(\alpha_{X/Y}) = 0$ and α_Y has completely positive entropy, where α_Y and $\alpha_{X/Y}$ are the $\mathbb{Z}^d$-actions induced by α on Y and X/Y.

Theorem 5.11. (Theorem 1.1 in [51] and Theorem 20.8 in [57]). *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $M = \widehat{X}$ be its dual module. The following conditions are equivalent.*

(1) *α has completely positive entropy;*
(2) *α is Bernoulli.*

Special cases of Theorem 5.11 appear in [68] and [69]. For $d = 1$ Theorem 5.11 yields the classical fact that ergodic automorphisms of compact abelian groups are Bernoulli (cf. [3], [27], [39], [43]).

Examples 5.12. (1) For any irreducible $f \in R_1$, Lemma 5.7 and Theorem 5.9 (1) show that

$$h(\alpha_{R_1/(f)}) = \log|c_s| + \sum_{\xi} \log^+|\xi|, \tag{5.13}$$

where the sum is taken over the (nonzero) roots of f, and where c_s is the highest nonzero coefficient of f.

If $f \in R_1$ is reducible, Theorem 5.9 (2) with an appropriately chosen prime filtration of $M = R_1/(f)$ shows that $h(\alpha_{R_1/(f)})$ is again given by (5.13).

(2) Let α be the toral automorphism determined by a matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ with characteristic polynomial $f \in R_1$. From Example 5.5 (2) it is clear that $h(\alpha) = h(\alpha_{R_1/(f)})$ (cf. (5.13)).

(3) Let α be the $\mathbb{Z}^2$-action defined in Example 5.6. Then

$$h(\alpha) = \log \mathsf{M}(1 + u_1 + u_2) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3),$$

where

$$\chi_3(m) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 3, \\ 1 & \text{if } m \equiv 1 \pmod 3, \\ -1 & \text{if } m \equiv 2 \pmod 3, \end{cases}$$

for every $m \in \mathbb{Z}$ and

$$L(s, \chi_3) = \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^s}.$$

For a proof of this and related results see [7], [63] and [57], p. 156ff.

(4) Let $f = 1 + u_1 + u_2 + u_3 \in R_3$, and let $\alpha = \alpha_{R_3/(f)}$ be the $\mathbb{Z}^3$-action determined by the module $R_3/(f)$. Then

$$h(\alpha) = \log \mathsf{M}(f) = \frac{7}{2\pi^2}\zeta(3),$$

where $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$. For a proof we refer to [7], [63] and [57], p. 159f.

(5) Let $f = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1} \in R_2$, and let $\alpha = \alpha_{R_2/(f)}$. Then

$$h(\alpha) = \log \mathsf{M}(f) = \int_0^1 \int_0^1 (4 - 2\cos 2\pi s - 2\cos 2\pi t)\, ds\, dt.$$

By comparing this with (2.9) we see that α has the same entropy as the *even* domino-shift $\mathbf{n} \to \sigma_{W_D}^{2\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^{\mathbf{2}}$ in Example 4.8 (cf. also [64]).

(6) Let $A, B \in \mathrm{GL}(n, \mathbb{Z})$ be commuting automorphisms of $\mathbb{T}^n$, and let α be the $\mathbb{Z}^2$-action on $\mathbb{T}^n$ defined by A and B. Then $h(\alpha) = 0$, as can be seen either from Theorem 5.9, or from the fact that smooth $\mathbb{Z}^d$-actions, $d > 1$, on a finite-dimensional manifold must have zero entropy.

5.4. Higher order mixing

Since ergodic automorphisms of compact abelian groups have completely positive entropy, they are mixing of every order. For algebraic $\mathbb{Z}^d$-actions with $d > 1$ the situation is far more complex.

Theorem 5.13. (Theorem 27.2, Corollary 27.4 and Corollary 27.6 in [57]). *Let $d \geq 2$, and let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$.*

(1) *The following conditions are equivalent for every integer $r \geq 2$.*
   (a) *α is mixing of order r,*
   (b) *$\alpha_{R_d/\mathfrak{p}}$ is r-mixing for every prime ideal $\mathfrak{p} \subset R_d$ associated with M.*
(2) *If X is connected then α is mixing of every order if and only if α is mixing.*
(3) *If X is zero-dimensional* (*i.e. totally disconnected*) *then α is mixing of every order if and only if it has completely positive entropy.*

Example 5.14. Theorem 5.13 implies that algebraic $\mathbb{Z}^d$-actions on compact abelian zero-dimensional groups may be mixing without being mixing of every order. The first such example appeared in [37]: let $M = R_2/(2, 1 + u_1 + u_2)$, and let $\alpha = \alpha_M$ be the $\mathbb{Z}^2$-action on $X = X_M \subset \mathbb{T}^{\mathbb{Z}^2}$ defined in Example 5.6 (2). Then α is mixing (cf. Example 5.6 (2)), but not three-mixing.

Indeed, $(1+u_1+u_2)^{2^n}\cdot a=0$ for every $n\geq 0$ and $a\in M$. For $a=1+(2,1+u_1+u_2)\in M$ our identification of M with $\widehat{X}$ in Example 5.1 (2) implies that $x_{(0,0)}+x_{(2^n,0)}+x_{(0,2^n)}=0 \pmod 1$ for every $x\in X$ and $n\geq 0$. For $B=\{x\in X: x_{(0,0)}=0\}$ it follows that

$$B\cap\alpha^{-(2^n,0)}(B)\cap\alpha^{-(0,2^n)}(B)=B\cap\alpha^{-(2^n,0)}(B),$$

and hence that

$$\lambda_X(B\cap\alpha^{-(2^n,0)}(B)\cap\alpha^{-(0,2^n)}(B))=\lambda_X(B\cap\alpha^{-(2^n,0)}(B))=1/4$$

for every $n\geq 0$. If α were three-mixing, we would obviously have that

$$\lim_{n\to\infty}\lambda_X(B\cap\alpha^{-(2^n,0)}(B)\cap\alpha^{-(0,2^n)}(B))=\lambda_X(B)^3=1/8.$$

In Example 5.14 higher order mixing breaks down in a very 'regular' way. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. A set $S\subset\mathbb{Z}^d$ is *mixing* if

$$\lim_{k\to\infty}\lambda_X\Bigl(\bigcap_{\mathbf{n}\in S}\alpha^{-k\mathbf{n}}(B_{\mathbf{n}})\Bigr)=\prod_{\mathbf{n}\in S}\lambda_X(B_{\mathbf{n}}) \tag{5.14}$$

for every collection $B_{\mathbf{n}}$, $\mathbf{n}\in S$, of Borel sets in X, and *nonmixing* otherwise. As we saw in Example 5.14, the set $S=\{(0,0),(1,0),(0,1)\}$ is nonmixing for the $\mathbb{Z}^2$-action considered there.

The collection of all mixing sets is obviously an invariant of measurable conjugacy (cf. [34] for a systematic discussion of this and related invariants).

We summarize some of the results listed in the Sections 5.2–5.4 in the form of a table on p. 41. In the second column of Figure 1 we assume that the R_d-module $M=\widehat{X}$ defining α is of the form $R_d/\mathfrak{p}$, where $\mathfrak{p}\subset R_d$ is a prime ideal (cf. Example 5.1 (2)), and describe the algebraic condition on $\mathfrak{p}$ equivalent to the dynamical condition on $\alpha=\alpha_{R_d/\mathfrak{p}}$ appearing in the first column. In the third column we consider a Noetherian R_d-module M and state the algebraic property of M corresponding to the dynamical condition on $\alpha=\alpha_M$ in the first column. The last column gives reference to the relevant results in this text.

5.5. Homoclinic points

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and 0_X denote the identity of X. A point $x\in X$ is *homoclinic* under α if

$$\lim_{\mathbf{n}\to\infty}\alpha^{\mathbf{n}}x=0_X. \tag{5.15}$$

The set $\Delta_\alpha(X)$ of α-homoclinic points is a subgroup of X.

For expansive algebraic $\mathbb{Z}^d$-actions both the existence and the abundance of homoclinic points are determined by entropy.

Theorem 5.15. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X.*

(1) *The following conditions are equivalent.*
   (a) $\Delta_\alpha(X)\neq\{0\}$,
   (b) $h(\alpha)>0$.
(2) *The following conditions are equivalent.*
   (a) *$\Delta_\alpha(X)$ is dense in X,*

Property of α	$\alpha = \alpha_{R_d/\mathfrak{p}}$: *property of* $\mathfrak{p}$	$\alpha = \alpha_M$ with M Noetherian: *property of* M	*Reference*
α satisfies the descending chain condition	Always true	Always true	Thm. 5.3
α is expansive	$V_{\mathbb{C}}(p) \cap \mathbb{S}^d = \varnothing$	$\alpha_{R_d/\mathfrak{p}}$ is expansive for every prime ideal $\mathfrak{p}$ associated with M	Thm. 5.3
The set of α-periodic points is dense	Always true	Always true	Thm. 5.3
$\lvert\mathrm{Fix}_\Lambda(\alpha)\rvert < \infty$ for a subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index	$V_{\mathbb{C}}(\mathfrak{p}) \cap \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1$ for every $\mathbf{n} \in \Lambda\} = \varnothing$	$\lvert\mathrm{Fix}_\Lambda(\alpha_{R_d/\mathfrak{p}})\rvert < \infty$ for every prime ideal $\mathfrak{p}$ associated with M	Thm. 5.3
$\alpha^{\mathbf{n}}$ is ergodic or topologically transitive for some $\mathbf{n} \in \mathbb{Z}^d$	$u^{k\mathbf{n}} - 1 \notin\subset p$ for every $k \geq 1$	$\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic for every prime ideal p associated with M	Thm. 5.3
α is ergodic or topologically transitive	$\{u^{k\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \not\subset p$ for every $k \geq 1$	$\alpha_{R_d/\mathfrak{p}}$ is ergodic for every prime ideal p associated with M	Thm. 5.3
α is mixing, either topologically or w.r.t. λ_X	$u^{\mathbf{n}} - 1 \notin \mathfrak{p}$ for every non-zero $\mathbf{n} \in \mathbb{Z}^d$	$\alpha_{R_d/\mathfrak{p}}$ is mixing for every prime ideal $\mathfrak{p}$ associated with M	Thm. 5.3
α is mixing of every order	Either $\mathfrak{p}$ is equal to pR_d for some rational prime p, or $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ and $\alpha_{R_d/\mathfrak{p}}$ is mixing	For every prime ideal $\mathfrak{p}$ associated with M, $\alpha_{R_d/\mathfrak{p}}$ is mixing of every order	Thm. 5.13
$h(\alpha) > 0$	$\mathfrak{p}$ is principal and not generated by a generalized cyclotomic polynomial	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for at least one prime ideal p associated with M	Thm. 5.10
$h(\alpha) < \infty$	$\mathfrak{p} \neq \{0\}$	If M is Noetherian: $\mathfrak{p} \neq \{0\}$ for every prime ideal $\mathfrak{p}$ associated with M	Thms. 5.9 and 5.10
α has completely positive entropy	$h(\alpha^{R_d/p}) > 0$	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for every prime ideal $\mathfrak{p}$ associated with M	Thm. 5.10
α is Bernoulli	$\alpha_{R_d/\mathfrak{p}}$ has completely positive entropy	α_M has completely positive entropy	Thm. 5.11
α has a unique measure of (finite) maximal entropy	$0 < h(\alpha_{R_d/\mathfrak{p}}) < \infty$	α_M has completely positive entropy and $h(\alpha_M) < \infty$	Thm. 6.14 in [42] or Thm. 20.15 in [57]

FIGURE 1

(b) *α has completely positive entropy.*

By using homoclinic points one can show that algebraic $\mathbb{Z}^d$-actions with completely positive entropy have an extremely strong form of specification.

Definition 5.16. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with homoclinic group $\Delta_\alpha(X)$, and let δ be a metric on X. Denote by $B(r)$ the ball in $\mathbb{R}^d$ of radius r. The action α has *homoclinic specification* if there exists, for every $\varepsilon > 0$, an integer $p(\varepsilon) \geq 1$ with the following property: for every rectangle $Q \subset \mathbb{Z}^d$ and every $x \in X$ there exists an α-homoclinic point $y \in \Delta_\alpha(X)$ with

$$\delta(\alpha^{\mathbf{n}}x, \alpha^{\mathbf{n}}y) < \varepsilon \text{ for all } \mathbf{n} \in Q,$$
$$\delta(0_X, \alpha^{\mathbf{n}}y) < \varepsilon \text{ for all } \mathbf{n} \in \mathbb{Z}^d \smallsetminus (Q + B(p(\varepsilon))).$$

With this terminology we have the following result.

Theorem 5.17. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group $X \neq \{0\}$. The following conditions are equivalent.*

(1) *α has completely positive entropy;*
(2) *α has weak specification;*
(3) *α has strong specification;*
(4) *α has homoclinic specification.*

For $d = 1$ there is a close connection between expansiveness and the existence of nontrivial homoclinic points: for example, if α is an irreducible ergodic automorphism of $X = \mathbb{T}^n$, then $\Delta_\alpha(X) \neq \{0\}$ if and only if α is expansive (Example 3.4 in [41]). For $d > 1$ the role of expansiveness is much more complex: nonexpansive $\mathbb{Z}^d$-actions may have no nonzero homoclinic points or uncountably many homoclinic points. For examples we refer to [41].

In addition to specification, homoclinic points also play a role in the construction of symbolic covers and almost topological conjugacies of algebraic $\mathbb{Z}^d$-actions (cf. [19], [58], [59], [60]). Since the results in this direction are very preliminary we restrict ourselves to a brief outline.

Definition 5.18. Let α be an expansive algebraic $\mathbb{Z}^d$-actions on a compact abelian group X. An α-homoclinic point $x \in X$ is *fundamental* if every homoclinic point $y \in X$ is of the form $y = f(\alpha)(x)$ for some $f \in R_d$ (cf. (5.7)).

Proposition 5.19. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X with completely positive entropy. Then α has a fundamental homoclinic point if and only if the dual module $M = \widehat{X}$ of α is of the form $M = R_d/(f)$ for some polynomial $f \in R_d$ with $V_\mathbb{C}(f) \cap \mathbb{S}^d = \varnothing$ (cf. (5.11)).*

For $d = 1$, Proposition 5.19 is Proposition 3.2 in [58]. The proof for $d > 1$ is very similar.

For the remainder of this section we assume that α is an expansive algebraic $\mathbb{Z}^d$-action with completely positive entropy, and with a fundamental homoclinic point x^Δ. Due to expansiveness, x^Δ decays very rapidly: if we use Proposition 5.19 and (5.6) to view X as a closed shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$, and if we write every $x \in X \subset \mathbb{T}^{\mathbb{Z}^d}$ as $(x_\mathbf{n})$ with $x_\mathbf{n} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$, then there exist positive constants $C > 0$ and $\eta < 1$ with

$$|x^\Delta_\mathbf{n}| < C\eta^{\|\mathbf{n}\|} \tag{5.16}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where $|t| = \min_{k\in\mathbb{Z}} |t+k|$ and $\|\mathbf{m}\| = \max\{|m_1|, \dots, |m_d|\}$ for every $t \in \mathbb{T}$ and $\mathbf{m} \in \mathbb{Z}^d$ (cf. Lemma 4.3 in [41]). We write $\ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ for the additive group of all bounded integer-valued maps on $\mathbb{Z}^d$ and obtain a surjective group homomorphism $\xi\colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \to X$, defined by

$$\xi(w) = \sum_{\mathbf{m}\in\mathbb{Z}^d} w_\mathbf{m} \alpha^\mathbf{m} x^\Delta \tag{5.17}$$

for every $w = (w_\mathbf{n}) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. Since the map ξ is continuous in the topology on $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ induced by the weak*-topology on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, there exists a bounded subset $W \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with $\xi(W) = X$.

The construction of a *symbolic representation* of α aims at a choosing bounded and shift-invariant subset $W \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with the following properties:

(a) $\xi(W) = X$,
(b) ξ is *almost one-to-one* in the sense that $|\xi^{-1}(\{x\}) \cap W| = 1$ for all x in a dense G_δ-subset of X,
(c) W is sofic or a shift of finite type in the obvious sense (as $W \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ is bounded, it can be viewed as a subshift of $\mathcal{A}^{\mathbb{Z}^d}$ for some finite set $\mathcal{A} \subset \mathbb{Z}$).

For $d = 1$, the existence such a sofic representation W for every expansive automorphism α of a compact abelian group X with a fundamental homoclinic point was shown in [58]. Other results in this direction can be found in [19], [28], [59] and [60]. For $d > 1$ only a few examples are known; here is one of them, taken from [19].

Example 5.20. Let $f = 3 - u_1 - u_2 \in R_2$, and let $\alpha = \alpha_{R_2/(f)}$ be the $\mathbb{Z}^2$-action on $X = X_{R_2/(f)}$ defined in (5.6). According to Proposition 5.19, α has a fundamental homoclinic point x^Δ. If $\xi\colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \to X$ is the homomorphism (5.17), then the restriction of ξ to $W = \{0, 1, 2\}^{\mathbb{Z}^2} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ is surjective and almost one-to-one in the sense of condition (b) above.

Since the same set W works for the polynomials $f_2 = 3 - u_1^{-1} - u_2$, $f_3 = 3 - u_1 - u_2^{-1}$, $f_4 = 3 - u_1^{-1} - u_2^{-1}$ in R_2, one obtains the following conclusion (cf. Corollary 5.1 in [19]): Let $f_1 = 3 - u_1 - u_2$, $f_2 = 3 - u_1^{-1} - u_2$, $f_3 = 3 - u_1 - u_2^{-1}$, $f_4 = 3 - u_1^{-1} - u_2^{-1}$, and define $X_i = X_{R_2/(f_i)}$, $\alpha_i = \alpha_{R_2/(f_i)}$ as in (5.6) for $i = 1, \dots, 4$. Then the $\mathbb{Z}^2$-actions α_i and α_j are measurably and almost topologically conjugate for $1 \le i < j \le 4$ in the sense that all these actions are almost one-to-one factors of the full three-shift, as described above. Note that these actions cannot be topologically conjugate by Theorem 5.9 in [57].

We remark in passing that this is one of currently very few examples of explicit nontrivial isomorphisms of $\mathbb{Z}^d$-actions (cf. Subsection 4.2.1).

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