ALGEBRA, ARITHMETIC AND MULTI-PARAMETER ERGODIC THEORY

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1. Introduction

While classical ergodic theory deals largely with single ergodic transformations or flows (i.e. with actions of $\mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$ on measure spaces), many of the lattice models in statistical mechanics (such as dimer models) have multi-dimensional symmetry groups: they carry actions of $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d > 1$. However, the transition from $\mathbb{Z}$- or $\mathbb{R}$-actions to multi-parameter ergodic theory presents considerable difficulties, even if one restricts attention to actions of $\mathbb{Z}^d$ with $d \geq 1$ (as we shall do throughout this article).

To illustrate this point, compare the classical theory of topological Markov chains (cf. e.g. [31]) with the complexities and undecidability problems arising in the study of cellular automata and more general multi-dimensional shifts of finite type (cf. [3], [49] or [24]). Even if undecidability is not an issue, multi-dimensional shift of finite type exhibit a markedly more complicated behaviour than their classical relatives (cf. e.g. [10, 11, 36, 41]).

Another feature of the transition from $d = 1$ to $d > 1$ is that smooth $\mathbb{Z}^d$-actions with $d > 1$ on compact manifolds have zero entropy, since individual elements of $\mathbb{Z}^d$ act with finite entropy. The powerful ideas and tools of smooth ergodic theory are thus of limited use for $\mathbb{Z}^d$-actions. Furthermore, smooth $\mathbb{Z}^d$-actions are not exactly abundant: all known examples arise from 'algebraic' constructions (commuting group translations, commuting automorphisms of finite-dimensional tori or solenoids, or actions of Cartan subgroups of semisimple Lie groups on homogeneous spaces). Again one should compare this with the richness of examples in classical smooth ergodic theory which contributes so much to the appeal of the subject.

Making a virtue out of necessity, let us briefly turn to commuting automorphisms of finite-dimensional tori. Toral automorphisms are among the longest and most intensively studied measure-preserving transformations (their investigation contributed much to the formulation and understanding of fundamental dynamical concepts like hyperbolicity and geometrical notions of entropy), and it came as a considerable surprise when Hillel Furstenberg [19] proved in 1967 that unexpected things may happen if one studies not one, but two commuting toral maps: he showed that the only closed infinite subset of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is simultaneously invariant under multiplication by 2 and by 3 is the circle itself (this is a statement about commuting surjective homomorphisms of $\mathbb{T}$, but it has an immediate extension to commuting automorphisms of the 6-adic solenoid). In contrast, each of the two maps consisting of multiplication by 2 and by 3, respectively, is very easily seen to have many infinite closed invariant subsets. In connection with this result Furstenberg asked the famous — and still unanswered — question whether Lebesgue measure is the only nonatomic probability measure on $\mathbb{T}$ which is simultaneously invariant under multiplication by 2 and by 3.

A partial answer to Furstenberg's question was given by D. Rudolph in [38], where he showed that Lebesgue measure is the only nonatomic probability measure on $\mathbb{T}$ which is ergodic under the $\mathbb{N}^2$-action generated by multiplication by 2 and by 3, and which has positive entropy under at least one of these maps. The results by Furstenberg on invariant sets and by Rudolph on invariant measures have subsequently been extended to commuting toral and solenoidal automorphisms by D. Berend ([1, 2]), A. Katok and R. Spatzier ([22]) and M. Einsiedler and E. Lindenstrauss in [15].

In 1978, Ledrappier [30] presented another surprising example: two commuting automorphisms of a compact abelian group such that the $\mathbb{Z}^2$-action generated by them is mixing, but not mixing of higher order (the problem whether there exists a *single* finite measure preserving transformation with this property is one of the famous unresolved questions in ergodic theory).

These examples by Furstenberg and Ledrappier sparked off a systematic investigation of $\mathbb{Z}^d$-actions by commuting automorphisms of compact groups (which will be referred to as *algebraic* $\mathbb{Z}^d$*-action* throughout this article). A key ingredient of this study, which began in 1989–1990 with the papers [25], [32], [39] and [40], is the connection of algebraic $\mathbb{Z}^d$-actions with commutative algebra and arithmetical algebraic geometry. By combining ideas and methods from these areas with standard tools of ergodic theory one can obtain a great deal of insight into these actions and effectively resolve some rather difficult problems like higher order mixing, entropy calculations or the Bernoulli property. For reasons of space I will not discuss the very intriguing *rigidity properties* of algebraic $\mathbb{Z}^d$-actions (such as scarcity of invariant probability measures and isomorphism rigidity). The interested reader can pursue these topics in the papers [4, 5, 6, 15, 21, 22, 27]. Instead I will focus on the many links between dynamics, algebra and arithmetic which become apparent in the investigation of these actions.

These notes are an expanded and updated version of the lecture [43] by the author at the Third European Congress of Mathematics in Barcelona.

I would like to end this introduction by thanking Michael Baake for bringing the reference [50] to my attention.

2. Algebraic $\mathbb{Z}^d$-actions and their dual modules

Let $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ be an action of $\mathbb{Z}^d$, $d \geq 1$, by continuous automorphisms of a compact abelian group X with Borel field $\mathcal{B}_X$ and normalized Haar measure λ_X. If β is a second algebraic $\mathbb{Z}^d$-action on a compact abelian group Y, then β is an *algebraic factor* of α if there exists a continuous surjective group homomorphism $\phi\colon X \longrightarrow Y$ with

$$\phi \cdot \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \cdot \phi \tag{2.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$. The actions α and β are *finitely equivalent* if each of them is a finite-to-one algebraic factor of the other. If the map ϕ in (2.1) is a group isomorphism then α and β are *algebraically conjugate*. If ϕ is a measure-preserving isomorphism of the measure spaces $(X, \mathcal{B}_X, \lambda_X)$ and $(Y, \mathcal{B}_Y, \lambda_Y)$, and if (2.1) holds λ_X-*a.e.*, then the actions α and β are *measurably conjugate*.

In [25] and [40], Pontryagin duality was shown to imply a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions (up to algebraic conjugacy) and modules over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ with integral coefficients in the commuting variables $u_1, \dots, u_d$ (up to module isomorphism). In order to explain this correspondence we write a typical element $f \in R_d$ as $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} u^{\mathbf{m}}$ with $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $f_{\mathbf{m}} \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $f_{\mathbf{m}} = 0$ for all but finitely many $\mathbf{m}$. A nonzero Laurent polynomial $f \in R_d$ is *irreducible* if it cannot be written as $f = f_1 f_2$ with $f_i \in R_d$ and $f_i \neq \pm u^{\mathbf{m}}$ for every $\mathbf{m} \in \mathbb{Z}^d$ and $i = 1, 2$.

If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the additively-written dual group $M=\widehat{X}$ is a module over the ring R_d with operation

$$f\cdot a=\sum_{\mathbf{m}\in\mathbb{Z}^d} f_{\mathbf{m}}\widehat{\alpha^{\mathbf{m}}}(a) \tag{2.2}$$

for $f\in R_d$ and $a\in M$, where $\widehat{\alpha^{\mathbf{m}}}$ is the automorphism of $M=\widehat{X}$ dual to $\alpha^{\mathbf{m}}$. In particular,

$$u^{\mathbf{m}}\cdot a=\widehat{\alpha^{\mathbf{m}}}(a) \tag{2.3}$$

for $\mathbf{m}\in\mathbb{Z}^d$ and $a\in M$. Conversely, any R_d-module M determines an algebraic $\mathbb{Z}^d$-action α_M on the compact abelian group $X_M=\widehat{M}$ with $\alpha_M^{\mathbf{m}}$ dual to multiplication by $u^{\mathbf{m}}$ on M for every $\mathbf{m}\in\mathbb{Z}^d$ (cf. (2.3)). Note that X_M is metrizable if and only if its *dual module* M is countable.

Examples 2.1. (1) Let $M=R_d$. Since R_d is isomorphic to the direct sum $\sum_{\mathbb{Z}^d}\mathbb{Z}$ of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}^d$, the dual group $X=\widehat{R_d}$ is isomorphic to the Cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. We write a typical element $x\in\mathbb{T}^{\mathbb{Z}^d}$ as $x=(x_{\mathbf{n}})$ with $x_{\mathbf{n}}\in\mathbb{T}$ for every $\mathbf{n}\in\mathbb{Z}^d$ and choose the identification

$$\langle x,f\rangle=e^{2\pi i\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}x_{\mathbf{n}}},\ \ x=(x_{\mathbf{n}})\in\mathbb{T}^{\mathbb{Z}^d},\ \ f=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}u^{\mathbf{n}}\in R_d, \tag{2.4}$$

of $X_{R_d}=\widehat{R_d}$ with $\mathbb{T}^{\mathbb{Z}^d}$. Under this identification the $\mathbb{Z}^d$-action α_{R_d} on $X_{R_d}=\mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action

$$(\alpha_{R_d}^{\mathbf{m}}x)_{\mathbf{n}}=(\sigma^{\mathbf{m}}x)_{\mathbf{n}}=x_{\mathbf{m}+\mathbf{n}}. \tag{2.5}$$

(2) For every $f=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}\in R_d$ we denote by $f(\sigma)\colon\mathbb{T}^{\mathbb{Z}^d}\longrightarrow\mathbb{T}^{\mathbb{Z}^d}$ the group homomorphism

$$f(\sigma)=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}\sigma^{\mathbf{n}}. \tag{2.6}$$

Suppose that $I\subset R_d$ is an ideal and $M=R_d/I$. Since M is a quotient of the additive group R_d by an $\widehat{\alpha_{R_d}}$-invariant subgroup (i.e. by a submodule), the dual group $X_M=\widehat{M}$ is the closed α_{R_d}-invariant subgroup

$$\begin{aligned} X_{R_d/I}&=\{x\in X_{R_d}=\mathbb{T}^{\mathbb{Z}^d}:\langle x,f\rangle=1\ \text{ for every }\ f\in I\}\\ &=\Bigl\{x\in\mathbb{T}^{\mathbb{Z}^d}:\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}x_{\mathbf{m}+\mathbf{n}}=0\pmod 1\\ &\qquad\qquad\text{for every }\ f\in I\ \text{ and }\ \mathbf{m}\in\mathbb{Z}^d\Bigr\}\\ &=\bigcap_{f\in I}\ker f(\sigma)\ \ \text{(cf. (2.6))}, \end{aligned} \tag{2.7}$$

and $\alpha_{R_d/I}$ is the restriction of the shift-action $\sigma=\alpha_{R_d}$ in (2.5) to the shift-invariant subgroup $X_{R_d/I}\subset\mathbb{T}^{\mathbb{Z}^d}$. Conversely, let $X\subset\mathbb{T}^{\mathbb{Z}^d}=\widehat{R_d}$ be a closed subgroup, and let

$$X^{\perp}=\{f\in R_d:\langle x,f\rangle=1\ \text{ for every }\ x\in X\}$$

be the annihilator of X in $\widehat{R_d}$. Then X is shift-invariant if and only if $X^{\perp}$ is an ideal in R_d.

Examples 2.2. (1) Let $d=1$, $c=\frac{1+\sqrt{5}}{2}$, and let $I=\{f\in R_1:f(c)=0\}$. Then I is the principal ideal generated by the irreducible polynomial $h(u)=u^2-u-1\in R_1$, $R_1/I\cong\{f(c):f\in R_1\}=\mathbb{Z}[c]$,

$$X_{R_1/I}=\{x=(x_n)\in\mathbb{T}^{\mathbb{Z}}:x_n+x_{n+1}-x_{n+2}=0\ \ (\text{mod }1)\ \text{ for every }\ n\in\mathbb{Z}\},$$

and $\alpha_{R_1/I}$ is the shift (2.5) on $X_{R_1/I}$.

We define a continuous group homomorphism $\phi\colon X_{R_1/I} \longrightarrow \mathbb{T}^2$ by setting $\phi(x) = \left(\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right)$ for every $x = (x_n) \in X_{R_1/I}$. It is easy to see that ϕ is actually a group isomorphism, and that $\phi \circ \alpha_{R_1/I} = \beta \circ \phi$, where β is the linear automorphism of $\mathbb{T}^2$ defined by the companion matrix $M_h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of the polynomial h. In other words, $\alpha_{R_1/(h)}$ is algebraically conjugate to (the algebraic $\mathbb{Z}$-action defined by) M_h.

(2) Let $n \geq 2$, and let $B \in \mathrm{GL}(n, \mathbb{Z})$ be an irreducible matrix (*irreducible* means that the characteristic polynomial $h = \chi_B \in R_1$ of B is irreducible). We write h as $h = h_0 + h_1 u + \dots + u^n$ and denote by

$$M_h = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & 0 & 1 \\ -h_0 & -h_1 & \cdots & & -h_{n-2} & -h_{n-1} \end{pmatrix}$$

the companion matrix of h. Then B and M_h are conjugate in $\mathrm{SL}(n, \mathbb{Q})$, i.e. there exists an nonsingular $n \times n$ integer matrix C with $C \cdot M_h = B \cdot C$.

Denote by $I = (h) \subset R_1$ the principal ideal generated by h and write $\alpha_{R_1/(h)}$ for the shift on the subgroup $X_{R_1/(h)} \subset \mathbb{T}^{\mathbb{Z}}$ in (2.7). Exactly as in Example (1) we consider the continuous group isomorphism $\phi\colon X_{R_1/I} \longrightarrow \mathbb{T}^n$ given by $\phi(x) = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$ and observe that $\phi \circ \alpha_{R_1/(h)} = M_h \circ \phi$, i.e. that $\alpha_{R_1/(h)}$ is algebraically conjugate to the toral automorphism M_h.

The matrix C defines a continuous, finite-to-one linear group homomorphism $\psi\colon \mathbb{T}^n \longrightarrow \mathbb{T}^n$ with $\psi \circ M_h = \beta \circ \psi$, where β is the linear automorphism of $\mathbb{T}^n$ defined by B. It follows that β is a finite-to-one algebraic factor of $\alpha_{R_1/(h)}$. Similarly one sees that $\alpha_{R_1/(h)}$ is a finite-to-one algebraic factor of β, i.e. that $\alpha_{R_1/(f)}$ and β are finitely equivalent.

(3) Let us call a polynomial $h = h_0 + \dots + h_{n-1}u^{n-1} + h_n u^n$ in R_1 a *unit polynomial* if $|h_0| = |h_n| = 1$. In Example (2) we saw that the automorphisms $\alpha_{R_1/(h)}$ arising from unit polynomials $h \in R_1$ are — up to finite equivalence — in one-to-one correspondence with the toral automorphisms.

Can we find equally familiar models for polynomials in R_1 which are not units? Consider, for example, the polynomial $h = 2 - u$. According to (2.7),

$$X_{R_1/(h)} = \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : x_{n+1} = 2x_n \text{ for every } n \in \mathbb{Z}\},$$

and the map $\phi\colon X_{R_1/(h)} \longrightarrow \mathbb{T}$ defined by $\phi(x) = x_0$ for every $x = (x_n) \in X_{R_1/(h)}$ satisfies that $\phi \circ \alpha_{R_1/(h)} = T_2 \circ \phi$, there $T_2\colon \mathbb{T} \longrightarrow \mathbb{T}$ is the surjective group homomorphism consisting of multiplication by 2. In other words, multiplication by 2 is a 'factor' of $\alpha_{R_1/(h)}$, and it is easy to see that $\alpha_{R_1/(h)}$ is — in an obvious sense — the 'smallest' extension of T_2 to a group automorphism.

Since $R_1/(h) \cong \mathbb{Z}[1/2]$, the group of rational numbers whose denominator is a power of 2, a little bit of classical harmonic analysis shows that

$$X_{R_1/(h)} = \widehat{R_1/(h)} \cong \widehat{\mathbb{Z}[1/2]} \cong (\mathbb{R} \times \mathbb{Q}_2)/\iota(\mathbb{Z}[1/2]) \cong (\mathbb{R} \times \mathbb{Z}_2)/\iota(\mathbb{Z}), \tag{2.8}$$

where $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic rationals and integers, respectively, and where ι denotes diagonal embedding. Under the above isomorphism between $X_{R_1/(h)}$ and $(\mathbb{R} \times \mathbb{Q}_2)/\iota(\mathbb{Z}[1/2])$ the shift $\alpha_{R_1/(h)}$ corresponds to diagonal multiplication by 2 on $\mathbb{R} \times \mathbb{Q}_2$.

For the polynomial $h = 3 - 2u$ we obtain a similar picture:

$$X_{R_1/(h)} = \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 2x_{n+1} = 3x_n \text{ for every } n \in \mathbb{Z}\}$$

and the map $\phi(x) = x_0$ defined above sends $\alpha_{R_1/(h)}$ to 'multiplication by $3/2$' on $\mathbb{T}$. As in (2.8) we find that

$$X_{R_1/(h)} = \widehat{R_1/(h)} \cong \widehat{\mathbb{Z}[1/6]} \cong (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/\iota(\mathbb{Z}[1/6]) \cong (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/\iota(\mathbb{Z}),$$

and that $\alpha_{R_1/(h)}$ corresponds to diagonal multiplication by $3/2$ on the 6-adic solenoid $(\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/\iota(\mathbb{Z}[1/6])$.

In order to understand the automorphism $\alpha_{R_1/(h)}$ for an arbitrary irreducible element $h \in R_1$ we cast our net a little wider and consider irreducible algebraic $\mathbb{Z}^d$-actions.

Definition 2.3. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *irreducible* if every closed α-invariant subgroup $Y \subsetneq X$ is finite.

Exercise 2.4. Let $h \in R_1$ be an irreducible Laurent polynomial. Show that $\alpha_{R_1/(h)}$ is irreducible. Show that this is no longer true if $d \geq 2$.

The following description of all irreducible algebraic $\mathbb{Z}^d$-actions is taken from [40] and [16].

Let K be an algebraic number field, i.e. a finite extension of $\mathbb{Q}$. For every valuation v of K, the completion K_v of K with respect to v is a locally compact, metrizable field. Choose a Haar measure λ_v on K_v (with respect to addition) and denote by $\mathrm{mod}_{K_v} \colon K_v \longrightarrow \mathbb{R}$ the map satisfying

$$\lambda_v(aB) = \mathrm{mod}_{K_v}(a)\lambda_v(B) \tag{2.9}$$

for every $a \in K_v$ and every Borel set $B \subset K_v$. The restriction of mod_{K_v} to K is a valuation which is equivalent to v and is denoted by $|\cdot|_v$. We write $P^{(K)}$, $P_f^{(K)}$, and $P_\infty^{(K)}$ for the sets of places, finite places and infinite places of K (the relevant terminology and results can be found in [12] or [51]).

For every $v \in P^{(K)}$, the sets

$$\mathcal{R}_v = \{a \in K_v : |a|_v \leq 1\}, \qquad \mathcal{R}_v^\times = \{a \in K_v : |a|_v = 1\} \tag{2.10}$$

are compact. If v is finite, then $\mathcal{R}_v$ is the unique maximal compact subring of K_v and is also open, and the ideal

$$\mathcal{P}_v = \{a \in K_v : |a|_v < 1\} \subset \mathcal{R}_v \tag{2.11}$$

is open, closed and maximal. The set

$$\mathfrak{o}_K = \bigcap_{v \in P_f^{(K)}} \{a \in K : |a|_v \leq 1\} \tag{2.12}$$

is the ring of integral elements in K.

Now suppose that $d \geq 1$ and $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^\times)^d$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ and $\bar{\mathbb{Q}}^\times = \bar{\mathbb{Q}} \smallsetminus \{0\}$. We set $K = K_{\mathbf{c}} = \mathbb{Q}(c_1, \dots, c_d) = \mathbb{Q}[c_1^{\pm 1}, \dots, c_d^{\pm 1}]$ and

$$S_{\mathbf{c}} = P_\infty^{(K)} \cup \{v \in P_f^{(K)} : |c_i|_v \neq 1 \text{ for some } i = 1, \dots, d\}. \tag{2.13}$$

The set $S_{\mathbf{c}}$ is finite by [51, Theorem III.3]. We denote by

$$\iota_{\mathbf{c}} \colon K \longrightarrow V_{\mathbf{c}} = \prod_{v \in S_{\mathbf{c}}} K_v \tag{2.14}$$

the diagonal embedding $a \mapsto (a, \dots, a)$, $a \in K$, and put

$$\mathcal{R}_{\mathbf{c}} = \{a \in K : |a|_v \leq 1 \text{ for every } v \in P^{(K)} \smallsetminus S_{\mathbf{c}}\} \supset \mathfrak{o}_K. \tag{2.15}$$

The set $V_{\mathbf{c}}$ is a locally compact algebra over K with respect to coordinate-wise addition, multiplication and scalar multiplication, and $\iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})$ is a discrete, co-compact, additive subgroup of $V_{\mathbf{c}}$, and we put

$$Y_{\mathbf{c}} = V_{\mathbf{c}} \,/\, \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}}). \tag{2.16}$$

According to [42, (7.6)] we may identify $Y_{\mathbf{c}}$ with the dual group of $\mathcal{R}_{\mathbf{c}}$, i.e.

$$Y_{\mathbf{c}} = \widehat{\mathcal{R}_{\mathbf{c}}}. \tag{2.17}$$

By definition,

$$c_i \in \mathcal{R}_{\mathbf{c}}^{\times} = \{a \in \mathcal{R}_{\mathbf{c}} : a^{-1} \in \mathcal{R}_{\mathbf{c}}\} \tag{2.18}$$

for every $1 \le i \le d$. We put, for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$,

$$\mathbf{c}^{\mathbf{n}} = c_1^{n_1} \cdots c_d^{n_d}, \tag{2.19}$$

write every $a \in V_{\mathbf{c}}$ as $a = (a_v) = (a_v,\, v \in S)$ with $a_v \in K_v$ for every $v \in S$, and define a $\mathbb{Z}^d$-action $\bar{\beta}_{\mathbf{c}}$ on $V_{\mathbf{c}}$ by setting

$$\bar{\beta}_{\mathbf{c}}^{\mathbf{n}} a = \iota_{\mathbf{c}}(\mathbf{c}^{\mathbf{n}}) a = (\mathbf{c}^{\mathbf{n}} a_v) \tag{2.20}$$

for every $a = (a_v) \in V_{\mathbf{c}}$ and $\mathbf{n} \in \mathbb{Z}^d$. As $\bar{\beta}_{\mathbf{c}}^{\mathbf{n}}(\iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})) = \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})$ for every $\mathbf{n} \in \mathbb{Z}^d$, $\bar{\beta}_{\mathbf{c}}$ induces an algebraic $\mathbb{Z}^d$-action $\beta_{\mathbf{c}}$ on the compact abelian group $Y_{\mathbf{c}}$ in (2.16) by

$$\beta_{\mathbf{c}}^{\mathbf{n}}(a + \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})) = \bar{\beta}_{\mathbf{c}}^{\mathbf{n}} a + \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}}) \tag{2.21}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in V_{\mathbf{c}}$, whose dual action $\hat{\beta}_{\mathbf{c}} \colon \mathbf{n} \mapsto \hat{\beta}_{\mathbf{c}}^{\mathbf{n}}$ is given by

$$\hat{\beta}_{\mathbf{c}}^{\mathbf{n}} b = \mathbf{c}^{\mathbf{n}} b \tag{2.22}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $b \in \mathcal{R}_{\mathbf{c}} = \widehat{Y_{\mathbf{c}}}$ (cf. (2.17)).

Before stating a description of all irreducible algebraic $\mathbb{Z}^d$-actions up to finite algebraic equivalence we recall two basic dynamical notions.

Definition 2.5. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with normalized Haar measure λ_X. The action α is *ergodic* if $\lambda_X(B) \in \{0, 1\}$ for every α-invariant Borel set $B \subset X$. The action α is *mixing*

$$\lim_{\mathbf{n} \to \infty} \lambda_X(B_1 \cap \alpha^{\mathbf{n}} B_2) = \lambda_X(B_1) \cdot \lambda_X(B_2)$$

for all Borel sets $B_1, B_2 \in X$.

Theorem 2.6 ([40], [16]). *Suppose that α is an algebraic $\mathbb{Z}^d$-action, $d \ge 1$, on an infinite compact connected abelian group X. Then α is irreducible if and only if it is finitely equivalent to the algebraic $\mathbb{Z}^d$-action $\beta_{\mathbf{c}}$ on $Y_{\mathbf{c}}$ for some $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^{\times})^d$ (cf. (2.21)–(2.22)).*

Suppose that α is irreducible.

(1) *The following conditions are equivalent.*
   (a) *α is ergodic,*
   (b) *$\beta_{\mathbf{c}}$ is ergodic,*
   (c) *At least one of the algebraic numbers c_i, $i = 1, \dots, d$, is not a root of unity.*

(2) *The following conditions are equivalent.*
   (a) *α is mixing,*
   (b) *$\beta_{\mathbf{c}}$ is mixing,*
   (c) *For every nonzero $\mathbf{n} \in \mathbb{Z}^d$, $c^{\mathbf{n}} \neq 1$.*

Example 2.7. If $d = 1$ and $\mathbf{c} = 2$, then $\mathcal{R}_{\mathbf{c}} = \mathbb{Z}[1/2]$ and $\beta_{\mathbf{c}}$ is multiplication by 2 on $(\mathbb{R} \times \mathbb{Q}_2)/\iota_{\mathbf{c}}(\mathbb{Z}[1/2])$ (This is, in fact, Example 2.2).

3. A DICTIONARY

The discussion in the Section 2 yields, for every ideal $I \subset R_d$ and, more generally, for every R_d-module M, an algebraic $\mathbb{Z}^d$-action (which will, in general, obviously not be irreducible). The correspondence between algebraic $\mathbb{Z}^d$-actions $\alpha = \alpha_M$ and R_d-modules M yields a correspondence (or 'dictionary') between dynamical properties of α_M and algebraic properties of the module M (cf. [42]). It turns out that some of the principal dynamical properties of α_M can be expressed entirely in terms of the prime ideals associated with the module M, where a prime ideal $\mathfrak{p} \subset R_d$ is *associated with* M if

$$\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$$

for some $a \in M$. The set of all prime ideals associated with M is denoted by $\operatorname{asc}(M)$; if M is Noetherian, then $\operatorname{asc}(M)$ is finite.

Figure 1 provides a small illustration of this correspondence; all the relevant results can be found in [42]. In the third column we assume that the R_d-module $M = \widehat{X}$ defining α is of the form $R_d/\mathfrak{p}$, where $\mathfrak{p} \subset R_d$ is a prime ideal, and describe the algebraic condition on $\mathfrak{p}$ equivalent to the dynamical condition on $\alpha = \alpha_{R_d/\mathfrak{p}}$ appearing in the second column. In the fourth column we consider a countable R_d-module M and state the algebraic property of M corresponding to the property of $\alpha = \alpha_M$ in the second column.

	Property of α	$\alpha = \alpha_{R_d/\mathfrak{p}}$	$\alpha = \alpha_M$
(1)	α is expansive	$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$	M is Noetherian and $\alpha_{R_d/\mathfrak{p}}$ is expansive for every $\mathfrak{p} \in \operatorname{asc}(M)$
(2)	$\alpha^{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$	$u^{k\mathbf{n}} - 1 \notin\subset \mathfrak{p}$ for every $k \geq 1$	$\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic for every $\mathfrak{p} \in \operatorname{asc}(M)$
(3)	α is ergodic	$\{u^{k\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \not\subset \mathfrak{p}$ for every $k \geq 1$	$\alpha_{R_d/\mathfrak{p}}$ is ergodic for every $\mathfrak{p} \in \operatorname{asc}(M)$
(4)	α is mixing	$u^{\mathbf{n}} - 1 \notin \mathfrak{p}$ for every non-zero $\mathbf{n} \in \mathbb{Z}^d$	$\alpha_{R_d/\mathfrak{p}}$ is mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$
(5)	α is mixing of every order	Either $\mathfrak{p}$ is equal to pR_d for some rational prime p, or $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ and $\alpha_{R_d/\mathfrak{p}}$ is mixing	For every $\mathfrak{p} \in \operatorname{asc}(M)$, $\alpha_{R_d/\mathfrak{p}}$ is mixing of every order
(6)	$h(\alpha) > 0$	$\mathfrak{p}$ is principal and $\alpha_{R_d/\mathfrak{p}}$ is mixing	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for at least one $\mathfrak{p} \in \operatorname{asc}(M)$
(7)	$h(\alpha) < \infty$	$\mathfrak{p} \neq \{0\}$	If M is Noetherian: $\mathfrak{p} \neq \{0\}$ for every $\mathfrak{p} \in \operatorname{asc}(M)$
(8)	α has completely positive entropy (or is Bernoulli)	$h(\alpha^{R_d/\mathfrak{p}}) > 0$	$h(\alpha_{R_d/\mathfrak{p}}) > 0$ for every $\mathfrak{p} \in \operatorname{asc}(M)$

FIGURE 1: A POCKET DICTIONARY

The notation in Figure 1 is as follows. In (1),

$$V_{\mathbb{C}}(\mathfrak{p}) = \{c \in (\mathbb{C} \smallsetminus \{0\})^d : f(c) = 0 \ \text{ for every } \ f \in \mathfrak{p}\}$$

is the *variety* of $\mathfrak{p}$, and $\mathbb{S} = \{c \in \mathbb{C} : |c| = 1\}$. From (2)–(4) in Figure 1 it is clear that α is ergodic if and only if $\alpha^{\mathbf{n}}$ is ergodic for *some* $\mathbf{n} \in \mathbb{Z}^d$, and that α is mixing if and only if $\alpha^{\mathbf{n}}$ is ergodic for *every nonzero* $\mathbf{n} \in \mathbb{Z}^d$. In (5), α is mixing of order $r \geq 2$ if

$$\lim_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_r \in \mathbb{Z}^d \\ \|\mathbf{n}_i - \mathbf{n}_j\| \to \infty \ \text{ for } \ 1 \leq i < j \leq d}} \lambda_X\left(\bigcap_{i=1}^r \alpha^{-\mathbf{n}_i} B_i\right) = \prod_{i=1}^r \lambda_X(B_i)$$

for all Borel sets $B_i \subset X$, $i = 1, \dots, r$. In (6)–(8), $h(\alpha)$ stands for the topological entropy of α (which coincides with the metric entropy $h_{\lambda_X}(\alpha)$). For background, details and proofs of these and further results we refer to [42] and the original articles cited there.

The following sections are devoted to two specific notions appearing in Figure 1: the entropies and the mixing behaviour of algebraic $\mathbb{Z}^d$-actions.

4. Entropy and Mahler measure

In [32] and [42] there is an explicit entropy formula for algebraic $\mathbb{Z}^d$-actions. In the special case where $\alpha = \alpha_{R_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R_d$ this formula reduces to

$$h(\alpha) = \begin{cases} |\log \mathsf{M}(f)| & \text{if } \mathfrak{p} = (f) = fR_d \text{ is principal,} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathsf{M}(f) = \begin{cases} \exp\bigl(\int_{\mathbb{S}^d} \log |f(\mathbf{s})|\, d\mathbf{s}\bigr) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0, \end{cases}$$

is the *Mahler measure* of the polynomial f. Here $d\mathbf{s}$ denotes integration with respect to the normalized Haar measure on the multiplicative subgroup $\mathbb{S}^d \subset \mathbb{C}^d$. This connection between entropy and Mahler measure is intriguing for a number of reasons (cf. e.g. [13, 14]).

For our first result on entropy we recall that an element $f \in R_d$ is a *generalized cyclotomic polynomial* if it is of the form $f = u^{\mathbf{m}} c(u^{\mathbf{n}})$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, where $\mathbf{n} \neq \mathbf{0}$ and $c(\cdot)$ is a cyclotomic polynomial in a single variable. The following proposition, taken from [7], [29] and [45], is a direct extension of Kronecker's theorem [28] (cf. also [42]).

Proposition 4.1. *Let $f \in R_d$, $d \geq 1$. Then $h(\alpha_{R_d/(f)}) = \log \mathsf{M}(f) = 0$ if and only if $\pm f$ is a product of generalized cyclotomic polynomials.*

The following examples are taken from [8] and [46] (cf. also [42]). Recall that a *character* (mod q) is a homomorphism $\chi \colon \mathbb{Z} \longmapsto \mathbb{Z}$ with $\chi(0) = 0$, $\chi(1) = 1$, $\chi(m+q) = \chi(m)$, and $\chi(mm') = \chi(m)\chi(m')$ for all $m, m' \in \mathbb{Z}$. The symbols χ_q, $q = 3, 4$, will denote the unique non-trivial characters (mod q) given by

$$\chi_3(m) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 3, \\ 1 & \text{if } m \equiv 1 \pmod 3, \\ -1 & \text{if } m \equiv 2 \pmod 3, \end{cases}$$

$$\chi_4(m) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 2, \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

The *L-function* $L(s, \chi)$ associated with a character χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Examples 4.2. (1) Let $k \in \mathbb{Z}$, and let $f_k = u_1 + u_2 + k \in R_2$. Then

$$h(\alpha_{R_2/(f_k)}) = \log \mathsf{M}(f_k) = \begin{cases} 0 & \text{if } k = 0, \\ \frac{3\sqrt{3}}{4\pi} L(2, \chi_3) & \text{if } |k| = 1, \\ \log |k| & \text{if } |k| \geq 2, \end{cases}$$

(2) Let $k \in \mathbb{Z}$, and let $f_k = (u_1 + u_2)^2 + k \in R_2$. Then

$$h(\alpha_{R_2/(f_k)}) = \log \mathsf{M}(f_k) = \begin{cases} 0 & \text{if } k = 0, \\ \frac{3\sqrt{3}}{2\pi} L(2, \chi_3) & \text{if } |k| = 1, \\ \frac{1}{2}\log 2 + \frac{2}{\pi} L(2, \chi_4) & \text{if } |k| = 2, \\ \frac{2}{3}\log 3 + \frac{\sqrt{3}}{\pi} L(2, \chi_3) & \text{if } |k| = 3, \\ \log |k| & \text{if } |k| \geq 4, \end{cases}$$

(3) Let $f = 1 + u_1 + u_2 + u_3 \in R_3$. Then

$$h(\alpha_{R_3/(f)}) = \log \mathsf{M}(f) = \frac{7}{2\pi^2}\zeta(3),$$

where $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$.

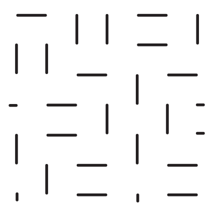
According to Figure 1 (8), all the actions in Example 4.2 with positive entropies are Bernoulli.

The connection between Mahler measure and entropy extends beyond algebraic $\mathbb{Z}^d$-actions. Certain dimer models in statistical mechanics also have topological entropies which are Mahler measures. Why this is so is something of a mystery at this stage.

Examples 4.3. (1) Let $f = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1} \in R_2$. Then

$$h(\alpha_{R_2/(f)}) = \log \mathsf{M}(f) = 4 \cdot h(\sigma_D),$$

where σ_D is the shift-action of $\mathbb{Z}^2$ on the space of 'dimers' consisting of all infinite configurations of exact pairings of elements in $\mathbb{Z}^2$ of the form

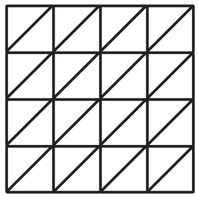


(cf. [20]). In [9] is was shown that this dimer model is Bernoulli with respect to its unique measure of maximal entropy. Since entropy is a complete invariant for measurable conjugacy of $\mathbb{Z}^d$-actions by [23] or [37], $\alpha_{R_2/(f)}$ is measurably conjugate to the 'even' shift-action of $\mathbb{Z}^2$ on the space of dimers, furnished with its measure of maximal entropy (the *even* shift action consists of all shifts by even amounts in the horizontal and vertical direction). In [47] a computational reason for this coincidence of entropies was given, but there is still no satisfactory explanation for the connection between these systems.

(2) This example was pointed out to me by M. Baake. Let $f = 3 - u_1 - u_1^{-1} - u_2 - u_2^{-1} + u_1u_2 + u_1^{-1}u_2^{-1} \in R_2$. Then

$$h(\alpha_{R_2/(f)}) = \log \mathsf{M}(f) = h(\sigma_\Delta),$$

where σ_Δ is the shift-action of $\mathbb{Z}^2$ on the space X_Δ of 'ground states' of the triangular antiferromagnetic lattice, i.e. the closed, shift-invariant subset $X_\Delta \subset \{1, -1\}^{\mathbb{Z}^2}$ consisting of all configurations which have at least two distinct symbols ± 1 on the vertices of each triangle in the infinite triangular lattice



(cf. [50]). The action $\alpha_{R_2/(f)}$ is Bernoulli by Figure 1 (8). Is the action σ_Δ Bernoulli with respect to its (presumably unique) measure of maximal entropy? If so, is there a 'natural' connection between these two measurably conjugate $\mathbb{Z}^2$-actions?

5. Higher order mixing and additive relations in fields

In this section we describe the connection between higher order mixing properties of algebraic $\mathbb{Z}^d$-actions and certain diophantine results on additive relations in fields due to Kurt Mahler ([33]), Masser ([34], [26]) and Schlickewei, W. Schmidt and van der Poorten ([18], [48]). In the discussion below we shall use the following elementary consequence of Pontryagin duality:

Lemma 5.1. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module M. Then X is connected if and only if* no *prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant, and X is zero-dimensional if and only if* every *$\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant.*

Let $\mathfrak{p} \subset R_d$ be a prime ideal, and let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action with dual module $M = R_d/\mathfrak{p} = \widehat{X}$. If α is not mixing, then there exist Borel sets $B_1, B_2 \subset X$ and a sequence $(\mathbf{n}_k,\ k \geq 1)$ in $\mathbb{Z}^d$ with $\lim_{k\to\infty} \mathbf{n}_k = \infty$ and

$$\lim_{k\to\infty} \lambda_X(B_1 \cap \alpha^{-\mathbf{n}_k} B_2) = c$$

for some $c \neq \lambda_X(B_1)\lambda_X(B_2)$. Fourier expansion implies that the latter condition is equivalent to the existence of nonzero elements $a_1, a_2 \in M$ such that

$$a_1 + u^{\mathbf{n}_k} \cdot a_2 = 0$$

for infinitely many $k \geq 1$. In particular,

$$(u^{\mathbf{m}} - 1) \cdot a_2 = 0 \tag{5.1}$$

for some nonzero $\mathbf{m} \in \mathbb{Z}^d$ (cf. Figure 1 (4)). A very similar argument shows that α is not mixing of order $r \geq 2$ if and only if there exist elements $a_1, \dots, a_r$ in M, not all equal to zero, and a sequence $((\mathbf{n}_k^{(1)}, \dots, \mathbf{n}_k^{(r)}),\ k \geq 1)$ in $(\mathbb{Z}^d)^r$ such that $\lim_{k\to\infty} \|\mathbf{n}_k^{(i)} - \mathbf{n}_k^{(j)}\| = \infty$ for all i, j with $1 \leq i < j \leq r$, and with

$$u^{\mathbf{n}_k^{(1)}} \cdot a_1 + \dots + u^{\mathbf{n}_k^{(r)}} \cdot a_r = 0 \tag{5.2}$$

for every $k \geq 1$.

Below we shall see that higher order mixing of an algebraic $\mathbb{Z}^d$-action α on a compact abelian group X can break down in a particularly regular way (cf. Examples 5.7 and 5.10). We call a nonempty finite subset $S \subset \mathbb{Z}^d$ *mixing* under α if

$$\lim_{k\to\infty} \lambda_X\Bigl(\bigcap_{\mathbf{n}\in S} \alpha^{-k\mathbf{n}} B_{\mathbf{n}}\Bigr) = \prod_{\mathbf{n}\in S} \lambda_X(B_{\mathbf{n}}) \tag{5.3}$$

for all Borel sets $B_{\mathbf{n}} \subset X$, $\mathbf{n} \in S$, and *nonmixing* otherwise. If α is r-mixing, then every set $S \subset \mathbb{Z}^d$ with cardinality $|S| = r$ is obviously mixing. The reverse implication for algebraic $\mathbb{Z}^d$-actions is the subject of Theorem 5.11.

As in (5.3) one sees that a nonempty finite set $S \subset \mathbb{Z}^d$ is nonmixing if and only if there exist elements $a_{\mathbf{n}} \in M$, $\mathbf{n} \in S$, not all equal to zero, such that

$$\sum_{\mathbf{n}\in S} u^{k\mathbf{n}} \cdot a_{\mathbf{n}} = 0 \tag{5.4}$$

for infinitely many $k \geq 1$.

The mixing behaviour of an algebraic $\mathbb{Z}^d$-action α with dual module M is again completely determined by that of the actions $\alpha_{R_d/\mathfrak{p}}$ with $\mathfrak{p} \in \operatorname{asc}(\mathfrak{M})$.

Theorem 5.2. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$.*

(1) *For every $r \geq 2$, the following conditions are equivalent:*
  (a) *α is r-mixing,*
  (b) *$\alpha_{R_d/\mathfrak{p}}$ is r-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

(2) *For every nonempty finite set $S \subset \mathbb{Z}^d$, the following conditions are equivalent:*
  (a) *S is α-mixing,*
  (b) *S is $\alpha_{R_d/\mathfrak{p}}$-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

In order to exhibit the connection between mixing properties and additive relations in fields we begin with a celebrated theorem by Kurt Mahler.

Theorem 5.3 ([33])**.** *Let K be a field of characteristic 0, $r \geq 2$, and let $x_1, \dots, x_r$ be nonzero elements of K. If we can find nonzero elements $c_1, \dots, c_r$ such that the equation*

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

has infinitely many solutions $k \geq 0$, then there exist integers $s \geq 1$ and i, j with $1 \leq i < j \leq r$ such that $x_i^s = x_j^s$.

We denote by K the field of fractions of the integral domain $R_d/\mathfrak{p}$, choose a finite set $S = \{\mathbf{n}_1, \dots, \mathbf{n}_r\} \subset \mathbb{Z}^d$ with $r \geq 2$, and set $x_i = u^{\mathbf{n}_i}$ for $i = 1, \dots, r$. In view of Figure 1 (4)–(5), Lemma 5.1, (5.1), (5.4) and Theorem 5.2, Theorem 5.3 implies (and is, in fact, equivalent to) the following statement:

Theorem 5.4 ([39])**.** *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then every nonempty finite subset $S \subset \mathbb{Z}^d$ is mixing.*

If an algebraic $\mathbb{Z}^d$-action α is not mixing of every order, then there exists a smallest integer $r \geq 2$ such that α is not r-mixing. As a consequence of Lemma 5.1 and (5.2) one obtains the equivalence of the Theorems 5.5 and 5.6 below.

Theorem 5.5 ([18], [48])**.** *Let K be a field of characteristic 0 and G a finitely generated multiplicative subgroup of $K^{\times} = K \smallsetminus \{0\}$. If $r \geq 2$ and $(c_1, \dots, c_r) \in (K^{\times})^r$, then the equation*

$$\sum_{i=1}^{r} c_i x_i = 0 \tag{5.5}$$

has only finitely many solutions $(x_1, \dots, x_r) \in G^r$ such that no sub-sum of (5.5) *vanishes.*

Theorem 5.6 ([44])**.** *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then α is mixing of every order.*

The 'absolute' version of the S-unit Theorem 5.5 in [18] and [17] contains a bound on the number of solutions of (5.5) without vanishing subsums which is expressed purely in terms of the integer r and the rank of the group G (in our setting: the order of mixing and the rank of the group $\mathbb{Z}^d$). This bound could be used, for example, to obtain quite remarkable uniform statements on the speed of multiple mixing for all irreducible and mixing algebraic $\mathbb{Z}^d$-actions (cf. Definition 2.3).

For algebraic $\mathbb{Z}^d$-actions on disconnected groups the situation is considerably more complicated due to the possible presence of nonmixing sets (cf. (5.3)).

Example 5.7 ([30])**.** Let $\mathfrak{p} = (2, 1 + u_1 + u_2) = 2R_2 + (1 + u_1 + u_2)R_2$, $M = R_2/\mathfrak{p}$, and let $\alpha = \alpha_M$ be the algebraic $\mathbb{Z}^2$-action on $X = X_M = \widehat{M}$ defined in Example 2.1 (2). Then α is mixing by Figure 1 (4), but not three-mixing.

Indeed, $(1+u_1+u_2)^{2^n} \cdot a = 0$ for every $n \geq 0$ and $a \in M$. For $a = 1+(2, 1+u_1+u_2) \in M$ our identification of M with $\widehat{X}$ in Example 2.1 (2) implies that $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0 \pmod 1$ for every $x \in X$ and $n \geq 0$. For $B = \{x \in X : x_{(0,0)} = 0\}$ it follows that

$$B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B) = B \cap \alpha^{-(2^n,0)}(B),$$

and hence that

$$\lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B \cap \alpha^{-(2^n,0)}(B)) = 1/4$$

for every $n \geq 0$. If α were three-mixing, we would have that

$$\lim_{n\to\infty} \lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B)^3 = 1/8.$$

By comparing this with (5.3) we see that the set $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ is nonmixing.

A mixing algebraic $\mathbb{Z}^d$-action α on a disconnected compact abelian group X has nonmixing sets if and only if it is not Bernoulli (cf. Figure 1 (8), [26] and [42, Section 27]). In particular, if α is an ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X with zero entropy, then α has nonmixing sets. The description of the nonmixing sets of such an action α is facilitated by a Theorem of D. Masser ([26], [34]), which should be seen as an analogue in positive characteristic of Theorem 5.3.

Theorem 5.8. *Let K be an algebraically closed field of characteristic $p > 0$, $r \geq 2$, and let $(x_1, \dots, x_r) \in (K^\times)^r$. The following conditions are equivalent:*

(1) *There exists an element $(c_1, \dots, c_r) \in (K^\times)^r$ such that*

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

for infinitely many $k \geq 0$;

(2) *There exists a rational number $s > 0$ such that the set $\{x_1^s, \dots, x_r^s\}$ is linearly dependent over the algebraic closure $\bar{F}_p \subset K$ of the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$.*

Corollary 5.9. *Let $\mathfrak{p} \subset R_d$ be a prime ideal containing a rational prime $p > 1$, and let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action on $X = X_{R_d/\mathfrak{p}}$ defined in Example 2.1 (2). We denote by $K = Q(R_2/\mathfrak{p}) \supset R_2/\mathfrak{p}$ the quotient field of $R_d/\mathfrak{p}$, write $\bar{K}$ for its algebraic closure, and set $x_{\mathbf{n}} = u^{\mathbf{n}} + \mathfrak{p} \in R_d/\mathfrak{p} \subset K \subset \bar{K}$ for every $\mathbf{n} \in \mathbb{Z}^d$. If $S \subset \mathbb{Z}^d$ is a nonempty finite set, then the following conditions are equivalent:*

(1) *S is not α-mixing;*

(2) *There exists a rational number $s > 0$ such that the set $\{x_1^s, \dots, x_r^s\} \subset \bar{K}$ is linearly dependent over $\bar{F}_p \subset K$.*

Example 5.10 ([26]). In the notation of Examples 5.7 and 2.1 (2) we set $f = 1+u_1+u_2+u_1^2+u_1u_2+u_2^2 \in R_2$ and put $\mathfrak{p} = (2, f) \subset R_2$, $M = R_2/\mathfrak{p}$, $\alpha = \alpha_M$ and $X = X_M = \widehat{M}$. We claim that the set $S = \{(0,0), (1,0), (0,1)\}$ is nonmixing.

In order to verify this we define $\{x_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\} \subset K = Q(R_2/\mathfrak{p})$ as in Corollary 5.9 and choose $\omega \in \bar{F}_2 \subset \bar{K}$ with $1 + \omega + \omega^2 = 0$. Since

$$f = (1 + \omega u_1 + \omega^2 u_2)(1 + \omega^2 u_1 + \omega u_2),$$

we obtain that $x_{(0,0)} + \omega x_{(1,0)} + \omega^2 x_{(0,1)} = 0$, so that S is nonmixing by Corollary 5.9.

Since the element $\omega' = \frac{1+u_1}{u_1+u_2} + \mathfrak{p} \in K$ satisfies that $1 + \omega' + \omega'^2 = 0$, we can recover (5.4) from the fact that

$$(u_1 + u_2) + (1 + u_2)u_1^{3k} + (1 + u_1)u_2^{3k} \in \mathfrak{p}$$

for every $k \geq 0$.

In the paper [35] David Masser proved a (somewhat technical) analogue of the S-unit Theorem 5.5 for fields with positive characteristic, which has the following remarkable dynamical consequence.

Theorem 5.11. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $r \geq 2$. If every subset $S \subset \mathbb{Z}^d$ of cardinality r is mixing, then α is r-mixing.*

The significance of Theorem 5.11 is that it reduces the difficult dynamical problem of determining the precise order of mixing to the slightly more manageable problem of finding nonmixing sets of small cardinality (cf. Corollary 5.9). The latter problem is investigated in [26].

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