Representation theory of a small ramified partition algebra

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Abstract

Here $n \in \mathbb{N}$, k is a field, S_n is the symmetric group, $\delta \in k$, and $P_n(\delta)$ is the partition algebra over k. Our aim in this note is to study the representation theory of a subalgebra $P_n^{\ltimes}$ of $kS_n \otimes_k P_n(\delta)$ with certain interesting combinatorial and representation theoretic properties.

In Section 1 we discuss the motivating combinatorial background. In Section 2 we define $P_n^{\ltimes}$ (see Proposition 1). In Section 3 we determine its complex representation theory.

1 Introduction

The Young graph [11] has vertex set the set Λ of all finite Young diagrams (equivalently of all integer partitions), and encodes the induction and restriction rules for ordinary irreducible modules of the sequence $... \subset S_n \subset S_{n+1} \subset ...$ of symmetric groups [9]. That is, the Young graph is the Bratteli diagram for this sequence [11, §1.1]. It can be considered to lie at the heart of the analysis of these groups, and much of combinatorics [13, 11]. The multiplicity free graph (see Figure 1) and simple associated combinatorics allows a gentle build up of what, eventually, becomes a deep and powerful representation theory [8, 9, 12, 24]. In various areas of Physics [2, 17], algebra [9, 4, 25] and analysis [1, 23] one is led also to study the *wreath products* of symmetric groups:

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \cup & & \cup & & \\ S_2 \wr S_1 & \subset & S_2 \wr S_2 & \subset ... \\ \cup & & \cup & & \\ S_1 \wr S_1 & \subset & S_1 \wr S_2 & \subset ... \end{array}$$

Here however, no such multiplicity free Young graph can exist in general (at least, without further refinement), and one confronts a much more rapid onset of combinatorial complexity. By working at the level of suitable Morita equivalents,

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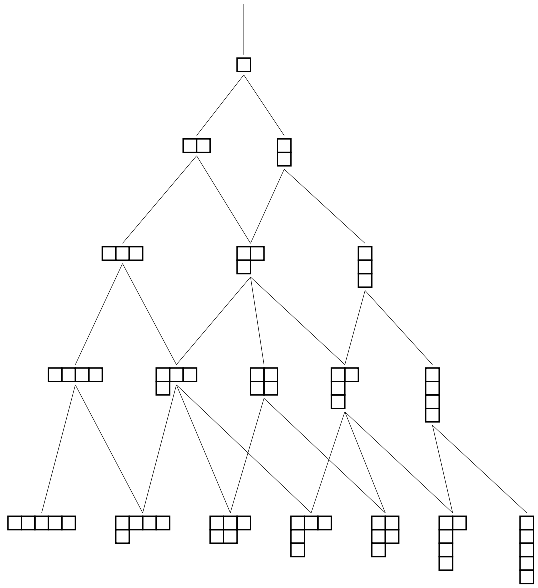
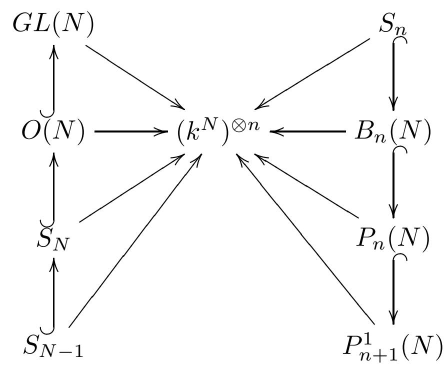


Figure 1: The Young graph up to rank 5

we aim to bypass this obstruction and assemble an analogue of Young's theory of comparable reach. The challenge is to find a sequence of algebras with suitable properties.

For example, as we shall see here, in the Bratteli diagram for the sequence $... \subset P_n^{\ltimes} \subset P_{n+1}^{\ltimes} \subset ...$, the vertex set Λ is replaced by Λ^{Λ^*}, the set of functions from Λ^* to Λ (where Λ^* is the set of finite Young diagrams excluding the empty diagram); while the combinatorial representation theory can also be closely tied to that of wreaths. (We do not claim here that this sequence, encountered by chance while working on a different problem, is the ideal tool for this purpose, but at least that it is worth studying.) The original idea for this approach (inflating, in the sense of [14], from one combinatorial structure to another through Morita equivalences) comes serendipitously from observations on the representation theory of the ramified partition algebra [21].

Partition algebras play potentially important roles in Statistical Mechanics, in combinatorics, and in invariant theory. This is partly captured by the Schur-Weyl duality diagram here:



where the groups/algebras in each layer give a dual pair of (left) actions on tensor space. In each successive layer the action shown on the left-hand side is included in the one above (and the action on the right correspondingly includes the one above). Thus $O(N)$ and $B_n(N)$ are the orthogonal group and the Brauer algebra respectively [6]; S_N acts by permuting the standard ordered basis of k^N and $P_n(N)$ acts by the Potts representation [18, §8.2.1],[10]. The S_{N-1} layer corresponds to breaking the global S_N symmetry of the N-state Potts model by applying a magnetic field[20]. (From an invariant theory perspective this dual pair sequence has been extended below $(S_N, P_n(N))$ in a number of ways. See for example [4, 22].)

The complex reductive representation theory (i.e. Cartan decomposition matrices and so forth, in case $k = \mathbb{C}$) of all the algebras appearing in this diagram is reasonably well understood. It has been noted that *ramified* partition algebras (RPAs) have applications in similar areas [21], but these are much less well understood. Particularly intriguing is the relationship between RPAs and wreaths (which, independently, also have roles in Physics [2, 17] and combinatorics [16]). The ramified partition algebras $P^{\mathbf{2}}_n(\delta', \delta)$ are physically motivated subalgebras of $P_n(\delta') \otimes_k P_n(\delta)$ (see [21] for a definition; δ, δ' are independently chosen parameters). As we shall see in Section 2, we have algebra inclusions

$$P_n(\delta') \otimes_k P_n(\delta) \quad \begin{array}{ccc} \supset & kS_n \otimes_k P_n(\delta) & \supset \\ \supset & P^{\mathbf{2}}_n(\delta', \delta) & \supset \end{array} \quad P_n^{\ltimes}$$

and the representation theory of $P_n^{\ltimes}$ provides, from one perspective, a kind of approximation to that of $P^{\mathbf{2}}_n$ (and hence also to that of the assembly of wreaths). Here, focussing on the representation theory of $P_n^{\ltimes}$, we are able to get pleasingly complete results on this representation theory (see the Theorems in the main section, §3.4). The Bratteli diagram motivating the connection to wreath combinatorics is then discussed in the final section.

2 Definitions

Set $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n}' = \{1', 2', ..., n'\}$ and so on. Write

$$\text{add}' : \underline{n} \cup \underline{n}' \to \underline{n}' \cup \underline{n}''$$

for the map that adds a prime; and $\text{cor}^{-\prime} : \underline{n} \cup \underline{n}'' \to \underline{n} \cup \underline{n}'$ for the map that removes a prime when necessary (i.e. when there are two).

For S a set, $\mathcal{P}_S$ is the set of partitions of S, and $\mathfrak{P}(S)$ the power set. Thus $|\mathcal{P}_{\underline{n}}| = B_n$, the Bell number [15]. We write $(\mathcal{P}_S, >)$ for the usual refinement order on $\mathcal{P}_S$, that is $p > q$ if each part of p is a union of parts of q. This order is a lattice.

Define $\mathcal{P}_n = \mathcal{P}_{\underline{n} \cup \underline{n}'}$. The propagating number

$$\# : \mathcal{P}_n \to \mathbb{N}$$

is the number of parts containing both primed and unprimed elements. We write $\mathcal{P}'_n$ for the subset of partitions in $\mathcal{P}_n$ in which every part contains both primed and unprimed elements.

The algebra $P_n(\delta)$ has a basis $\mathcal{P}_n$. We now briefly recall the algebra product. (We refer the reader to [19] or [21] for a gentler exposition.) For $a \subset \mathfrak{P}(S)$ (some S) write $\overline{a} \in \mathcal{P}_S$ for the most refined (lowest) partition such that each part of $\overline{a}$ is a union of elements of a. Thus for example $a = \{\{1, 2\}, \{2, 3\}, \{4\}\}$ gives $\overline{a} = \{\{1, 2, 3\}, \{4\}\}$. Note that if $p, q \in \mathcal{P}_{\underline{n} \cup \underline{n}'}$ then $p \cup \text{add}'(q) \subset \mathfrak{P}(\underline{n} \cup \underline{n}' \cup \underline{n}'')$ and we can define

$$p \nabla q \ := \ \overline{p \cup \text{add}'(q)} \in \mathcal{P}_{\underline{n} \cup \underline{n}' \cup \underline{n}''}.$$

For $r \in \mathcal{P}_{\underline{n} \cup \underline{n}' \cup \underline{n}''}$ we write $\text{res}(r)$ for the restriction of this partition to $\mathcal{P}_{\underline{n} \cup \underline{n}''}$ (so that $\text{cor}^{-\prime}(\text{res}(r)) \in \mathcal{P}_{\underline{n} \cup \underline{n}'}$); and $c(r)$ for the number of parts containing only elements of $\underline{n}'$. Then the multiplication in $P_n(\delta)$ is defined on pairs p, q from the basis $\mathcal{P}_n$ by

$$p.q \ = \ \delta^{c(p \nabla q)} \ \text{cor}^{-\prime}(\text{res}(p \nabla q)).$$

Note from this construction that the set $\mathcal{P}'_n$ forms a submonoid in $P_n(\delta)$, and that this submonoid contains an isomorphic image of S_n, defined by identifying the transposition $\sigma_i = \ (i, i+1) \in S_n$ with the partition

$$\sigma_i = \{\{1, 1'\}, \{2, 2', \}, ..., \{i, (i+1)'\}, \{(i+1), i'\}, ..., \{n, n'\}\}$$

Write diag-$\mathcal{P}_n$ for the subset of $\mathcal{P}'_n$ of partitions such that i, i' are in the same part for all i. Such partitions are in natural bijection with the partitions of $\underline{n}$, so $|\text{diag-}\mathcal{P}_n| = |\mathcal{P}_{\underline{n}}|$. For example

$$A^{i,j} \ := \{\{1, 1'\}, \{2, 2', \}, ..., \{i, i', j, j'\}, ..., \{n, n'\}\}$$

is in diag-$\mathcal{P}_n$.

We write M^b_n for the monoid generated by $\{A^{ij}\}_{ij}$ and M^d_n for that generated by $S_n \cup \{A^{ij}\}_{ij}$. Define subalgebras of $P_n(\delta)$ generated by subsets: $P^d_n =$

$k\langle S_n, A^{i,j}\rangle_{i,j}$ and $P_n^b = k\langle A^{i,j}\rangle_{i,j}$. (Note that neither subalgebra depends on δ.) These are simply the monoid algebras of the monoids above.

From the form of the partition algebra product we have

Lemma 1 *P_n^b is a commutative algebra with basis diag-$\mathcal{P}_n$ of idempotents. Indeed P_n^b is isomorphic (via the natural bijection) to the monoid algebra of the monoid $(\mathcal{P}_{\underline{n}}, \wedge)$, where $\wedge$ is the meet operation on $(\mathcal{P}_{\underline{n}}, >)$.* $\square$

Note that this remark completely determines the reductive representation theory of P_n^b (as for any finite commutative monoid of idempotents).

The tensor product algebra $kS_n \otimes_k P_n(\delta)$ has basis $S_n \times \mathcal{P}_n$. Just as for S_n [9] and P_n^b, the complex representation theory of $P_n(\delta)$ is well understood [20], and hence so are the tensor products [5]. We get a more challenging new algebra, however, if we proceed as follows. Define an injective map

$$\ltimes : S_n \times \text{diag-}\mathcal{P}_n \to S_n \times \mathcal{P}_n$$

$$(a, b) \mapsto (a, ba)$$

Write $P_n^{\ltimes}$ for the free k-submodule of $kS_n \otimes_k P_n^d$ with basis $\ltimes(S_n \times \text{diag-}\mathcal{P}_n)$.

Proposition 1 *The k-submodule $P_n^{\ltimes}$ is a subalgebra of $kS_n \otimes_k P_n^d$.*

Proof: Multiplication is given by $(a, ba)(c, dc) = (ac, badc)$, but $badc = bada^{-1}ac$, and $bada^{-1} \in P_n^b$. $\square$

Proposition 2 *The algebra $P_n^{\ltimes}$ is generated by $(1, A^{ij})$ and (σ_i, σ_i) (all i, j), and hence by $(1, A^{12})$ and (σ_i, σ_i).* $\square$

We will write $[a, b] = \ltimes(a, b)$. Thus $[a, b][c, d] = [ac, bada^{-1}]$ and in particular

$$[a, 1][1, d] = [a, ada^{-1}] \tag{1}$$

Note that $A^{ij} \mapsto (1, A^{ij})$ defines a natural injection of P_n^b into $P_n^{\ltimes}$; and $\sigma_i \mapsto (\sigma_i, \sigma_i)$ a natural injection of kS_n into $P_n^{\ltimes}$.

Define the set of ($\underline{2}$-)ramified partitions

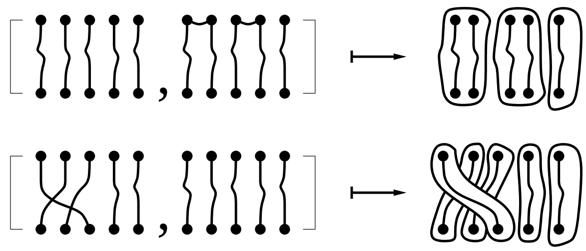
$$\mathcal{P}_n^{\underline{2}} = \{(a, b) \mid a, b \in \mathcal{P}_n;\ a < b\}$$

From [21] this is a basis for the RPA $P_n^{\underline{2}}(\delta, \delta') \subset P_n(\delta) \otimes_k P_n(\delta')$. Note also from the definition of $P_n^{\underline{2}}(\delta, \delta')$ in [21] that $kS_n \otimes_k P_n(\delta')$ is not a subalgebra of $P_n^{\underline{2}}(\delta, \delta')$ (for example, any non-identical pair of permutations lies outside $\mathcal{P}_n^{\underline{2}}$). However

Proposition 3 *We have an algebra inclusion $P_n^{\ltimes} \hookrightarrow P_n^{\underline{2}}$.*

Proof: It is easy to see that elements of form $[1, b]$ and $[s, 1]$ are ramified, and these generate $P_n^{\ltimes}$. $\square$

Remark: We shall not make explicit use of it here, but for those comfortable with the ramified diagram calculus (see in particular [21, Fig.2]) it might well be helpful to note that the diagrams for these generators may be exemplified as follows (in case $n = 5$):



3 Representation theory of $P_n^{\ltimes}$

3.1 Shapes and combinatorics

The *shape* of a set partition is the list of sizes of parts in non-increasing order. Thus the shape of a partition of $\underline{n}$ is an integer partition of n. We will write $b \Vdash \mu$ if set partition b has shape μ.

By convention we shall express shapes in power notation:

$$\mu = (\underbrace{\lambda_1, \lambda_1, ..., \lambda_1}_{p_1}, \underbrace{\lambda_2, \lambda_2, ..., \lambda_2}_{p_2}, ...) \rightsquigarrow \qquad \lambda^p = (\lambda_1^{p_1}, \lambda_2^{p_2}, ...)$$

In particular

$$\lambda^p{}_i = \lambda_i^{p_i}$$

Via this notation a shape can be considered as a pair of a strictly descending integer partition $(\lambda_1, \lambda_2, ...)$ and a composition $(p_1, p_2, ...)$ of the same length.

There is a natural action of S_n on $\mathcal{P}_{\underline{n}}$. For each $b \in \mathcal{P}_{\underline{n}}$ define $S(b)$ as the subgroup that fixes b. We mention two subgroups in $S(b)$: $S^0(b)$ is the group the permutes within parts: $S^0(b) \cong (S_{\lambda_1})^{\times p_1} \times (S_{\lambda_2})^{\times p_2} \times ... \subset S_n$ (in case $b \Vdash \lambda^p$); and $S^1(b)$ permutes parts of equal order: $S^1(b) \cong S_{p_1} \times S_{p_2} \times ... \subset S_n$. We have

$$S(b) \cong \times_i (S_{\lambda_i} \wr S_{p_i}) \tag{2}$$

Considering $S(b)$ or otherwise, the number of parts of given shape is, from [19],

$$\mathcal{D}_{\lambda^p} = \frac{n!}{\prod_i ((\lambda_i!)^{p_i} p_i!)} = \frac{n!}{|S(b \Vdash \lambda^p)|} \tag{3}$$

Write T_b^L (resp. T_b^R) for a traversal of the left (resp. right) cosets of $S(b)$ in S_n. I.e. $\cup_{w \in T_b^L} wS(b)$ is a partition of S_n.

3.2 On representations of wreaths

We shall establish later a construction of irreducible representations of our algebra $P_n^{\ltimes}$ directly in terms of representations of $S(b)$. Accordingly we mention these now. (However the reader may safely skip all the standard material in this section.)

Write Λ for the set of all integer partitions including the empty partition, and Λ_n for the subset of partitions of degree n. For G a group, write $\Lambda_{\mathbb{C}}(G)$ for an index set for ordinary irreducible representations (together, in principle, with a map to explicit representations), so that $\Lambda_{\mathbb{C}}(S_n) = \Lambda_n$. (We will use the analogous notation, $\Lambda_{\mathbb{C}}(A)$, for any algebra A over the complex field.) Set $s_G = |\Lambda_{\mathbb{C}}(G)|$ and assume there is a natural counting. For S, T any sets, write $\mathrm{Hom}(S,T)$ for the set of maps $f : S \to T$. Thus an element V of $\mathrm{Hom}(\Lambda_{\mathbb{C}}(G), \Lambda)$ may be expressed as an s_G-tuple $(V_1, V_2, ..., V_{s_G})$ of integer partitions (a *multipartition*). For any $\mathrm{Hom}(S, \Lambda)$, write $\mathrm{Hom}(S,\Lambda)_n$ for the subset of multipartitions of total degree n.

The ordinary irreducible representation theory of $S(b)$ is, in effect, fairly well understood. Since $\mathbb{C}$ is a splitting field it is enough to study the wreath factors. Now see [9]. In particular we have

Theorem 1 (Cf. [9, COR.4.4.4] or [16, §1.Appendix B])

$$\Lambda_{\mathbb{C}}(G \wr S_n) = Hom(\Lambda_{\mathbb{C}}(G), \Lambda)_n$$

The construction of irreducible L_V, $V \in \Lambda_{\mathbb{C}}(S_l \wr S_n)$ is then as follows. The datum V consists (see [9] or, say, [2]) of a map $V : \Lambda_{\mathbb{C}}(G = S_l) \to \Lambda$ such that $\sum_i |V_i| = n$. We set $v = (v_1, v_2, ...) = (|V_1|, |V_2|, ...)$ and form a traversal T_v of the left cosets of S_v in S_n. Let B^i be a basis for the irreducible representation $\mathcal{S}_i$ in our numbering scheme for irreducible representations of S_l (*lex* order of Λ_l, say); and B^{V_i} a basis for the irreducible representation $\mathcal{S}(V_i)$ of S_{v_i} (note that $V_i \vdash v_i$, so this is via the usual labelling scheme). Thus

$$B^v = \times_i ((B^i)^{\times v_i})$$

is a basis for the irreducible representation of $S_l^{\times n}$ obtained from the representations $(\underbrace{\mathcal{S}_1, \mathcal{S}_1, .., \mathcal{S}_1}_{v_1\ copies}, \underbrace{\mathcal{S}_2, \mathcal{S}_2, .., \mathcal{S}_2}_{v_2\ copies}, ..., \mathcal{S}_s)$ of S_l. Set

$$B_V^v = \times_i ((B^i)^{\times v_i} \times B^{V_i})$$

or rather in the order

$$B_V^v = (\times_i (B^i)^{\times v_i}) \times (\times_i B^{V_i})$$

Then $B_V^v \times T_v$ can be equipped with the property of basis for an (irreducible) representation of $S_l \wr S_n$.

Let $b_1 \otimes ... \otimes b_n \otimes (b_{n+1}..) \otimes [t]$ be an element of this basis. If $\sigma \in S_v$, $t' \in T_v$, then the action of $(g_1, g_2, .., g_n; t'\sigma)$ is given by

$$(g_1, g_2, .., g_n; t'\sigma) \;\; b_1 \otimes ... \otimes b_{v_1} \otimes b_{v_1+1} \otimes ... \otimes b_n \otimes (b_{n+1}..) \otimes [t]$$

$$= g_1 b_{\sigma^{-1}(1)} \otimes ... \otimes g_n b_{\sigma^{-1}(n)} \otimes \sigma(b_{n+1}..) \otimes [t'\sigma t]$$

where $[t'\sigma t]$ is understood as the coset representative of the coset containing this element. (See [9] for a much more detailed exposition, but) Note that the dimension of L_V is clear:

$$\dim L_V = \frac{n!}{\prod_i v_i!} \prod_i d_{V_i}(d_i)^{v_i} \tag{4}$$

where we write d_λ for the dimension of the S_{v_i} Specht module $\mathcal{S}_\lambda$, and d_i for Specht dimensions for S_l labeled using our numbering scheme.

Recall

$$n! = \sum_{\lambda \vdash n} d_\lambda^2 \tag{5}$$

3.3 Useful decompositions of Λ^{Λ^*}

The following will be useful later.

Another way to express an integer partition in an (ascending) power notation is simply as an element α of $\mathrm{Hom}(\mathbb{N}, \mathbb{N}_0)$ of finite support. The construct $(1^{\alpha(1)}, 2^{\alpha(2)}, ...)$ determines an integer partition in ordinary power notation on omitting all terms i such that $\alpha(i) = 0$ and then reversing the order of the remaining terms.
For example $\alpha : (1, 2, 3, 4, ...) = (2, 4, 0, 0, ...)$ becomes $(2^4, 1^2)$.

More generally, to specify a function $\mu \in \mathrm{Hom}(S, T)$, given $\underline{x}$ an ordered list of the elements of S, we may write $\mu : \underline{x} = \underline{y}$, meaning $\mu(x_i) = y_i$ (as in the example immediately above). But if almost all $\mu(x_i) = t_0$, with t_0 some given element of T, then it is convenient to write $\mu = (x_{i_1}, \mu(x_{i_1}))(x_{i_2}, \mu(x_{i_2}))...$ where $\{i_1, i_2, ...\}$ is the set of i such that $\mu(x_i) \neq t_0$. Depending on circumstances, the alternative layout

$$\mu = \frac{x_{i_1}}{\mu(x_{i_1})} \frac{x_{i_2}}{\mu(x_{i_2})} ... \tag{6}$$

may also be useful.
In this notation our example above becomes $\alpha = \frac{2}{4}\frac{1}{2}$ (with $t_0 = 0$).

Let us write $\mathrm{Hom}^f(\Lambda^*, \Lambda)$ for the set of functions

$$\mu : \Lambda^* \to \Lambda$$

with only finitely many $\lambda \in \Lambda^*$ such that $\mu(\lambda) \neq \emptyset$. We also emphasise that Λ is the set of integer partitions of finite integers. Thus the *degree* of $\mu \in \mathrm{Hom}^f(\Lambda^*, \Lambda)$

$$|\mu| = \sum_\lambda |\lambda||\mu(\lambda)|$$

is well defined. Write $\mathrm{Hom}_N(\Lambda^*, \Lambda)$ for the subset of $\mathrm{Hom}^f(\Lambda^*, \Lambda)$ of functions of degree N.
For example, $\mathrm{Hom}_3(\Lambda^*, \Lambda) = \{\frac{(3)}{(1)}, \frac{(21)}{(1)}, \frac{(1^3)}{(1)}, \frac{(2)}{(1)}\frac{(1)}{(1)}, \frac{(1^2)}{(1)}\frac{(1)}{(1)}, \frac{(1)}{(3)}, \frac{(1)}{(21)}, \frac{(1)}{(1^3)}\}$

The *shape* of $\mu \in \mathrm{Hom}^f(\Lambda^*, \Lambda)$ is an integer partition $\kappa(\mu)$ defined as follows. We specify via ascending power notation, in terms of which the partition is given by

$$\alpha(i) = \sum_{\lambda \vdash i} |\mu(\lambda)|$$

and then recast in ordinary power notation as described above.
Example: $\mu : (\emptyset, (1), (2), (1^2), (3), ...) = (\emptyset, \emptyset, (1), (1^3), \emptyset, ...)$ has $\kappa(\mu) = (2^4)$.
We write $\mathrm{Hom}_{\lambda^p}(\Lambda^*, \Lambda)$ for the subset of functions of shape λ^p. We have

$$\mathrm{Hom}_N(\Lambda^*, \Lambda) = \bigcup_{\lambda^p \vdash N} \mathrm{Hom}_{\lambda^p}(\Lambda^*, \Lambda)$$

In the simple case in which κ has just a single 'factor' i^m then $\mathrm{Hom}_{(i^m)}(\Lambda^*, \Lambda)$ is just the set of maps from Λ_i to Λ of total degree m. By Theorem 1 then,

$$\Lambda_{\mathbb{C}}(S_n \wr S_m) = \mathrm{Hom}_{(n^m)}(\Lambda^*, \Lambda)$$

Thus with $b \Vdash \lambda^p$

$$\Lambda_{\mathbb{C}}(S(b)) = \mathrm{Hom}_{\lambda^p}(\Lambda^*, \Lambda)$$

We now have the notation to assert (as we shall show in Theorem 4)

$$\Lambda_{\mathbb{C}}(P_n^{\ltimes}) = \mathrm{Hom}_n(\Lambda^*, \Lambda)$$

3.4 Decomposing the regular $P_n^{\ltimes}$-module

We will say that the shape of $[a, b] = (a, ba)$ is the shape of b. It follows from (1) that the shape of $[a, b] = (a, ba)$ is unchanged by left or right multiplication by $[\sigma_i, 1] = (\sigma_i, \sigma_i)$.

As shapes of set partitions, integer partitions inherit a partial order from the order on set partitions themselves. E.g.

$$\begin{array}{ccccccc} (1^4) & < & (2,1^2) & < & (3,1) & < & (4) \\ & & & < & (2^2) & < & \end{array}$$

Thus left or right multiplication by $[1, A^{ij}]$ either acts like 1 or takes the shape up in this order. Altogether, then, the left regular $P_n^{\ltimes}$-module is filtered by a poset of submodules (indeed ideals) labelled by shape. Set

$$e_{\lambda^p} := \sum_{b \Vdash \lambda^p} [1, b]$$

and note that these are central elements in $P_n^{\ltimes}$. For example $e_{1^n} = [1, 1]$. We have

$$P_n^{\ltimes} e_{\lambda^p} \subset P_n^{\ltimes} e_{\lambda^{p\prime}} \iff \lambda^p > \lambda^{p\prime}$$

The sections $\mathcal{M}_{\lambda^p}$ of this poset each have basis the set of elements of $\ltimes(S_n \times \text{diag-}\mathcal{P}_n)$ of fixed shape. The number of basis elements of shape λ^p is $n! \mathcal{D}_{\lambda^p}$.

We want to decompose the sections as far as possible.

As a vector space we have

$$\mathcal{M}_{\lambda^p} = \bigoplus_{b \Vdash \lambda^p} k[S_n, b] = \bigoplus_{b \Vdash \lambda^p} \bigoplus_{w \in T_b^R} k[S(b)w, b] \tag{7}$$

Note that the $S(b)$-module $k[S_n, b]$ is isomorphic to kS_n as an $S(b)$-module, and hence is simply $\frac{n!}{|S(b)|}$ copies of the regular module.

Consider the quotient algebra of $P_n^{\ltimes}$ by all the ideals $P_n^{\ltimes} e_{\lambda^{p\prime}}$ below λ^p. The central element e_{λ^p} is idempotent in this quotient. Thus we can regard $\mathcal{M}_{\lambda^p}$ as an idempotent subalgebra of the quotient, with identity element e_{λ^p}. The category of left $\mathcal{M}_{\lambda^p}$-modules thus fully embeds in the category of left $P_n^{\ltimes}$-modules [7, §6.2], with the simple modules not hit by this embedding coming from the other $\mathcal{M}_{\lambda^{p\prime}}$.

Now consider the idempotent $[1, b_0]$, $b_0 \Vdash \lambda^p$, and note that in the algebra $\mathcal{M}_{\lambda^p}$ we have $[1, b_0][1, b] = \delta_{b_0,b}[1, b_0]$. We have

$$[1, b_0]\mathcal{M}_{\lambda^p} = [1, b_0] \bigoplus_{w \in S_n; b \Vdash \lambda^p} [w, b] = \bigoplus_{w \in S_n; b \Vdash \lambda^p} [1, b_0][w, b] = \bigoplus_{w \in S_n} [w, b_0]$$

Thus

$$[1, b_0]\mathcal{M}_{\lambda^p}[1, b_0] = \bigoplus_{w \in S_n} [w, b_0][1, b_0] = \bigoplus_{w \in S_n} [w, b_0 w b_0 w^{-1}] = \bigoplus_{w \in S(b_0)} [w, b_0] \cong kS(b_0)$$

and

$$\mathcal{M}_{\lambda^p}[1, b_0]\mathcal{M}_{\lambda^p} = \mathcal{M}_{\lambda^p} \bigoplus_{w \in S_n} [w, b_0] = \left(\bigoplus_{x \in S_n;\, b \Vdash \lambda^p} [x, b] \right) \bigoplus_{w \in S_n} [w, b_0]$$

$$= \bigoplus_{x \in S_n;\, b \Vdash \lambda^p} \bigoplus_{w \in S_n} [x, b][w, b_0] = \bigoplus_{x \in S_n;\, b \Vdash \lambda^p;\, w \in S_n} [xw, bxb_0x^{-1}] = \mathcal{M}_{\lambda^p}$$

Thus

Theorem 2 *The algebras $\mathcal{M}_{\lambda^p}$ and $kS(b_0)$ (with $b_0 \Vdash \lambda^p$) are Morita equivalent.*

Recall that $P_n^{\ltimes}$ has a subalgebra isomorphic to P_n^b. By restricting to this we see that no two sections contain any isomorphic factors. Thus each simple factor will appear in its section with multiplicity given by the dimension of its projective cover (with this dimension bounded from below, ab initio, by the dimension of the simple itself).

It also follows that

$$\Lambda_{\mathbb{C}}(P_n^{\ltimes}) = \dot{\bigcup}_{\lambda^p \vdash n} \Lambda_{\mathbb{C}}(\mathcal{M}_{\lambda^p})$$

so we have determined $\Lambda_{\mathbb{C}}(P_n^{\ltimes})$ (by Theorem 2 and the results in §3.2 – equation(2) and Theorem 1). We will unpack the details shortly.

Next we compute the dimensions of these simple modules, and the overall algebra structure.

Consider the left submodule generated by an arbitrary non-zero element $\sum_{ij} c_{ij}[x_i, y_j]$ of the λ^p-th section, $\mathcal{M}_{\lambda^p}$. Choosing l so that some scalar $c_{il} \neq 0$, then in the section,

$$[1, y_l] \sum_{ij} c_{ij}[x_i, y_j] = \sum_{ij} c_{ij}[1, y_l][x_i, y_j] = \sum_{ij} c_{ij}[x_i, y_l y_j] = \sum_{i} c_{il}[x_i, y_l]$$

Thus this submodule itself contains a submodule generated by $\sum_i c_{il}[x_i, y_l]$. Further, by (1) this submodule contains, for every partition of shape λ^p, an element of this form whose partition part is that partition. (These elements are of course all linearly independent.) Thus

Lemma 2 *Any submodule of $\mathcal{M}_{\lambda^p}$ contains a non-vanishing element of form $\sum_i c_i[x_i, b]$, with $b \Vdash \lambda^p$.*

How does $P_n^{\ltimes} = \langle [1, A^{12}], [S_n, 1] \rangle$ act on this element? As noted, $[1, A^{12}]$ acts as 1 or 0. We consider the action of $[S_n, 1]$ in two parts: $[S(b), 1]$; and a traversal. The first part is simply a copy of $S(b) \hookrightarrow P_n^{\ltimes}$, so the element in Lemma 2 generates at least a simple $S(b)$-module. But since $S(b)$ fixes b, the $S(b)$-module generated will be spanned by elements of this form, so there will be an element of this form which generates precisely a simple $S(b)$-module. Meanwhile the action of an element w of a traversal is

$$w \sum_i c_i[x_i, b] = [w, 1] \sum_i c_i[x_i, b] = \sum_i c_i[wx_i, b^w]$$

Note that the right hand side generates an $S(b^w)$-module that is isomorphic (via the natural group isomorphism) to the original $S(b)$-module. This tells us that every $P_n^{\ltimes}$-submodule of $\mathcal{M}_{\lambda^p}$ decomposes as a vector space in to summands, indexed by $b \Vdash \lambda^p$, the b-th of which is an $S(b)$-module isomorphic (via the various group isomorphisms) to all the other summands. Clearly then, in particular every *simple* $P_n^{\ltimes}$-submodule is at least a sum (as a vector space) of $\mathcal{D}_{\lambda^p}$ spaces each of which is an (isomorphic) simple module for $S(b)$ for the appropriate b.

In particular

Proposition 4 *For each inequivalent simple $S(b)$-module L_μ (i.e. with $\mu \in Hom_{\lambda^p}(\Lambda^*, \Lambda)$ and $\lambda^p \dashv\!\!\!\dashv b$) of dimension m_μ and basis $\{g_i^\mu x_\mu | i = 1, .., m_\mu\}$, say (see §3.2), there is a simple $P_n^{\ltimes}$-module $L_\mu^{\ltimes}$ of dimension*

$$\dim L_\mu^{\ltimes} = m_\mu \mathcal{D}_{\lambda^p} \tag{8}$$

and basis $\{[wg_i^\mu x_\mu, b^w] \mid i = 1, .., m_\mu,\ w \in T_b^L\}$. The modules $\{L_\mu^{\ltimes}\}$ are pairwise inequivalent. □

Similarly,

Theorem 3 *The decomposition of the b-th summand (any b) of $\mathcal{M}_{\lambda^p}$ itself, $S(b)[1, b]$, into a series of simple $S(b)$-modules passes to a complete decomposition of $\mathcal{M}_{\lambda^p}$ into a series of simple $P_n^{\ltimes}$-modules of this construction.*

That is, every simple $P_n^{\ltimes}$-module arises this way (for some λ^p).

Working over k such that $kS(b)$ is split semisimple for every shape (e.g. over the complex numbers), the multiplicity of L_μ in the b-th summand is $m_\mu \mathcal{D}_{\lambda^p}$, since the summand is $\mathcal{D}_{\lambda^p}$ copies of the regular $S(b)$-module. Thus (or by Theorem 2) each $\mathcal{M}_{\lambda^p}$ is semisimple (cf. [3, §1.7] for example), and hence

Theorem 4 *Let $n \in \mathbb{N}$. Over k as above, $P_n^{\ltimes}$ is split semisimple. The simple modules may be indexed by the set $Hom_n(\Lambda^*, \Lambda)$. The dimensions of the simple modules are given by (8), using (2) and (4).* □

We have

$$n!\mathcal{D}_\lambda = \sum_\mu (m_\mu^\lambda \mathcal{D}_\lambda)^2$$

$$n! = \sum_\mu (m_\mu^\lambda)^2 \mathcal{D}_\lambda \tag{9}$$

$$\prod_i ((\lambda_i!)^{p_i} p_i!) = \sum_\mu (m_\mu^{\lambda^p})^2$$

This is not a trivial identity, but it is simply the $S(b)$ version of the hook dimension formula (cf. [13]). Note that the solution to this when $\mathcal{D}_\lambda = 1$ is given *by* the hook dimension formula.

3.5 Examples and combinatorial restriction

First we unpack Theorem 4 a little. The explicit simple module index sets for the first few algebras $P_n^{\ltimes}$ $(n = 1, 2, 3, 4, ...)$ are given in figure 3. In the figure they appear as the vertices in the n-th layer of a certain directed graph. In a vertex, i.e. in an element $\mu \in \mathrm{Hom}_n(\Lambda^*, \Lambda)$, we use here the notation $\frac{\lambda^{|\mu(\lambda)|}}{\mu(\lambda)}$ for each 'factor' (i.e. each remaining $(\lambda, \mu(\lambda))$-pair after removing pairs of form $\frac{\lambda}{\emptyset}$). (We further omit the brackets, if there is only a single factor in μ.) The slight redundancy here (the exponent on the 'numerator' – a redundant addition to our notation in (6)) facilitates some useful consistency checking in practical calculations.

Next we turn attention to restriction rules. The graph in the figure shows the Bratteli diagram of the sequence $P_{n-1}^{\ltimes} \subset P_n^{\ltimes}$ for $n \leq 4$. We define another directed graph $\mathcal{G}$ with vertex set $\mathrm{Hom}(\Lambda^*, \Lambda)$ as follows. Consider an element μ. Each non-trivial factor is of the form $\frac{\lambda}{\mu(\lambda)}$ (or $\frac{\lambda^{|\mu(\lambda)|}}{\mu(\lambda)}$ in the redundant notation) as noted. We first define a linear map M from $\mathbb{Z}\mathrm{Hom}(\Lambda^*, \Lambda)$ to itself by

$$M\mu = \sum_\lambda{}' \sum_j \frac{\lambda}{\mu(\lambda) - e_j} \sum_k \sum_l \frac{\lambda - e_k}{\mu(\lambda - e_k) + e_l} \mu|_{\lambda, \lambda - e_k} \tag{10}$$

where the sum $\sum'_\lambda$ is over partitions not mapped to $\emptyset$ by μ; and all the sums involving rows of partitions are restricted to the appropriate addable or subtractable rows as usual (if $\lambda = (1)$ then the $\sum_k$ nominally consists in a single summand contributing a factor with 'numerator' $(1) - e_1 = \emptyset$ and 'denominator' undefined — this overall-undefined factor is omitted, but the term is kept); and $\mu|_{\lambda,\lambda-e_k}$ means μ with the images of $\lambda, \lambda - e_k$ both omitted (NB, they are replaced by the explicitly given factors). We draw an edge between μ and μ' in $\mathcal{G}$ if μ' appears in $M\mu$ above.

The edges up to level 4 are as in figure 3. For example $M\frac{(2)}{(1)} = \frac{(2)}{\emptyset}\frac{(1)}{\emptyset+e_1} = \frac{(1)}{(1)}$, and $M\frac{(1)^2}{(2)} = \frac{(1)}{(1)}\frac{\emptyset}{-} = \frac{(1)}{(1)}$ omitting the undefined factor. A more challenging example is $\frac{(2)^2(1)^2}{(2)(1^2)}$. Here we have

$$M\mu = \frac{(2)(1)^3}{(1)(21)} + \frac{(2)(1)^3}{(1)(1^3)} + \frac{(2)^2(1)}{(2)(1)} \tag{11}$$

One finds by direct calculation of characters that $\mathcal{G}$ coincides with the Bratteli diagram of the sequence $P^{\ltimes}_{n-1} \subset P^{\ltimes}_n$ for $n \leq 4$ (i.e. as shown in the figure). We conjecture that the graphs coincide.

We do not discuss a proof here, but a heuristic explanation for the form of $\mathcal{G}$ (i.e. the form of this conjecture) is as follows. A representation of the 'right-hand end' of a diagram in a typical simple $P^{\ltimes}_n$-module is as in Figure 2. Here we shall call a collection of 'strings' with symmetrisation λ a λ-string. Thus the last string in line in any diagram (i.e. string n) is involved in a λ-string for some λ. The action of $P^{\ltimes}_{n-1}$ on this $P^{\ltimes}_n$-module excludes the last string, which has the following effect. (We assume that the reader is familiar with induction and restriction rules for S_n.) Firstly it 'destroys' a λ-string into $\mu(\lambda)$, so we need to restrict $\mu(\lambda) \rightsquigarrow \sum_j \mu(\lambda) - e_j$ (cf. the first term in (10)). At the same time this creates a new $\lambda - e_k$-string for each suitable k. And for each such k this extra string gives rise to an induction on $\mu(\lambda - e_k)$, hence $\mu(\lambda - e_k) \rightsquigarrow \sum_l \mu(\lambda - e_k) + e_l$ for each suitable l. (Note that the overall degree of every term produced in this way is $n - 1$, as required.)

To explicitly illustrate the application of the structure Theorem we compute the dimensions of the modules in the example (11) above. For $\frac{(2)^2(1)^2}{(2)(1^2)}$ itself we need $\mathcal{D}_{2^2 1^2} = \frac{6!}{(2!)^2 2!(1!)^2 2!} = 45$, $\dim L_{\frac{(2)}{(2)}} = \frac{2!}{2!} d_{(2)} d^2_{(2)} = 1$, and $\dim L_{\frac{(1)}{(1^2)}} = \frac{2!}{2!} d_{(1^2)} d^2_{(1)} = 1$ (from the formula (4)), giving $\dim L^{\ltimes}_{\mu} = 45$. For the summands on the right we have $\mathcal{D}_{21^3} = \frac{5!}{2!3!} = 10$, and $\mathcal{D}_{2^2 1} = \frac{5!}{(2!)^2 2!} = 15$. We have $\dim L_{\frac{(1^3)}{(21)}} = \frac{3!}{3!} d_{(21)} d^3_{(1)} = 2$, and all the other $\dim L_V$s are 1. Altogether then $\dim L^{\ltimes}_{\frac{(2)}{(1)}\frac{(1^3)}{(21)}} = 20$, $\dim L^{\ltimes}_{\frac{(2)}{(1)}\frac{(1^3)}{(1^3)}} = 10$, and $\dim L^{\ltimes}_{\frac{(2)^2}{(2)}\frac{(1)}{(1)}} = 15$. Note finally, then, that these dimensions indeed obey $45 = 20 + 10 + 15$.

3.6 Discussion

We remark that there is an established setting in which the Faa di Bruno coefficients $\mathcal{D}_{\lambda^p}$ appear *ensemble* in a way intriguingly analogous to their role in $P^{\ltimes}_n$. This is in the combinatorics of *Bell matrices* (that is, of Taylor series of composite functions) [1]. Let $g(x) = \sum_{i=1} g_i \frac{x^i}{i!}$ be a formal power series with

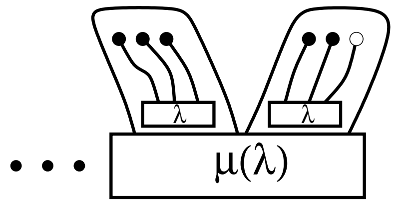


Figure 2: Right-hand end of a representative diagram in a simple module over $\mathbb{C}$, with the last string marked.

$g(0) = 0$. The *Bell matrix* is the matrix whose j^{th} column contains the coefficients of the corresponding power series for $\frac{g^j(x)}{j!}$ (see e.g. [1, (13.66)]). This begins

$$B[g] = \begin{pmatrix} g_1 & 0 & & & & & \\ g_2 & g_1^2 & 0 & & & & \\ g_3 & 3g_1g_2 & g_1^3 & 0 & & & \\ g_4 & 4g_1g_3 + 3g_2^2 & 6g_1^2g_2 & g_1^4 & 0 & & \\ g_5 & 5g_1g_4 + 10g_2g_3 & 10g_1^2g_3 + 15g_1g_2^2 & 10g_1^3g_2 & g_1^5 & 0 & \\ g_6 & \begin{matrix} 6g_1g_5 + 15g_2g_4 \\ +10g_3^2 \end{matrix} & \begin{matrix} 15g_1^2g_4 + 60g_1g_2g_3 \\ +15g_2^3 \end{matrix} & \begin{matrix} 20g_1^3g_3 \\ +45g_1^2g_2^2 \end{matrix} & 15g_1^4g_2 & g_1^6 & 0 \\ \ldots & & & & & & \end{pmatrix}$$

The coefficients (*within* the entries) are the Faa di Bruno coefficients. The intriguing point is that these coincide with the multiplicities $\mathcal{D}_{\lambda^p}$ from (3) (i.e. with $\mathcal{D}_{\lambda^p}$ the coefficient of $\prod_i g_{\lambda_i}^{p_i}$). It would be interesting to extend this connection to the dimensions of the simple modules of $P_n^{\ltimes}$.

The sum of coefficients within the n, k-th entry in $B[g]$ is the Stirling number of the second kind [15], [18, §8.3.2]. The sum of coefficients in the n-th row is B_n. Now, these last two facts give a connection with the partition algebra (using the combinatorial analysis in [19]). But a deeper connection between representation theory and this branch of analysis would be quite interesting to explore. (We include a brief report on some partial results in this direction in the Appendix.) One feature of the Bell matrix is that it gives a kind of linear realisation of the semigroup of compositions of suitable functions. Working formally over finite commutative rings one can study the subgroup of permutations of the ring realised by polynomial functions. Once again wreaths arise in this context, so it would be interesting to articulate a connection along these lines.

Refering back to our generalised Schur-Weyl duality diagram, we note that, besides S_{N-1}, the centralizers for various other actions of subgroups of S_N on tensor space, including wreath subgroups, have been studied (although not, as yet, with physical motivation). Any such commutator is, by construction, an extension of the partition algebra action; and several have been explicitly cast as such, and used to study invariant theory. See for example [4, 22].

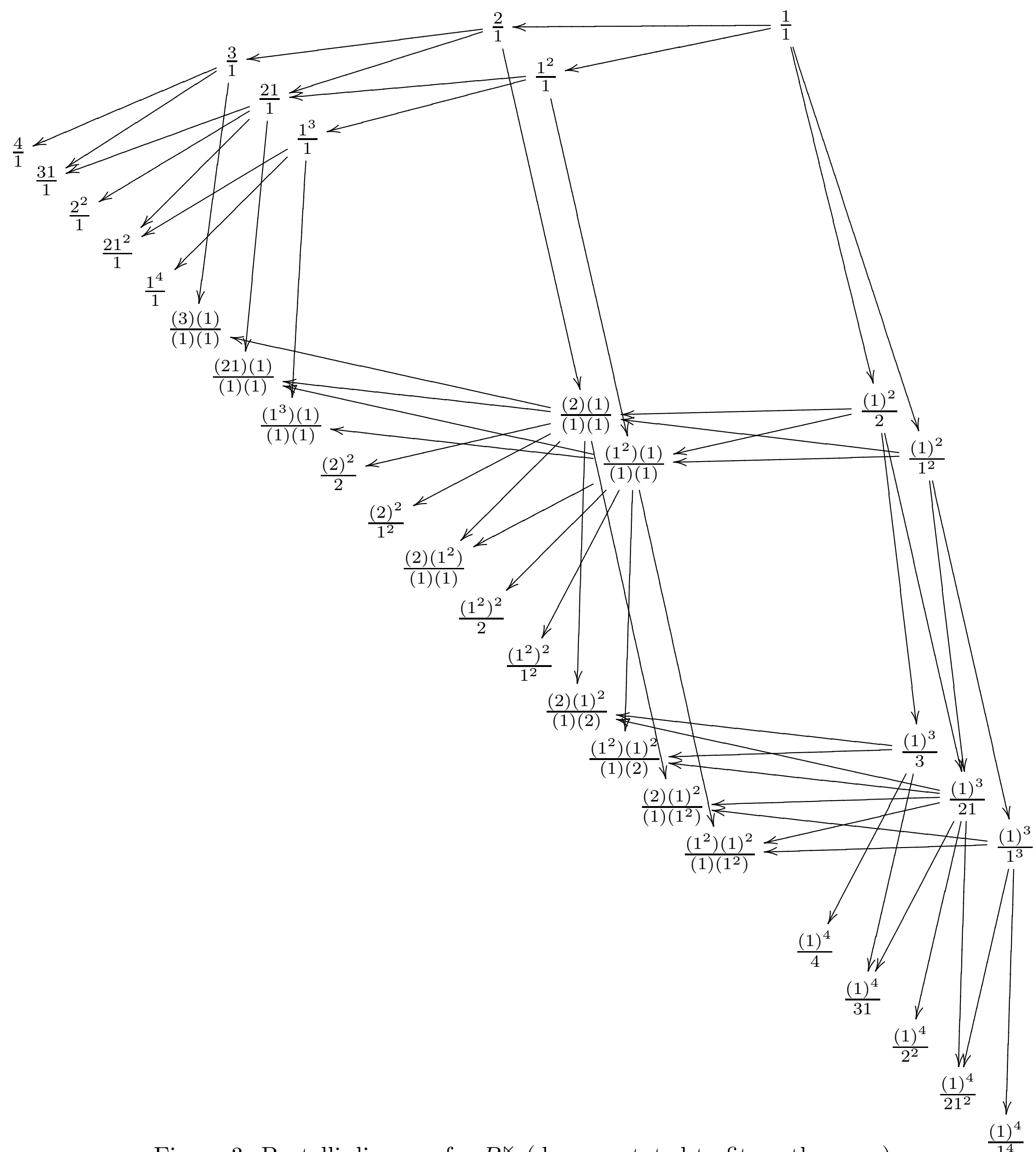


Figure 3: Bratelli diagram for $P_n^{\ltimes}$ (drawn rotated to fit on the page).

From the original physical perspective (i.e. of statistical mechanical transfer matrix algebras [18]), the fixing of a particular wreath subgroup, for example, seems somewhat arbitrary [17]. Here, as we see, we are able to treat large collections of types of wreath all together. This both removes the arbitrariness, and also provides a means to study some universal properties. In particular studying restriction rules for the algebras $P^{\ltimes}_{n-1} \subset P^{\ltimes}_n$ manifests a kind of universal restriction for wreaths (see Section 3.5), much closer to the beautiful restriction rules for ordinary symmetric groups [9, 24] than are available for specific wreaths (i.e. via $G' \subset G$ or $S_{n-1} \subset S_n$ in $G \wr S_n$). However, this is pure representation theory. The question of possible physical interpretations for $P^{\ltimes}_n$ remains open.

Acknowledgements.
The earlier version of this paper that has been hosted on my webpage for the last year or so (http://www.maths.leeds.ac.uk/∼ppmartin/pdf/baby08.pdf) is not fun to read, so warm thanks to Robert Marsh for reading it, and for many valuable comments; and to Mark Wildon for reading the newer draft.

Appendix: On the Bell matrix and graph walks

Again following [19], let $\{v_i \mid i \in \mathbb{N}\}$ be a basis for a seminfinite vector space, and define semiinfinite matrices by

$$\mathcal{N} v_i = i v_i, \qquad \mathcal{D}_- v_i = v_{i-1} \qquad (set\ \ v_0 = 0)$$

$$\mathcal{M} = \mathcal{N} + \mathcal{D}_-$$

Then the first column of $\mathcal{M}^n$ gives the level-n Stirling numbers of the second kind (whose sum is B_n).

Recall that Λ is the set of all integer partitions. Define another semiinfinite vector space $\mathbb{C}^\Lambda$ with basis $\{w_\lambda \mid \lambda \in \Lambda\}$. Define a matrix $\mathcal{L}_+$ by

$$\mathcal{L}_+ w_\lambda = \sum_i w_{\lambda + e_i}$$

where the sum is over rows of λ such that $\lambda + e_i \in \Lambda$. This is the incidence matrix of the directed Young graph (see for example [11]). Thus the λ-th entry in the first column of $\mathcal{L}_+^n$ gives the number of walks of length n from the empty diagram to λ, and hence the dimension of the corresponding irreducible $\mathbb{C}S_n$-module.

Aiming to address the restriction conjecture, our task here is to come up with an analogous graph-walk construction for the Faa di Bruno coeffients.

For each integer partition define the set of associated *broken partitions* as follows. A broken partition λ^+ is a partition together with a choice of a marked row from each collection of rows of equal length. (For this purpose we consider there to be precisely one row of length zero at the bottom of each partition. Thus this row is always marked, and is not combinatorially significant.) We further require that at most one marked row of λ^+ is other than the top row among its collection of rows of equal length. For example $(2, 1^4)$ has four associated

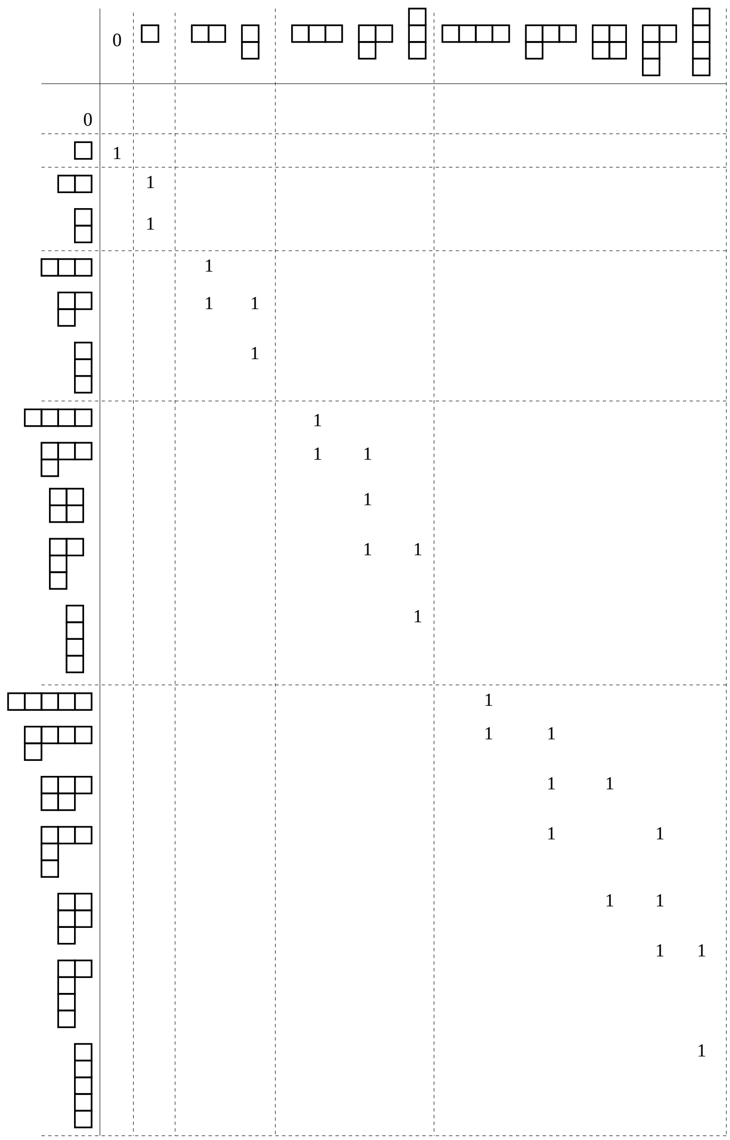


Figure 4: The Young matrix up to rank 5

broken partitions, while $(2^2, 1^2)$ has three. Define Λ^+ as the set of all broken partitions.

If a broken partition has underlying partition λ and marked rows $i, j, ..., k$, we may write $\lambda^{i,j,...,k}$ for this broken partition. Thus $(2,2,1,1)^{1,3} \in \Lambda^+$.

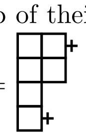
If λ^+ is a broken partition, and row i is marked, then adding a box to this row: $\lambda \to \lambda + e_i$ does not give a partition unless the marked row is the highest among those of length equal to itself. Given such a composition $\lambda + e_i$, we define $\lambda + e_i'$ as the partition that lies in the same orbit (in the usual orbit of compositions) as $\lambda + e_i$.

Define another semiinfinite vector space $\mathbb{C}^{\Lambda^+}$ with basis $\{w^+_\lambda \mid \lambda \in \Lambda^+\}$. Define a matrix $\mathcal{L}^+_+$ acting on $\mathbb{C}^\Lambda \otimes \mathbb{C}^{\Lambda^+}$ by

$$\mathcal{L}^+_+ w_\lambda = \sum_{\lambda+} w^+_{\lambda+}$$

where the sum is over broken partitions associated to λ; and

$$\mathcal{L}^+_+ w^+_{\lambda+} = \sum_i w_{\lambda+e_i'}$$

where the sum is over marked rows i of $\lambda+$, but restricted to those with the property that all *other* marked rows of $\lambda+$ are at the top of their collection of rows of equal length. For example, if λ^+ is $(2,2,1,1)^{1,4} =$  then

$$\mathcal{L}^+_+ w^+_{\lambda+} = w_{(2,2,1,1)+e_4'} = w_{(2,2,1,2)'} = w_{(2,2,2,1)}$$

while $\mathcal{L}^+_+ w^+_{(2,2,1,1)^{1,3}} = w_{(3,2,1,1)} + w_{(2,2,2,1)}$. We postpone explicit analysis of these matrices to another work, but see Figure 5 and compare with $B[g]$.

References

[1] R Aldrovandi, *Special matrices of mathematical physics*, World Scientific, 2001.

[2] P Bantay and G Zala, *Ultrametric matrices and representation theory*, cond-mat/9705041v1 (1997).

[3] D J Benson, *Representations and cohomology I*, Cambridge, 1995.

[4] M Bloss, *g-colored partition algebras as centralizer algebras of wreath products*, Journal of Algebra **265** (2003), 690–710.

[5] C W Curtis and I Reiner, *Representation theory of finite groups and associative algebras*, Wiley Interscience, New York, 1962.

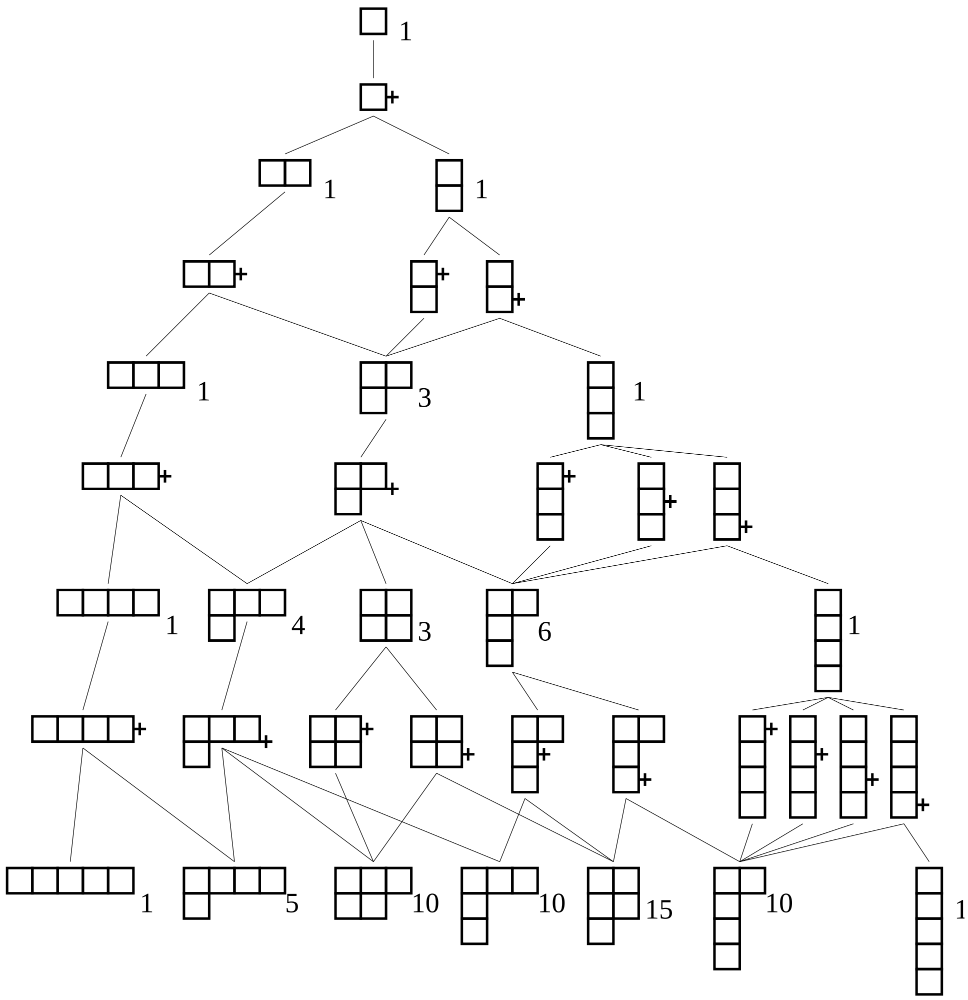


Figure 5: Illustration of the modified Young graph. The number of walks from the root to λ are shown for each 'ordinary' layer.

[6] R Goodman and N R Wallach, *Representations and invariants of the classical groups*, Cambridge, 1998.

[7] J A Green, *Polynomial representations of GL_n*, Springer-Verlag, Berlin, 1980.

[8] M Hamermesh, *Group theory*, Pergamon, Oxford, 1962.

[9] G D James and A Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.

[10] V F R Jones, *The Potts model and the symmetric group*, in Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras, World Scientific, Singapore, 1994.

[11] S V Kerov, *Asymptotic representation theory of the symmetric group and its applications in analysis*, Translations, vol. 219, AMS, 2003.

[12] A Kleshchev, *Branching rules for modular representations of symmetric groups, IV*, J. Algebra **201** (1998), 547–572.

[13] D E Knuth, *Sorting and searching*, 2 ed., The Art of Computer Programming, vol. 3, Addison Wesley, 1998.

[14] S Koenig and C C Xi, *Cellular algebras: inflations and Morita equivalences*, Journal of the LMS **60** (1999), 700–722.

[15] C L Liu, *Introduction to combinatorial mathematics*, McGraw–Hill, New York, 1968.

[16] I Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford, 1995.

[17] P P Martin, *Block spin transformations in the operator formulation of two–dimensional Potts models*, J Phys A **22** (1989), 3991–4005.

[18] ———, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[19] ———, *Temperley–Lieb algebras for non–planar statistical mechanics — the partition algebra construction*, Journal of Knot Theory and its Ramifications **3** (1994), no. 1, 51–82.

[20] ———, *The partition algebra and the Potts model transfer matrix spectrum in high dimensions*, J Phys A **32** (2000), 3669–3695.

[21] P P Martin and A Elgamal, *Ramified partition algebras*, Math. Z. **246** (2004), 473–500, (math.RT/0206148).

[22] M Parvathi, *Signed partition algebras*, Commun. Algebra **32** (2004), 1865–1880.

[23] P Sniady, *Gaussian fluctuations of representations of wreath products*, QUANTUM PROBABILITY AND RELATED TOPICS **9** (2006), 529.

[24] A M Vershik and A Y Okunkov, *An inductive method of expounding the representation theory of symmetric groups*, Russian Mathematical Surveys **51** (1996), 1237–1239.

[25] Weiqiang Wang, *Vertex algebras and the class algebras of wreath products*, Proceedings of the London Mathematical Society **88** (2004), 381.