Notes on blob algebras

P P Martin

Mathematics Department, City University,
Northampton Square, London EC1V 0HB, UK.

1 Introduction

Blob algebras [4] are generalisations of the TL algebra [6] regarded as a diagram algebra [3]. They share with TL and with each other a number of combinatorial structures. We collect and summarize some of these here.

Note that this document is just intended to be a useful collection of formulae. Some appropriate references may be missing.

2 Structure

Exclude this for now.

3 Matrices

For each $d > 1$ define $d \times d$ matrix

$$M_{a,b,c}(d, L, R) = \begin{pmatrix} a & b & 0 & & & & & \\ L & [2] & 1 & 0 & & & & \\ 0 & 1 & [2] & 1 & 0 & & & \\ 0 & 0 & 1 & [2] & 1 & 0 & & \\ & & & & \ddots & & & \\ 0 & \dots & 0 & 0 & 1 & [2] & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & [2] & R \\ 0 & \dots & 0 & 0 & 0 & 0 & c & c \end{pmatrix}$$

Thus

$$\det(M_{a,a,c}(d, L, R)) = ac \det(M_{1,1,1}(d, L, R))$$

and

$$\det(M_{a,1,1}(d, L, R)) = a \det(M_{[2],1,1}(d-1, 1, R)) - L \det(M_{[2],1,1}(d-2, 1, R))$$

In particular

$$\det(M_{[2],1,1}(d, 1, R)) = [2] \det(M_{[2],1,1}(d-1, 1, R)) - \det(M_{[2],1,1}(d-2, 1, R))$$

As is well known, the recurrence

$$M(d) = [2]M(d-1) - M(d-2)$$

is solved by $M(d) = \alpha[r+d]$ for any constants r, α. Noting that $\det(M_{[2],1,1}(2,1,R)) = [2] - R$, if we parametrise by $R = \frac{[r]}{[r+1]}$ we get

$$\det(M_{[2],1,1}(d,1,R)) = \frac{[r+d]}{[r+1]}$$

Altogether then

$$\det(M_{1,1,1}(d,L,R)) = \frac{[r+d-1]}{[r+1]} - L\frac{[r+d-2]}{[r+1]}$$

and parameterising by $L = \frac{[l]}{[l+1]}$ we get

$$\det(M_{1,1,1}(d,L,R)) = \frac{[l+1][r+d-1] - [l][r+d-2]}{[r+1][l+1]} = \frac{[l+r+d-1]}{[r+1][l+1]}$$

and

$$\det(M_{L,L,R}(d,L,R)) = \frac{[r][l][l+r+d-1]}{[r+1]^2[l+1]^2}$$

We also note

$$\det(M_{L,1,1}(d,1,R)) = \frac{[l][r+d-1] - [l+1][r+d-2]}{[r+1][l+1]} = \frac{[l-(r+d-2)]}{[r+1][l+1]} \tag{1}$$

Define

$$M'(d,L,R) = \begin{pmatrix} L & L & 1 & & & & & \\ L & [2] & 1 & 0 & & & & \\ 1 & 1 & [2] & 1 & 0 & & & \\ 0 & 0 & 1 & [2] & 1 & 0 & & \\ & & & & \ddots & & & \\ 0 & \dots & 0 & 0 & 1 & [2] & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & [2] & R \\ 0 & \dots & 0 & 0 & 0 & 0 & R & R \end{pmatrix}$$

Thus

$$\det(M'(d,L,R)) = R([2]-L)\det(M_{L,1,1}(d-1,1,R)) = \frac{[r][l+2]}{[r+1][l+1]}\frac{[l-(r+d-3)]}{[r+1][l+1]}$$

4 Combinatorics and generating functions

4.1 Bracket sequences and trees

Let B_n denote the set of properly nested bracket sequences of n brackets. This begins

$$\emptyset,\ \{()\},\ \{()(),(())\},\ \dots$$

Write

$$G(x) = \sum_{n=0} x^n |B_n| \;=\; x^0 + x^2 + 2x^4 + 5x^6 + \ldots$$

for the generating function for the degrees of this sequence of sets — the Catalan numbers [1].

The set of rooted plane trees with n edges is in bijection with B_n. Noting that every tree with at least one edge may be 'factored' as a tree growing from the root together with a tree growing from the vertex at the end of this edge we have

$$G(x) = 1 + x^2 (G(x))^2$$

so

$$G(x) = \frac{1 - \sqrt{1-4x^2}}{2x^2}$$

Let B_n^l denote the set of composites of nested sequences with l propagating lines, with a total of n objects. This array begins

$$\begin{array}{cccccc} \emptyset & & & & & \\ & \{|\} & & & & \\ \{()\} & & & \{||\} & & \\ & \{()|,\ |()\} & & & \{|||\} & \end{array}$$

(see [**?**]).

The tree version of this is a forrest of $l+1$ trees with walls between. Accordingly we have

$$\sum_{n=0} x^n |B_n^l| \;=\; G(x)(xG(x))^l \;=\; (G(x))^{l+1} x^l$$

4.2 Exclude the rest for now!

Exercise: explain the relevance of all this to blob representation theory.

5 James-Murphy Gram determinants

The following recursion was introduced by James and Murphy [2] in case $q = 1$. Let μ be an integer partition (or equivalently a Young diagram), I_μ the set of row positions of μ from which a box may be removed, and for $i \in I_\mu$, let μ^i be the corresponding subdiagram (we follow [5, Appendix B]). Define a function $\dim -$ from integer partitions to integers by

$$\dim(1) = 1$$

and

$$\dim \mu = \sum_{i \in I_\mu} \dim \mu^i$$

For $i \in I_\mu$ let J_i be the set of hook lengths of μ in the column above the removable box. Define a function from integer partitions to functions of q recursively by

$$D_{(1)} = 1$$

and

$$D_\mu = \prod_{i\in I_\mu} D_{\mu^i} \left(q^{x(\mu^i)} \prod_{j\in J_i} \frac{[j]}{[j-1]} \right)^{\dim \mu^i}$$

(here x is a function whose details need not concern us for now — see [5, Appendix B] for this, and also for a number of examples).

The point of James-Murphy's contruction is that D_μ is the Gram determinant for the S_n Specht module with label μ or (as noted by James and Mathas) the corresponding Hecke algebra module for general q.

Theorem 1 *This recursion is solved by the following explicit form in case* $\mu = (\mu_1, \mu_2)$ *(and* $x = 0$*):*

$$D'_\mu = \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l + 1]}{[\mu_2 - l]} \right)^{\binom{\mu_1+\mu_2}{l} - \binom{\mu_1+\mu_2}{l-1}}$$

Proof: (We will just do the cases in which $|I_\mu| = 2$.) In this case

$$\dim(\mu_1, \mu_2) = \binom{\mu_1+\mu_2}{\mu_1} - \binom{\mu_1+\mu_2}{\mu_1+1} = \binom{\mu_1+\mu_2}{\mu_2} - \binom{\mu_1+\mu_2}{\mu_2-1}$$

Sustituting D' for D in the recursion, we require to compute

$$\mathcal{K} = D'_{(\mu_1-1,\mu_2)} D'_{(\mu_1,\mu_2-1)} \left(\frac{[\mu_1-\mu_2+2]}{[\mu_1-\mu_2+1]} \right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1} - \binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

$$= \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l]}{[\mu_2 - l]} \right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} \prod_{l=0}^{\mu_2-2} \left(\frac{[\mu_1 - l + 1]}{[\mu_2 - l - 1]} \right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} \left(\frac{[\mu_1-\mu_2+2]}{[\mu_1-\mu_2+1]} \right)^{\bullet}$$

We need to show that this can be equated with D'_μ. The first factor has numerator

$$\prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l]}{1} \right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} = \prod_{l=1}^{\mu_2} \left(\frac{[\mu_1 - l + 1]}{1} \right)^{\binom{\mu_1+\mu_2-1}{l-1} - \binom{\mu_1+\mu_2-1}{l-2}}$$

$$= \chi \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l + 1]}{1} \right)^{\binom{\mu_1+\mu_2-1}{l-1} - \binom{\mu_1+\mu_2-1}{l-2}} \qquad \left(\frac{[\mu_1 - \mu_2 + 1]}{1} \right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1} - \binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

Here we have shifted the dummy l to get the argument as in D'_μ, then applied appropriate correcting factors to get the range of the product right. In particular we have a correcting factor for the lower limit of the product

$$\chi = \left(\frac{1}{[\mu_1+1]} \right)^{\binom{\mu_1+\mu_2-1}{-1} - \binom{\mu_1+\mu_2-1}{-2}} = 1$$

The second factor has numerator

$$\prod_{l=0}^{\mu_2-2} \left(\frac{[\mu_1 - l + 1]}{1} \right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} = \gamma \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l + 1]}{1} \right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}}$$

where

$$\gamma = \left(\frac{1}{[\mu_1 - \mu_2 + 2]}\right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1} - \binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

Thus collecting these numerators we have

$$\prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l + 1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l-1} - \binom{\mu_1+\mu_2-1}{l-2} + \binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} \qquad \left(\frac{[\mu_1 - \mu_2 + 1]}{[\mu_1 - \mu_2 + 2]}\right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1} - \binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

The second factor has denominator

$$\prod_{l=0}^{\mu_2-2} \left(\frac{1}{[\mu_2 - l - 1]}\right)^{\binom{\mu_1+\mu_2-1}{l} - \binom{\mu_1+\mu_2-1}{l-1}} = \prod_{l=1}^{\mu_2-1} \left(\frac{1}{[\mu_2 - l]}\right)^{\binom{\mu_1+\mu_2-1}{l-1} - \binom{\mu_1+\mu_2-1}{l-2}}$$

Noting that

$$-\binom{\mu_1 + \mu_2 - 1}{l - 2} + \binom{\mu_1 + \mu_2 - 1}{l} = \binom{\mu_1 + \mu_2}{l} - \binom{\mu_1 + \mu_2}{l - 1}$$

altogether we have

$$\mathcal{K} = \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1 - l + 1]}{[\mu_2 - l]}\right)^{\binom{\mu_1+\mu_2}{l} - \binom{\mu_1+\mu_2}{l-1}}$$

as required. □

References

[1] H W J Blote and M P Nightingale, *Critical behaviour of the two–dimensional Potts model with a continuous number of states; a finite size scaling analysis*, Physica **112A** (1982), 405–465.

[2] G D James and G E Murphy, *The determinant of a Gram matrix for a Specht module*, J. Algebra **59** (1979), 222–235.

[3] P P Martin, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[4] P P Martin and H Saleur, *The blob algebra and the periodic Temperley–Lieb algebra*, Lett. Math. Phys. **30** (1994), 189–206, (hep-th/9302094).

[5] P P Martin and B W Westbury, *On quantum spin–chain spectra and the representation theory of Hecke algebras by augmented braid diagrams*, J Phys A **30** (1997), 5471–5495.

[6] H N V Temperley and E H Lieb, *Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: some exact results for the percolation problem*, Proceedings of the Royal Society A **322** (1971), 251–280.