A lecture on the representation theory of partial Brauer algebras *

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Abstract

We construct the Specht modules and determine the corresponding decomposition matrix, and the Cartan decomposition matrix, for the partial Brauer algebras $R_n(\delta, \delta')$ ($n \in \mathbb{N}$) in all cases over the complex field. We also determine the Specht module restriction rules for the restriction $R_{n-1} \hookrightarrow R_n$.

1 Introduction

Let $\Bbbk$ be a commutative ring, and $\Bbbk^\times$ its group of units. For each choice of $\delta, \delta' \in \Bbbk$, and $n \in \mathbb{N}_0$, the partial Brauer algebra $R_n(\delta, \delta')$ is a $\Bbbk$-algebra [9, 16, 17, 11] with a finite basis of certain set partitions (or partition diagrams [12]). Specifically the partial Brauer algebra is the subalgebra of the partition algebra [12] with basis only of partitions into pairs and singletons. We show that these algebras are generically semisimple over $\Bbbk = \mathbb{C}$, and construct 'Specht' modules — modules over $\Bbbk = \mathbb{Z}[\delta, \delta']$ that pass to a full set of generic simple modules over $\mathbb{C}$. In the remaining non-semisimple cases over $\mathbb{C}$ the decomposition matrices for these Specht modules, and hence the Cartan decomposition matrices, become very complicated, however we determine them via a string of Morita equivalences that end up with direct sums of Brauer algebras, whose decomposition matrices are known by [5, 14].

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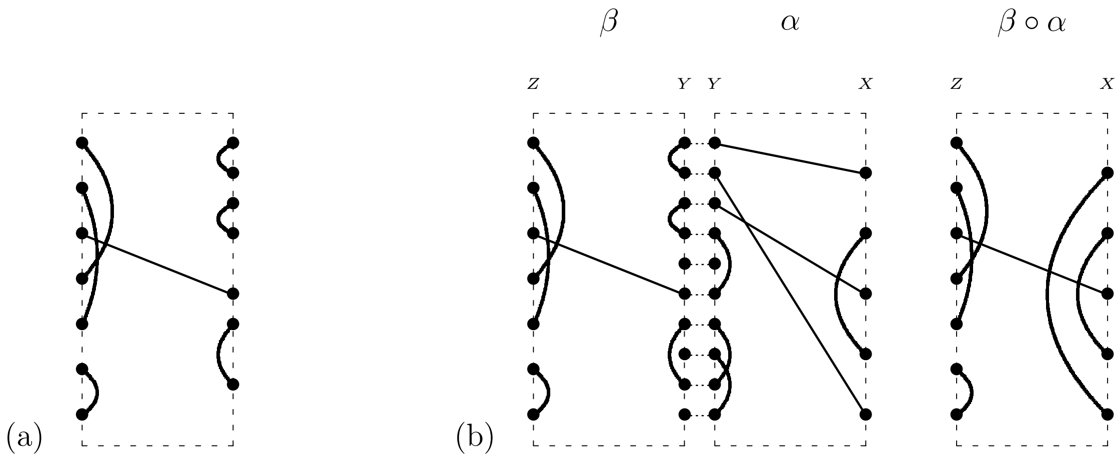

Figure 1: (a) Brauer diagram; (b) Basic composition of partial Brauer partitions (the straightening factor here is $\delta\delta'^2$).

1.1 Between the Brauer algebra and the partition algebra

The partial Brauer algebra is a unital diagram subalgebra of the partition algebra [12] (i.e. a subalgebra with basis a subset of partition diagrams); itself containing the Brauer algebra [3] as a unital diagram subalgebra. It is convenient to define the partial Brauer algebra in terms of these classical objects. A *Brauer diagram* depicts two rows of n vertices, connected in pairs, as in Figure 1(a). The Brauer algebra B_n over $\Bbbk$ is a $\Bbbk$-algebra with a basis of Brauer diagrams. The composition is by first juxtaposing diagrams and then applying a *straightening rule*, i.e. a δ-dependent rule for writing any juxtaposition as an element of the $\Bbbk$-span of basis diagrams (see [3]). The Brauer algebra is a subalgebra of the partition algebra P_n, which has a basis consisting of all partitions of the same vertex set (see [12]). There is a larger subalgebra of P_n with basis the set of partitions into pairs and singletons. This has a 2-parameter version $R_n(\delta, \delta')$, with a two-parameter straightening rule: a factor δ for each loop and a factor δ' for each open string removed. The rule is well illustrated by Figure 1(b), which shows an example from the corresponding diagram category (see also Mazorchuk [16]). This $R_n(\delta, \delta')$ is the partial Brauer algebra. We call the diagrams in the corresponding diagram basis *partial Brauer diagrams*.

1.2 The result

We will use $\equiv$ to denote Morita equivalence. Our key theorem is the following.

(**1.1**) Theorem. For $\delta', \delta - 1 \in \Bbbk^\times$ we have

$$R_n(\delta, \delta') \;\equiv\; B_n(\delta - 1) \oplus B_{n-1}(\delta - 1)$$

In §2 we prove this theorem (using a direct analogue of the method used for the partition algebra variation treated in [13]). In §3 we combine the theorem with results on the representation theory of B_n from [14] to describe the complex representation theory of $R_n(\delta, \delta')$. In particular we give an explicit construction for a complete set of Specht modules for $R_n(\delta, \delta')$, and show that these are images under the Morita equivalence of corresponding Brauer Specht modules (or Δ-modules). This means in particular that we can use the Brauer Δ-decompositon matrices from [14]. Provided that δ' is a unit then it can be 'scaled out' of representation theoretic calculations, as the theorem suggests. However we also deal with the non-unit cases excepted in the theorem (see §§4, 5.1).

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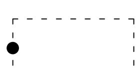
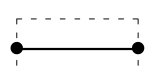
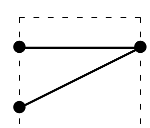
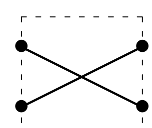
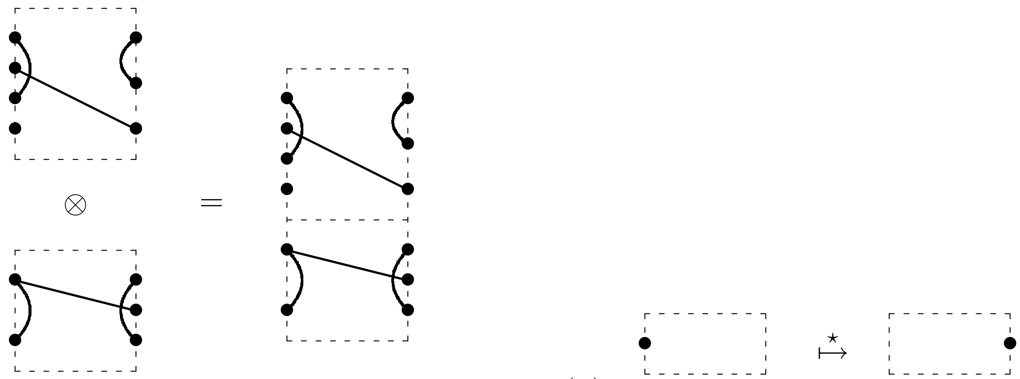
2 Constructing the Morita equivalences

In order to construct the Morita equivalences in Theorem 1.1 we will introduce a little more notation.

2.1 Partition categories

As a matter of expository efficiency (rather than necessity) we note the following. The partition and Brauer algebras extend in an obvious way to $\Bbbk$-linear categories [12, 14], here denoted P and B respectively. The partial Brauer category R is defined similarly. The category P is a monoidal category with monoidal composition $a \otimes b$ defined as in figure 2; and an involutive antiautomorphism $\star$ (in terms of diagrams as drawn here, the $\star$ operation is reflection in a vertical line — see e.g. [14], and Figure 2(b)). It is then generated as a $\Bbbk$-linear category with $\otimes$ and $\star$ by

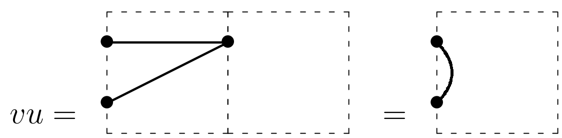
$u =$  , $1=$  , $v =$  , $x =$ 



(a) (b)

Figure 2: (a) Tensor product of partition diagrams; (b) $\star$ operation.

that is to say, the minimal $\Bbbk$-linear subcategory closed under $\otimes$ and $\star$ and containing these four elements is P itself (this follows immediately from [12, Prop.2]). Similarly, B is generated by 1, x and



Finally, R is generated by B and u.

2.2 Some basics of the partition diagram calculus

For S a set we write $\mathtt{P}_S$ for the set of partitions of S. For $n, m \in \mathbb{N}_0$, $\underline{n} := \{1, 2, ..., n\}$, $\underline{n}' := \{1', 2', ..., n'\}$, and $\mathtt{P}_{n,m} := \mathtt{P}_{\underline{n} \cup \underline{m}'}$. A partition in which each part (each subset) has two elements is called a *pair partition* or Brauer partition. A partition in which each part has at most two elements is here called a *partial partition*. Define $\mathtt{B}_{n,m}$ as the subset of $\mathtt{P}_{n,m}$ of pair partitions; and $\mathtt{R}_{n,m}$ as the subset of partial partitions. That is, the algebra $R_n := R_n(\delta, \delta')$ has basis $\mathtt{R}_{n,n}$.

Let S be a set as before. Any graph G with a 'structure map' λ from S to the vertex set of G (a partial labelling of vertices by labels from S) defines an element $\pi(G) \in \mathtt{P}_S$ as follows: two elements of S are in the same part in $\pi(G)$ if and only if their image vertices are in the same connected component of G. A *partition diagram* for a partition in $\mathtt{P}_{n,m}$ is such a graph drawn in the interior of a rectangular frame, in which the vertices are arranged on two opposite edges of the frame, and λ is a bijection.

In a partition diagram or Brauer diagram d, say, a part with vertices in both edges is called a *propagating part* or a propagating line. The number $\#^p(d)$ of propagating lines is called the *propagating number* (in the literature this is sometimes also known as *rank*).

When two diagrams d, d' are concatenated in composition, we call the 'middle' layer formed (before straightening) the *equator*. We write $d|d'$ for the concatenated unstraightened 'diagram'.

The following elementary exercise will illustrate the diagram calculus machinery, and also be useful later on. Define $\mathtt{R}_{n,n}^{+(l)}$ as the subset of partial partitions with l singletons. For $d \in \mathtt{R}_{n,n}^{+(l)}$ and $d' \in \mathtt{R}_{n,n}^{+(l')}$ then the singletons from d and d' appear in $d|d'$ in three possible ways: (i) in the exterior (becoming singletons of dd'); (ii) as endpoints of open strings in the equator, i.e. in pairs connected by a (possibly zero length) chain of pair parts; (iii) as endpoints of chains terminating in the exterior.

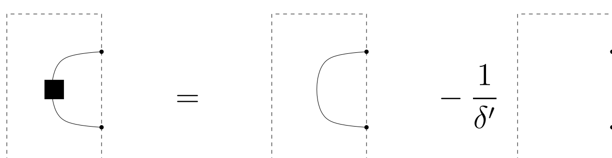
Let $2m$ be the number of singletons in $d|d'$ of type-(ii). Then

$$dd' \in \Bbbk \delta'^m \mathtt{R}_{n,n}^{+(l+l'-2m)} \tag{1}$$

One should keep in mind that every partial Brauer diagram d encodes a partition. Thus the assertion $\{i, j\} \in d$ means that there is a line between vertices i and j in d. If $\{i, j\} \in d$ then one can decompose partition d as

$$d = \{\{i, j\}\} \cup \ d' \qquad \text{where} \qquad d' = \ d - \{i, j\} \tag{2}$$

A useful diagram shorthand for certain linear combinations of diagrams is to decorate a line $\{i, j\}$ with a box. In case $\delta' \in \Bbbk^{\times}$ the resultant decorated diagram denotes the combination $d - (1/\delta')d_0$ where $d_0 = \ \{\{i\}, \{j\}\} \cup d'$. For example



It will be clear from (2) that this diagram shorthand extends naturally to decorations on any number of lines.

2.3 Idempotents

It is convenient to treat the special case of $R_n(\delta, \delta')$ with $\delta' = 0$ separately — see §5.1. Thus we will assume for now that $\delta' \in \Bbbk^{\times}$.

(**2.1**) For $\delta' \in \Bbbk^\times$, define a map $\langle\rangle : \mathtt{R}_{n,n} \to R_n$ that takes diagram d to the linear combination $\langle d \rangle$ obtained by decorating every line with a box.

For example, with $U := uu^\star \in \mathtt{R}_{1,1}$ we have:

$$\langle 1 \otimes 1 \rangle \;=\; (1 - \frac{1}{\delta'}U) \otimes (1 - \frac{1}{\delta'}U) \;=\; 1 \otimes 1 \;-\; \frac{1}{\delta'}(1 \otimes U + U \otimes 1) \;+\; \frac{1}{\delta'^2} U \otimes U$$

We can alternatively represent $\langle d \rangle$ simply using the diagram for d itself, with the boxes *implicit*. We call this the 'ψ-realisation'. Of course in this case diagrams will have a different composition rule — in this role, we call them ψ-diagrams.

(**2.2**) Lemma. For $\delta' \in \Bbbk^\times$, another basis for $R_n(\delta, \delta')$ is $\langle \mathtt{R}_{n,n} \rangle$.

Proof. The coefficient matrix for $\langle \mathtt{R}_{n,n} \rangle$ in the $\mathtt{R}_{n,n}$ basis is upper-unitriangular in any order refining the partial order by number of pair parts of d. □

(**2.3**) Lemma. For $d, d' \in \mathtt{R}_{n,n}$, the product $\langle d \rangle \langle d' \rangle$ in $R_n(\delta, \delta')$ is given by:

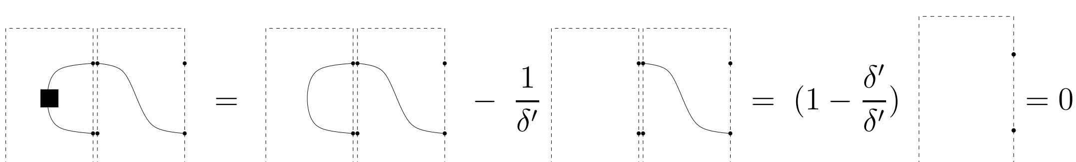
$$\langle d \rangle \langle d' \rangle = \begin{cases} 0 & \text{if, when the (decorated) diagrams } \langle d \rangle, \langle d' \rangle \text{ are} \\ & \text{concatenated, a singleton part meets a pair part,} \\ \langle dd' \rangle|_{\delta \rightsquigarrow \delta - 1} & \text{otherwise} \end{cases}$$

Here $\delta \rightsquigarrow \delta - 1$ means that the factor associated to a closed loop is $\delta - 1$ not δ.

Proof. Let $d, d \in \mathtt{R}_{n,n}$. Recall that if a line from d meets a line from d' in concatenation then the composite is a line in the product dd' (as per Figure 1). Passing to $\langle d \rangle, \langle d' \rangle$ these two lines are replaced by decorated lines. But by the idempotent property

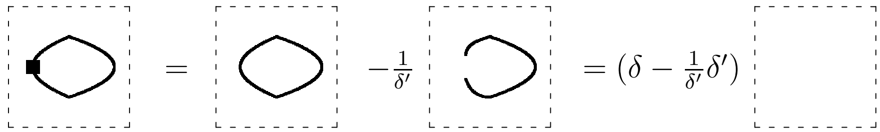
$$(1 - \frac{1}{\delta'}U)(1 - \frac{1}{\delta'}U) = (1 - \frac{1}{\delta'}U)$$

the corresponding single line in dd' is replaced by a single decorated line in $\langle dd' \rangle$. Case (i) is verified in one example by Equation(3):



(3)

It is easy to see however, using the decomposition (2), that this example is representative. Case (ii) may be verified as follows:



□

(**2.4**) Notice that a diagram in $\mathtt{R}_{n,n}$ with l propagating lines and $n-l$ odd has an odd number of singletons on both the top and bottom edges. It follows immediately that the product $\langle d\rangle\langle d'\rangle$ in R_n is zero unless d, d' both have even or both have odd number of propagating lines.

We call a partition/diagram d *odd* if $n - \#^p(d)$ is odd, and *even* otherwise. Let $\mathtt{R}^o_{n,n}$ denote the subset of odd diagrams, and $\mathtt{R}^e_{n,n}$ the subset of even diagrams.

For example, the identity diagram in R_n, denoted 1_n (or 1 where this is unambiguous), is even.

(**2.5**) Proposition. *Suppose $\delta' \in \Bbbk^\times$. The subset $\langle \mathtt{R}^o_{n,n}\rangle$ (respectively $\langle \mathtt{R}^e_{n,n}\rangle$) of $\langle \mathtt{R}_{n,n}\rangle$ is a basis for an idempotent subalgebra, denoted R^1_n (respectively R^0_n). The algebra R_n is a direct sum of odd and even subalgebras:*

$$R_n = R^0_n \oplus R^1_n$$

Proof. We just need to show for all d, d' that if $\langle d\rangle\langle d'\rangle$ nonzero then the parity of dd' is the same as the parity of d and d'.

By (2.3)(i) the product $\langle d\rangle\langle d'\rangle$ is zero unless the singletons in the equator match up. Consider the contribution of a propagating line in d, say, to dd'. This either forms part of a propagating line in dd' (possibly via a chain of arcs in the equator) in combination with precisely one propagating line from d'; or else is joined, via a chain of arcs in the equator, to another propagating line in d, and hence does not contribute to a propagating line in dd'. □

Suppose $\delta' \in \Bbbk^\times$. For any given n let

$$\chi = \langle 1\rangle.$$

(**2.6**) Proposition. *Suppose $\delta' \in \Bbbk^\times$. For any given n:*
(i) χ is idempotent.
(ii) For any $d \in \mathtt{R}_{n,n}$ we have $\chi d\chi \neq 0$ if and only if $d \in \mathtt{B}_{n,n}$.
(iii) The idempotent subalgebra $\chi R_n \chi$ has basis $\{\chi d \chi \mid d \in \mathtt{B}_{n,n}\}$.
(iv) The map

$$\gamma : \chi R_n(\delta, \delta')\chi \xrightarrow{\sim} B_n(\delta - 1)$$

given on the basis by $\chi d\chi \mapsto d$ is a $\Bbbk$-algebra isomorphism.

Proof. (i) follows from the definition. (ii) follows from Lemma 2.3. (iii) follows from (ii); and (iv) from (iii) and Lemma 2.3. □

2.4 Idempotent induction functors

(**2.7**) Consider the usual idempotent induction functor construction (see e.g. [8])

$$\chi R_n \chi - \mathrm{mod} \underset{F_\chi}{\overset{G_\chi}{\rightleftarrows}} R_n - \mathrm{mod}$$

$$G_\chi : M \mapsto R_n\chi \otimes_{\chi R_n \chi} M \qquad\qquad F_\chi : N \mapsto \chi R_n \otimes_{R_n} N \cong \chi N$$

By (2.6), this construction relates $B_n(\delta - 1)$ and $R_n(\delta)$. This is directly analogous to the partition algebra case [13]. However a significant difference with the partition algebra case is that $R_n\chi \otimes \chi R_n \not\cong R_n$, so B_n-mod fully embeds in but is not Morita equivalent to R_n-mod. Our main task in determining the structure of R_n is to deal with this difference. In fact, by the construction of R_n^1,

$$\chi R_n^1 = R_n^1 \chi = 0$$

and we have

(**2.8**) Proposition. *Let $\mu : R_n^0\chi \otimes_{\chi R \chi} \chi R_n^0 \rightarrow R_n^0 \chi R_n^0$ be the multiplication map. For $\delta', \delta - 1 \in \Bbbk^\times$*

$$R_n\chi \otimes_{\chi R\chi} \chi R_n = R_n^0\chi \otimes_{\chi R\chi} \chi R_n^0 \overset{\mu}{\cong} R_n^0 \chi R_n^0 = R_n^0$$

is an isomorphism of R_n^0-bimodules. That is, the functors F_χ and G_χ induce a Morita equivalence between $R_n^0(\delta)$ and $B_n(\delta - 1)$.

For $\delta = 1$ the failure of isomorphism is a degeneration (in the sense of §4); and the Morita equivalence is replaced by a saturated full embedding.[1]

Proof. By a general argument (see e.g. [1, §21 Ex.6] or [6]) it is enough to prove the last identity. We do this by way of the following Lemma. □

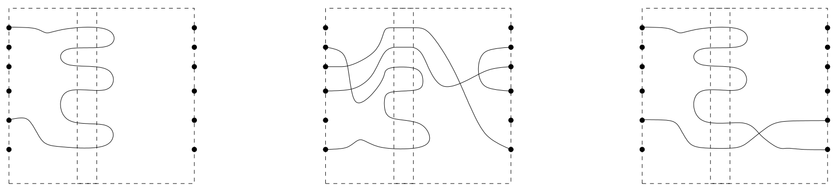
We write s_n for the all-singletons diagram $s_n = U^{\otimes n}$ in R_n.

[1]That is, Specht modules are taken to Specht modules (or at least 'combinatorial Specht modules' — modules with the same composition factors as Specht modules), with no gaps; but a simple module is killed by F_χ. See §4.

(**2.9**) LEMMA. For any $\langle d \rangle$ in $\langle \mathtt{R}^e_{n,n} \rangle$ there exist $x, y \in \langle \mathtt{R}^e_{n,n} \rangle$ such that

$$\langle d \rangle = \begin{cases} x\chi y & d \neq s_n \\ \frac{1}{\delta-1} x\chi y & d = s_n \end{cases}$$

Proof. It is convenient to describe our proof using ψ-diagrams. In this realisation χ is drawn like the ordinary identity diagram. Thus it is enough to show that the ψ-diagram for $\langle d \rangle$ can be drawn with the middle band of this form. First note that any line can be extruded to 'meander' across (repeatedly traverse) a middle band of the diagram — as illustrated on the left here:



(NB, picture drawn in the ψ-realisation). If there are propagating lines then χ may be realised as a mixture of meander and propagating lines, as illustrated in the other two pictures above.

The exception — the all-singletons diagram in case n even — can be expressed in the form $\frac{1}{\delta-1} d\chi d'$ by adding a loop which traverses the middle in the same way. □

We now have the following reformulation of our Theorem.

(**2.10**) THEOREM. For $\delta' \neq 0$, $R_n = R^0_n \oplus R^1_n$ and then for $\delta \neq 1$

$$R^0_n(\delta, \delta') \;\equiv\; B_n(\delta - 1)$$

$$R^1_n(\delta, \delta') \;\equiv\; R^0_{n-1}(\delta, \delta')$$

that is we have the Morita equivalence

$$R_n(\delta, \delta') \;\equiv\; B_n(\delta - 1) \oplus B_{n-1}(\delta - 1)$$

(We will deal with the excepted cases later.)

Proof. Only the second Morita equivalence is still to prove. We do this via the following Lemma. Here $U = uu^\star$ and $U_1 = uu^\star \otimes 1_{n-1} \in R_{n,n}$.

(**2.11**) LEMMA. For $\delta' \neq 0$, $U_1 R^1_n U_1 \cong R^0_{n-1}$

Proof. Consider the augmented basis $\langle \mathtt{R}^o_{n,n} \rangle$ of R^1_n. Since this is the odd part, each diagram d has an odd number of singleton vertices top and bottom (left and right as we are drawing it). But in each case one of these vertices must match up with the singleton in U_1, else $U_1 d U_1 = 0$. The map is to ignore these vertices, so that we end up with a diagram in R_{n-1}, which then has an even number of singletons, and hence is an even diagram. It is clear that all even $(n-1)$-diagrams arise in this way. □

(**2.12**) Lemma. For $\delta' \neq 0$, the multiplication map $\mu : R^1_n U_1 \otimes_{U R^1_n U} U_1 R^1_n \xrightarrow{\sim} R^1_n$ is a R^1_n-bimodule isomorphism.

Proof. Since R^1_n is spanned by diagrams with at least one singleton, and hence with no more than $n-1$ propagating lines, we can always express them in the form $aU_1 b$, that is, $R^1_n U_1 R^1_n = R^1_n$. Now use the same argument as for Proposition 2.8. □

It follows from (2.11) that $R^1_n U_1$ is a left-R^1_n right-R^0_{n-1}-module; and from (2.12) that $G_U = (R^1_n U_1 \otimes_{R^0_{n-1}} -)$ is a Morita equivalence functor. This concludes the proof of the Theorem. □

In short this, together with the construction of R_n-Specht modules which we give in §3.4, reduces the representation theory of R_n to a problem whose solution is known. The remainder of the paper is concerned with extracting this representation theory in practice. (And dealing with the couple of special cases excluded above.)

2.5 Applying the Morita equivalence

(**2.13**) Recall that if A, B are Morita equivalent finite dimensional algebras over a field then there is a bijection between the sets of equivalence classes of simple modules; and this induces an identification of Cartan decomposition matrices [2].

(**2.14**) For each Brauer algebra over $\mathbb{C}$ the Cartan decomposition matrix C is determined in [14] (using heavy machinery such as [5, **?**, **?**]). Thus theorem 2.10 and theorem 2.13 determine the Cartan decomposition matrix C^R for R_n over $\mathbb{C}$ for $\delta', \delta - 1 \in \mathbb{k}^\times$.

However, knowledge of C^R does not lead directly to constructions for simple or indecomposable projective modules, or simple characters. To facilitate this for R_n it is convenient to introduce an intermediate class of modules with a concrete construction, and to tie these also to the Brauer algebra case. In [14] the Cartan decomposition matrix is determined in the framework of a splitting π-modular system (in the sense of Brauer's modular representation theory, although the prime π here is a linear monic not a prime number). That is, the Brauer–Specht module decomposition matrix D, which gives the simple content of 'modular' reductions of the lifts of generic (i.e. in this case δ-indeterminate) simple modules, is determined. The Morita equivalence gives a correspondence between Brauer–Specht modules $\Delta_n^B(\lambda)$ of B_n and corresponding modules for R_n, thus it only remains to cast the partial Brauer algebras in the same framework and construct their Brauer–Specht modules $\Delta_n^R(\lambda)$; and show that each $G_\chi.\Delta_n^B(\lambda) = \Delta_n^R(\lambda)$.

3 Specht modules for R_n

We call the case of R_n over $\mathbb{Z}[\delta, \delta']$ the integral case. We want to construct a set of modules in the integral case that pass, on extending to a suitable field, to a complete set of simple modules. Our construction for these partial Brauer Specht modules (or Δ-modules) is closely analogous to the construction for the partition and Brauer algebras.

(**3.1**) It is convenient to write $\circ$ for the bare composition of diagrams (i.e. ignoring factors of δ). Define

$$\mathtt{R}^l_{n,m} := \mathtt{R}_{n,l} \circ \mathtt{R}_{l,m} \ \subset \mathtt{R}_{n,m}$$

Note that this is the subset of partial partitions with at most l propagating lines.

Define

$$\mathtt{R}^{=l}_{n,m} = \mathtt{R}^l_{n,m} \setminus \mathtt{R}^{l-1}_{n,m}$$

(**3.2**) Examples: $\mathtt{R}^{=1}_{1,1} = \{1_1\}$; while $\mathtt{R}^{=2}_{2,2} = \{1_2, x\}$ (with $1_2 := 1_1 \otimes 1_1$). Note that $(\mathtt{R}^{=n}_{n,n}, \circ)$ gives a copy of the symmetric group S_n.

(**3.3**) Proposition. *For any $\delta, \delta' \in \Bbbk$ we have that*

$$\Bbbk\mathtt{R}_{n,n} \supset \Bbbk\mathtt{R}^{n-1}_{n,n} \supset \Bbbk\mathtt{R}^{n-2}_{n,n} \supset \ldots \supset \Bbbk\mathtt{R}^0_{n,n} \tag{4}$$

is a chain of two-sided ideals in R_n. The l-th ideal is generated by $U^l = U^{\otimes l}$. That is

$$\Bbbk\mathtt{R}^{n-l}_{n,n} = R_n U^l R_n$$

The section $\Bbbk\mathtt{R}^l_{n,n}/\Bbbk\mathtt{R}^{l-1}_{n,n}$ in (4) has basis $\mathtt{R}^{=l}_{n,n}$. Let us write $R^{l/}_{n,n}$ for this section of the regular bimodule. □

(**3.4**) Note that a bimodule $R^{l/}_{n,m}$ may be defined similarly starting from $\mathtt{R}_{n,m}$. In particular $R^{l/}_{n,l}$ has the nice property that its basis consists of all diagrams in $\mathtt{R}^l_{n,l}$ such that each vertex on the bottom edge is in a distinct propagating part.

(**3.5**) We may consider the parts of $p \in \mathtt{P}_{n,m}$ that meet the top set of vertices to be totally ordered by the natural order of their lowest numbered elements (from the top set). We may define a corresponding order for parts that meet the bottom set of vertices. We say that p is *non-permuting* if the subset of propagating parts has the same order from the top and from the bottom.

For a partition in $\mathtt{R}_{n,m}$, the property of being non-permuting is the same as having a diagram with no crossing propagating lines.

(**3.6**) Let $\mathtt{R}^{||}_{l,n}$ denote the subset of $\mathtt{R}^{=l}_{l,n}$ of non-permuting partitions.

(**3.7**) Lemma. As a left-module

$$R^{l/}_{n,n} \cong \bigoplus_{w \in R^{||}_{l,n}} R^{l/}_{n,l} w$$

Every summand is isomorphic to $R^{l/}_{n,l}$. We have that $\mathtt{R}^{=l}_{n,l}$ is a basis for $R^{l/}_{n,l}$, and

$$\mathtt{R}^{=l}_{n,l} = \mathtt{R}^{||}_{n,l} \circ \mathtt{R}^{=l}_{l,l} \tag{5}$$

where $\mathtt{R}^{=l}_{l,l} = S_l$. □

(**3.8**) Note that $R^{l/}_{n,l}$ is also a free right $\Bbbk S_l$-module in a natural way. For each $\lambda \vdash l$ let us choose an element $f_\lambda \in \Bbbk S_l$ such that

$$\mathcal{S}_\lambda = \Bbbk S_l f_\lambda \tag{6}$$

is the corresponding Specht module [10]. Then define 'inflation'

$$\Delta_n(\lambda) \; := \; R^{l/}_{n,l} f_\lambda$$

Including $f_\lambda \in \Bbbk S_l$ in R_l in the obvious way allows us to draw a picture for this — see for example Fig.3.

(**3.9**) Proposition. *For each basis $b(\lambda)$ of $\mathcal{S}_\lambda$ there is a basis*

$$b^R(\lambda) \;=\; \{rb \mid (r,b) \in \mathtt{R}^{||}_{n,l} \times b(\lambda)\}$$

of $\Delta_n(\lambda)$.

Proof. Note that the module is spanned by elements of form abf_λ where $a \in \mathtt{R}^{||}_{n,l}$ and $b \in S_l$ (consider (5)). □

(**3.10**) Note that the regular module has a filtration by Δ-modules (cf. [14, Prop.3.4] — a sufficient condition for well-defined filtration multiplicities is char.$\Bbbk = p > 3$). Indeed the regular module has a filtration in which all the modules labelled with partitions of a given degree l (say) are consecutive. Indeed, if $\Bbbk = \mathbb{C}$ (or at least contains $\mathbb{Q}$ so that $\Bbbk S_l$ is semisimple) the modules $\{\Delta_n(\lambda) \mid \lambda \vdash l\}$ do not extend each other (so can be arranged, among themselves, in any order in the filtration).

(**3.11**) Define quotient algebra

$$R_n^l = R_n / R_n U^{l+1} R_n$$

(**3.12**) Note that if $\delta' \neq 0$ and f_λ idempotent (as can always be chosen over $\mathbb{C}$) then $R_n U^{n-|\lambda|} f_\lambda$ is a projective R_n-module; and $R_n^{n-|\lambda|} U^{n-|\lambda|} f_\lambda$ is an indecomposable projective $R_n^{n-|\lambda|}$-module.

(**3.13**) PROPOSITION. *For $\delta' \in \Bbbk^\times$, $R_n(\delta, \delta')$ is quasihereditary over $\mathbb{C}$.*

Proof. In case $\delta' = 1$ one readily checks that $U^n, U^{n-1}, ..., U^0$ (as in Prop.3.3) are a set of heredity idempotents [6]. Other cases are similar. $\square$

3.1 Images of Δ-modules under the Morita equivalence

We have given a simple concrete construction for the Δ-modules of $R_n(\delta, \delta')$. Let Λ_l denote the set of integer partitions of l; Λ the set of all integer partitions; and $\Lambda^n = \cup_{l \in \{n, n-2, ...\}} \Lambda_l$. We will see that over $\mathbb{C}$ the modules $\{\Delta_n(\lambda) \; : \; \lambda \in \Lambda^n \cup \Lambda^{n-1}\}$ are, for generic $\delta, \delta' \in \mathbb{C}$, a complete set of simple modules for R_n. Thus if D^R is the decomposition matrix for these modules for some k and $\delta, \delta' \in k$ then $C = D^R (D^R)^T$ is the Cartan decomposition matrix (see e.g. [2]). Next we show that these modules are the images of the Δ-modules of the appropriate Brauer algebra (generally denoted $\Delta_n^B(\lambda)$ here) under the Morita equivalence, and hence determine D^R for $k = \mathbb{C}$.

Note that the modules $R_{n,l}^{l/}$ and so on have direct correspondents in the Brauer algebra case, so long as $n - l$ is even. Accordingly we use the notation $B_{n,l}^{l/}$ and so on there. Thus (see e.g. [5]) for each $l \in \{n, n-2, ...\}$, for each $\lambda \vdash l$:

$$\Delta_n^B(\lambda) \; = \; B_{n,l}^{l/} f_\lambda$$

with basis

$$b^B(\lambda) \; = \; \{db \;\mid d \in \mathtt{B}_{n,l}^{||}, \; b \in b(\lambda)\}. \tag{7}$$

Applying the (exact) ME functors we get the following. Firstly

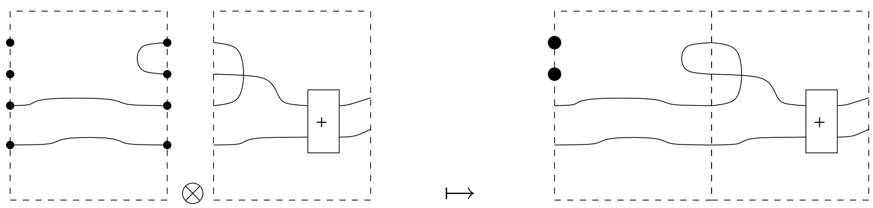
$$\chi R_n \otimes_{R_n} \Delta_n(\lambda) \;=\; \chi \otimes_{R_n} \chi\Delta_n(\lambda) \;=\; \chi \otimes_{R_n} \chi\Delta_n^B(\lambda) \;\cong\; \Delta_n^B(\lambda) \tag{8}$$

where we note that χ kills all the diagrams with singletons in the basis $b^R(\lambda)$ for $\Delta_n(\lambda)$; and in effect simply decorates all the lines on other diagrams. The last isomorphism then follows on comparing bases, noting that B_n is to be considered in its $\chi R_n \chi$ realisation (in which χ acts like 1).

(**3.14**) Lemma. For $\delta - 1 \in \Bbbk^\times$

$$R_n\chi \otimes_{\chi R_n \chi} \Delta_n^B(\lambda) \;\cong\; \Delta_n(\lambda)$$

Proof. Equating $\Delta_n^B(\lambda)$ with $\chi\Delta_n(\lambda) \subset \Delta_n(\lambda)$ as before gives us a well-defined multiplication map $\mu : a \otimes b \mapsto ab$, with ab in $\Delta_n(\lambda)$. For example:



(9)

— here we understand the left-hand factor to be drawn in the ψ-realisation; while the right-hand factor is a basis element of $\Delta_n^B(\lambda)$, regarded via the isomorphism as a $\chi R_n \chi$-module, so this is also, in a suitable sense, in the ψ-realisation. The outcome ab is then also in the ψ-realisation (remark: because of the quotient used in the definition of $\Delta_n(\lambda)$, ψ-decorations on *propagating* lines are irrelevant).

Besides $b^R(\lambda)$ another basis is $\psi(b^R(\lambda))$ (one draws the same set of pictures, but considers them to be in the ψ-realisation). Note that every basis element in $\psi(b^R(\lambda))$ can be realised (up to a scalar) as an image under μ, so μ is surjective. (The restriction to $\delta - 1 \in \Bbbk^\times$ arises because of the case $\lambda = \emptyset$ ($n \geq 2$), where multiplication involves a loop.)

To see that the μ-map is injective we proceed as follows. For T a set let $P_{even}(T)$ denote the set of subsets of T of even order; and $P_m(T)$ the set of subsets of order m. Note from (**??**) and (7) that $\{d_L(a) \otimes db \mid a \in P_{n-l}(\underline{n}),\ d \in \mathtt{B}^{||}_{n,l},\ b \in b(\lambda)\}$ is a spanning set for $R_n^0\chi \otimes \Delta_n^B(\lambda)$. Note that μ takes elements of this set to (scalar multiples of) diagrams. Distinct diagrams are independent, so it is enough to show that two elements only pass to the same diagram if they are equal. Apart from cases where $d_L(a) \otimes db = 0$ (due to too few propagating lines), no two diagrams $d_L(a)$ give rise to the same image since they have singletons in different positions. On the other hand, if two elements have the same $d_L(a)$ factor then they are the same only if their 'Brauer diagram part' (the lines from $d_L(a)$ and the factor db) is the same. But this part passes through the tensor product so they are then (possibly up to a scalar) the same element. □

3.2 Representation theory examples

We conclude with a few examples illustrating how to import results from $B_n(\gamma)$ representation theory in practice. (We use γ rather than the usual δ as the parameter here to avoid confusion with *our* δ.)

A convenient way to summarize the complex representation theory of $B_n(\gamma)$ ($\gamma \in \mathbb{C}$) is for each integer partition λ to describe the nonzero entries in the corresponding row of the Δ^B-decomposition matrix D. That is,

$$D_{\lambda,\mu} \;=\; [\Delta^B(\mu) : L^B(\lambda)]_\gamma$$

gives the multiplicity of simple $B_n(\gamma)$-module $L^B(\lambda)$ (the simple head of $\Delta^B(\lambda)$) as a composition factor of $\Delta^B(\mu)$. Characterised in this way, we can treat all n at once. We restrict attention to $\gamma \in \mathbb{Z}$ since all other cases are semisimple. For such a γ, matrix D may then be given as follows [15].

First we define the map $e : \mathbb{R} \times \Lambda \to \mathbb{R}^{\mathbb{N}}$ by

$$e(\gamma, \lambda) \;=\; \lambda - (0, 1, 2, ...) - \frac{\gamma}{2}(1, 1, 1, ...)$$

The image $e(\mathbb{R}, \Lambda)$ is a set of strongly decreasing sequences, such as

$$e(2, (4, 1)) = (3, -1, -3, -4, ...),$$

but it includes sequences with pairs of terms of equal magnitude (as in the example). Those sequences without any such pairs are said to be *regular*. For v a sequence in $e(\mathbb{R}, \Lambda)$ we define $Reg(v)$ as the (regular) sequence obtained by deleting all such pairs. For v regular we define $o(v)$ as the list of 'signed' positions in the magnitude order of terms in v. The sign of $o(v)_i$ is the sign of v_i unless $v_i = 0$ in which case it is chosen so that there are an even number of positives. Note in any case that $o(v)$ is now a descending signed permutation of $(-1, -2, -3, ...)$. We define a subset of $\mathbb{N}$ from this $o(v)$, denoted $o(v)|_+$, by keeping the positive terms, except toggling the presence of 1 if necessary to have a set of even order. Using this we define

$$o_\gamma(\lambda) = o(Reg(e(\gamma, \lambda)))|_+.$$

For example, $o_2((3^2)) = \{1,2\}$. Note that $o_\gamma(\lambda)$ is a (finite) element of the power set $P(\mathbb{N})$. To any $q \in P(\mathbb{N})$ we associate a binary sequence $b \in \{0,1\}^{\mathbb{N}}$ by $b_i = 1$ if $i \in q$; and $b_i = 0$ otherwise. We hence associate a binary sequence $\mathbf{b}_\gamma(\lambda) \in \{0,1\}^{\mathbb{N}}$ to $o_\gamma(\lambda)$ (one may omit the infinite string of 0s on the right).

The next step in the determination of the decomposition matrix D for $B_n(\gamma)$ is, for each λ, to pair certain terms in the $\{0,1\}$-sequence $\mathbf{b}_\gamma(\lambda)$, as follows. Every adjacent 01 is paired. Every 0...1 between which are only paired terms is paired; and this step is iterated. If there is a 1...1 between which are only paired terms, where the first 1 is the first unpaired term, then these 1s are paired; and this step is iterated (see [14] for examples). Using this construction we may define a certain hypercubical digraph on a subset of $P(\mathbb{N})$, with $o_\gamma(\lambda)$ at the head. Each descending edge corresponds to toggling a pair, either 01 to 10, or 11 to 00. The resultant collection of elements of $P(\mathbb{N})$ occur at most once each in this construction. Each element is $o_\gamma(\mu)$ for some $\mu \in [\lambda]_\gamma$, where $[\lambda]_\gamma$ is the block of λ (this can be characterised as the orbit of elements in Λ whose images under $e(\gamma, -)$ are related by a sequence of signed permutations [5]). Indeed fixing a block by a choice of λ we may define $o^\lambda_\gamma : P_{even}(\mathbb{N}) \to \Lambda$ to take $o_\gamma(\mu)$ to μ [15].

Finally let $h_\gamma(\lambda)$ denote the hypercube regarded as a (partially ordered) set of these μs. For example, $(3^2) \in [\emptyset]_2$ and $o_2((3^2)) = \{1,2\}$, so the corresponding hypercube contains only (3^2) and $o_2^\emptyset(\emptyset) = \emptyset$.

For $\lambda, \mu \in \Lambda$, define $(h_\gamma(\lambda) : \mu)$ as the number of times μ appears in $h_\gamma(\lambda)$ (i.e. either 1 or 0). Then

(**3.15**) THEOREM. [15] For given γ,

$$D_{\lambda^T \mu^T} \;=\; [\Delta^B(\mu^T) : L^B(\lambda^T)]_\gamma = \; (h_\gamma(\lambda) : \mu)$$

□

Combining with Theorem 2.10 and the results of Section 3.1 we have

(**3.16**) THEOREM. Fix $\delta' \in \Bbbk^\times$. The λ^T-row of the $R_n(\delta, \delta')$ Δ-decomposition matrix D^R for $\delta = \gamma + 1$ contains a 1 in the μ^T-column for each $\mu \in h_\gamma(\lambda)$; and zero otherwise. That is

$$[\Delta(\mu^T) : L(\lambda^T)]_\delta \quad = \quad (h_{\delta-1}(\lambda) : \mu)$$

(NB, besides $\delta = \gamma + 1$, the other difference from $B_n(\gamma)$ is the range of values of λ, μ). □

(**3.17**) Consider the example $o_2((3^2))$ again. One readily checks that among λs with $|\lambda| \leq 6$, $h_\gamma((3^2))$ is the only $h_\gamma(\lambda)$ containing $\mu = \emptyset$ (besides $h_\gamma(\emptyset)$ itself); and since

$\delta - 1 = \gamma = 2$ here, giving $\delta = 3$ in the Morita equivalent part of $R_n(\delta, \delta')$, this tells us, for example, that

$$0 \to L_6((2^3)) \to \Delta_6(\emptyset) \to L_6(\emptyset) \to 0$$

is a short exact sequence of $R_6(3, \delta')$-modules; where $L_n(\lambda)$ is the corresponding simple head of $\Delta_n(\lambda)$. The dimensions of $L_6((2^3)) = \Delta_6((2^3))$ and $\Delta_6(\emptyset)$ are clear from our construction, so this determines also the dimension of the other simple.

This concludes the analysis of the main body of complex representation theory of R_n.

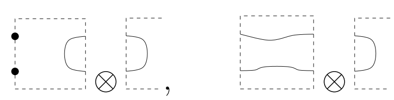
We now turn to the excepted cases. This is not only for completeness. They exhibit some very interesting properties, as we shall now start to explain.

4 Representation theory in the case $\delta = 1$, $\delta' \neq 0$

The Brauer algebra in case $\gamma = 0$ is special in that (for n even) there is one fewer simple module than Δ^B-modules (noting that simple modules are counted up to isomorphism, whereas the set of Δ^B-modules has a construction formally independent of such checks). Indeed for $n = 2$, $\gamma = 0$, $\Delta_2^B((2)) \xrightarrow{\sim} \Delta_2^B(\emptyset)$.

On the other hand $\gamma = 0$ is also the case $\delta - 1 = \gamma = 0$ where our Morita equivalence in Prop.2.8 fails degenerately (in the sense described there: — the claim is that F_χ, G_χ preserve 'combinatorial' Δ-modules, but $F_\chi = \chi R_n \otimes -$ kills a simple module). This is manifested, for example, as follows.

(**4.1**) Example. Case $n = 2$. First note that $R_2\chi \otimes \Delta_2^B(\emptyset)$ has basis



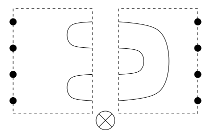
The first element spans a submodule, upon which $s_2 = U^{\otimes 2}$ acts like δ'^2. There is no such submodule in $\Delta_2(\emptyset)$, so $R_2\chi \otimes \Delta_2^B(\emptyset) \ncong \Delta_2(\emptyset)$. Indeed $R_2\chi \otimes \Delta_2^B((2)) \ncong \Delta_2((2))$ either ($\Delta_2((2))$ has the same rank as $\Delta_2^B((2))$). Thus $\Delta_2^B((2)) \xrightarrow{\sim} \Delta_2^B(\emptyset)$ does *not* induce an isomorphism of R_2 Δ-modules; and indeed there are the *same* number of simple R_n-modules as Δ-modules.

(**4.2**) What happens to Specht modules under functors F_χ, G_χ in case $\delta = 1$ ($\delta' \in \Bbbk^\times$) in general? As a first step we see how the Morita equivalence itself degenerates. To do this we look at what happens to Proposition 2.8 when $\delta = 1$.

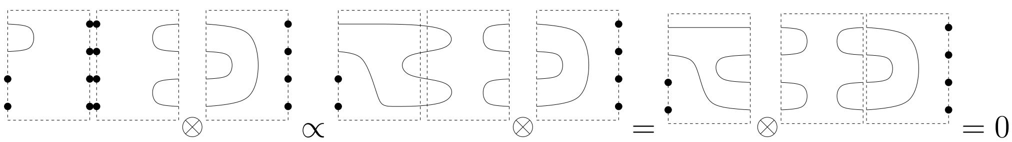
Write $R^{even'}_{n,n}$ for $R^{even}_{n,n}$ excluding the all-singletons diagram $s_n = U^{\otimes n}$. For $\delta = 1$ we see that $\Bbbk\psi(R^{even'}_{n,n}) = R_n\chi R_n$ is a proper ideal in R^0_n. We have a non-split short exact sequence of bimodules

$$0 \to R_n\chi R_n \to R^0_n \to R^0_n/R_n\chi R_n \to 0 \tag{10}$$

where $R^0_n/R_n\chi R_n$ has basis $\{\bar{s}_n\}$ (where $\bar{s}_n = s_n + R_n\chi R_n$). On the other hand there is an element in $R_n\chi \otimes \chi R_n$ whose image under μ formally contains a single loop (so it lies in the kernel of μ). For example



Call this element s'_n. It spans the sub-bimodule it generates since, for example,



Then

$$0 \to \Bbbk s'_n \to R_n\chi \otimes \chi R_n \to R_n\chi \otimes \chi R_n/\Bbbk s'_n \to 0$$

is a non-split short exact sequence of bimodules, with

$$R_n\chi \otimes \chi R_n/\Bbbk s'_n \cong R_n\chi R_n$$

We see immediately from (10) that $\Bbbk\{\bar{s}_n\}$ is a 1-dimensional simple R_n-module. It will be evident that this lies in (and indeed is) the head of $\Delta(\emptyset)$, so we call it $L(\emptyset)$. The multiplicity of $L(\emptyset)$ in the regular module is $\dim(\Delta(\emptyset))$, and hence this is the only Δ-module containing $L(\emptyset)$ as a composition factor (and the composition multiplicity is 1).

Consider the composite functor $G_\chi F_\chi = (R_n\chi \otimes \chi R_n \otimes_{R^0_n} -)$. For $\delta \neq 1$ we have $G_\chi F_\chi = (R_n\chi \otimes \chi R_n \otimes_{R^0_n} -) \cong (R^0_n \otimes_{R^0_n} -)$ which is an isomorphism functor, so that G_χ, F_χ are Morita equivalences. For $\delta = 1$ as we have just seen $R_n\chi \otimes \chi R_n$ and R^0_n are not isomorphic as bimodules. However they are 'close', in the following sense. The difference is that R^0_n has $L(\emptyset)$ in the head (restricting to the left regular module it is the head of indecomposable projective $P(\emptyset)$), while $R_n\chi \otimes \chi R_n$ has $L(\emptyset)$ in the socle instead, i.e. it is otherwise the same, and has the same composition factors.

We have $F_\chi L(\emptyset) \cong \chi \Bbbk \bar{s}_n \cong 0$ and, as before, $F_\chi \Delta(\lambda) \cong \Delta^B(\lambda)$ (any λ).

WHAT'S THIS?:

For $\lambda \neq \emptyset$ (and $n \neq 2$) we have $G_\chi \Delta^B(\lambda) \cong \Delta(\lambda)$. Finally $G_\chi \Delta^B(\emptyset) \ncong \Delta(\emptyset)$, however $G_\chi \Delta^B(\emptyset)$ and $\Delta(\emptyset)$ have the same composition factors.

Thus (for $\delta - 1 = \gamma = 0$) the rows of the decomposition matrix not involving $\emptyset$ are the same between D and D^R (see e.g. [8, §6.6]). The difference is that D^R has a row labelled by $\emptyset$. However, as already noted, this row is $(1, 0, 0, 0, ...)$. We have thus shown that for any n

$$D^R = D_{formal}$$

(as defined in (4.4)).

4.1 Examples: $\delta = 1$

(**4.3**) With $\gamma = 0$ we get $o_0(\emptyset) = \emptyset$ and $o_0((1^2)) = \{1, 2\}$, so hypercube $h_0((1^2))$ contains only (1^2) and $\emptyset$. Formally this implies a Δ^B-module homomorphism $\Delta^B_2((2)) \to \Delta^B_2(\emptyset)$, but in case $n = 2$ both modules are rank-1, and so the implied Δ^B-module homomorphism is actually an isomorphism, as noted above.

(**4.4**) Note that $(h_\gamma(\lambda) : \mu)$ makes sense for any $\lambda, \mu \in \Lambda$. We thus have a *formal 'decomposition matrix'* for $\gamma = 0$ in case $n = 2$ given by

$$D_{formal} = \left(\begin{array}{c|ccc} & \emptyset & (2) & (1^2) \\ \hline \emptyset & 1 & & \\ (2) & 1 & 1 & \\ (1^2) & & & 1 \end{array} \right)$$

However since there is no separate simple $B_2(0)$-module associated to $\lambda = \emptyset$ here the corresponding row is certainly spurious, and indeed the decomposition matrix Theorem tells us that we have $B_2(0)$-decomposition matrix

$$D_{n=2} = \left(\begin{array}{c|ccc} & \emptyset & (2) & (1^2) \\ \hline (2) & 1 & 1 & \\ (1^2) & & & 1 \end{array} \right)$$

(which coincides with the matrix obtained by deleting that row). This gives

$$C = \begin{pmatrix} 1 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$$

Note that the corresponding $\lambda = \emptyset$ *column* of $D_{n=2}$ is *not* spurious in this Brauer modular treatment. The module $\Delta^B((2))$ is not isomorphic to $\Delta^B(\emptyset)$ over $\mathbb{C}[\gamma]$ or $\mathbb{C}(\gamma)$ and both are needed as elements of the complete set of simples over $\mathbb{C}(\gamma)$. The $\gamma = 0$ projective $P_{(2)}$, which looks like a self-extension here, lifts to a combination of non-isomorphic integral modules. Indeed the primitive idempotent decomposition of 1 in case $\gamma = 0$ is $1 = (1 + (12))/2 + (1 - (12))/2$, which lifts trivially to $\mathbb{C}[\gamma]$; but the idempotent $(1+(12))/2$ does not pass to a primitive idempotent over $\mathbb{C}(\gamma)$, instead passing to a combination of two inequivalent idempotents.

—

GO THROUGH BENSON 1.9.6 in THIS CASE!!!

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In contrast there is a simple $R_2(0)$-module associated to $\lambda = \emptyset$, and in fact $D^R = D_{formal}$, as noted in §4.

For one final example, consider the block of $\emptyset$ in case $n = 4$. For $B_4(0)$ we have

$$D_{formal}|_{\emptyset} = \left(\begin{array}{c|ccc} & \emptyset & (2) & (31) \\ \hline \emptyset & 1 & & \\ (2) & 1 & 1 & \\ (31) & & 1 & 1 \end{array} \right)$$

We have $\dim(\Delta^B(\emptyset)) = 3$; $\dim(\Delta^B((2))) = 6$; $\dim(\Delta^B((31))) = 3$. We deduce that $\dim(L^B((2))) = 6 - 3 = 3$, and since this is a composition factor of $\Delta^B(\emptyset))$ they are isomorphic. Thus

$$D_{n=4}|_{\emptyset} = \left(\begin{array}{c|ccc} & \emptyset & (2) & (31) \\ \hline (2) & 1 & 1 & \\ (31) & & 1 & 1 \end{array} \right)$$

and

$$C|_{\emptyset} = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & 1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

On the other hand for $R_4(1)$, we have $\dim(\Delta(\emptyset)) = 10$; $\dim(\Delta((2))) = 12$; $\dim(\Delta((31))) = 3$. We deduce that $\dim(L((2))) = 12-3 = 9$, and hence $\dim(L(\emptyset)) = 10 - 9 = 1$. Again we confirm that $D^R = D_{formal}$.

5 The case $\delta' = 0$ and δ generic

(**5.1**) Define $\mathtt{R}^+_{n,m} \subset \mathtt{R}_{n,m}$ as the subset of partial partitions with at least one singleton.

(**5.2**) Lemma. Let $\Bbbk$ be any commutative ring. Then

$$R_n \mathtt{R}^+_{n,n} R_n \;=\; \Bbbk \mathtt{R}^+_{n,n} \;\cup\; \delta' R_n$$

Proof. Any product involving a diagram with a singleton either produces a diagram with a singleton, or else an open string. This is elementary — see (1). □

(**5.3**) In the $\delta' = 0$ case one sees from (1) that the diagrams with one or more singleton generate a nilpotent ideal. In determining the simple modules of R_n one can quotient by this ideal. The quotient can be identified with the Brauer subalgebra, whose generic representation theory has been studied in [4]; and general representation theory over $\mathbb{C}$ in [5, 14].

(**5.4**) In §3.4 we give a construction for *Specht modules* for each partial Brauer algebra. By this we mean modules for the algebra over $\mathbb{Z}[\delta, \delta']$ that can be made well-defined over a PID either by fixing $\delta' = 1$ or by fixing δ to a suitable value; such that, over a suitable extension to a field in either case, the set of modules passes to a complete set of simple modules in a semisimple algebra. This means that we can use the tools of Brauer's modular representation theory (here to study non-semisimple specialisations of the parameters, rather than fields of finite characteristic).

If we consider $\delta' = 0$ and δ generic, so that the Brauer algebra is semisimple, then we can describe the simple content of Specht modules (and hence the Cartan decomposition matrix). We do this in §**??**.

5.1 Representation theory in case $\delta' = 0$

The following proposition on the structure of Specht modules completely determines the Cartan decomposition matrix in case $\delta' = 0$ in all cases in which the Brauer subalgebra is semisimple (e.g. for $\delta >> n$).
(Other cases over $\mathbb{C}$ are also determined in principle, but one must combine with the appropriate Brauer algebra representation theory. This is known, but we do not give an explicit description here.)

(**5.5**) Recall from (5.3) that the Specht modules of the Brauer algebra are the generically simple modules of $R_n(\delta, 0)$. That is to say, they lift to $R_n(\delta, 0)$-modules with this property.

We assume the reader is familiar with these modules.

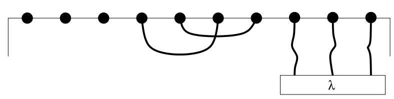
(**5.6**) PROPOSITION. *Let $\lambda \vdash l \leq n$ and let $B_n^\Delta(\lambda)$ be the basis for the Specht module $\Delta_n(\lambda)$ of $R_n(\delta, 0)$ (any δ) of form $R^{||}(n,l) \times b(\lambda)$, associated to some definite choice of $b(\lambda)$. We may decompose*

$$B_n^\Delta(\lambda) \;=\; \bigsqcup_k B_n^\Delta(\lambda, k)$$

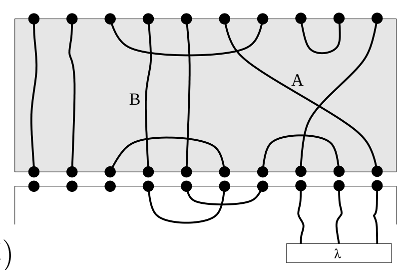
where $B_n^\Delta(\lambda, k)$ is the subset with k singletons ($k \equiv n - l$ mod.2).

Then $\bigsqcup_{k=0}^{i} B_n^\Delta(\lambda, k)$ is a basis for a submodule for each $i \in \{n-l, n-l-2, ..., 0/1\}$. Further, over field $K = \mathbb{C}$ the section with basis $KB_n^\Delta(\lambda, k)$ is a sum of Brauer Specht modules with labels determined by the S_n-simple module content of Ind$(\lambda \otimes (k))$; that is, labels obtained from λ by adding k boxes in all possible ways such that none of the new boxes are in the same column.

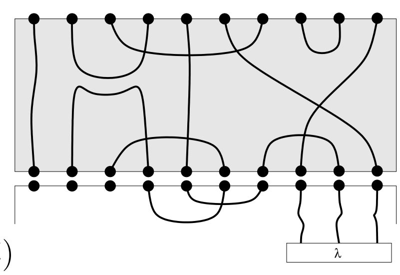
Proof. The basis decomposition is trivial. The submodule structure follows from noting that non-vanishing actions on $B_n^\Delta(\lambda, k)$ require $l + k$ propagating lines, else an open line is created in composition, giving $\delta' = 0$; and that if there is such a non-vanishing action at least k of the propagating lines will result in singletons after composition.
(An indicative example is given by the following figures. First a basis element, drawn to accept an action of algebra diagrams from above:



— this is for $n = 10$ with $k = 3$ and $l = 3$. Then:



(*i*)



(*ii*)

shows the action of two algebra elements from above. In (i) the number of singletons in the outcome is the same as for the original basis element. In (ii) the number of singletons is nominally reduced, but at cost an open string, giving a factor $\delta' = 0$.)

For the final part note that singletons are acted on as if they are propagating lines, except that they are symmetrised (they are indistinguishable under interchange). The result then follows from classical S_n representation theory. $\square$

6 Branching rules for Specht modules

Here $\mathcal{M}_+$ is the incidence matrix of the directed Young graph (this has vertex set Λ; and edge $\mu \triangleleft \lambda$ if the Young diagrams differ by a single box).

(**6.1**) THEOREM. The map from R_n to R_{n+1} given by $d \mapsto d \otimes 1$ is an algebra injection. The restriction rule for R_{n+1} Δ-modules to R_n is given by the short exact sequence

$$0 \to \; \Delta_n(\lambda) \oplus B_\lambda \; \to \mathrm{res}_n \Delta_{n+1}(\lambda) \to C_\lambda \to 0$$

(here and throughout we take $\Delta_n(\mu) = 0$ if $|\mu| > n$), where

$$B_\lambda \;=\; \oplus'_{\mu \triangleleft \lambda} \Delta_n(\mu); \qquad\qquad C_\lambda \;=\; \oplus'_{\mu \triangleright \lambda} \Delta_n(\mu)$$

($\oplus'$ denotes a direct sum if $k = \mathbb{C}$, but a not necessarily direct sum for fields of finite characteristic). That is

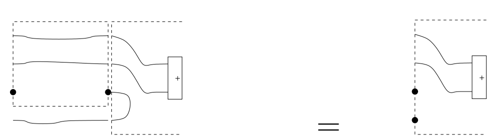
$$\mathrm{res}_n \Delta_{n+1}(\lambda) \;=\; \left(\underbrace{\Delta_n(\lambda)}_{A} \oplus \underbrace{(\oplus'_{\mu \triangleleft \lambda} \Delta_n(\mu))}_{B} \right) + (\oplus'_{\mu \triangleright \lambda} \Delta_n(\mu))$$

Proof. We assume familiarity with induction and restriction rules for the symmetric group [10]. Consider a basis as in Prop.3.9. Separate the basis into three subsets:
(1) elements in which the last vertex is a singleton;
(2) elements in which the last vertex starts a propagating line;
(3) elements in which the last vertex ends an arc from some other vertex.

Now consider the action of R_n. Clearly (1) is a basis for A. Meanwhile (2) is a basis for the inflation $R^{l-1/}_{n,l-1} \mathrm{res}_{l-1} \Delta_\lambda$, which gives B. Finally, modulo A and B, (3) is a basis for the inflation $R^{l+1/}_{n,l+1} \mathrm{ind}_{l+1} \Delta_\lambda$.

For example the basis for $\Delta_4((2))$ is given in fig.3. Here the six leftmost diagrams are in (2); the remaining diagrams in the top row are in (3); and the rest are in (1).

The action of the included R_n on an element of (3) is illustrated by the following diagram (NB. this is *not* a ψ-diagram):



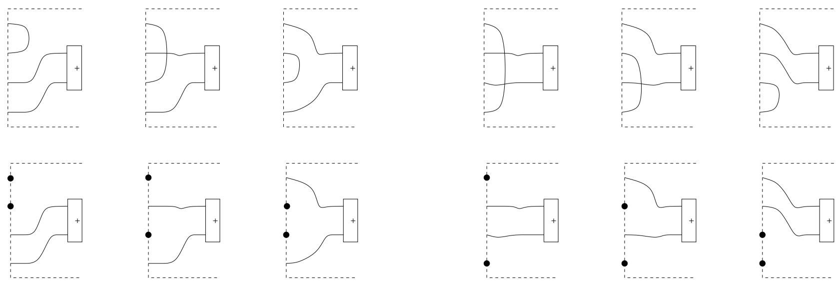


Figure 3: Basis for $\Delta_4((2))$.

(note that the right-hand side lies in (1) and so vanishes in the quotient C). $\square$

In particular

$$\dim \Delta_n(\lambda) = ((\mathcal{M}_* + 1)^n)_{\emptyset,\lambda} \tag{11}$$

where $\mathcal{M}_* = \mathcal{M}_+ + \mathcal{M}_+^t$.

7 Schur–Weyl duality

Recall that an index set for simple modules of $O(V)$ arising in arbitrary tensor products of V and $\mathbb{C}$ (the trivial module) is the set of integer partitions (or Young diagrams). In particular $V = R_\square$ and $\mathbb{C} = R_\emptyset$. If $V = \mathbb{C}^N$ then this index set can be restricted to the set $\Lambda(N)$ of integer partitions λ such that $\lambda_1' + \lambda_2' \leq N$ (see for example [7, Th.10.2.5]). The product is given by

$$V \otimes R_\lambda = (\oplus'_{\mu \triangleleft \lambda} R_\mu) \ \oplus \ (\oplus'_{\mu \triangleright \lambda} R_\mu) \tag{12}$$

where the sums are restricted to allowed partitions. Meanwhile obviously

$$\mathbb{C} \otimes R_\lambda = R_\lambda$$

Let F_N be the truncated Fock space $\mathbb{C}^{\Lambda(N)}$. Then each $O(N)$-module M generated as above corresponds to an element $v(M)$ of $\mathbb{N}_0^{\Lambda(N)}$ in $\mathbb{C}^{\Lambda(N)}$. For example $v(V) = e_\square = (0, 1, 0, 0, ...)$.

Let $\mathcal{M}_N \in \text{End}(\mathbb{C}^{\Lambda(N)})$ denote the matrix such that

$$v(V \otimes M) = \mathcal{M}_N v(M)$$

Note that this matrix is well-defined and directly determined by (12). Then of course

$$v((V \oplus \mathbb{C}) \otimes M) = (\mathcal{M}_N + 1)v(M)$$

Also

(**7.1**) THEOREM. The vector $(\mathcal{M}_N + 1)^n$ gives the list of dimensions of simple modules of the commutant $\mathrm{End}_{O(N)}((V \oplus \mathbb{C})^n)$.

(**7.2**) THEOREM. For $N >> n$ we have $R_n(N) \cong \mathrm{End}_{O(N)}((V \oplus \mathbb{C})^n)$.

Proof. cf. (11), Theorem 7.1. □

(**7.3**) The action of R_n on $(V \oplus \mathbb{C})^n$ is.... WHAT?

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