Notes in representation theory

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Contents

Chapter 1

Introduction

Chapters 1 and 2 give a brief introduction to representation theory, and a review of some of the basic algebra required in later Chapters. A more thorough grounding may be achieved by reading the works listed in §1.2: *Notes and References*.

Section 1.1 (upon which later chapters do not depend) attempts to provide a sketch overview of topics in the representation theory of finite dimensional algebras. In order to bootstrap this process, we use some terms without prior definition. We assume you know what a vector space is, and what a ring is (else see Section 2.1.1). For the rest, either you know them already, or you must intuit their meaning and wait for precise definitions until after the overview.

1.1 Representation theory preamble

1.1.1 Matrices

Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices over a ring R, with additive identity $0_{m,n}$. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. Define a block diagonal composition (matrix direct sum)

$$\begin{aligned} \oplus : M_m(R) \times M_n(R) &\rightarrow M_{m+n}(R) \\ (A, A') &\mapsto A \oplus A' = \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & A' \end{pmatrix} \end{aligned}$$

Define Kronecker product

$$\otimes : M_{a,b}(R) \times M_{m,n}(R) \rightarrow M_{am,bn}(R) \tag{1.1}$$

$$(A, B) \mapsto \begin{pmatrix} a_{11}B & a_{12}B & ... \\ a_{21}B & a_{22}B & ... \\ \vdots & & \end{pmatrix} \tag{1.2}$$

In general $A \otimes B \neq B \otimes A$, but (if R is commutative then) for each pair A, B there exists a pair of permutation matrices S, T such that $S(A \otimes B) = (B \otimes A)T$ (if A, B square then $T = S$ — the *intertwiner* of $A \otimes B$ ans $B \otimes A$).

1.1.2 Groups

(**1.1.1**) A matrix representation of a group G over a commutative ring R is a map

$$\rho : G \to M_n(R) \tag{1.3}$$

such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. In other words it is a map from the group to a different system, which nonetheless respects the extra structure (of multiplication) in some way. The study of representations — models of the group and its structure — is a way to study the group itself.

(**1.1.2**) The map ρ above is an example of the notion of representation that generalises greatly. A mild generalisation is the representation theory of R-algebras that we shall discuss, but one could go further. Physics consists in various attempts to model or represent the observable world. In a model, Physical entities are abstracted, and their behaviour has an image in the behaviour of the model. We say we understand something when we have a model or representation of it mapping to something we understand (better), which does not wash out too much of the detailed behaviour.

(**1.1.3**) Representation theory itself seeks to classify and construct representations (of groups, or other systems). Let us try to be more explicit about this.

(I) Suppose ρ is as above, and let S be an arbitrary invertible element of $M_n(R)$. Then one immediately verifies that

$$\rho_S : G \quad \to \quad M_n(R) \tag{1.4}$$
$$g \quad \mapsto \quad S\rho(g)S^{-1} \tag{1.5}$$

is again a representation.
(II) If ρ' is another representation (by $m \times m$ matrices, say) then

$$\rho \oplus \rho' : G \quad \to \quad M_{m+n}(R) \tag{1.6}$$
$$g \quad \mapsto \quad \rho(g) \oplus \rho'(g) \tag{1.7}$$

is yet another representation.
(III) For a finite group G let $\{g_i \ : \ i = 1, ..., |G|\}$ be an ordering of the group elements. Each element g acts on G, written out as this list $\{g_i\}$, by multiplication from the left (say), to permute the list. That is, there is a permutation $\sigma(g)$ such that $gg_i = g_{\sigma(g)(i)}$. This permutation can be recorded as a matrix,

$$\rho_{Reg}(g) = \sum_{i=1}^{|G|} \epsilon_{i\ \sigma(g)(i)}$$

(where $\epsilon_{ij} \in M_{|G|}(R)$ is the i, j-elementary matrix) and one can check that these matrices form a representation, called the *regular representation*.

Clearly, then, there are unboundedly many representations of any group. However, these constructions also carry the seeds for an organisational scheme...

(**1.1.4**) Firstly, in light of the ρ_S construction, we only seek to classify representations *up to isomorphism* (i.e. up to equivalences of the form $\rho \leftrightarrow \rho_S$).

Secondly, we can go further (in the same general direction), and give a cruder classification, by *character*. (While cruder, this classification is still organisationally very useful.) We can briefly explain this as follows.

Let c_G denote the set of classes of group G. A *class function* on G is a function that factors through the natural set map from G to the set c_G. Thus an R-valued class function is completely specified by a c_G-tuple of elements of R (that is, an element of the set of maps from c_G to R, denoted R^{c_G}). For each representation ρ define a *character* map from G to R

$$\chi_\rho : G \to R \tag{1.8}$$
$$g \mapsto \mathrm{Tr}(\rho(g)) \tag{1.9}$$

(matrix trace). Note that this map is fixed up to isomorphism. Note also that this map is a class function. Fixing G and varying ρ, therefore, we may regard the character map instead as a map χ_- from the collection of representations to the set of c_G-tuples of elements of R.

Note that pointwise addition equips R^{c_G} with the structure of abelian group. Thus, for example, the character of a sum of representations isomorphic to ρ lies in the subgroup generated by the character of ρ; and $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$ and so on.

We can ask if there is a small set of representations whose characters '$\mathbb{N}_0$-span' the image of the collection of representations in R^{c_G}. (We could even ask if such a set provides an R-basis for R^{c_G} (in case R a field, or in a suitably corresponding sense — see later). Note that $|c_G|$ provides an upper bound on the size of such a set.)

(**1.1.5**) Next, conversely to the direct sum result, suppose $R_1 : G \to M_m(R)$, $R_2 : G \to M_n(R)$, and $V : G \to M_{m,n}(R)$ are set maps, and that a set map $\rho_{12} : G \to M_{m+n}(R)$ takes the form

$$\rho_{12}(g) = \begin{pmatrix} R_1(g) & V(g) \\ 0 & R_2(g) \end{pmatrix} \tag{1.10}$$

(a matrix of matrices). Then ρ_{12} a representation of G implies that both R_1 and R_2 are representations. Further, $\chi_{\rho_{12}} = \chi_{R_1} + \chi_{R_2}$ (i.e. the character of ρ_{12} lies in the span of the characters of the smaller representations). Accordingly, if the isomorphism class of a representation contains an element that can be written in this way, we call the representation *reducible*.

(**1.1.6**) For a finite group over $R = \mathbb{C}$ (say) we shall see later that there are only a finite set of 'irreducible' representations needed (up to equivalences of the form $\rho \leftrightarrow \rho_S$) such that every representation can be built (again up to equivalence) as a direct sum of these; and that all of these irreducible representations appear as direct summands in the regular representation.

We have done a couple of things to simplify here. Passing to a field means that we can think of our matrices as recording linear transformations on a space with respect to some basis. To say that ρ is equivalent to a representation of the form ρ_{12} above is to say that this space has a G-subspace (R_1 is the representation associated to the subspace). A representation is irreducible if there is no such proper decomposition (up to equivalence). A representation is *completely reducible* if for every decomposition $\rho_{12}(g)$ there is an equivalent identical to it except that $V(g) = 0$ — the direct sum.

Theorem [Mashke] Let ρ be a representation of a finite group G over a field K. If the characteristic of K does not divide the order of G, then ρ is completely reducible.

Corollary Every complex irreducible representation of G is a direct summand of the regular representation.

Representation theory is more complicated in general than it is in the cases to which Mashke's Theorem applies, but the notion of irreducible representations as fundamental building blocks survives in a fair degree of generality. Thus the question arises:
Over a given R, what are the irreducible representations of G (up to $\rho \leftrightarrow \rho_S$ equivalence)?
There are other questions, but as far as physical applications (for example) are concerned, this is arguably the main interesting question.

(**1.1.7**) Examples: In this sense, of constructing irreducible representations, the representation theory of the symmetric groups S_n over $\mathbb{C}$ is completely understood! (We shall review it.) On the other hand, over other fields we do not have even so much as a conjecture as to how to organise the statement of a conjecture! So there is work to be done.

1.1.3 Group algebras

(**1.1.8**) Remark: When working with R a *field* it is natural to view $M_n(R)$ as the ring of linear transformations of vector space R^n expressed with respect to a given ordered basis. The equivalence $\rho \leftrightarrow \rho_S$ corresponds to a change of basis, and so working up to equivalence corresponds to demoting the matrices themselves in favour of the underlying linear transformations (on R^n). In this setting it is common to refer to the linear transformations by which G acts on R^n as the representation (and to spell out that the matrices are a *matrix* representation, regarded as arising from a choice of ordered basis).

Such an action of a group G on a set makes the set a G-set.[1] However, given that it is a set with extra structure (in this case, a vector space), it is a small step to want to try to take advantage of the extra structure. For example, we can define RG to be the R-vector space with basis G (see Exercise 1.3.1), and define a multiplication on RG by

$$\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right) = \sum_{ij} (r_i r'_j)(g_i g_j) \tag{1.11}$$

which makes RG a ring (see Exercise 1.3.2). One can quickly check that

$$\rho : RG \quad \to \quad M_n(R) \tag{1.12}$$

$$\sum_i r_i g_i \quad \mapsto \quad \sum_i r_i \rho(g_i) \tag{1.13}$$

extends a representation ρ of G to a representation of RG in the obvious sense. Superficially this construction is extending the use we already made of the multiplicative structure on $M_n(R)$, to make use not only of the additive structure, but also of the particular structure of 'scalar' multiplication (multiplication by an element of the centre), which plays no role in representing the group multiplication *per se*. The construction *also* makes sense at the G-set/vector space level, since linear transformations support the same extra structure.

The same formal construction of RG works when R is an arbitrary commutative ring, except that RG is not then a vector space. Instead it is called (in respect of the vector-space-like aspect

[1] For a set S, a map $\psi : G \times S \to S$ (written $\psi(g, s) = gs$) such that $(gg')s = g(g's)$, equips S with the property of left G-set.

of its structure) a free R-module with basis G. The idea of matrix representation goes through unchanged. If one wants a generalisation of the notion of G-set for RG to act on, the additive structure is forced from the outset. This is called a (left) RG-module. This is, then, an abelian group $(M,+)$ with a suitable action of RG defined on it: $r(x+y) = rx + ry$, $(r+s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$ $(r, s \in RG,\ x, y \in M)$, just as the original vector space R^n was. What is new at this level is that such a structure may not have a basis (a *free* module has a basis), and so may not correspond to any class of matrix representations.

(**1.1.9**) Exercise. Construct an RG-module without basis.
(Possible hints: 1. Consider $R = \mathbb{Z}$, G trivial, and look at §6.3. 2. Consider the ideal $\langle 2, x\rangle$ in $\mathbb{Z}[x]$.

From this point the study of representation theory may be considered to include the study of both matrix representations and modules.

(**1.1.10**) What other kinds of systems can we consider representation theory for?
A natural place to start studying representation theory is in Physical modeling. Unfortunately we don't have scope for this in the present work, but we will generalise from groups at least as far as rings and algebras.

The generalisation from groups to *group algebras* RG over a commutative ring R is quite natural as we have seen. The most general setting within the ring-theory context would be the study of arbitrary ring homomorphisms from a given ring. However, if one wants to study this ring by studying its modules (the obvious generalisation of the RG-modules introduced above) then the parallel of the matrix representation theory above is the study of modules that are also free modules over the centre, or some subring of the centre. (For many rings this accesses only a very small part of their structure, but for many others it captures the main features. The property that *every* module over a commutative ring is free holds if and only if the ring is a field, so this is our most accessible case. We shall motivate the restriction shortly.) This leads us to the study of *algebras*.

To introduce the general notion of an algebra, we first write $\mathrm{cen}\,(A)$ for the centre of a ring A

$$\mathrm{cen}\,A = \{a \in A \mid ab = ba\ \forall b \in A\}$$

(**1.1.11**) An algebra A (over a commutative ring R), or an R-algebra, is a ring A together with a homomorphism $\psi : R \to \mathrm{cen}\,(A)$, such that $\psi(1_R) = 1_A$.

Examples: Any ring is a $\mathbb{Z}$-algebra. Any ring is an algebra over its centre. The group ring RG is an R-algebra by $r \mapsto r1_G$. The ring $M_n(R)$ is an R-algebra.

Let $\psi : R \to \mathrm{cen}\,(A)$ be a homomorphism as above. We have a composition $R \times A \to A$:

$$(r, a) = ra = \psi(r)a$$

so that A is a left R-module with

$$r(ab) = (ra)b = a(rb) \tag{1.14}$$

Conversely any ring which is a left R-module with this property is an R-algebra.

(**1.1.12**) An R-representation of A is a homomorphism of R-algebras

$$\rho : A \to M_n(R)$$

(**1.1.13**) The study of RG depends heavily on R as well as G. The study of such R-algebras takes a relatively simple form when R is an algebraically closed field; and particularly so when that field is $\mathbb{C}$. We shall aim to focus on these cases. However there are significant technical advantages, even for such cases, in starting by considering the more general situation. Accordingly we shall need to know a little ring theory, even though general ring theory is not the object of our study.

Further, as we have said, neither applications nor aesthetics restrict attention to the study of representations of groups and their algebras. One is also interested in the representation theory of more general algebras.

1.1.4 Modules and representations

The study of algebra-modules and representations for an algebra over a field has some special features, but we start with some general properties of modules over an arbitrary ring R.

(**1.1.14**) A left R-module (for R an arbitrary ring) is *simple* if it has no non-trivial submodules...

Let M be a left R-module. A *composition series* for M is a sequence of submodules $M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_l = 0$ such that M_i/M_{i+1} is simple.

(**1.1.15**) **Theorem.** (Jordan–Holder) Let M be a left R-module. The following are equivalent:
(I) M has a composition series;
(II) every ascending and descending chain of submodules of M stops (these two stopping conditions separately are known as *ACC* and *DCC*);
(III) every sequence of submodules of M can be refined to a composition series.

(**1.1.16**) A module M is *semisimple* if equal to the sum of its simple submodules.

A *left ideal* of R is a submodule of R regarded as a left-module for itself. The *Jacobsen radical* of R is the intersection of its maximal left ideals. Ring R itself is a *semisimple ring* if its radical vanishes.

(**1.1.17**) Ring R is *Artinian* (resp. *Noetherian*) if it has the DCC (resp. ACC) as a left module for itself.

(**1.1.18**) **Theorem.** (Schur's Lemma) Suppose M, M' are nonisomorphic simple R modules. Then the ring $hom_R(M, M)$ of R-module homomorphisms from M to itself is a division ring; and $hom_R(M, M') = 0$.

(**1.1.19**) **Theorem.** (Artin–Wedderburn) Suppose R is semisimple and Artinian. Then R is a direct sum of rings of form $M_{n_i}(R_i)$ $(i = 1, 2, ..., l$, some $l)$ where each R_i is a division ring.

(**1.1.20**) Suppose M', M'' submodules of R-module M. They *span* M if $M' + M'' = M$; and are *independent* if $M' \cap M'' = 0$. If they are both independent and spanning we write $M = M' \oplus M''$ (*direct sum*). A module is *indecomposable* if it has no proper direct sum decomposition.

(**1.1.21**) **Theorem.** (Krull–Schmidt) If R is Artinian then as a left-module for itself it is a finite direct sum of indecomposable modules; and any two such decompositions may be ordered so that the i-th summands are isomorphic.

(**1.1.22**) If $x : M \to M'$, $x' : M' \to M$ are R-module homomorphisms such that $x \circ x' = 1_{M'}$ then x is a *split surjection* (and x' a split injection).

(**1.1.23**) An R-module is *projective* if it is a direct summand of a free module.

(**1.1.24**) **Theorem.** TFAE
(I) R-module P is projective;
(II) whenever there is an R-module surjection $x : M \to M'$ and a map $y : P \to M'$ then there is a map $z : P \to M$ such that $x \circ z = y$;
(III) every R-module surjection $t : M \to P$ splits.

(**1.1.25**) If R is Artinian and J_R its radical then

$$R/J_R \;=\; \oplus_{i=1}^{l} M_{n_i}(R_i)$$

There is a simple R/J_R-module (L_i say) for each factor, so that *as a left module*

$$R/J_R \cong \oplus_i n_i L_i$$

(i.e. n_i copies of L_i). There is a corresponding decomposition of 1 in R/J_R:

$$1 = \sum_i e_i$$

into orthogonal idempotents. One may find corresponding idempotents in R itself (see later) so that $1 = \sum_i e_i'$ there. This gives left module decomposition

$$R = \oplus_i n_i P_i$$

where (by (1.1.21)) the P_is are a complete set of indecomposable projective modules up to isomorphism.

(**1.1.26**) TO DO:
Finish overview of modules
Grothendieck group
Tensor product
induction
—

(**1.1.27**) **Operators acting on a space; their eigenvectors and eigenvalues.**
Here we remark very briefly and generally on the kind of Physical problem that can lead us into representation theory.
A typical Physical problem has a linear operator Ω acting on a space H, with that action given by the action of the operator on a (spanning) subset of the space. One wants to find the eigenvalues of Ω.
The eigenvalue problem may be thought of as the problem of finding the one-dimensional subspaces of H as an $\langle\Omega\rangle$-module, where $\langle\Omega\rangle$ is the (complex) algebra generated by Ω. That is, we want to find elements h_i in H such that:

$$\Omega h_i = \lambda_i h_i$$

— noting only that, usually, the object of primary physical interest is λ_i rather than h_i. If H is finite dimensional then (the complex algebra generated by) Ω will obey a relation of the form

$$\prod_i (\Omega - \lambda_i)^{m_i} = 0$$

Of course the details of this form are *ab initio* unknown to us. But, proceeding formally for a moment, if any $m_i > 1$ (necessarily) here, so that $S = \prod_i (\Omega - \lambda_i) \neq 0$, then S generates a non-vanishing nilpotent ideal (we say, the algebra has a radical). Obviously any such nilpotent object has 0-spectrum, so two operators differing by such an object have the same spectrum. In other words, the image of Ω in the quotient algebra by the radical has the same spectrum $\{\lambda_i\}$. An algebra with vanishing radical (such as the quotient of a complex algebra by its radical) has a particularly simple structural form, so this is a potentially useful step.

However, gaining *access* to this form may require enormously greater arithmetic complexity than the original algebra. In practice, a balance of techniques is most effective, even when motivated by physical ends. This balance can often be made by analysing the regular module (in which every eigenvalue is manifested), and thus subquotients of projective modules, but not more exotic modules. (Of course Mathematically other modules may well also be interesting — but this is a matter of aesthetic judgement rather than application.)
It may also be necessary to find the subspaces of H as a module for an algebra generated by a set of operators $\langle \Omega_i \rangle$. A similar analysis pertains.

A particularly nice (and Physically manifested) situation is one in which the operators Ω_i (whose unknown spectrum we seek to determine) are known to take the form of the representation matrices of elements of an abstract algebra A in some representation:

$$\Omega_i = \rho(\omega_i)$$

Of course any reduction of Ω_i in the form of (1.10) reduces the problem to finding the spectrum of $R_1(\omega_i)$ and $R_2(\omega_i)$. Thus the reduction of ρ to a (not necessarily direct) sum of irreducibles:

$$\rho(\omega_i) \cong +_\alpha \, \rho_\alpha(\omega_i)$$

reduces the spectrum problem in kind. In this way, Physics drives us to study the representation theory of the abstract algebra A.

1.2 Notes and references

The following texts are recommended reading: Jacobson[25, 26], Bass[4], Maclane and Birkoff[31], Green[22], Curtis and Reiner[16, 17], Cohn[12], Anderson and Fuller[3], Benson[5], Adamson[2], Cassels[9], Magnus, Karrass and Solitar[**?**], and references therein. .

1.3 Exercises

(**1.3.1**) Let R be a commutative ring and S a set. Then RS denotes the 'free R-module with basis S', the R-module of formal finite sums $\sum_i r_i s_i$ with the obvious addition and R action. Show that this is indeed an R-module.

(**1.3.2**) Let R be a commutative ring and G a finite group. Show that the multiplication in (1.11) makes RG a ring.

Hints: We need to show associativity. We have

$$\left(\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right)\right)\left(\sum_k r''_k g_k\right) = \left(\sum_{ij}(r_i r'_j)(g_i g_j)\right)\left(\sum_k r''_k g_k\right) = \sum_{ijk}((r_i r'_j)r''_k)((g_i g_j)g_k) \tag{1.15}$$

and

$$\left(\sum_i r_i g_i\right)\left(\left(\sum_j r'_j g_j\right)\left(\sum_k r''_k g_k\right)\right) = \left(\sum_i r_i g_i\right)\left(\sum_{jk}(r'_j r''_k)(g_j g_k)\right) = \sum_{ijk}(r_i(r'_j r''_k))(g_i(g_j g_k)) \tag{1.16}$$

These are equal by associativity of multiplication in R and G separately.

(**1.3.3**) Show that RG is still a ring as above if G is a not-necessarily finite monoid and RG means the free module of finite support as above.

Hints: Multiplication in monoid G is also associative.