Chapter 4

Categories

4.1 Categories I

ss:cati

(**4.1.1**) A CATEGORY is a triple

$$A = (\mathrm{Ob}\,A,\ \hom_A(-,-),\ \circ)$$

(one sometimes writes simply A for $\mathrm{Ob}\,A$ and $A(-,-)$ for $\hom_A(-,-)$) where
(i) $\mathrm{Ob}\,A$ is a collection of 'objects'; [1,2]
(ii) for each ordered pair (M,N) of objects $A(M,N)$ is a set of 'morphisms';
(iii) $\circ$ is an associative composition

$$A(M,N) \times A(L,M) \to A(L,N)$$

such that for each object M there is an identity $1_M \in A(M,M)$.

(**4.1.2**) REMARK. It is not uncommon to find a variant of (iii) used instead:

$$A(N,M) \times A(M,L) \to A(N,L)$$

This is ultimately just a matter of organisation (we have reversed the order of writing of all the pairs). The first formulation is natural in some settings (examples coming up), and the second formulation in others.

Set (**4.1.3**) Examples: Let **Set** be the collection of all sets, and for $M,N \in$ **Set** let **Set**(M,N) be the set of maps from M to N. The usual composition of maps is associative and has identities, so this is a category.

Ab Let **Ab** be the collection of all abelian groups and **Ab**(M,N) the set of group homomorphisms from M to N. This is a category.

Grp is the obvious extension of **Ab** to arbitrary groups.

[1](the possible failure of this collection to be a set will not concern us here [8])

[2]The notation $\mathrm{Ob}\,A$ is used, for example, in [26]; the notation $\hom_A(-,-)$ is used, for example, in [26] and in [31].

Let A be a category. Consider a triple A' consisting of any subclass of $\mathrm{Ob}\, A$; and a subset of $A(M,N)$ for each pair M, N in the subclass, such that $1_M \in A'(M,M)$, and the composition from A closes on these subsets; and the composition from A. This is a category — a *subcategory* of A. A subcategory is *full* if every $A'(M,N) = A(M,N)$.

Let $\mathbf{Set}^f$ be the full subcategory of $\mathbf{Set}$ consisting of finite sets.

For R a ring let $\mathbf{Mat}_R$ be the category of R-valued matrices. That is $\mathrm{Ob}\,\mathbf{Mat}_R = \mathbb{N}$, and $\hom_{\mathbf{Mat}_R}(m,n)$ is the set of $m \times n$ matrices, and composition is matrix multiplication. [1]

For R a ring R-mod is the category of left R-modules and their homomorphisms.

monoid (**4.1.4**) Proposition. *Let A be a category and N an object in A. Then the full subcategory of A induced on the single object N consists essentially in the set $A(N,N)$ with its unital associative composition, and hence is a monoid.*

Conversely any monoid is a category on (essentially) any single object.

(**4.1.5**) Example. The monoid $\mathbf{Set}(\underline{2},\underline{2}) = \underline{2}^{\underline{2}}$ is the one studied in Section 2.3.1.

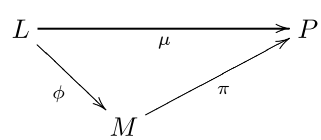
(**4.1.6**) Given a category A there is a Dual category (or opposite category) A^o which has the same objects, and $A^o(M,N) = A(N,M)$, and composition is reversed.

Note that a category and its dual can be very different. For example, $\mathbf{Set}(S,\emptyset)$ is empty unless $S = \emptyset$, while $\mathbf{Set}(\emptyset, S) = \mathbf{Set}^\circ(S,\emptyset)$ contains precisely one element for each S (the appropriate empty relation).

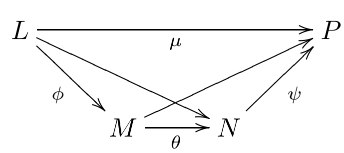
(**4.1.7**) Remarks: In a given category we may write

$$M \xrightarrow{\theta} N$$

for $\theta \in A(M,N)$. Thus a (small) category is a directed graph with some extra data. To say that a triangle of such homs/arrows



commutes is to say that the long arrow (in the obvious sense) is the composite of the shorter ones. Then associativity says that commutativity of any three of the triangles here:



implies commutativity of the fourth.

(**4.1.8**) A morphism $f \in A(M,N)$ is an Isomorphism if there exists $g \in A(N,M)$ such that $gf = 1_M$ and $fg = 1_N$.

Example: The isomorphisms in $\hom_{\mathbf{Set}}(\underline{n},\underline{n})$ form a submonoid which is a subgroup — the symmetric group S_n.

padprpddcat (**4.1.9**) Given a set $\{C_i\}_{i\in I}$ of categories, the triple $\times_i C_i$ consisting of object class $\times_i \mathrm{Ob}\, C_i$, I-tuples of morphisms, and the corresponding pointwise composition, is a category.

brauerexmoBt (**4.1.10**) Exercise. Show that

$$\mathbf{Br} = (\mathbb{N}, \mathbf{Br}(-,-), \circ)$$

where $\mathbf{Br}(m,n) = J_{m,n}$ (from (2.2.10)), is a category.

4.1.1 Functors

(**4.1.11**) Concrete Category. If there is a map und : $A \to \mathbf{Set}$ from the object collection of category A to **Set**, such that $A(M,N) \subseteq \mathbf{Set}(\mathrm{und}(M), \mathrm{und}(N))$ for each $M, N \in A$, with $1_M = 1_{\mathrm{und}(M)}$, and composition is the usual composition of maps, then A is a concrete category.

Examples: **Ab**, **Grp** are a concrete categories for which 'und' is simply inclusion. **Br** is not a concrete category by inclusion (indeed its objects are not sets).

(**4.1.12**) For A, B categories, a (covariant) Functor $F : A \to B$ is a map on objects together with a map on morphisms which preserves composition and identities.

A Contravariant functor from A to B is a functor from A^o to B (examples later).

pa:func Examples: As noted, **Grp** is concrete. Thus there is an 'und' functor $U : \mathbf{Grp} \to \mathbf{Set}$. Indeed for each $n \in \mathbb{N}$ there is a 'pointwise' functor $D_n : \mathbf{Grp} \to \mathbf{Set}$ given by

$$D_n(A \xrightarrow{f} B) = A^n \xrightarrow{f^n} B^n$$

p:morexf (**4.1.13**) More examples: (The following simple examples will come up again later, when we develop the notions of *natural transformation* and of *adjoint pairs* of functors.) Let S be a set. Then there is a functor $F_S : \mathbf{Set} \to \mathbf{Set}$ given by $F_S(T) = T \times S$ and

$$F_S(T \xrightarrow{f} T')(t,s) = (f(t), s)$$

And a functor $F^S : \mathbf{Set} \to \mathbf{Set}$ given by $F^S(T) = \hom(S,T)$ and $F^S(f) : g \mapsto f \circ g$.

exe:preadj (**4.1.14**) Exercise. Since $F^S(T) = \hom(S,T)$ is a set as well as a hom set, we may consider the hom set $\hom(U, F^S T) = \hom(U, \hom(S,T))$ in **Set**. Let U', T' be two further sets. A pair of maps $u' : U' \to U$ and $t : T \to T'$ define a map from $\hom(U, F^S T)$ to $\hom(U', F^S T')$ by

$$g \mapsto (u, F^S t)(g) = (F^S t) \circ g \circ u'$$

(note the direction of the map u'!).
Show that this gives rise to a functor

$$\hom(-, F^S -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(cf. (4.1.9)), and construct an analogous functor

$$\hom(F_S -, -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(Remark: Functors from products are sometimes called bifunctors.)
Answer to last part: the map from $\hom(F_S U, T)$ to $\hom(F_S U', T')$ is given by $f \mapsto (F_S u', t)(f) = t \circ f \circ F_S u'$.

(**4.1.15**) A *forgetful functor* is a functor to a category whose objects have some structure (binary operation; inverses; etc) from a category whose objects have this and additional structure. The functor simply forgets the additional structure.

Our 'und' functors are examples of forgetful functors. Another example would be the functor from **Fld** (the category of fields) to the category of integral domains and injective ring maps (call it C), inside the category **Rng** of rings.
(The restriction to injective maps is just because every field homomorphism is injective.)

4.1.2 Natural transformations

de:natt (**4.1.16**) Let A, B be categories, and T, S be functors from A to B. A 'natural transformation' $a : T \to S$ is a family $a = (a_M)_{M \in A}$ of B–morphisms

$$a_M : TM \to SM$$

such that for each $f \in A(M, N)$ we have $Sf a_M = a_N T f$.

Example: The functors from A to B are the objects of a category B^A with morphisms the (set of!) natural transformations.

(**4.1.17**) As we have noted, a group is an example of an algebraic system — one with a binary operation. That is, for each group, group multiplication is a function

$$\kappa_G : G \times G \to G$$

The collection of all group multiplications $\kappa = (\kappa_G)$ is thus a candidate to be a natural transformation κ from D_2 to U (the functors **Grp** $\to$ **Set** defined in (4.1.12)):

$$\begin{array}{ccc} G \times G & \xrightarrow[\kappa_G]{} & G \\ \downarrow{\scriptstyle f^2} & & {\scriptstyle f}\downarrow \\ G' \times G' & \xrightarrow[\kappa_{G'}]{} & G' \end{array}$$

and the commutativity condition $Sf a_M = a_N T f$ is $U f \kappa_G(a, b) = \kappa_{G'}(D_2 f)(a, b)$ which is simply

$$f(ab) = f(a)f(b)$$

That is, group multiplication (collectively) is a natural transformation.
Other operations in categories of algebraic systems are viewable as natural transformations similarly.

exa:natiso (**4.1.18**) Example. Recall the functors F_S, F^S from (4.1.13), and $\hom(-, F^S-)$, $\hom(F_S-, -)$ from (4.1.14). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S, U))$ by $(\psi x)(v)(s) = x((v, s)) \in U$. On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v, s) = (y(v))(s)$.

Comparing with (4.1.14) one finds that ψ and ψ' are natural transformations between the functors $\hom(-, F^S-), \hom(F_S-,-) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$. For example, for each object (V, U) in $\mathbf{Set}^o \times \mathbf{Set}$ we have $\psi_{U,V}$ such that the diagram

$$\begin{array}{ccc} \hom(F_S V, U) & \xrightarrow{\psi_{U,V}} & \hom(V, F^S U) \\ \downarrow & & \downarrow \\ \hom(F_S V', U') & \xrightarrow{\psi_{U',V'}} & \hom(V', F^S U') \end{array}$$

commutes for vertical maps built from any $(f,g) = (V' \xrightarrow{f} V, U \xrightarrow{g} U') \in \hom_{\mathbf{Set}^o \times \mathbf{Set}}((V,U),(V',U')))$. To see this note that going to the right first we have

$$((\hom(-, F^S-)(f,g))\psi_{U,V})(V \times S \xrightarrow{x} U) = (\hom(-, F^S-)(f,g))(V \xrightarrow{\psi x} \hom(S,U))$$

$$= (V' \xrightarrow{f} V \xrightarrow{\psi x} \hom(S,U) \xrightarrow{F^S g} \hom(S,U'))$$

so this way round the image of x is a map in which $v' \in V'$ is taken to a map which takes s in S to $g(x(f(v'),s))$. The other way round

$$(\psi_{U',V'}(\hom(F_S-,-)(f,g)))(V \times S \xrightarrow{x} U) = (\psi_{U',V'})((V' \times S \xrightarrow{f \otimes 1} V \times S \xrightarrow{x} U \xrightarrow{g} U')$$

which eventually gives the same thing.

4.2 R-linear and ab-categories

ss:ab

(**4.2.1**) Let R be a commutative ring. An R-linear category is a category in which each hom set is an R-module, and the composition map is bilinear.

A basis for an R-linear category C is a subset $\hom_C^o$ of $\hom_C$ such that

$$\hom_C^o(m,n) = \hom_C^o \cap \hom_C(m,n)$$

is a basis for $\hom_C(m,n)$.

Any category C extends R-linearly to an R-linear category RC.

(**4.2.2**) If C is an R-linear category then each $\hom_C(m,m)$ is an R-algebra.

(**4.2.3**) Remark. A good working aim for this course is to compute the dimensions of the irreducible modules for the $\mathbb{C}$-algebras contained in $\mathbb{C}\mathbf{Br}$ (as defined in Exercise (4.1.10)).

(**4.2.4**) A category C is called an *ab-category* if there is a + operation on each $\hom_C(A,B)$ making it an abelian group; and morphism composition distributes over +:

$$f(g+h) = fg + fh \qquad \text{and} \qquad (g+h)f = gf + hf$$

exa:abadd

(**4.2.5**) Example: We can define a + for any $\hom_{\mathbf{Ab}}(A,B)$ pointwise:

$$(g+h)(a) = g(a) + h(a)$$

(this defines an element of $\hom_{\mathbf{Set}}(A,B)$, but $(g+h)(a+b) = g(a+b)+h(a+b) = g(a)+g(b)+h(a)+h(b) = (g+h)(a)+(g+h)(b)$, so $(g+h) \in \hom_{\mathbf{Ab}}(A,B)$ as required).
Thus **Ab** is an ab-category.

de:additive (**4.2.6**) A functor $F : A \to B$ between ab-categories is *additive* if for $f, g \in \hom_A(X,Y)$:

$$F(f+g) = F(f) + F(g)$$

(**4.2.7**) If there is an object 0 in a category C such that $|\hom_C(0,A)| = |\hom_C(A,0)| = 1$ for all A then 0 is called a *zero object*.

(**4.2.8**) Consider L, M, N objects in an ab-category C. If there are morphisms $a : L \to N$, $a' : N \to L$, $b : M \to N$, $b' : N \to M$ such that $a'a = 1_L$, $b'b = 1_M$ and

$$aa' + bb' = 1_N$$

then we write $N \cong L \oplus M$.
If there is an object $N \cong L \oplus M$ for any two objects L, M we say C *has direct sums*.

pa:ac (**4.2.9**) An *additive category* is an ab-category with direct sums and zero object.

Example: **Ab** with the trivial group as zero object.

4.2.1 Abelian categories

See for example Freyd's 1964 book [21]. Abelian categories can be regarded as abstractions of the class of module categories, and so are useful in representation theory.

(**4.2.10**) An additive category A is an *abelian category* if
(I) every $f \in \hom_A(M,N)$ has a kernel and a cokernel.
(II) every monomorphism is a kernel; every epimorphism is a cokernel.

4.3 Categories II

ss:catii

(**4.3.1**) A functor $F : A \to B$ is:
full (respectively *faithful*) if all hom set maps

$$F : \hom_A(S,T) \to \hom_B(FS, FT)$$

are surjective (respectively injective);
isomorphism dense if for every object T in B there is an object S in A such that $F(S)$ is isomorphic to T.

(**4.3.2**) A skeleton for a category is a full isomorphism dense subcategory in which no two objects are isomorphic.

(**4.3.3**) Example. The assembly of sets in $\mathbf{Set}^f$ into cardinality classes induces a corresponding set of isomorphisms between hom sets

$$\begin{array}{rcl} f_S : S & \xrightarrow{\sim} & S' \qquad (4.1) \\ f : \hom(S,T) & \to & \hom(S',T') \qquad (4.2) \\ g & \mapsto & f_T \circ g \circ f_S^{-1} \qquad (4.3) \end{array}$$ 1

Associate a representative element of each class to each cardinality. We may then construct a category $C_{\mathbb{N}}$ whose objects are the set $\mathbb{N}$ of finite cardinalities, and with $\hom_{C_{\mathbb{N}}}(m,n) = \hom(\underline{m},\underline{n})$. The functor

$$F : C_{\mathbb{N}} \to \mathbf{Set}^f$$

which takes object n to object $\underline{n}$ and identifies the corresponding hom sets is isomorphism dense and full. This $C_{\mathbb{N}}$ is thus a subcategory of $\mathbf{Set}^f$, from which the rest of $\mathbf{Set}^f$ can easily be constructed. We have:

(**4.3.4**) Proposition. *This* $C_{\mathbb{N}}$ *is a skeleton for* $\mathbf{Set}^f$*.*

(**4.3.5**) Note that the set of isomorphisms in an end set form a group. The set of isomorphisms in $\hom(\underline{n},\underline{n})$ form the symmetric group S_n.

(**4.3.6**) A *congruence relation* I on a category C is an equivalence relation on each hom set such that $f' \in [f]_I$ and $g' \in [g]_I$ implies $f'g' \in [fg]_I$ (compositions of morphisms).
The quotient category C/I has the same object class as C but $\hom_{C/I}(F,G) = \hom_C(F,G)/I$, with the obvious composition well-defined by congruence.

4.3.1 Adjunctions

de:adjunction1 (**4.3.7**) An *adjunction* between categories A, B is a pair of functors $F : A \to B$ and $G : B \to A$ such that for all objects (U,V) in $A \times B$ there is a bijection

$$\psi_{U,V} : \hom_A(GV,U) \to \hom_B(V,FU)$$

such that

$$\psi : \hom_A(G-,-) \to \hom_B(-,F-)$$

is a natural isomorphism of bifunctors.
That is, we have

$$\begin{array}{ccc} \hom_A(GV,U) & \xrightarrow{\psi_{U,V}} & \hom_B(V,FU) \\ \downarrow & & \downarrow \\ \hom_A(GV',U') & \xrightarrow{\psi_{U',V'}} & \hom_B(V',FU') \end{array}$$

commutative for each $f \in \hom_{A^o \times B}(V,U)$ (and hence each pair of vertical maps, cf. (4.1.16)).
(**4.3.8**) Example. Recall the functors F_S, F^S from (4.1.13). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S,U))$ by $(\psi x)(v)(s) = x((v,s)) \in U$.

On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v, s) = (y(v))(s)$.

Comparing with (4.1.14) one checks that ψ and ψ' are natural transformations (the diagram above commutes for vertical maps built from $\hom_{\mathbf{Set}^o \times \mathbf{Set}}(V, U)$) and hence isomorphisms. Thus (F_S, F^S) is an adjunction.

(**4.3.9**) The left adjoint to a forgetful functor is usually something interesting!

4.4 Categories III

4.4.1 Tensor/monoidal categories

ss:tc1

See Section 9.4. See also Joyal–Street [**?**], Kassel [30], Reshetikhin–Turaev [37].

Let A be a category and $A \times A$ the product category as in (4.1.9). Whenever we have a functor $F : A \times A \to A$ we have in particular an association of an object $F(m, n)$ to each pair of objects. If this binary operation is associative and unital (so that an object set becomes a monoid) then (A, F) is a *strict tensor category*. If the binary operation is associative and unital up to (certain suitable) natural isomorphisms

$$a_{LMN} : F(F(L, M), N) \to F(L, F(M, N))$$

$$l_M : F(1, M) \to M$$

$$r_M : F(M, 1) \to M$$

(see later for axioms) then $(A, F) = (A, F, 1, a, l, r)$ is a *tensor category*.

Suppose there are additional natural isomorphisms

$$g_{LM} : F(L, M) \to F(M, L)$$

Then we can reorder and move brackets in any expression of form $F(M_1, F(F(M_2, F(M_3, M_4)), M_5))$ by applying suitable a_{LMN} and g_{LM}s. Suppose we associate such a manipulation to an element of the braid group by associating each $g_{M_i M_j}$ to a braiding in that position. If the manipulation morphism depends only on the associated braiding, then the tensor category A together with (the collection) g is a *braided tensor category*.

A natural example is the category of modules of a finite group algebra, where $F(M, N) = M \otimes N$. (Indeed later we will write $F(M, N)$ as $M \otimes N$ quite generally.)