ON THE EXCEPTIONAL STRUCTURE OF $End_{U_q sl_2}(V_N^{\otimes n})$

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Abstract

We show that the algebras $T_n^N(q) = End_{U_q sl_2}(V_N^{\otimes n})$ and $T_m^M(q) = End_{U_q sl_2}(V_M^{\otimes m})$ are Morita equivalent for any N, M, for $n > 1$ provided that $m = \frac{n(N-1)}{M-1}$. Since all the generic structures are known, and the full structure of the $N = 2$ case is known, this completely determines the representation theory of all these algebras, including all unitarizable and non-unitarizable exceptional cases.

1 Introduction

Let $Ann_{Nn} = ann_{U_q sl_2} V_N^{\otimes n}$ [1], where V_N is the simple N dimensional module, and let $U_n^N = U_q sl_2 / Ann_{Nn}$ (that is, the algebra *faithfully* represented on $V_N^{\otimes n}$). It is well known, and a straightforward combinatorial proof exists [2], that the generic Temperley-Lieb algebra $T_n(q) = T_n^2(q)$ and the quantum group $U_q sl_2$ centralize each other on $V_2^{\otimes n}$ (when each has its standard action). It has been shown that this is still true in any specialization of q, including roots of unity [3]. In the same paper it was shown that $T_n(q)$ and $U_q sl_2/Ann_{2n}$ are Morita equivalent in any specialization (although an explicit functor was not given). This property is always (trivially) true for dual pairs of semi-simple algebras (i.e. dual pairs are Morita equivalent). Here this property, together with the fact that $U_q sl_2/Ann_{Nn}$ and $U_q sl_2/Ann_{Mm}$ are obviously related by a well understood relaxation of the quotient (generically $U_n^N \cong U_m^M$ if $m = \frac{n(N-1)}{M-1}$), enables us to deduce that $End_{U_q sl_2}(V_N^{\otimes n})$ and $End_{U_q sl_2}(V_M^{\otimes m})$ are Morita equivalent generically. *We show that it is still true in any specialization.* Since the category of left $T_n(q)$ modules is understood in every specialization ([2] and references therein) we can then simply read off the structure of each $T_m^M(q)$!

Recall [4, 2] that Morita equivalence is an equivalence of categories of (algebra left) modules. It is weaker than isomorphism of algebras, but since it takes morphisms to morphisms as well as modules to modules it is frequently strong enough that, given the structure of one algebra, that of a Morita equivalent algebra may be deduced. We will exploit this device.

The property that dual pairs are Morita equivalent is a category theoretic rather than algebraic property (i.e. it is independent of the details of the representations). It is rare in general (see [5] and references therein), but since it happens here at level $N = 2$, the continued equivalence between $U_q sl_2/Ann_{Nn}$ and $U_q sl_2/Ann_{Mm}$ would allow us to complete a commuting square of Morita equivalences between different M and N:

$$\begin{array}{ccccc} T_n(q) & \overset{ME}{\leftrightarrow} & T^3_{n/2}(q) & \overset{ME}{\leftrightarrow} & T^M_m(q) \\ \updownarrow & & \updownarrow & & \updownarrow \\ U_n^2 & \leftrightarrow & U^3_{n/2} & \leftrightarrow & U^M_m \end{array} \tag{1}$$

where vertical arrows denote 'dual and Morita equivalent', ME denotes Morita equivalent, and unmarked horizontal arrows are algebra isomorphisms. The only issue, then, is whether these algebra isomorphisms survive in every specialization. Since the underlying algebra is the same ($U_q sl_2$), the specializations of the generically isomorphic irreducibles are the same in U_n^N and U_m^M (it is well known that in $U_q sl_2$ they either remain irreducible or else develop a simple invariant subspace in any specialization [6]), thus the only problem would be if these generic irreducibles become glued into indecomposable projective modules in different ways in the Nn and Mm cases. It is generally true that the decompositions of the generic irreducibles actually determine the decompositions of the indecomposable projectives up to the nesting of invariant subspaces (c.f. modular representation theory of the symmetric group, the exceptional structure of T-L itself, and so on [7, 2]) and

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in this relatively simple case the nesting is also fixed. Thus the algebra isomorphisms survive, and the commuting squares are complete, establishing the Morita equivalences in the top row.

Note that the commuting square is broken for $T_m^M(q)$ if $m = 1$, since the vertical arrow requires duality (i.e. requires the tensor representation to *exist*!). This means that nothing can be said about the $m = 1$ case directly via our device. This is not really a problem, since the structure of $T_1^M(q)$ is straightforward!

As an illuminating exercise, and to illustrate the consequences of this result, we checked the result by an explicit calculation of the structure of the $N = 3$ case for $n = 1, 2, 3, 4, 5, ...$, using a basis described below (and some category theory tools). We also confirmed that the result gives the correct irreducible dimensions in all the known (unitarizable) cases - see [8], and in particular the seminal work of Vladimir Rittenberg et al, [9]. For example we have

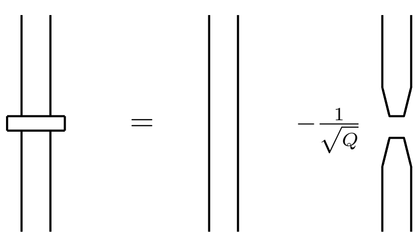
$$S_{n,N}(0) = \frac{1}{2} \sum_{j=1}^{l+1} sin^2(j\pi/(l+2)) \left(\frac{sin(Nj\pi/(l+2))}{sin(j\pi/(l+2))} \right)^n \tag{2}$$

for the dimension of the spine representation when $q = exp(i\pi/(l+2))$ (we got this particular formulation from [10]).

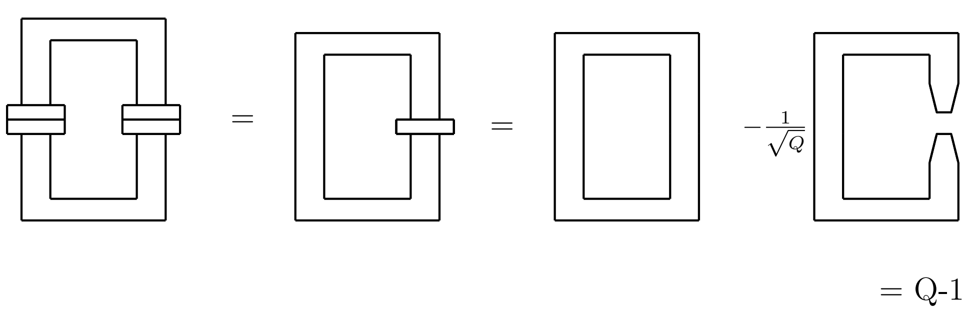
The result has applications in the spectrum analysis of higher spin $U_q sl_2$ vertex models (c.f. Martin and Rittenberg [11]), and in conformal field theory. The technique has applications in the analysis of $U_q G$, for G any Lie group. These issues will be dealt with elsewhere.

2 Gram matrices

A basis for $T_n^N(q)$, and for each irreducible representation, is given in [2]. The idea is to take $N-1$ fold cabled T-L diagrams (see also, for example, Rittenberg et al [12] and references therein) with a projector 'pumped' down the cable [13, 14] to kill all but the N dimensional irreducible (the cabling gives $\square^{\otimes N-1}$, as it were). For $N = 3$ the basic propagating line may thus be expanded in T-L diagrams (putting $Q = (q + q^{-1})^2$) as

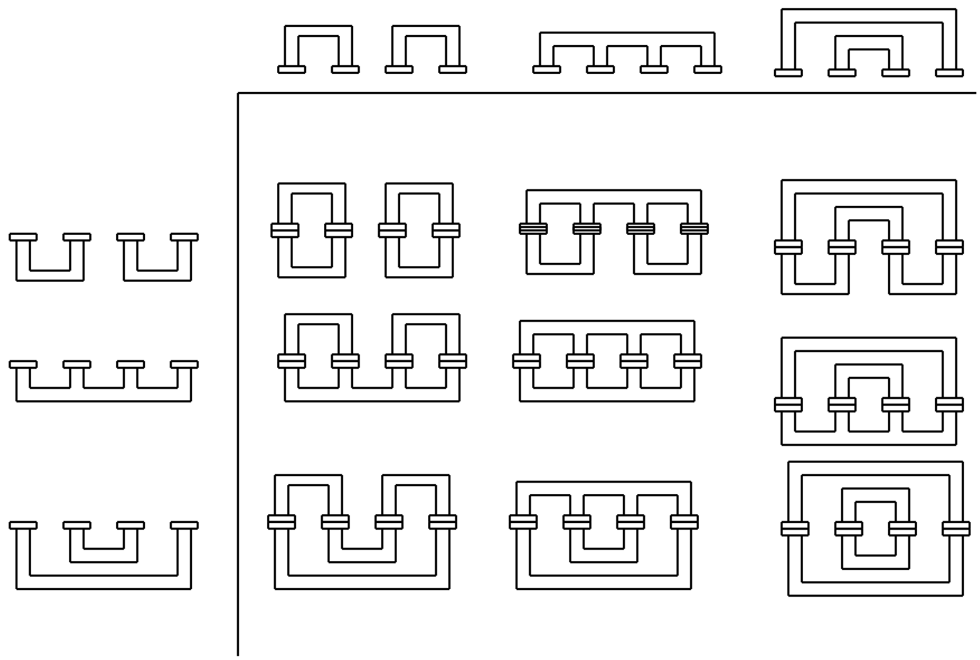


giving.....



for a cabled closed loop, using the usual T-L rule (single T-L closed loop $\rightarrow$ factor of $\sqrt{Q}$).

In the Temperley-Lieb diagrammatic version, invariant subspaces of $T_n^N(q)$ are indexed by the number of lines l propagating through the diagram, just as they are in the T-L algebra itself. Also as in $T_n^2(q)$, bases for irreducibles are obtained by focussing on the top or bottom parts of these diagrams. For example, with $N=3, n=4$ and $l=0$ we have left and right bases (that is, with diagrams acting from the top/bottom respectively) and their inner product illustrated as follows:



The explicit Gram matrix for the inner product on this basis is then

$$M_{3,4}(0) = \begin{pmatrix} (Q-1)^2 & \frac{(Q-1)^2}{\sqrt{Q}} & Q-1 \\ \frac{(Q-1)^2}{\sqrt{Q}} & \frac{(Q-1)(Q^2-3Q+3)}{Q} & \frac{(Q-1)^2}{\sqrt{Q}} \\ Q-1 & \frac{(Q-1)^2}{\sqrt{Q}} & (Q-1)^2 \end{pmatrix}$$

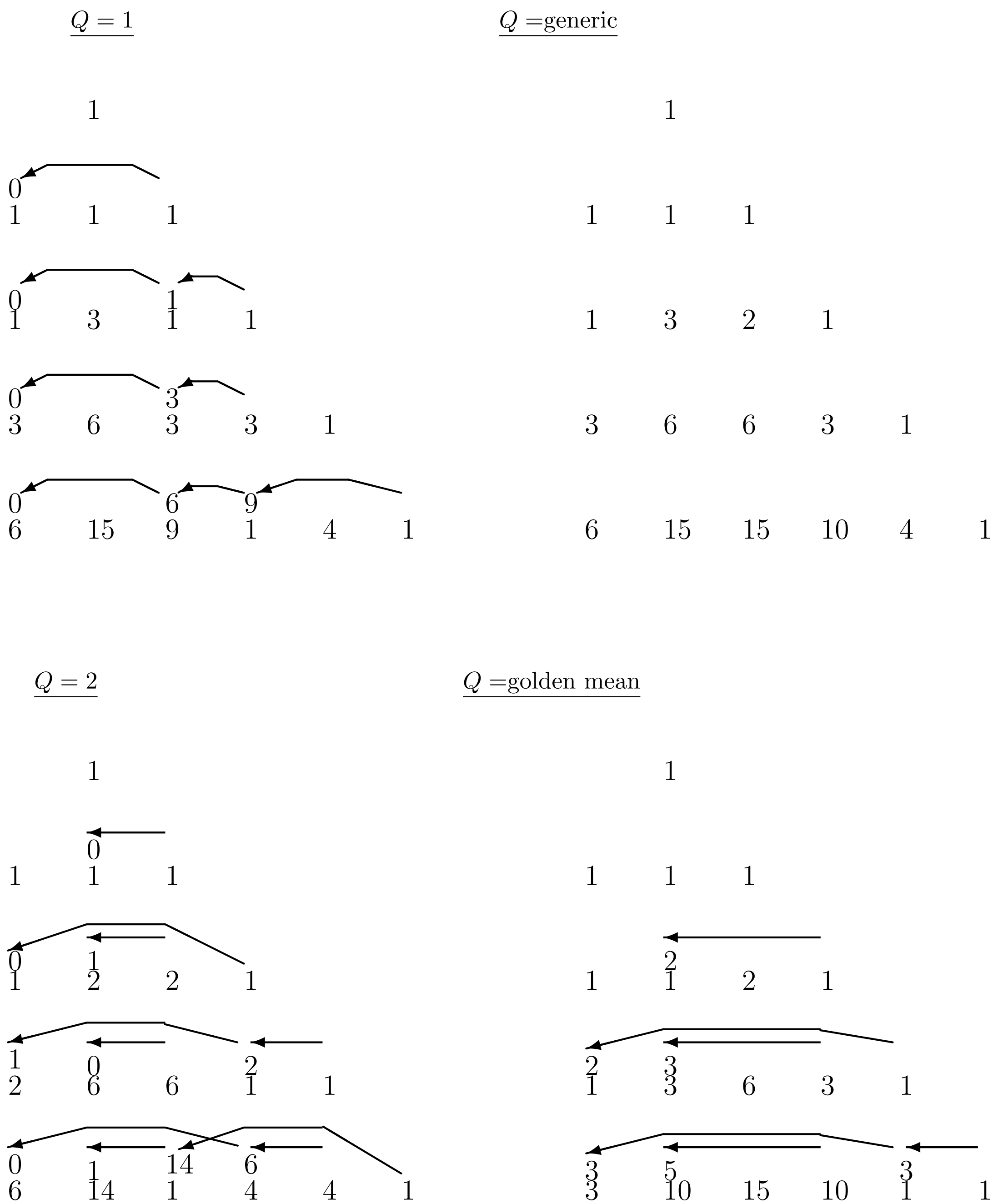
with determinant

$$|M_{3,4}(0)| \;=\; (Q-1)^3(Q-2)^2(Q^2-3Q+1)/Q.$$

Following the arguments in [15] we deduce that this module has a 2 dimensional invariant subspace for $Q=2$, a one dimensional invariant subspace at $Q=$golden mean, and so on.

Altogether, such simple calculations, and categorical embedding of successive layers as in the T-L case [2], lead us to the following list of quivers for generically simple modules (we write each module as its Loewy decomposition, i.e. with the dimension of the invariant subspace on the bottom). [3] In these tables the rows describe the generically simple modules for $T_n^3(q)$ for $n=1,2,3,4,5$ for each of the Q values given:

[3]In simpler terms, the generically simple modules break up to contain smaller simple parts at exceptional Q values. The tables show how they break up, identify isomorphisms among the new smaller simple parts, and show which appear as invariant subspaces and which as quotients in the decomposition.



and so on. For example, the second row (describing the generically simple modules for $T_2^3(q)$) consists generically of three inequivalent one dimensional modules. At $Q = 1$ two of these become isomorphic, since there is a homomorphism shown from the rightmost to the leftmost which the 0 on the top indicates to be surjective. Effectively, then, in this case we just have two distinct simple modules altogether, rather than three. However it is still worth presenting the data as above, since the structure of indecomposable projective modules can be deduced (and, for example, the dimension of the algebra checked). The total dimension of this algebra is still 3 (c.f. 1+1+1 generically), since the regular representation has structure $1 \oplus \begin{matrix} 1 \\ 1 \end{matrix}$. Although the two dimensional projective is actually made of two isomorphic modules, one glued on top of the other (in the sense of [5]), the dimension count is 1 for the simple projective, 1 for the doubled up module, and 1 for the glue.

To summarize: If a generic irreducible representation has dimension D (say) and, at some special Q, has an invariant subspace of dimension d, then (this is signalled by an arrow incoming and) the quotient has dimension $D-d$. In the language of solvable models this is the dimension of what one might call the 'restricted irreducible', in the sense that it is the dimension of the representation to be found in restricted models such as Andrews-Baxter-Forrester or Potts at q a root of unity, which is smaller than the corresponding dimension in unrestricted (generic) cases. A standard example of this is Temperley-Lieb, i.e. $T_n^2(q)$, for the left hand irreducible (in the sense of our tables) at $Q = 1$ - the 1 state Potts model! Here $d = D - 1$ for all n, so the restricted representation has dimension=1 (as required for a 1 state transfer matrix). For $Q = 0$ then $d = D$ in this case. Rather than say that there is a 0 dimensional quotient representation here we should say there is no representation left after quotienting by the invariant subspace (i.e. the subspace is the whole thing). Again this is right physically, since there is *no* Potts representation for $Q = 0$ states! Our $T_2^3(Q = 1)$ example above is analogous to this.

To see that the results above are Morita equivalent to the $N = 2$ case in general, consider the diagrams on pages 169-174 of [2]. The results for $n = 2, 3, 4, 5$ above may be checked via our commuting square against $T_n^2(q)$ for $n = 4, 6, 8, 10$ respectively. These may be read from [2], although it must be noted that we have switched to 'braid' notation since [2] was written. That is, the n in $T_n^2(q)$ refers to n strings as a braid quotient, whereas in [2] $T_n(q)$ refers to n *generators*. Thus for direct comparison $n = 3, 5, 7, 9$ in the numering used in [2] are the relevant rows ($n = 9$ is not written out explicitly in the examples in [2]). Note also that, as explained in the introduction, the case $n = 1$ above should not be compared directly with $T_2^2(q)$ (and indeed we see that it has one fewer module).

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