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ON SCHUR-WEYL DUALITY, A_n HECKE ALGEBRAS AND QUANTUM $sl(N)$ ON $\otimes^{n+1}\mathbb{C}^N$

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Abstract We prove that an N-state vertex model representation of the A_n Hecke algebra quotient $NH_n(q)$ is faithful for all q. We use the result to examine the indecomposable content of these representations, and hence the structure of the centraliser algebra, which is generically a quotient of $U_q sl(N)$, at q a root of unity. We achieve a complete analysis in the case $N = 2$, finding a number of Morita self-dual algebraic structures.

1 Introduction

It is well known, and a straightforward combinatorial proof exists, that a given quotient of the A_n type Hecke algebra over the complex numbers $H_n(q)$ (called $NH_n(q)$ - see section 2) and an appropriate quotient of the so called quantum group $U_q sl(N)$ (Jimbo 1985, Drinfeld 1986) are in Schur-Weyl duality on $\otimes^{n+1}\mathbb{C}^N$ if q is not a root of unity. We will briefly review these results shortly. For an introductory review see, for example, Martin (1991).

The simple proof does not work for the physically crucial (and mathematically most interesting) case of q a root of unity, even though the actions of both algebras remain well defined (care must be taken with the definition of $U_q sl(N)$ - see appendix A). The key issue in this case is the faithfulness of the representation of $NH_n(q)$. It is important to know if this result holds in order to compare our partial knowledge of the structures of the two algebras (see e.g. Dipper and James 1989, Lusztig 1989). Our main result in this paper (section 3) is that it does!

In order to illustrate the importance of this result explicitly we then consider some aspects of the mechanism of its application (section 4). In the process we find that, in all the cases we can check, i.e. all of $N = 2$, the 2 algebras remain Schur-Weyl dual for all q, and furthermore are Morita equivalent (for a general reference on Morita equivalence try Pierce 1982). This is a trivial result for all N for generic q, but is very strong, if true, in general.

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The faithfulness result, and the techniques involved in its proof, are also crucial for the computation of the spectrum of quantum spin chains at q a root of unity, and in providing the means to analyse $H_n(q)$ through other sequences of quotients besides $NH_n(q)$, as we will see in a subsequent paper (Martin and Rittenberg 1991).

Let A^o be an associative algebra over the complex numbers and M a finite dimensional left A^o-module. Then $B = End_{A^o}(M)$ is the centraliser algebra of A^o on M (that is, the algebra of endomorphisms of M which commute with the action of A^o), and $A = End_B(M)$ and B are said to be in Schur-Weyl duality on M. That is

$$B = End_{End_B(M)}(M).$$

In this case A and B are sometimes called a dual pair (see e.g. Zelevinskii 1987) or a Howe pair on M.

Let K be the smallest double sided ideal of A^o such that M is a faithful A^o/K module (i.e. $K = ann_A M$). Then $A \supset A^o/K$, and in general we have a composite morphism

$$A^o \to A^o/K \to A$$

For example, if A^o/K gives all upper triangular matrices on M then B is just scalars and A is all matrices on M.

We say that A^o itself has a Schur-Weyl dual on M if and only if

$$A^o/K = A.$$

A sufficient, but not necessary, condition for this is that A^o/K is semi-simple (this happens in our case when q is not a root of unity). We will discuss necessary conditions in section 4.

If $A^o = H_n(q)$ and M is the usual N-state vertex model representation (see section 2), then B is generically a certain (n-dependent) quotient of $U_q sl(N)$ (section 4.2), called $U_q^n sl(N)$, and $A^o/K = A = NH_n(q)$. The representation M of $NH_n(q)$ is block diagonal for all q, with the blocks being q-permutation modules in the sense of a q deformation of the symmetric group permutation modules (see section 4). This means that the indecomposable content of these modules is known generically, and in particular that some of them are themselves faithful representations of the algebra (see Robinson 1962 for example). It is the multiplicities and morphisms of the indecomposables which tell us the structure of the centraliser algebra in general, so these are crucial results. We can construct a counterexample (see section 4) showing that the faithfulness of generic blocks is not necessarily preserved in the case q a root of unity! The faithfulness of the whole representation is thus by no means a trivial result, giving an important clue to its indecomposable and irreducible content (which will be discussed, and in some cases determined, using this property).

In the next section we define $NH_n(q)$ and the action of $H_n(q)$ on $M = \otimes^{n+1}\mathbb{C}^N$. We then establish our main result - that this gives a faithful representation of the algebra $NH_n(q)$. In the following section we illustrate the use of these results with some applications. We determine the structure of the $U_q sl(2)$ quotient from that of $2H_n(q)$, and note in particular a Morita equivalence. In appendix A we define (from Jimbo 1985, Drinfeld 1986, Lusztig 1989) the appropriate action of $U_q sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$, and note commutativity with the representation of $NH_n(q)$. Finally appendix B contains a technical remark.

2 Hecke algebra

The main result of this section is a theorem given on page 8.

Following existing notation (for a recent introductory review and full references see, for example, Martin 1991 -hereafter called I, or Westbury 1990) we have:

Definition 1 (Hecke algebra) *For n a positive integer, $q \in \mathbb{C} - \{0\}$ and*

$$\sqrt{Q} = q + q^{-1}$$

$H_n(q)$, or simply H_n, is the unital associative algebra over $\mathbb{C}$ defined by generators $\{U_i : i = 1, 2, .., n\}$ and relations

$$\begin{aligned} U_i U_i &= \sqrt{Q} U_i && (1) \\ U_i U_{i\pm1} U_i - U_i &= U_{i\pm1} U_i U_{i\pm1} - U_{i\pm1} && (2) \\ U_i U_{i+j} &= U_{i+j} U_i \qquad (j \neq 1) && (3) \end{aligned}$$

Note that there is a natural inclusion of $H_{n-1}(q)$ as a subalgebra in $H_n(q)$.

Proposition 1 *There is an isomorphism of left H_{n-1} modules*

$$H_n \cong H_{n-1} \oplus \bigoplus_{m=1}^{n} H_{n-1} \left(\prod_{i=1}^{m} U_{n+1-i} \right).$$

Proof: By induction: The proposition holds at level $n = 1$. Now suppose the proposition is true at level n, then trivially there is a natural mapping

$$H_n \to H_{n-1} \oplus H_{n-1} U_n H_{n-1}$$

(remark: this is an isomorphism of H_{n-1} bimodules) so, using this and the defining relations,

$$H_n U_{n+1} H_n U_{n+1} H_n \to H_n U_{n+1} (H_{n-1} + H_{n-1} U_n H_{n-1}) U_{n+1} H_n \cong H_n U_{n+1} H_n \oplus H_n.$$

Now obviously

$$H_{n+1} \cong H_n + H_n U_{n+1} H_n + H_n U_{n+1} H_n U_{n+1} H_n + \; ....$$

so in fact

$$H_{n+1} \cong H_n \oplus H_n U_{n+1} H_n .$$

This may now be combined with the proposition at level n to give the inductive step to level $n+1$.

Corollary 1.1 *The proposition also holds with U_i replaced by $U_i' = \alpha + \beta U_i$, for any complex number α and non-zero complex number β.*

Proof: as above.

In particular we define alternative generators

$$g_i = 1 - qU_i .$$

We also obtain a well known result:

Corollary 1.2 *The dimension of $H_n(q)$ is $(n+1)!$, and if $\{b_i \;:\; i = 1, 2, .., n!\}$ is a basis for H_{n-1} then*

$$\{b_i, b_i U_n', b_i U_n' U_{n-1}', .., b_i U_n' U_{n-1}' .. U_1' \;:\; i = 1, 2, .., n!\}$$

is a basis for H_n.

Definition 2 *Let $B_n(U')$ be the basis of words in $\{U_i'\}$ obtained by iterating this process from $B_0(U') = \{1\}$.*

Remark 1 *The words in $B_n(U')$ cannot be written as shorter words in $\{U_i'\}$ by applying the defining relations.*

Outline proof: Suppose true for $n-1$. The only relation which can shorten every word in the output is the first, but each new word has at most one factor of U_n, and at least one factor of U_i between each factor of U_{i-1} by construction.

Corollary 1.3 *Words in $B_n(U')$ of length l or less span all such words in $\{U_i'\}$.*

For many purposes we break the analysis of $H_n(q)$ down through a sequence of quotients $NH_n(q)$. To define these we need first to define some special elements of $H_n(q)$ (which will generate the kernels of the these quotients).

We can summarize our procedure (given in the next section) as the q-analogue of the following observations on the familiar $q = 1$, i.e. symmetric group, case. There we have two obvious central idempotents in the group algebra of S_N, the Young symmetriser and antisymmetriser:

$$E_{N+2} = \frac{1}{N!} \sum_{w \in S_N} w \ ; \qquad\qquad F_{N+2} = \frac{1}{N!} \sum_{w \in S_N} (-1)^{l(w)} \ w$$

corresponding to the trivial and sign (alternating) representations respectively. Quotienting a symmetric group algebra by F_{N+2} excludes irreducible representations corresponding to Young diagrams with more than N rows.

For q generic our quotient will have the equivalent effect. Up to normalisation (which we will discuss shortly) the q-analogues of E_{N+2}, F_{N+2} in our definition of the Hecke algebra become

$$E_{N+2} \propto \sum_{w \in B_N(g)} (q^2)^{-l(w)} \ w \ ; \qquad\qquad F_{N+2} \propto \sum_{w \in B_N(g)} (-1)^{l(w)} \ w.$$

Readers familiar with the formulation used, for example, by Dipper and James will note a difference, which comes from rescaling g_i, and a redefinition of q. We will now introduce and use two alternative constructions.

2.1 Special elements of $H_n(q)$

This subsection is taken from Martin and Westbury (1991) and references therein. We first reintroduce the central idempotents starting from scratch.

Definition 3 (Idempotents) *For each* $m = 1, 2, 3, .., n+2$ *define an idempotent* $E_m \in H_n(q)$ *by*

$$E_1 = E_2 = 1$$

and then

$$E_m \in H_{m-2}(q) \subset H_n(q)$$

and

$$E_m E_m = E_m$$

and for $i = 1, 2, .., m-2$

$$E_m U_i = U_i E_m = 0.$$

There can be at most one such element, since if $E_m, E'_m \in H_{m-2}(q)$ both have the above properties then $E_m E'_m = E_m = E'_m$.

Let us consider the existence of such an element. We need

Definition 4 *For each positive integer* n *define* k_n*, a function of* Q*, by* $k_1 = 0$ *and*

$$k_{n+1} = 1/(\sqrt{Q} - k_n).$$

Definition 5 *For s an integer and q given*

$$[s] = \frac{q^s - q^{-s}}{q - q^{-1}}$$

and for N a positive integer

$$[N]! = \prod_{s=1}^{N} [s].$$

For example, with $\sqrt{Q} = q + q^{-1}$ as before:

$$[0] = 0$$

$$[1] = 1$$

$$[2] = \sqrt{Q}$$

$$[3] = Q - 1$$

and

$$k_n = \frac{[n-1]}{[n]}$$

from definition 4.

Definition 6 *Define $I[m-2] \in H_{m-2}(q)$ by $I[0] = 1$ and*

$$I[m-2] = I[m-3](1 - k_{m-1}U_{m-2})I[m-3]$$

The existence of $I[m-2]$ for a given value of q is guaranteed unless some k_n required in its construction has a pole at that point.

Proposition 2 (see I) *If $I[m-2]$ exists then*

$$E_m = I[m-2]$$

Under the automorphism $D : H_n(q) \to H_n(q)$ defined by

$$U_i \mapsto \sqrt{Q} - U_i$$

we have another idempotent

$$D(E_m) = F_m.$$

For $X \in H_n(q)$ we define $X^{(t)} \in H_{n+t}(q)$ by the translation

$$U_i^{(t)} = U_{i+t}.$$

Definition 7 *For $n \geq b \geq a > 0$ and $c = 3 - a + b$ define $F_{ab} \in H_n(q)$ by*

$$F_{ab} = F_c^{(a-1)}.$$

Consequently, if $a \leq i \leq b$ then

$$U_i\, F_{ab} = F_{ab}\, U_i = \sqrt{Q}\, F_{ab}. \tag{4}$$

For example,

$$F_{11} = \frac{U_1}{\sqrt{Q}}$$

$$F_{12} = \frac{U_1 U_2 U_1 - U_1}{\sqrt{Q}(Q-1)}.$$

With $c = 3 - a + b$ again, we similarly define

$$E_{ab} = E_c^{(a-1)}$$

so that

$$U_i\, E_{ab} = E_{ab}\, U_i = 0$$

if $a \leq i \leq b$. For example $E_{1\ -1} = 1$.

Note that, as with E_{ij}, F_{ij} may not be well defined for all Q (consider our examples). However, note the following

Definition 8 *For n a positive integer and $Y_1 = 1$*

$$Y_{n+1} = -[n-1]\ Y_n + Y_n \left(\sum_{m=1}^{n} [m-2]\ (U_n U_{n-1} ... U_m) \right).$$

The element $Y_{n+1} \in H_n$ is clearly finite for all Q, with the coefficient of the longest word (c.f. proposition 1) equal to one. It is established in Martin and Westbury (1991) that

$$Y_n = [n]!\ F_{n+1}. \tag{5}$$

2.2 The quotient algebras $NH_n(q)$

We define a sequence of quotient algebras of $H_n(q)$ as follows. The quotient $NH_n(q)$, or simply NH_n, is obtained by imposing the quotient relations

$$Y_{N+1} = 0.$$

For example, with $Q \neq 0$ then $1H_n(q)$ can be obtained by putting $F_{11} = U_1/\sqrt{Q} = 0$, whilst with $Q = 0$ the quotient relation is $\sqrt{Q}\ F_{11} = U_1 = 0$ (the first expression is, of course, purely formal at $Q = 0$). Note that the case $N = 2$ corresponds to the Temperley-Lieb algebra. As we have already indicated, the case $NH_n(1)$ is the quotient of the group algebra of the symmetric group on $n+1$ objects to exclude irreducible representations with Young diagrams of more than N rows (c.f. Robinson 1962).

Proposition 3 *For q a nonzero complex number the dimension of $NH_n(q)$ equals the dimension of $NH_n(1)$.*

We prove this proposition in the course of proving our main theorem.

2.3 Representations

Let $V_N = \{1, 2, .., N\}$ be shorthand for the standard ordered basis for $\mathbb{C}^N$, and I_N be the $N \times N$ identity matrix, and M the $N^2 \times N^2$ matrix with action on V_N^2 given by

$$M\ (a,b) = 0 \qquad\qquad if\ \ a = b \tag{6}$$

and otherwise, with $p = sign(b-a)$,

$$M\ (a,b) = q^p\ (a,b)\ +\ (b,a). \tag{7}$$

Then for $N < n$ and V the space spanned by V_N^{n+1} we can check by direct computation that there is a representation $R_N : H_n(q) \mapsto End_{\mathbb{C}}(V)$ given by

$$R_N(U_i) = I_N \otimes I_N \otimes ... \otimes M \otimes ... \otimes I_N$$

where M appears in the i^{th} position in the product. It may be useful to note that this action of $H_n(q)$ on V is not the same as those used by Jimbo, Lusztig and others, being essentially the twist of Jimbo's action by the automorphism D.

Note that the objects $R_{P,M}(U_i)$, with $P + M = N$, defined by replacing equation 6 by

$$M\ (a,b) = q + q^{-1}(a,b) \qquad\qquad if\ \ a = b > P \tag{8}$$

if $a = b > P$ (so $R_{N,0} = R_N$), provide a host of useful representations, which are studied in Martin and Rittenberg 1991.

Theorem 1 (Main theorem) *$R_N H_n(q)$ is a faithful representation of $NH_n(q)$ for all q.*

3 Proof of the Theorem

3.1 Part 1: R_N a representation

We need to show that $R_N(F_{N+2}) = 0$ when q is an indeterminate (this is sufficient, since $R_N(kF_{N+2}) = kR_N(F_{N+2})$ and $k = [N+1]!$ is well defined in any specialisation). We proceed by induction.

The result is true for $N = 1$ by equation 6. If true for $N = m$ then for $a \in V_{m+1}^{n+1}$

$$R_{m+1}(F_{m+3})\ a = R_{m+1}(F_{m+3}F_{m+2})\ a = R_{m+1}(F_{m+3})R_{m+1}(F_{m+2})\ a \qquad (9)$$

can only possibly be non-vanishing if $a_1a_2...a_{m+1}$ are distinct (and $a_2a_3...a_{m+2}$ distinct, by symmetry), so $a_1 = a_{m+2}$. We then have

$$R_{m+1}(F_{m+3})\ a = R_{m+1}(F_{m+3}U_1/\sqrt{Q})\ a$$

which, using equation 7 becomes

$$R_{m+1}(F_{m+3})\left(\frac{q}{\sqrt{Q}}a \ + \ a_2a_1a_3a_4...a_{m+1}a_1...\right).$$

Since the second through $(m+2)^{th}$ components of the latter vector are no longer distinct it is killed by the F_{m+3} ($= F_{m+3}F_{m+2}^{(1)}$ as above) leaving

$$R_{m+1}(F_{m+3})\ a = R_{m+1}(F_{m+3})\frac{q}{\sqrt{Q}}a = 0.$$

Q.E.D.

3.2 Part 2: R_N faithful

We construct a basis for $NH_n(q)$ and prove explicitly that every element is linearly independent in R_N. The idea is to use the very limited off-diagonal action of the generators in R_N in constructing a partially ordered set of words which have non-zero matrix elements further and further (in a certain sense) from the diagonal.

We will need some properties of $R_N(F_{N+1})$. Note from equations 6 and 7 that each U_i mixes between basis vectors in V with a fixed number of 1's, 2's, ..., N's appearing as its components. Consequently the representation R_N is block diagonal up to permutations of the basis, and equivalence classes of the direct summand representations may be associated with a subset of the *partitions* (see e.g. Macdonald 1979, James and Kerber 1981) of $n + 1$.

Definition 9 *Define the set $\mathcal{D}_n^N$, of partitions of $n+1$ into at most N parts, as N-tuples of non-negative integers*

$$\alpha = (\alpha_1, \alpha_2, .., \alpha_N)$$

with the properties

$$\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_N$$

and

$$\sum_i \alpha_i = n + 1.$$

We will denote by α' the complement of a partition α.

Define a total order on $\mathcal{D}_n^N$ by $\alpha > \beta$ if there exists integer j such that

$$\alpha_i = \beta_i$$

for $N \geq i > j$, and

$$\alpha_j - \beta_j < 0.$$

This is the L'_n order defined in Macdonald 1979. For example, the N-tuple

$$\nu_N = (1, 1, 1, .., 1)$$

is the least element and $(N, 0, 0, .., 0)$ the greatest in this order in the set of all partitions of N, $\mathcal{D}_{N-1}^N$.

We will regard $\mathcal{D}_n^M \subset \mathcal{D}_n^N$ for $M < N$ by extending the M-tuples to N-tuples by adding zeros on the right.

For given n and N, and $\alpha \in \mathcal{D}_n^N$, we define R_N^α, or simply R^α, as the representation on subspace with basis vectors containing α_1 1's, α_2 2's, and so on. For example, $R_3^{(1,1,1)}$ has basis $\{123, 213, 132, 312, 231, 321\}$.

Proposition 4 *The matrix elements of $R_N^{\nu_N}(Y_N) = R_N^{\nu_N}([N]!F_{N+1})$ are all integer powers of q (and hence non-zero).*

Proof:

It follows from the defining relations 1- 3 and corollary 1.2 that $R_N^{\nu_N}$ is the regular representation of $H_{N-1}(q)$ (see I for details). The relations

$$U_i\ [N]!F_{N+1} = [N]!F_{N+1}\ U_i = \sqrt{Q}\ [N]!F_{N+1}$$

then imply

$$rank(R_N^{\nu_N}([N]!F_{N+1})) = 1$$

(consider the action of $[N]!F_{N+1}$ on each element of the algebra). The matrix $R_N^{\nu_N}([N]!F_{N+1})$ is also symmetric by construction, so there exists some row vector h such that

$$R_N^{\nu_N}(kF_{N+1}) = h^t h.$$

The relations above then further imply that in this representation

$$U_i\ h^t = \sqrt{Q}\ h^t.$$

Now U_i mixes basis vectors in pairs so this may be broken up into sub-equations of the form

$$\begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix} \begin{pmatrix} s_i \\ s_j \end{pmatrix} = \sqrt{Q} \begin{pmatrix} s_i \\ s_j \end{pmatrix}$$

which implies

$$s_i = qs_j.$$

Using all the U_i's all the components of the vector are connected in this way. Finally, from definition 8 note that $[N]!F_{N+1} \neq 0$. This completes the proof of proposition 4.

To review, we have seen that with $p_{ij} \subset V_N^{n+1}$ the subset of basis vectors $\{p \in V_N^{n+1} : p_i, p_{i+1}, .., p_j$ distinct elements of $V_N\}$, and introducing bra and ket notation to make row vectors easier to spot, then

$$R_N(kF_{ij})|a\rangle = 0 \qquad\qquad if \;\; a \in V_N^{n+1} - p_{ij} \tag{10}$$

so R_N *is* a representation of NH_n, and from proposition 4 that

$$R_N([j-i+2]!F_{ij})|p\rangle = \sum_{r \in p_{ij}} q^{f(p,r)} \, |r\rangle \qquad\qquad if \;\; p \in p_{ij} \tag{11}$$

where f is a finite integer. We do not need to know this integer for our purposes, it is given in appendix B.

In what follows we will use the notion of standard tableau (see e.g. James and Kerber 1981). These are the insertions of numbers $1, 2, .., n+1$ into the Young diagrams of partitions of $n+1$ such that each row and column has the natural order.

Definition 10 *For α a partition define $\mathcal{D}(\alpha)$ as the set of standard tableau of shape α.*

Definition 11 *Define a function*

$$F : \mathcal{D}(\alpha) \to V_N^{n+1}$$

by setting $(F\{a\})_i$ equal to the number of the row in which the number i appears in the standard tableau $\{a\}$.

Note that F is not surjective unless $n = 0$ and $N = 1$, but is injective. The range is the set of 'lattice permutations' of length $n+1$ in the alphabet $\{1, 2, .., N\}$. We will make frequent use of this (reverse) Yamanouchi notation for standard tableaux (c.f. Chen 1989, for example). We write $\{a\}$ for a standard tableau, and simply a for its Yamanouchi word (and $|a\rangle$ for the corresponding ket of V_N^{n+1}). For example, $\{a\} = \{e_{(1,1,1)}\}$, the unique (1^3) standard tableau, gives $F\{a\} = a = 123$.

If i is in a row above $i+1$ in $\{a\}$ then we say $F\{a\} = a$ has a maximum at i.

If $\{s\}$ is a standard tableau with i in a lower row than $i+1$, and the tableau obtained by interchanging $i, i+1$ in $\{s\}$ is standard, then this tableau may be called $\{s^i\}$.

Definition 12 *Define a partial order $\leq$ on standard tableau $\{s\}$ in $\mathcal{D}(\alpha)$ by*

$$\{s\} \leq \{t\}$$

if and only if $\{t\}$ can be obtained from $\{s\}$ by a sequence of moves of the form

$$\{s\} \to ... \to \{v\} \to \{v^i\} \to ... \to \{t\}.$$

The lowest sequence in the partial order in $\mathcal{D}(\alpha)$, call it $\{e_\alpha\}$, is the unique standard tableau with unit increases down the columns.

The poset is a lattice.

We also use another partial order. Let a sequence of moves from s to t in Definition 12 be recorded by $t = s^{i_1 i_2 .. i_m}$, i.e.

$$s \to s^{i_1} \to s^{i_1 i_2} \to ... \to t = s^{i_1 i_2 .. i_m},$$

then:

Definition 13 *Define a partial order $(\mathcal{D}(\alpha), \preceq)$ by $\{s\} \preceq \{t\}$ iff for some list $i_1 i_2 .. i_m$ such that $t = e_\alpha^{i_1 i_2 .. i_m}$ there exists a sublist of $i_1 i_2 .. i_m$, say $i_{j_1} i_{j_2} .. i_{j_k}$ $(k \leq m)$, with $j_1 < j_2 < .. < j_k$ and such that $s = e_\alpha^{i_{j_1} i_{j_2} .. i_{j_k}}$.*

This is similar to the Bruhat order on elements of the symmetric group.

Proposition 5 *If $\{s\} \preceq \{t\}$ and $\{s^i\}, \{t^i\}$ are well defined, then $\{s^i\} \preceq \{t^i\}$.*

Proof: With $\{t\}, \{s\}$ defined as above then $\{t^i\}, \{s^i\}$ well defined implies their lists may be written $i_1 i_2 .. i_m i$ and $i_{j_1} i_{j_2} .. i_{j_k} i$ respectively. QED.

Note that $\{s\} \leq \{t\}$ implies $\{s\} \preceq \{t\}$.

Definition 14 *Define extended partial orders $\leq$ and $\preceq$ on all standard tableau of $n+1$ boxes by the total order on partitions if the shapes are different, and by definitions 12 and 13 respectively otherwise.*

Definition 15 *Define $P^N(n)$ as the disjoint union of sets $\mathcal{D}(\alpha) \times \mathcal{D}(\alpha)$ over all values of $\alpha \in \mathcal{D}_n^N$.*

Definition 16 *Define partial orders $(P^N(n), \geq)$ and $(P^N(n), \succeq)$ by*

$$(a, b) \geq (c, d)$$

iff $a \geq c$ and $b \geq d$, and by a corresponding extension of $\succeq$, respectively.

Note that the former is a lattice.

3.2.1 A basis for $NH_n(q)$

Let us adopt the convention that F_{ij} may be regarded as a word in the generators of $H_n(q)$, and that for W a word in $H_n(q)$ then W^T is obtained by writing the generators in reverse order (note that $F_{ij} = F_{ij}^T$). We need the following definitions:

For each partition α of $k+1$ let us define elements $(e_\alpha, e_\alpha) \in H_k(q)$ iteratively as follows. Firstly take $(e_{(1)}, e_{(1)}) = 1$ and suppose $(e_{\alpha_-}, e_{\alpha_-})$ defined for all α_- partitions of k or less. For each α a partition of $k+1$ there is some α_- a partition of $l < k+1$ obtained from α by removing the first column *of partition shape α. This column has length $m = k+1-l$ (note that $m \geq (\alpha_-)'_1$, the length of the first column in α_-). Then*

$$(e_\alpha, e_\alpha) \;=\; Y_m\,\left((e_{\alpha_-}, e_{\alpha_-})^{(m)}\right)$$

(recall that $X^{(m)}$ denotes the translation of X by $U_i \mapsto U_{i+m}$).

Note that if α has only one part,

$$(e_\alpha \circ e_\alpha) = 1.$$

Note also that

$$U_i(e_\alpha \circ e_\alpha) = \sqrt{Q}(e_\alpha \circ e_\alpha)$$

if i is above $i+1$ in the standard tableau $\{e_\alpha\}$. For example,

$$\begin{aligned}(e_{(5,3,1)} \circ e_{(5,3,1)}) &= Q\sqrt{Q}(Q-1)\; F_{12}\; F_{44}\; F_{66} \\ &= (U_1U_2U_1 - U_1)\; U_4\; U_6\;.\end{aligned}$$

Note that $(e_\alpha \circ e_\alpha) = 0$ in $NH_n(q)$ if α has more than N rows.

Definition 17 *If $(\{s\},\{t\})$ is in $P^N(n)$, the set of pairs of standard tableau of the same shape and at most N rows, then $(s \circ t)$ is a word in the generators of $H_n(q)$ (counting F_{ij} as a word) obtained iteratively from $(e_\alpha \;\circ\; e_\alpha)$ as follows:*

$$(s^i \;\circ\; t) = (1 - qU_i)\;(s \;\circ\; t) \tag{12}$$

and

$$(t \circ s) = (s \circ t)^T. \tag{13}$$

If $(t \circ s)$ is obtained from $(e_\alpha \circ e_\alpha)$ in this way we call $(e_\alpha \circ e_\alpha)$ the *root* of $(t \circ s)$. For example, $(e_{(2,1)} \circ e_{(2,1)}) = U_1$ is the root of $(U_1)(1 - qU_2)$.

The path taken to construct a word here is not unique in general. To check that the construction is independent of the path taken let us assume it is so for all pairs below some pair P (it is trivially true for the lowest pair, since there is only one path), and suppose without loss of generality that we have two distinct paths to P from below, giving at the last step two nominally distinct words, $W_1 g_m$ and $W_2 g_l$.

Consider, case 1, that g_m and g_l commute (take $l > m+1$, say). We need only consider the right hand side of the pair P, and the changes in the $(m, m+1)$ and $(l, l+1)$ positions, so we might as well take them (12)(34) (in Yamanouchi notation). Then in W_1 they are (21)(34) and in W_2 they are (12)(43) with all other positions the same as in P. By our assumption there then exist W_3, W_4 such that $W_3 g_l = W_1$ and $W_4 g_m = W_2$, so W_3, W_4 both have (21)(43), and are equal. Altogether we have two constructions, $W_3 g_m g_l$ and $W_3 g_l g_m$ which are equal by commutation.

Now consider, case 2, that g_m and g_l do not commute (take $l = m+1$, say). Then the relevant part of P is in the $(m, m+1)$ and $(m+1, m+2)$ positions, say (123). In W_1 we have (213) and in W_2 we have (132), with other positions the same. By assumption there exist W_3, W_4 such that $W_3 g_m g_{m+1} = W_1$ and $W_4 g_{m+1} g_m = W_2$, and W_3, W_4 both have (321). Altogether we have $W_3 g_m g_{m+1} g_m$ and $W_3 g_{m+1} g_m g_{m+1}$, which are equal by the defining relations.

Proposition 6 (see I) *The set of elements in definition 17 spans $NH_n(q)$.*

The definition corresponding to definition 17 in I is not unique in general. It becomes unique on choosing a unique path of construction for each element. In fact it is proved in I that these elements form a basis, but we will of necessity prove linear independence in what follows, so it is sufficient to note that this set has order $dim(NH_n(1))$. Since we show linear independence for each N then in particular the large N limit (i.e. $N \geq n+1$) shows that *overall* we have $(n+1)!$ linearly independent elements and hence a basis for $H_n(q)$ (by corollary 1.2). But within this full basis $(n+1)! - dim(NH_n(1))$ linearly independent elements are *manifestly* taken to zero under the $Y_{N+1} = 0$ quotient (by definition 17, and specifically the definition of $(e_\alpha \circ e_\alpha)$). Overall then

$$dim(H_n Y_{N+1} H_n) \geq (n+1)! - dim(NH_n(1)).$$

Meanwhile $dim(NH_n(1)) = order(NP(n))$ elements will be shown linearly independent in $NH_n(q)$ so that $dim(NH_n(q)) \geq dim(NH_n(1))$. Clearly the bounds are saturated. This also proves proposition 3.

3.3 Proof of faithfulness (conclusion)

For given N and n, the set of elements in $NH_n(q)$ from definition 17 will be called S. With $w, v \in V_N^{n+1}$ and $X \in NH_n(q)$ we write $X_{wv} = \langle w|X|v\rangle$ for the w, v matrix element of $R_N(X)$. For example, if $w = v = 11111$ and $W \in S$ then $W_{wv} = 0$ unless $W = 1$.

Note that

$$\langle w|1|v\rangle = \delta_{wv}.$$

Then

$$U_i|v\rangle = 0 \qquad\qquad if \;\; v_i = v_{i+1} \tag{14}$$

and with $v^i = v$ except for

$$v^i_i = v_{i+1}$$

and

$$v^i_{i+1} = v_i$$

we have

$$U_i|v\rangle = q^{\pm 1}|v\rangle + |v^i\rangle \qquad\qquad if \;\; v_i \lessgtr v_{i+1}. \tag{15}$$

Definition 18 *Let $(w, v) \in V \times V$, then define a set*

$$J_{(w,v)} = \{W \in S \;:\; W_{wv} \neq 0\}.$$

Here are some illustrative examples of the above ideas:

Example 1. $n = 1$

Here $H_1(q)$ is spanned by $\{1, U_1\}$ and in Yamanouchi notation

$$(11 \circ 11) = 1$$

$$(12 \circ 12) = U_1$$

$$J_{(11,11)} = \{1\}$$

$$J_{(12,12)} = \{1, U_1\}$$

(note that $(11, 11) > (12, 12)$).

Example 2. $n = 2$

Here $H_2(q)$ is in fact spanned by $\{1, g_2U_1g_2, U_1g_2, g_2U_1, U_1, F_{12}\}$ and in decreasing order down the page we have pairs

$$(111, 111)$$

$$(112, 112)$$

$$(121, 112) \qquad\qquad (112, 121)$$

$$(121, 121)$$

$$(123, 123)$$

and, for example,

$$J_{(111,111)} = \{1\}$$

$$J_{(112,112)} = \{1, g_2 U_1 g_2\}$$

$$J_{(121,112)} = \{g_2 U_1 g_2, U_1 g_2\}$$

and so on.

Proposition 7 *For any partition $\alpha \in \mathcal{D}_n^N$ let $E_\alpha = (e_\alpha \circ e_\alpha)$. Then:*

$$E_\alpha \in J_{(e_\alpha, e_\alpha)}.$$

Proof:

Up to constants E_α is of the form

$$E_\alpha = \prod F_{ij}$$

and

$$e_\alpha = 123...\alpha'_1 123...\alpha'_2 123...\alpha'_3.....$$

Making repeated application of proposition 4 (as in equation 11) we see that $\langle e_\alpha | E_\alpha | e_\alpha \rangle$ is a finite power of q.

Proposition 8 *Let $u \in F(\mathcal{D}(\alpha) - \{\{e_\alpha\}\})$, then*

$$R_N(E_\alpha)|u\rangle = 0.$$

Proof:

Here $R_N(E_\alpha)|u\rangle \neq 0$ implies $u_1 u_2 ... u_{\alpha'_1}$ distinct, and so on, by equation 10. But $u \in F(\mathcal{D}(\alpha))$ implies $u_1 = 1$ and that no number can appear as a component of u until every lower positive integer has appeared at least once more often. The *only* possibility for $u \in F(\mathcal{D}(\alpha))$ is $u = e_\alpha$.

More generally we have:

Proposition 9 *For pairs $(a,b), (w,v) \in P^N(n)$ if*

$$\langle w | (a \circ b) | v \rangle \neq 0$$

then $(a,b) \succeq (w,v)$.

Proposition 10 *For $(a,b) \in P^N(n)$*

$$\langle a | (a \circ b) | b \rangle \neq 0.$$

These last two propositions imply the theorem. To see this, suppose $R_N(S)$ is not linearly independent. Then there exists

$$R_N(X) = \sum_{W_i \in S} \alpha_i R_N(W_i) = 0$$

with some α_i non-vanishing. Consider in particular a maximal pair (w, v) in the partial order $(P^N(n), \succeq)$ such that $\alpha_j \neq 0$ for $W_j = (w \circ v)$. Then

$$\langle w|X|v\rangle \overset{prop.9}{=} \sum_{(a,b)\succeq(w,v)} \alpha_{(a,b)} \langle w|(a \circ b)|v\rangle \overset{maximality}{=} \alpha_j \langle w|W_j|v\rangle = 0,$$

so

$$\alpha_j = 0$$

by proposition 10, giving a contradiction. Therefore $R_N(S)$ is linearly independent.

Proof of proposition 9:

There are two cases to consider. Either (w, v) and (a, b) have the same root (i.e. tableau shape), or they do not.

In the latter case we need only consider $(a, b) \prec (w, v)$. The proposition follows from the observation that kF_{ij} is zero on any basis vector in which the i^{th} to j^{th} entries are not all distinct. This means that from their definitions the *roots* of all elements $(a \circ b)$ are already zero on the whole $R^{shape(v)}$ subspace of V_N^{n+1} (i.e. that containing the basis state given by the Yamanouchi word v). To see this explicitly first note that each root E_α takes the form

$$E_\alpha = \prod_{i=1}^{m} F_{a_i\ a_{i+1}-2}$$

where $a_1 = 1$ and

$$a_{i+1} - a_i > a_{j+1} - a_j \Rightarrow i < j.$$

The first F_{ij} in the product tells us that for $E_\alpha|v\rangle \neq 0$ the components of v

$$v_{a_1} v_{a_1+1} ... v_{a_2-1}$$

must be distinct (i.e. v is associated to the partition of $n+1$ whose Young diagram has as its left hand edge a column of boxes of length at least $a_2 - 1$). Without loss of generality we can make these components of v

$$1\ 2\ ...\ a_2 - 1.$$

The next F_{ij} in E_α tells us that

$$v_{a_2} v_{a_2+1} ... v_{a_3-1}$$

are distinct. It follows that the partition associated to a v satisfying $E_\alpha|v\rangle \neq 0$ will be the greatest possible one in the total order if in fact

$$\{v_{a_2}, v_{a_2+1}, ..., v_{a_3-2}\} \subset \{1\ 2\ ...\ a_2 - 1\}$$

(i.e. v has partition with second column at least this length). Iterating we obtain the desired result. In fact we obtain the stronger result that $R^\lambda(E_\alpha) \neq 0$ only if (λ, α) appears in the *natural* (partial) order of partitions (c.f. Dipper and James 1989).

In the former case we want to prove that $(a \circ b)_{wv} = 0$ for all $(a, b) \not\succeq (w, v)$ (but with the same root, i.e. the same tableau shape, α, say). We can work by an induction on the poset $(\mathcal{D}(\alpha) \times \mathcal{D}(\alpha), \succeq)$. Consider a pair of standard tableaux (c, d), and assume that the proposition is true for all pairs $(a, b) \prec (c, d)$ with *all* (w, v) (it is true for $(a, b) = (e_\alpha, e_\alpha)$, i.e. $(a \circ b) = E_\alpha$, the root of any $(w \circ v)$, by proposition 8). Without loss of generality we may assume that this includes a case $(a, b) = (f, h)$ such that $(f, h^i) = (c, d)$ for some i. We want to establish now that $(c \circ d) \in J_{(w,v)}$ implies $(c, d) \succeq (w, v)$ for all (w, v), i.e. $(c \circ d)_{wv} = 0$ if $(c, d) \not\succeq (w, v)$, so it is sufficient to consider those pairs $(w, v) \not\preceq (c, d)$. Now $(w, v) \preceq (f, h)$ and $(f, h) < (c, d)$ implies $(w, v) \preceq (c, d)$, so by counterpositivity we have $(w, v) \not\preceq (f, h)$ and hence

$$\langle w|(f \circ h)|v\rangle = 0$$

by assumption. We want to compute

$$\langle w|(c \circ d)|v\rangle = \langle w|(f \circ h^i)|v\rangle = \langle w|(f \circ h)g_i|v\rangle.$$

This vanishes by equation 10 and the assumption unless v has a maximum or minimum at i. In the latter cases it takes the form

$$\langle w|(f \circ h)g_i|v\rangle = (1 - q.q^{\pm 1})\langle w|(f \circ h)|v\rangle - q\langle w|(f \circ h)|v'\rangle = -q\langle w|(f \circ h)|v'\rangle$$

where $|v'\rangle$ is $|v\rangle$ with 2 components interchanged. If it was a minimum then $(w, v') = (w, v^i)$ and since $(w, v) \not\preceq (f, h)$ then $(w, v^i) \not\preceq (f, h)$, and the proposition is true by assumption. If it was a maximum then $(w, (v')^i) = (w, v)$ so $(f, h^i) \not\succeq (w, (v')^i)$ which implies $(f, h) \not\succeq (w, v')$ (by the counterpositive argument to proposition 5) and again the proposition is true by assumption.

This completes the proof of proposition 9.

Proof of proposition 10:

We proceed by induction using the $(\mathcal{D}(\alpha) \times \mathcal{D}(\alpha), \geq)$ sublattice. Let $E_\alpha = (e_\alpha \circ e_\alpha)$ be the root of some $(a \circ b)$ and assume the proposition is true for all $(a', b') < (a, b)$ (it is true for the universal lower bound $(a' \circ b') = E_\alpha$ by proposition 7). By symmetry, there is no loss of generality in assuming the proposition true for (c, d) such that $(c, d) < (a, b)$ and $(c, d^i) = (a, b)$ for some i. Then noting that $c = a$ here we have the inductive assumption

$$\langle a|(a \circ d)|d\rangle \neq 0.$$

Noting that $(a, b) = (a, d^i)$ (so $(a \circ b) = (a \circ d)g_i$) implies $|b\rangle$ has a maximum at i we have, from equations 14 and 15,

$$(a \circ b)|b\rangle = (a \circ d)\ ((1 - q^2)\ |b\rangle - q|d\rangle)$$

so that

$$(a \circ b)_{ab} = \langle a|(a \circ b)|b\rangle = (1 - q^2)\langle a|(a \circ d)|b\rangle - q\langle a|(a \circ d)|d\rangle.$$

The first term on the right vanishes by proposition 9, so we have

$$\langle a|(a \circ b)|b\rangle = -q\langle a|(a \circ d)|d\rangle \neq 0$$

by assumption.

This completes the proof of proposition 10 and the main theorem.

4 Applications

To review: with $A^o = H_n(q)$, $M = V$ the A^o-module defined above, $B = End_{A^o}(M)$, and $A = End_B(M)$, we have established that $A^o/K = NH_n(q)$ so $NH_n(q) \subset A$ for all q. The next question is ... Are there any other matrices which commute with B in $End(M)$? Clearly not for q indeterminate. We will see shortly that, at least for $N = 2$ there are not for any q. We do this by computing B from ${}_{NH_n}M$, and thence A. The final question is ... Is B given by $U_q^n sl(N)$, i.e. by the quotient of $U_q sl(N)$ faithfully represented on M? The structure of $U_q^n sl(N)$ is known (see Lusztig 1989 and appendix A), so this will be answered in the process of answering the previous question.

Some Algebra

The abstract algebraic problems associated with centralisers of non-semisimple algebras are interesting in their own right. We do not wish to get bogged down with what are, in the present context, technical details, so we will merely quote the results applicable here and refer to a companion paper - Martin and McAnally 1991.

Note that if P_i (resp. Q_i) are indecomposable A^o (resp. B) modules and R_i (resp. S_i) simple A^o (resp. B) modules then

$$_{A^o}M = \bigoplus_i m_i P_i$$

(see e.g. Curtis and Reiner 1962) implies $dim(S_i) = m_i$.

Let $J = rad(B)$ so B/J is the maximal semi-simple quotient (as an B module) of B, and $B/J \subset B$ as an algebra. In the case of B semi-simple ($J = 0$) then so is A^o/K and M may be regarded as a $B \otimes A^o/K$ module

$$M = \bigoplus_i S_i \otimes P_i.$$

In our case each simple and projective module of the algebra B/J may be associated to some simple B module S_i, and

$$_{B/J\otimes A^o}M = \bigoplus_i S_i \otimes P_i,$$

where $dim(P_i)$ is then the multiplicity of each simple module in $_{B/J}M$ (or $_BM$, although not in this case as a direct summand). Obviously if $A^o/K = A = End_B(M)$ there is a similar result with A and B interchanged.

These essentially combinatorial results provide the first stage in establishing the structure of the centraliser from that of $_{NH_n}M$. What remains is the effect of the internal structure of the indecomposables on the quiver diagram - which can then be computed up to Morita equivalence. We will give explicit examples shortly.

4.1 The case $N = 2$

Our main result is that $_{NH_n}M$ is faithful. Since we know the structure of $2H_n$ (we will review it now), the content of M in the case $N = 2$ can be deduced as follows:

Recall from I that each block (i.e. each connected piece of the quiver diagram) in $2H_n$ takes the form either of a single simple module or, for some m, has Loewy structure

$$\begin{matrix} s_1 \\ s_2 \end{matrix} \;\oplus\; \left(\bigoplus_{i=2}^{m} \begin{matrix} & s_i & \\ s_{i-1} & & s_{i+1} \\ & s_i & \end{matrix} \right) \oplus \begin{matrix} s_{m+1} \\ s_m \\ s_{m+1} \end{matrix} \qquad (16)$$

From the definition M is a direct sum of permutation modules (see I - taking the large imaginary limit of x in the definition there, or Dipper and James 1986,1989), and these may be written as a nested sequence of invariant subspaces $M \supset M_1 \supset M_2 ...$ such that M_i/M_{i+1} is a given Specht module (a module with structure $\begin{matrix} s_i \\ s_{i+1} \end{matrix}$ in the labelling convention above). A faithful representation which has this property must contain at least one copy of each indecomposable projective except possibly the leftmost one above (in order that the glue between copies of s_i be represented). The faithfulness, the symmetric property of the generators in M, and the defining relations, ensure that indecomposables must look the same (in Loewy decomposition) upside down, so no other glue can be omitted.

On the other hand a simple counting argument shows that s_{m+1} is also a direct summand of M. This means that there are at least as many inequivalent indecomposables as simples in ${}_{NH_n}M$.

The symmetry and Specht properties allow no other indecomposable configurations. Let us be more explicit (thanks are due to the referee for suggesting the following phrasing): From equation 16 indecomposable projectives except the leftmost one in a block are also injective modules. Thus we can extend morphisms from Specht modules to projectives to morphisms from M to those projectives (see e.g. Adamson p.83). If such a morphism is surjective it splits. If not then it leads to the possibility that the Specht module is a direct summand, which contradicts the symmetry property of M, except for the cases s_{m+1}.

We see that M is 'almost projective', consisting of a direct sum of almost all indecomposable projectives (only $\begin{matrix} s_1 \\ s_2 \end{matrix}$ type of multiplicity zero) plus copies of s_{m+1} type modules, which alone are not projective. Note that these could be quotiented out without destroying the faithfulness property, so that M/L for some (known) invariant subspace L is a faithful projective module.

These observations determine the NH_n module content of M completely. We are now in a position to read off the structure of the centraliser algebra.

4.2 Examples

The situation is best illustrated by some examples. We write $q = e^{i\pi/r}$. In fact the situation differs in no qualitative way for different rational r values within $N = 2$, so any one well illustrates the procedure. Here are the first few cases for $N = 2$, $r = 4$:

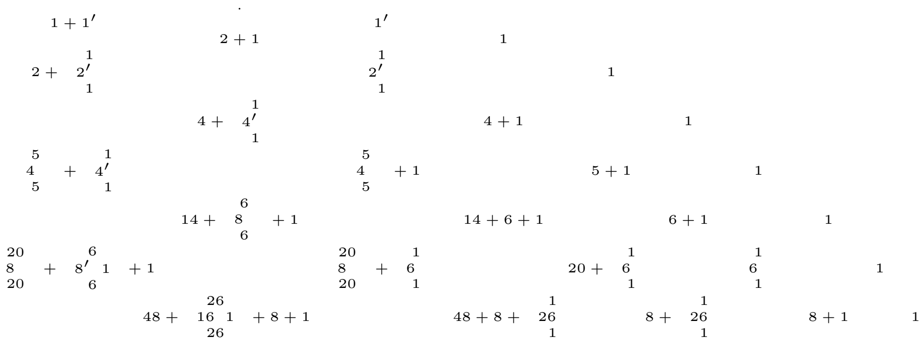
The successive rows of the table below give the generic irreducible dimensions of $2H_n$ for $n = 0, 1, 2, .., 8$ respectively.

	1								
1		1							
	2		1						
2		3		1					
	5		4		1				
5		9		5		1			
	14		14		6		1		
14		28		20		7		1	
	42		48		27		8		1

Next we give the (corresponding) dimensions for the various 'permutation' modules in the representation R_2. The representations down the left hand spine of the diagram occur only once (equal numbers of 1's and 2's in the basis vectors), all the others twice in each R_2 (so $dim(R_2) = 2^{n+1}$). Note that each representation generically contains the corresponding irreducible in the table above plus a copy of each irreducible to the right in that row (see e.g. Robinson 1962).

	.								
2		1							
	3		1						
6		4		1					
	10		5		1				
20		15		6		1			
	35		21		7		1		
70		56		28		8		1	
	126		84		36		9		1

The following table gives the indecomposable content of each of the permutation modules above for $r = 4$, as forced by the symmetry, faithfulness and Specht conditions. The organisational key to this diagram lies in associating the row of entries at level n (from the top) with the various 2 row partitions λ of $n+1$ ($\lambda_1 - \lambda_2$ increasing from left to right). Baring in mind the Specht module content of each permutation module (one copy of the module with the same diagram, plus one copy of each one to the right), the block structures of equation 16 may then be meshed together on the table by noting (c.f. Dipper and James 1986 and references therein) that blocks are characterised by the common r-cores (4-cores in this case) of their diagrams.



Note that the $n = 7$ spine block is not faithful, and that quotienting by the trivial representation in the larger algebra followed by a vertical move is identical to the action of the standard Morita equivalence functor from $2H_{n+2} \rightarrow 2H_n$ (from Martin and Westbury 1991).

Finally, in the following table we form the centraliser structure corresponding to that in the diagram above (in the form (multiplicity).(dim. of indecomposable)). The dimensions of simples come from the multiplicities of indecomposables in the diagram above using the results quoted in the previous section (not forgetting that each block has multiplicity 2 unless on the left hand spine), and so on. The structures of the indecomposables come from the observation that the quiver blocks of ${}_{NH_n}M$ from above contain indecomposables with Loewy structure

$$\left(\bigoplus_{i=2}^{m} \begin{matrix} & s_i & \\ s_{i-1} & & s_{i+1} \\ & s_i & \end{matrix} \right) \oplus \begin{matrix} s_{m+1} \\ s_m \\ s_{m+1} \end{matrix} \oplus s_{m+1}.$$

We thus read off the non-trivial morphisms between the indecomposables $P_i \subset {}_{2H_n}M$ (which commute with the action of $2H_n$ and are in the radical of B) as (ignoring multiplicities - see Martin and McAnally 1991)

$$P_2 \underset{\overrightarrow{\beta_2}}{\overset{\alpha_2}{\leftarrow}} P_3 \underset{\overrightarrow{\beta_3}}{\overset{\alpha_3}{\leftarrow}} \cdots \underset{\overrightarrow{\beta_m}}{\overset{\alpha_m}{\leftarrow}} P_{m+1} \underset{\overrightarrow{\beta_{m+1}}}{\overset{\alpha_{m+1}}{\leftarrow}} s_{m+1}$$

where

$$\alpha_i \beta_i = \gamma_{i+1} \neq 0 \qquad\qquad i = 2, 3, .., m$$

$$\beta_i \alpha_i = \gamma_i \neq 0 \qquad\qquad i = 2, 3, .., m+1$$

and all other composite morphisms are zero. Altogether this glues the simples in B

together as follows (with the multiplicities now forced)

	.								
1		3							
	2.2		1.4						
				3					
2.1		2.3		1. 2					
				3					
					2				
	4.2		4.4		1. 4′				
					2				
				3		1			
4.1		4.3		5. 2		1. 6			
				3		1			
					2				
	8.2		14.4		6. 4′		1.8		
					2				
				3		1		6	
8.1		8.3		20. 2		6. 6		1. 1 3′	
				3		1		6	
					2				4′
	16.2		48.4		26. 4′		8.8		1. 2 6
					2				4′

Here again the block structures can be picked out conveniently by mapping diagrams to their r-cores.

This is the structure of $U_q sl(2)$ on M (c.f. Lusztig 1989 and appendix A). It is a straightforward combinatorial exercise to check that the correspondence continues for higher n. Note that the centraliser algebra $U_q^n sl(2)$ is Morita equivalent to the original algebra $2H_n(q)$, so the centraliser of the centraliser is the original algebra and the pair are in Schur-Weyl duality.

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Appendix A: Quantum $sl(N)$

Here we briefly review the properties of $U_q sl(N)$ for comparison with the results of section 4.

Definition 19 *For N a positive integer and q an indeterminate define $U_q sl(N)$ as a unital associative bialgebra over $\mathbb{C}[q^{\pm 1}]$ with generators $1, e_i, f_i, k_i^{\pm 1}$ $(i = 1, 2, .., N -$

1) *in the following way: Firstly, there exists a left $U_q sl(N)$ module with basis $\{v_1, v_2, .., v_N\}$ and action of $U_q sl(N)$ given by:*

$$e_i\ v_j = \delta_{ij-1} v_{j-1}$$
$$f_i\ v_{j-1} = \delta_{ij-1} v_j$$
$$k_i v_j = (\delta_{ij} q + \delta_{ij-1} q^{-1})\ v_j.$$

All finite dimensional indecomposable representations appear as constituents of those generated from this one by use of the coassociative comultiplication, which is given by

$$m(e_i) = e_i \otimes 1 + k_i \otimes e_i$$
$$m(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$
$$m(k_i) = k_i \otimes k_i.$$

Truncating this procedure at the n^{th} comultiplication gives the quotient algebra $U_q^n sl(N)$.

For example, for $N = 2$, the complete list of finite irreducible representations is as follows. There is a one dimensional represenation $e = f = 0, k = 1$, and then one of each dimension, p, called r_{p-1}, with the action on a basis $\{v_1, v_2, .., v_p\}$ given by

$$(e)\ v_i = [i-1]\ v_{i-1}$$
$$(f)\ v_i = [p-i]\ v_{i+1}$$
$$(k)\ v_i = q^{p+1-2i}\ v_i.$$

Note that r_1 is our defining representation. It follows that

$$r_1 \otimes r_m = r_{m-1} \oplus r_{m+1} \qquad (17)$$

and the content of the various comultiplications of r_1 can be deduced from this.

The above definitions hold for q specialised to any non-zero complex number other than a root of unity. There are some inequivalent choices available for the definition of $U_q sl(N)$ in the specialisation to q a root of unity. The one appropriate for its roll as a centraliser algebra (but which excludes the so called cyclic representations, c.f. Date et al 1990) is the following.

Definition 20 *For $q = e^{i\pi/r}$ and r integer we define $U_q sl(N)$ as before, except to include additional generators*

$$e_i^{(r)} = e_i^r/[r]!$$

and

$$f_i^{(r)} = f_i^r/[r]!$$

where it is to be understood that r is taken to its specialisation after *reducing the ratio to its lowest form.*

Note that in the specialisation to r integer equation 17 does not generally hold. However $U_q sl(N)$ is in fact a Hopf algebra, which ensures that the product of any module with a projective module is projective.

Proposition 11 *In the specialisation to r integer the representations r_p remain well defined and indecomposable, but develop an irreducible invariant subspace with basis*

$$\{v_{i+1} \;:\; i = 0, 1, 2, .., (p-1)_{mod.r} \;\; mod.r\}$$

and irreducible quotient. For $p+1 = mr+k$ the quotient is isomorphic to the invariant subspace for $p+1 = mr-k$.

Proof: In this case it follows from the definition of r_p and $[s]$ that all actions are zero except

$$\begin{array}{ccccc} e\, v_i & \propto & v_{i-1} & (i \neq 1 \;\; mod.r) \\ f\, v_i & \propto & v_{i+1} & (p-i \neq 0 \;\; mod.r) \\ e^{(r)}\, v_{i+4} & \propto & v_{i-1} & (i = 1 \;\; mod.r) \\ f^{(r)}\, v_i & \propto & v_{i+4} & (p-i = 0 \;\; mod.r). \end{array}$$

The representations are thus block upper triangular in two blocks when written with the specified basis elements first. The irreducibility and indecomposability properties follow from continuity with the generic case.

For example, with $r=4$ we have the following table: The top line gives the generic irreducible dimensions and below are the non-generic Loewy decompositions into simple modules:

$p+1:$	1	2	3	4	5	6	7	8	9	... mr	$mr+1$	$mr+2$	...
$r_p:$	1	2	3	4	3	2	1	8	6	... mr	$m(r-1)$	$m(r-2)$	...
					2	4	6		3		$m+1$	$2(m+1)$	

where the top representation is isomorphic to the first one of that dimension found by moving to the left, and the invariant subspace is not isomorphic to any representation to the left.

Note that r_{mr} (m a natural number) remains irreducible in proposition 11, so it will also be projective.

It is the representation on $M = \otimes^{n+1}\mathbb{C}^N$ (obtained, up to isomorphism, by repeating the comultiplication in definition 19 n times) which commutes with the action of $H_n(q)$. Strictly speaking the action of $H_n(q)$ required is the isomorphic action obtained from equation 7 by replacing $1 \rightarrow -1$ in the off-diagonal elements.

This may be readily verified by direct computation. In particular note that the additional generators in definition 20 must be included in the centraliser algebra B.

Appendix B: The vector h

Definition 21 *For each non-negative integer m define an $m+1$ component row vector*

$$h_m = (q^m, q^{m-1}, ..., q^0).$$

and then $s \in \mathbb{C}^{N!}$ by

$$s = \otimes_{m=1}^{N-1} h_m.$$

Remark 2 *There is an ordering of the ν_N basis such that $R_N^{\nu_N}(Y_N) = s^t s$.*

To see this introduce a basis B for the $N!$ dimensional space $L = \otimes_{m=1}^{N} \mathbb{C}_m$ (isomorphic to the ν_N subspace of V, which we will call V^{ν_N}) as follows:

$$B = \{(a_1 a_2 ... a_N) \ : \ a_j \in \{0, 1, 2, .., j-1\}\}.$$

The isomorphism is given by

$$J : V^{\nu_N} \to L$$

defined on the given bases by

$$J : \alpha \to a$$

where

$$a_i = No.(\alpha_{j<i} \ \ s.t. \ \ \alpha_j > \alpha_i).$$

Definition 22 *Define a length function*

$$l : B \to \mathbf{Z}$$

by

$$l(a) = N - \sum_i a_i.$$

We will also write $l(\alpha)$ for $\alpha \in V^{\nu_N}$ to mean $l(J(\alpha))$.

Then for $a \in B$ and $i = 1, 2, .., N-1$ define matrices $M_i \in End_{\mathbb{C}}(L)$ by

$$M_i a = qa + \overbrace{(a_1 a_2 ... a_{i-1} \ \ a_{i+1} \ \ a_i + 1 ... a_N)}^{b} \qquad ((J^{-1}a)_{i+1} > (J^{-1}a)_i).$$

$$M_i b = q^{-1} b + \overbrace{(b_1 b_2 ... b_{i-1} \ \ b_{i+1} - 1 \ \ b_i ... b_N)}^{a}.$$

It follows by direct computation that

$$R_N^{\nu_N} : H_{N-1}(q) \to End_{\mathbb{C}}(L)$$

is given by

$$R_N^{\nu_N}(U_i) = M_i$$

and the remark follows from this.