TEMPERLEY-LIEB ALGEBRAS FOR NON-PLANAR STATISTICAL MECHANICS - THE PARTITION ALGEBRA CONSTRUCTION

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Abstract We give the definition of the Partition Algebra $P_n(Q)$. This is a new generalisation of the Temperley-Lieb algebra for Q-state n-site Potts models, underpinning their transfer matrix formulation on arbitrary transverse lattices. In $P_n(Q)$ subalgebras appropriate for building the transfer matrices for all transverse lattice shapes (e.g. cubic) occur. For $Q \in \mathbb{C}$ the Partition algebra manifests either a semi-simple generic structure or is one of a discrete set of exceptional cases. We determine the Q-generic and Q-independent structure and representation theory. In all cases (except $Q = 0$) simple modules are indexed by the integers $j \leq n$ and by the partitions $\lambda \vdash j$. Physically they may be associated, at least for sufficiently small j, to $2j$ 'spin' correlation functions.

We exhibit a subalgebra isomorphic to the Brauer algebra.

1 Introduction

In the ordinary transfer matrix approach to computation in classical statistical mechanics an Euclidean space is resolved into one 'time' and $(d-1)$ 'space' directions (the Minkowskian labels are purely a notational convenience). The transfer matrix (TM) then describes the states of a complete space-like layer evolving through a single, or at least minimal, time step [1]. For example, in one common formulation the transfer matrix is a product of two types of single interaction matrices - those incorporating interactions which occur within a single time layer, and those which connect two adjacent time layers [2]. † It is often possible to resolve the single interaction transfer matrix into a scalar function of the temperature parameter plus a constant matrix [3]. The algebra of these matrices is the TM algebra.

In two dimensions the TM layer for the planar square lattice of width L sites may be thought of as a chain of L sites (each supporting the projection of a time-like bond) and $L-1$ space-like bonds. The transfer matrix algebra is generated by a corresponding chain of $2L-1$ matrices. A large class of statistical mechanical models inculding Q-state Potts, 6-vertex, IRF models and dichromatic polynomials is characterized by the fact that the interaction matrices for these models provide a representation of the Temperley-Lieb algebra $T_{2L}(Q)$ [4]. In general, for n a natural number, and Q a complex number or indeterminate, $T_n(Q)$ is a unital (associative) algebra over $\mathbb{C}$ with generators $< 1,\ U_i \quad (i = 1, 2, .., n-1) >$ and relations:

$$U_i^2 = \sqrt{Q}\, U_i \tag{1}$$

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†Then again in two dimensions the symmetry between the space and time projections facilitates a more symmetrical choice [4] - sometimes called left and right light cone interactions.

$$U_i U_{i\pm 1} U_i = U_i \tag{2}$$

$$[U_i, U_j] = 0 \qquad |i-j| \neq 1. \tag{3}$$

In this notation the odd numbered indices correspond to matrices building timelike (also called longitudinal) bond Potts interactions, those even numbered give spacelike (transverse) bonds. The rough parity between the two types is a special feature of two dimensions, so it will be convenient to replace it in what follows with the equivalent but more versatile notation:

$$U_{2i-1} = U_{i.} \qquad i = 1, 2, .., L$$

$$U_{2i} = U_{ii+1} \qquad i = 1, 2, .., L-1.$$

If we number the nodes of our transfer matrix chain from $1, 2, ..., L$ then we see that $U_{i.}$ is associated to the i^{th} node and U_{ii+1} to the bond between nodes i and $i+1$.

Consider now the complete unoriented graph of n nodes, here called $\underline{n}$, and all those subgraphs $G \subset \underline{n}$ obtained by removing bonds (edges) from the complete graph.

Definition 1 *We define $T_G(Q)$, the Full Temperley-Lieb algebra of the graph G [3], to be the unital algebra over $\mathbb{C}$ with generators*

$$< 1,\ U_{i.} \quad (i = 1, 2, .., n),\ U_{ij} = U_{ji} \quad (edge\ (i,j) \in G) >$$

and relations:

$$U^2 = \sqrt{Q} U \tag{4}$$

(any indices)

$$U_{i.} U_{ij} U_{i.} = U_{i.} \tag{5}$$

$$U_{ij} U_{i.} U_{ij} = U_{ij} \tag{6}$$

$$[U_{i.}, U_{j.}] = [U_{ij}, U_{kl}] = [U_{i.}, U_{kj}] = 0 \qquad i \neq k, j. \tag{7}$$

For example, with $G = A_n$, the n node chain graph, we recover the original Temperley-Lieb algebra $T_{2n}(Q)$.

It follows from the definition of the Potts model [2] and dichromatic polynomial [5] that the relations of $T_G(Q)$ are an appropriate generalization (of the transfer matrix algebra relations for the chain, $T_{A_n}(Q)$) for building a transfer matrix layer of shape G [3] - that is, for overall lattice shape $G \times \mathbb{Z}$. †

In other words, for those statistical mechanical models which *have* a suitable generalization onto a lattice with spacelike layer G, such as the Potts model (defined by Hamiltonian

$$\mathcal{H} = \beta \sum_{(ij) \in G \times \mathbb{Z}} \delta_{\sigma_i \sigma_j} \tag{8}$$

†This graph G corresponding to the shape of physical space is not to be confused with the *configuration space* graphs of Pasquier and Saleur [6], which work only for the two dimensional case. For example, G a square lattice here produces a cubic lattice statistical mechanical model.

where β is essentially an inverse temperature variable) the TM algebra provides a representation (abstractly, a quotient) of this algebra. The transfer matrix itself is a representation of the element

$$\mathcal{T}(v) = \prod_{i=1}^{n} \left(v1 + \sqrt{Q} U_{i.} \right) \prod_{(ij) \in G} \left(1 + \frac{v}{\sqrt{Q}} U_{ij} \right) \tag{9}$$

where $v = exp(\beta) - 1$. The Potts representation is given explicitly in [3] By well known arguments [1, 3, 7] the irreducible representations of $T_G(Q)$ which compose this representation are efficient blocks to use in computing the TM spectrum. The irreducible structure of $T_G(Q)$ is thus important for extending computation in statistical mechanics to three and higher dimensions.

On the other hand, whilst the $G = A_n$ algebra is finite dimensional for finite n, and typically faithfully represented by the finite dimensional physical transfer matrices, we will show that for general G the Full algebra is *usually* infinite dimensional (we will also examine the special conditions under which $dim(T_G)$ is finite). Since the physical transfer matrices usually remain finite dimensional in higher dimensions (for finite systems) one problem is to find explicitly the finite dimensional quotients of the Full algebra appropriate for these physical systems. In two dimensions the exceptional cases of Q, where the $G = A_n$ algebra is *not* faithfully represented in physical transfer matrices, constitute perhaps the most interesting sector of all, corresponding to models with unitary conformal field theory limits [8] (or, more simply, lots of extra symmetry in the long distance properties). By establishing the physically appropriate generic algebra in other dimensions we develop a procedure for investigating the analogous situation there.

In this paper we find the quotient algebra for several models. We introduce the Diagram algebra of G, $D_G(Q)$, which *is* finite dimensional for any finite G. Each TM algebra is either a quotient or the whole of $D_G(Q)$. We examine the structure and representation theory of this algebra from the point of view of someone wanting to optimise computation in statistical mechanics. This is without regard to the possibility of a star-triangle like diagonalization manoeuvre (which is in any case widely studied elsewhere [9], with great skill but somewhat limited success).

We begin (in section 2) by introducing a closely related algebra, the Partition algebra $P_n(Q)$, which is *not* a quotient of $T_G(Q)$, but which also has subalgebras indexed by a graph. This provides a key organisational link between the physical and abstract algebras we have described. We will indulge in a very careful abstract formulation, anticipating the need for a possible generalisation to encompass 'Full Hecke algebras'. In section 3 we prove some technical results which are central to the structure analysis of $P_n(Q)$, and hence $T_G(Q)$. It will come as no surprise to physicists to learn that the $G = \underline{n}$ or 'mean field limit' case is one of the easiest to analyse for any of these algebras. We deal with this in full detail (in section 4), as it is a useful envelope guide for the more complex subalgebra structures. These are addressed in [10]. In section 5 we generalise the construction from the TM to the partition vector formalism [3], in the process providing alternative (and hopefully illuminating) versions of some earlier definitions. We conclude with a discussion, pointing out an inclusion of the Brauer algebra [11] and mentioning some outstanding problems.

2 Set theory preamble

The Partition algebra $P_n(Q)$ is a finite dimensional algebra which includes a quotient of the Full algebra, and which will play the crucial role in our analysis. This algebra can be introduced in

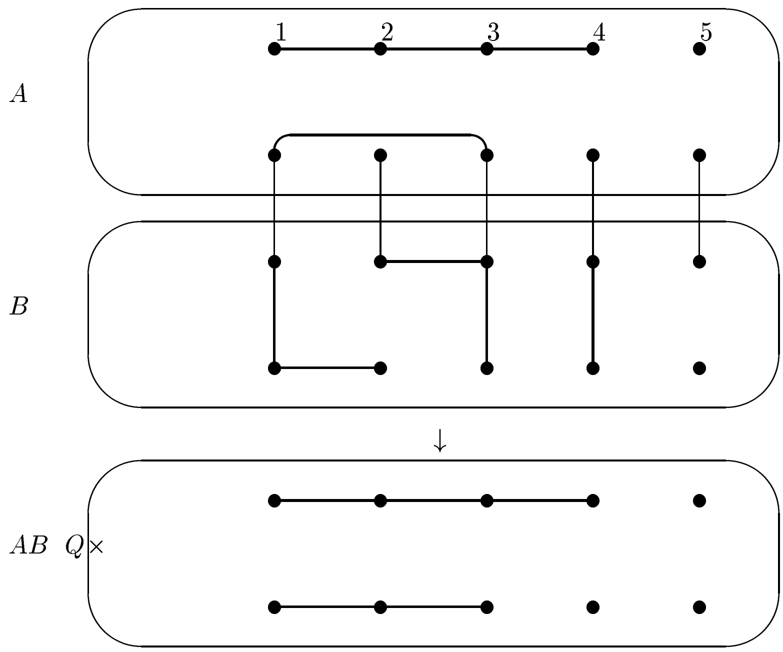

Figure 1: Pictorial realisations of parts in a partition of $\{1, 2, 3, 4, 5, 1', 2', 3', 4', 5'\}$ as clusters (A and B); and composition of partitions (AB) by juxtaposing clusters (c.f. page 868 of [11]).

a number of different ways, depending on the level of generality required (lattices can be 'grown' in more exotic ways than simple TM layering, and there exists a general algebraic framework to reflect this). Here we will stick to the ordinary TM formalism. In section 7 we will give a more general version.

The following ideas arose in considering the dichromatic polynomial formulation of the Potts model - see [2, 3, 5] and references therein. The formalism we use is abstracted far from this physical picture. It has the merit, however, of versatility, and of making proofs simple. Those wishing extra intuitive support might study the above references.

In short the Partition algebra is summarized by the example in figure 1. We now elaborate on this summary and introduce notation.

2.1 On Partitions of a set M

Recall

Definition 2 *For a set M the power set $\sqcup_M$ is the set of all subsets of M.*

so the order of the set $|\sqcup_M| = 2^{|M|}$ [12]. Let us introduce

Definition 3 *For k a natural number $\sqcup_M^k = \sqcup_{\sqcup_M^{k-1}}$ where $\sqcup_M^0 = M$.*

Definition 4 *Let $\sqcup\!\!\!\!\!\!\_\,_M$ be the subset of $\sqcup^2_M$ such that $A \in \sqcup\!\!\!\!\!\!\_\,_M$ implies*

$$\bigcup_{A_i \in A} A_i = M,$$

i.e. every element of M is an element of at least one element of A.

Definition 5 *Define the set S_M of equivalence relations on, or partitions of, a set M of m distinguished objects*

$$\begin{aligned} S_M = \{((M_1)(M_2)....(M_i)...) :& \\ & M_i \subseteq M \;\; s.t. \quad M_i \neq \emptyset, \quad \cup_i M_i = M, \quad M_j \cap M_k = \emptyset \;\; (j \neq k)\}. \end{aligned} \tag{10}$$

For example, if M is the set of the first m natural numbers

$$S_{\{1,2\}} = \{((12)), ((1)(2))\}$$

$$\begin{aligned} S_{\{1,2,3,4\}} = \{&((1234)), ((1)(2)(3)(4)), ((123)(4)), ((124)(3)), \\ &((134)(2), ((234)(1)), ((12)(34)), ((13)(24)), ((14)(23)), \\ ((12)(3)(4)), &((13)(2)(4)), ((14)(2)(3)), ((23)(1)(4)), ((24)(1)(3)), ((34)(1)(2))\}. \end{aligned}$$

Note that (up to redundant punctuation) S_M is a subset of $\sqcup\!\!\!\!\!\!\_\,_M$. In discussing general properties of S_M depending only on the order $|M| = m$ we may write S_m for S_M.

In an element of S_M we call the individual equivalenced subsets of the set of objects 'parts'. Thus $(M_1) = (123)$ is a part of the partition $((123)(4))$, and so on. Clearly the various partitions have 'shapes' like the m box Young diagrams, with the objects inserted into the shapes in all possible ways - ignoring order within a row, so the number of partitions of shape $\lambda = (\lambda_1^{p_1}, \lambda_2^{p_2}, \lambda_3^{p_3}, ...)$ (with $\lambda_i > \lambda_{i+1}$; $\sum_i p_i \lambda_i = m$) is

$$\mathcal{D}_\lambda = \frac{m!}{\prod_i \left(((\lambda_i)!)^{p_i} \, (p_i)!\right)}.$$

The set S_M is finite for finite m. Its order is the sum of Stirling numbers of the second kind at level m (see, for example, [13] and references therein). It is computed in a more general context - which will be useful later on - in [3, 14] (also see section 6.2.2). The first few values are:

m	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert S_M \rvert$	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597	27644437

We write $i \sim^A j$ in case objects i, j are in the same partition in $A \in S_M$, so the relation $\sim^A$ is transitive, reflexive and symmetric.

We will be mainly interested in the case $m = 2n$. We will then write our $2n$ objects simply as

$$M = \{1, 2, 3, ..., n, 1', 2', 3', ..., n'\}. \tag{11}$$

Before we proceed it will be useful to have a little more technical hardware. For M a set as in list 11 it will be convenient to write M' for the set obtained by priming all the elements of M (e.g. $\{1, 1'\}' = \{1', 1''\}$). If A is a partition of M we then write A' for the corresponding partition of M'.

Definition 6 (Transitive completion of $A \in \sqcup_M$) *Let*

$$\mathcal{Q} : \sqcup_M \to S_M$$

be defined by $i \sim^{\mathcal{Q}(A)} j$ if there exists $A_k \in A$ such that $i, j \in A_k$. †

For example $\mathcal{Q}(\{\{1, 2\}\{2, 4\}\{3\}\{4, 5\}\}) = ((1245)(3))$.

2.2 The Partition algebra $P_n(Q)$

For M as in equation 11

Definition 7 *Let*

$$f : S_M \times S_M \to \mathbb{Z}$$

be such that $f((A, B))$ is the number of parts of $\mathcal{Q}(A \cup B') \in S_{M \cup M'}$ (note that $|M \cup M'| = 3m$) containing exclusively elements with a single prime.

For example $\mathcal{Q}(((12)(1')(2')) \cup ((11')(2)(2'))') = ((12)(1'1'')(2')(2''))$ so $f((A, B)) = 1$.

Definition 8 *Let*

$$C : S_M \times S_M \to S_M$$

be such that $AB = C((A, B))$ is obtained by deleting all single primed elements of $\mathcal{Q}(A \cup B')$ (discarding the $f((A, B))$ empty brackets so produced), and replacing all double primed elements with single primed ones.

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†Transitivity of $\sim$ means that $\mathcal{Q}$ is such that for $A = (A_1, A_2, ..., A_j, ...) \in \sqcup_M$ then $\mathcal{Q}(A) = ((M_1), (M_2), ...)$, say (c.f. equation 10), is such that $A_j \bigcap A_k \neq \emptyset$ implies that there exists $M_i \supset A_j \bigcup A_k$; and for each M_i there exists a list $K = \{k_1, k_2, ...\}$ such that $M_i = \bigcup_{j \in K} A_j$ and there is no partition of K into 2 non-empty parts K_1, K_2 such that

$$\left(\bigcup_{j \in K_1} A_j \right) \bigcap \left(\bigcup_{l \in K_2} A_l \right) = \emptyset.$$

It is apposite to give a 'colouring' interpretation of $\mathcal{Q}$: Suppose we have any 'colouring' of M

$$g : M \to M$$

such that for each i all the elements of A_i have the same colour, i.e. $g(A_i)$ has a single element. Then since $a, b \in A_i$ now means a, b coloured the same, necessarily $g(M_j)$ has a single element for M_j any part of $\mathcal{Q}(A)$. Now suppose we choose g so that $g(M)$ has the maximum number of different colours consistent with the constraint. Then $g(M_i) = g(M_j)$ implies $i = j$.

Definition 9 *For Q an indeterminate and K the field of rational functions of Q we define a product [3, 14]*

$$\begin{aligned} \mathcal{P} &: S_M \times S_M \to KS_M \\ \mathcal{P} &: (A,B) \mapsto AB = Q^{f((A,B))} C((A,B)). \end{aligned} \tag{12}$$

An alternative form of this definition is given in a more general setting in section 7.

For example,

$$\begin{aligned} &((1234)(1'3')(2')(4')(5)(5'))\ \ ((11'2')(233')(44')(5)(5')) \xrightarrow{E_1} \\ &\qquad ((1234)(1'3')(2')(4')(5)(5')\ \ (1'1''2'')(2'3'3'')(4'4'')(5')(5'')) \\ &\qquad\qquad \xrightarrow{Q} ((1234)(1'3'1''2''3''2')(4'4'')(5)(5')(5'')) \\ &\qquad\qquad \to ((1234)(1''2''3'')(4'')(5)()(5'')) \to \ \ Q\,.\,((1234)(1'2'3')(4')(5)(5')). \end{aligned}$$

The quickest way to see this is with the picture - figure 1. There are some other pictorial examples in section 5.1.

Note that $C = C((A,B))$ is such that

$$\begin{aligned} i \sim^C j' \quad &iff \quad \exists \quad \text{sequence} \quad k_1, k_2, ..., k_{2l+1} \in M \quad s.t. \\ &i \sim^A k_1' \quad \text{and} \\ k_{2p-1} \sim^B k_{2p} \quad &k_{2p}' \sim^A k_{2p+1}' \quad \text{for } p = 1,2,...,l\,, \quad \text{and} \\ &k_{2l+1} \sim^B j' \end{aligned}$$

(a sequence of length 1, i.e. $l = 0$, is allowed); and

$$\begin{aligned} i \sim^C j \quad iff \quad i \sim^A j \quad &or \quad \exists \quad \text{sequence} \quad k_1, k_2, ..., k_{2l+1} \in M \quad s.t. \\ &i \sim^A k_1' \quad \text{and} \\ k_{2p-1} \sim^B k_{2p} \quad k_{2p}' \sim^A k_{2p+1}' \quad &\text{for } p = 1,2,...,l\,, \quad \text{and} \quad k_{2l}' \sim^A j \end{aligned}$$

and similarly

$$i' \sim^C j'$$

on interchanging A, B and primed and unprimed in the above 'connected path'.

Proposition 1 *The product $\mathcal{P}$ is associative.*

Proof: Let us drop, for the moment, the explicit distinction between primed and unprimed elements of M, but rather say that if an element a appears in both A and B in a product AB then it is to be understood primed in A. Then $a \sim^{A(BC)} b$ implies that there exists a sequence

$$i_1, i_2, ..., i_k$$

such that

$$a \sim^A i_1 \quad i_1 \sim^B i_2 \quad i_{2i} \sim^{A \text{ or } C} i_{2i+1} \quad i_{2i+1} \sim^B i_{2i+2} \quad i_{2k+2} \sim^C b$$

which in turn implies $a \sim^{(AB)C} b$, and vice versa. QED.

Definition 10 (Partition algebra) *Considering the vector space over K spanned by S_{2n}, the linear extension of the product $\mathcal{P}$ gives us a finite dimensional algebra over K which we call the partition algebra $P_n(Q)$.*

3 Relationship of $P_n(Q)$ to Full Temperley-Lieb Algebra

There are several realisations of the inclusion

$$P_{n-1}(Q) \subset P_n(Q).$$

Definition 11 *The* natural *inclusion $\mathcal{S}$ is defined by*

$$0 \to P_{n-1} \stackrel{\mathcal{S}}{\to} P_n$$

$$\mathcal{S} : ((...)...(..)) \mapsto ((...)...(..)(nn')). \tag{13}$$

It is convenient to introduce the following special elements of the partition algebra:

$$1 = ((11')(22')...(nn')) \tag{14}$$

$$1_{ij} = ((11')(22')..(ij')..(ji')..(nn')) \qquad i,j = 1,2,..,n \tag{15}$$

$$A_{i.} = \frac{1}{\sqrt{Q}} \; ((11')(22')...(i)(i')...(nn')) \tag{16}$$

$$A_{ij} = \sqrt{Q} \; ((11')(22')...(iji'j')...(nn')). \tag{17}$$

Proposition 2 *These elements generate $P_n(Q)$.*

Proof: (by induction on n) The proposition is true in case $n = 1$. Let us assume true for $n = k-1$, then show that all possible extensions of the partitions of $1, 2, ..., k-1, 1', 2', ...(k-1)'$ to include k, k' can be built using these special elements. Note that the set of special elements for P_{k-1} are (formally) a subset of those for P_k. For each $A \in P_{k-1}$ we then have $\mathcal{S}(A) \in P_k$ with the same expression as a word in the special elements, but given in full by $((...)...(..)(kk'))$ (c.f. equation 13). We will show that this subset of P_k can be extended to the whole set by using the extra special elements.

There are various cases to consider for the parts containing k, k'. In what follows we omit cases obviously consequent on symmetry grounds:
case 1: parts of the form

$$(...(..akk')) = A_{ak}(...(..a)(kk'))$$

case 2:

$$(...(..ak)(..b'k')) = A_{ak}A_{k.}(...(..a)(..b')(kk'))A_{bk}$$

case 3:

$$(...(..a'k)(..b'k')) = (...(..a')(..b')(kk'))A_{ak}A_{k.}A_{bk}$$

case 4:

$$(...(a'_1a'_2...a'_ik)(b_1b_2...b_jk')) = A_{b_1b_2}1_{bk}(...(a'_2a'_3...a'_i)(b_2b_3...b_j)(a'_1b_1)(kk'))A_{a_1a_2}.$$

QED.

It follows that 1, 1_{ii+1} $(i = 1, 2, ..., n-1)$, $A_{1.}$, A_{12} generate $P_n(Q)$.

Definition 12 *For $A \in P_n$ let $[A]$ denote the maximum number of distinct parts containing both primed and unprimed elements, over the S_m basis elements with a non-zero coefficient in A.*

For example $[1] = n$, $[A_{i.}] = n - 1$. Then

Corollary 2.1 *For $A, B \in P_n$*

$$[AB] \leq min([A],[B]).$$

Proof: It is sufficient to check for the cases where B is one of the special elements.

Proposition 3 *There is a homomorphism from the Full Temperley-Lieb algebra to the partition algebra given by*

$$H : T_{\underline{n}}(Q) \to P_n(Q)$$

$$H : 1 \mapsto 1$$

$$H : U_{i.} \mapsto A_{i.}$$

$$H : U_{ij} \mapsto A_{ij}.$$

Proof: Without loss of generality we may consider for example,

$$((11')(22')..(i)(i')..(nn')) \quad ((11')(22')..(i)(i')..(nn')) =$$

$$((11')(22')..(i)()(i')..(nn')) = Q \quad ((11')(22')..(i)(i')..(nn'))$$

and

$$(\ ((11')(22')..(i)(i')..(nn')) \ \ ((11')(22')..(ii'jj')..(nn')) \ ) \ \ ((11')(22')..(i)(i')..(nn'))$$

$$= ((11')(22')..(i)(i'jj')..(nn')) \ \ ((11')(22')..(i)(i')..(nn'))$$

$$= ((11')(22')..(i)(i')..(nn')).$$

We leave it as an exercise to check other relations (4) and (6).

Proposition 4 (see [3]) *The subalgebra of $P_n(Q)$ generated by*

$$< 1, \ \ A_{i.} \ \ (i = 1, 2, .., n), \ \ A_{ii+1} \ \ (i = 1, 2, .., n-1) >$$

is isomorphic to $T_{A_n}(Q)$.

Definition 13 *For given n we define Σ_n as the subalgebra of $P_n(Q)$ generated by*

$$< 1, 1_{ij} \ \ (i, j = 1, 2, ..., n) >$$

or, where appropriate, as the corresponding symmetric group.

4 General results for $P_n(Q)$

4.1 Full embedding of P_{n-1} in P_n

It is a useful feature of these algebras that we can largely determine the representation theory of $P_n(Q)$ in terms of $P_{n-1}(Q)$ and the symmetric group (and hence inductively from the trivial case $P_0(Q) = \mathbb{C}$). We will need the following simple but surprisingly powerful theorem:

Theorem 1 *For each n, $Q \neq 0$ and idempotent $e = e_n = A_{n.}/\sqrt{Q}$ there is an isomorphism of algebras*

$$e_n P_n e_n \cong P_{n-1}.$$

Proof:

Note that partitions in $e_n P_n e_n$ (ignoring factors of $\sqrt{Q}$ for the moment) can be thought of as partitions of P_{n-1} extended by the presence of $(n)(n')$ as isolated parts. The map $\mathcal{I}$ from left to right is to simply ignore these parts

$$\mathcal{I} : ((...)...(..)(n)(n')) \mapsto ((...)...(..)). \tag{18}$$

This is manifestly an injection. That it is a surjection comes from considering the image of $e_n P_{n-1} e_n$, noting that e_n and P_{n-1} commute. It also follows from this that the multiplication is preserved.

Corollary 1.1 *The category of left P_{n-1} modules is fully embedded in the category of left P_n modules. That is, there exist functors*

$$(P_{n-1} - mod) \xrightarrow{G} (P_n - mod) \xrightarrow{F} (P_{n-1} - mod)$$

such that FG is the identity map on $(P_{n-1} - mod)$.

Proof:

This is a standard result in case theorem 1 holds [15, 16]. There is then a standard functor

$$F : (P_n - mod) \to (P_{n-1} - mod) \tag{19}$$

with object map

$$F : M \mapsto e_n M$$

and morphism map constructed as follows. Suppose ψ is a morphism in $(P_n - mod)$:

$$\psi : M \to \psi(M)$$

$$\psi : y \mapsto \psi(y);$$

then

$$F(\psi) : e_n y \mapsto e_n \psi(y)$$

(the reader will readily confirm that composition of morphisms is preserved). Similarly we have

$$G : (P_{n-1} - mod) \to (P_n - mod)$$

with set map

$$G : N \mapsto P_n e_n \otimes_{P_{n-1}} N$$

and morphism map

$$G(\phi) : xe_n \otimes z \mapsto xe_n \otimes \phi(z).$$

We leave it as an exercise to check that FG acts as the identity functor on the appropriate category. For example, understanding by $\mathcal{I}(e_n x e_n)$ its image under the isomorphism in the full embedding theorem

$$F(G(\phi)) : (e_n x e_n) \otimes z \mapsto (e_n x e_n) \otimes \phi(z) \cong \phi(\mathcal{I}(e_n x e_n) z)$$

(we have used that ϕ is a morphism of left P_{n-1} modules). QED.

Similarly

$$GF(M) = P_n e_n M$$

and

$$G(F(\psi)) : xe_n \otimes_{P_{n-1}} e_n y \mapsto xe_n \otimes e_n \psi(y) \cong \psi(xe_n y).$$

These give us the range of GF, which will tell us (in Proposition 6) which pieces of information about the regular representation we are missing from P_n in $G(P_{n-1})$. These can then be added by explicit computation.

Some of the power of this result will be revealed when we apply it, in section 6.2.2. It is also useful in analysing the non-generic cases, which we will discuss elsewhere [10].

Let us denote by $F_n(M) = e_n M$ the object map from the isomorphism of categories in the above corollary at level n (equation 19).

Proposition 5 *Let f_n be the object map of categories defined by restriction of left P_n modules to left P_{n-1} modules through the inclusion $\mathcal{S}$,*

$$f_n : (P_n - mod) \to (P_{n-1} - mod)$$

$$f_n : M \mapsto {}_{P_{n-1}}\!\downarrow M.$$

Then the following diagram of object maps of categories commutes:

$$\begin{array}{ccc} (P_n - mod) & \xrightarrow{F_n} & (P_{n-1} - mod) \\ {}_{f_n}\downarrow & & \downarrow_{f_{n-1}} \\ (P_{n-1} - mod) & \xrightarrow{F_{n-1}} & (P_{n-2} - mod) \end{array} \quad . \tag{20}$$

Proof:

We must show that for each left P_n module M

$${}_{P_{n-2}}\!\downarrow e_n M \cong e_{n-1}\, {}_{P_{n-1}}\!\downarrow M$$

that is

$$e_n M \cong e_{n-1} M$$

is an isomorphism of left P_{n-2} modules. But this follows from the observation that the definitions of P_n and P_{n-2} are both unaffected by the interchange of labels n and $n-1$. QED.

The commutative diagram 20 may be extended to a diagram of functors.

Proposition 5 implies that, up to edge effects caused by the difference between P_n and $P_n e_n P_n$, the Bratteli restriction diagram for the algebras P_n (see section 6.2.2 onwards) has the same structure on each level n. But then

Proposition 6 *The following is a short exact sequence of algebras*

$$0 \to P_n e P_n \to P_n \to \Sigma_n \to 0.$$

Proof:

Clearly we have an injection $P_n/P_n e_n P_n \to \Sigma_n$, the group algebra of the symmetric group on n objects, since in this quotient $A_{i.} = A_{ij} = 0$. That this is surjective follows from the corollary to proposition 2 since $P_n e_n P_n$ is spanned by

$$\{A \,:\, A \in S_{2n},\ [A] < n\}$$

QED.

Thus, at least for P_n semi-simple, a knowledge of the structure of P_{n-1} essentially determines for us the structure of P_n.

Corollary 6.1 *In case $P_n(Q)$ semi-simple the distinct equivalence classes of irreducible representations may be indexed by the list of all standard partitions of every integer from* 0 *(understood to have one standard partition) to n.*

Proof: In this case the exact sequence splits [12] and P_n thus has $Card\{\lambda : \lambda \vdash n\}$ more irreducibles than P_{n-1}.

We will in fact show later that $P_n(Q)$ is semi-simple for Q indeterminate and for all $Q \in \mathbb{C}$ except for the roots of a finite order polynomial in Q for any finite n. We will also show that in any case the same classification is appropriate for *any* specialisation of $Q \neq 0$ (including non-semi simple cases).

We will apply these results repeatedly from section 6.2.2 onwards.

5 Diagram algebra for a graph G

Let us return to proposition 4. More generally we have

Definition 14 *For graph G the Diagram algebra $D_G(Q)$ is defined as the subalgebra of the partition algebra generated by*

$$< 1,\ \ A_{i.}\ \ (i = 1, 2, .., n),\ \ A_{ij}\ \ (i, j \in G) > .$$

Note that $D_{\underline{n}}(Q) \subset P_n(Q)$, as 1_{ij} cannot be built with these generators. However, under certain conditions it can be substituted, for example,

$$1_{23} A_{1.} = A_{1.} A_{12} A_{2.} A_{23} A_{3.} A_{13} A_{1.}. \tag{21}$$

In fact we are more interested here in $D_{\underline{n}}(Q)$ than $P_n(Q)$ (compare proposition 3 with equation 9), but $P_n(Q)$ provides a more versatile general setting. We will see shortly that it is straightforward to move from one to the other.

The relationship between the algebra types T, P and D is summarized by saying that the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ T & \xrightarrow{H_1} & D & \longrightarrow & 0 \\ & {\scriptstyle H} \searrow & \downarrow {\scriptstyle H_2} & & \\ & & P & & \end{array}$$

is commutative and exact at D.

Proposition 7 *The subalgebra $D_{\underline{n}}(Q) \subset P_n(Q)$ is invariant under conjugation by elements of the group Σ_n, i.e.*

$$b^{-1} D_{\underline{n}}(Q) b = D_{\underline{n}}(Q) \quad \forall b \in \Sigma_n.$$

Proof: W.l.o.g. consider $b^{-1}Ab$ with A a word in $D_{\underline{n}}(Q)$ and insert $1 = b^{-1}b$ between each letter of A. This just takes each letter to another letter. Specifically, if b is given as a permutation

$$b : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

$$b : a \mapsto b(a)$$

then

$$b^{-1} A_{i.} b = A_{b(i).} \qquad\qquad b^{-1} A_{ij} b = A_{b(i)b(j)}$$

(consider equations 16 and 17).QED.

Corollary 7.1 *Every word in $P_n(Q)$ can be written in the form AB where $A \in \Sigma_n$ and $B \in D_{\underline{n}}(Q)$.*

Clearly we have an inclusion structure

$$G \supset G' \quad \Rightarrow \quad D_G(Q) \supseteq D_{G'}(Q)$$

as for the Full algebras.

It also follows that $D_G(Q)$, and indeed $P_n(Q)$, obeys a number of quotient relations in addition to the Temperley-Lieb relations. For example, with $W \in D_G(Q)$ there exists $X(W)$ a certain (known) scalar function of Q (see [3]) such that

$$\left(\prod_i A_{i.} \right) \; W \; \left(\prod_i A_{i.} \right) = X(W) \; \left(\prod_i A_{i.} \right).$$

Specifically, if $W \in S_m$ with b_W parts

$$X(W) = Q^{b_W}.$$

This relation is suitable for at least *part* of the set appropriate for physical systems, as it corresponds to the existence of disorder at very high temperatures (there is also a dual corresponding to order at low temperatures). At the level of the dichromatic polynomial it corresponds to isolating b_W clusters (c.f. [2], for example). Several analogous relations have also been found [3].

5.1 Graphical realisation of $D_G(Q)$: Connectivities

Here the *order of a graph G*, written $|G|$, is the number of nodes.

Definition 15 *For a graph G let $\mathcal{B}_G$ be the set of all (not necessarily proper) subgraphs of G of the same order.*

For example, representing graphs by incidence matrices,

$$\mathcal{B}_{\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)} = \left\{ \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right) \right\}.$$

Note that elements of $\mathcal{B}_G$ need not be connected graphs [2].

Definition 16 *For T a natural number we write $G \times T$ for the graph $G \times A_{T+1}$, and write $G \times \mathbb{Z}$ for $G \times T$ in the limit of large T.*

Consider the graph $\underline{n} \times T$ (c.f. figure 2). Explicitly number the nodes of the lateral subgraph $\underline{n}$ at 'time' $t = 0$ (written $(\underline{n}, 0)$) from $1, 2, ..., n$ and number the nodes of $(\underline{n}, T)$ correspondingly from $1', 2', ..., n'$. Then introduce the map

$$\mathcal{F}_T : \mathcal{B}_{\underline{n} \times T} \to P_n(Q)$$

$$\mathcal{F}_T : B_o \mapsto Q^b \; B$$

where $B \in S_m$ is such that $i \sim^B j$ iff i, j (primed, unprimed or mixed) are connected by a path of bonds present in the subgraph B_o, and b is the number of isolated connected components in B_o not connected to any point in either of the layers $t = 0$ or $t = T$. Note that the definition of $\mathcal{F}_T$ does not depend on T except in the domain, so we can extend it to a map $\mathcal{F}$ on $\bigcup_T \mathcal{B}_{\underline{n} \times T}$. Then we have a relation ρ on this new domain defined by $(a, b) \in \rho$ iff $\mathcal{F}(a) = \mathcal{F}(b)$. For finite n there exists some finite T beyond which (range$\mathcal{F}_T$)$\cap S_m$ does not increase.

The range of $\mathcal{F}_T$ does not include the whole of S_m however large we make T (see the remark after definition 1). We can extend to the whole of S_m by, for example, building our 'connectivities' on $\underline{n+1} \times \mathbb{Z}$ (but only labelling the 'first' n nodes, see figure 3).

This complication is connected to the nature of the lattice and the TM formalism, it will be discussed further in [10]. In general, different choices of G in $\mathcal{B}_{G \times \mathbb{Z}} \subset \mathcal{B}_{\underline{n} \times \mathbb{Z}}$, realise different sets of conectivities, i.e. different ranges for the restricted map $\mathcal{F}(\mathcal{B}_{G \times \mathbb{Z}})$. This is, in fact, the essence of the physically important problem of finding irreducible representations of $D_G(Q)$ (see later, and [10]).

We may extend $\bigcup_T \mathcal{B}_{\underline{n+1} \times T}/\rho$ or $\bigcup_T \mathcal{B}_{\underline{n} \times T}/\rho$ to an algebra (over rational functions in Q). We define a product $B_o \,.\, C_o = (BC)_o$ by joining B_o and C_o, identifying the layer $t = T$ in B_o with $t = 0$ in C_o. It is a simple exercise to check that the product is also well defined in the quotient ρ, whereupon the map $\mathcal{F}$ becomes an algebra homomorphism.

The explicit pictorial realization is particularly neat (but sufficiently general for illustration) if we distribute the nodes of $\underline{n}$ linearly, as in A_n. Then for example with $n = 12$ the ρ class of A_{ii+1} has a simple representative with $T = 0$:

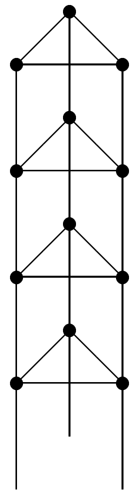


Figure 2: Part of the graph $\hat{A}_3 \times \mathbb{Z} = \underline{3} \times \mathbb{Z}$.

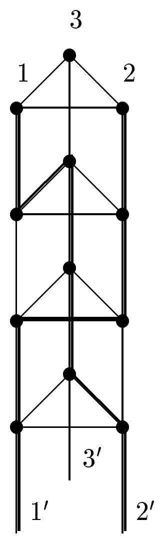
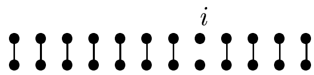


Figure 3: Diagram for the connectivity $1_{12}U_{3.} = ((12')(21')(3)(3'))$ which restricts to 1_{12} for $n = 2$.

$A_{i\ i+1}/\sqrt{Q}\ \leftarrow$ 

The ρ class of $A_{i.}$ has $T = 1$ representative

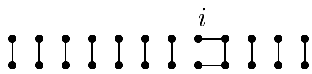
$\sqrt{Q}\,A_{i.}\ \leftarrow$ 

The composition rule is to identify the top row of dots in the second diagram with the bottom row in the first. Clusters then isolated from both top and bottom rows of the new diagram so formed may be removed, contributing a factor Q.

Finally, then, for example, the TL relation 5

$$A_{i\ i+1}\ \ A_{i.}\ \ A_{i\ i+1} = A_{i\ i+1}$$

amounts to the statement that the subgraph



has the same list of connections within and between the top and bottom layers as the ρ representative of $A_{i\ i+1}$ above.

Note that no composition of diagrams increases the number of *distinct* connected clusters connecting between the top and bottom layers (c.f. corollary 2.1). This means that the subset of ρ cosets with no connections top to bottom form a basis for a $P_n(Q)$ bimodule. Furthermore, the subset with $\leq p$ distinct connections top to bottom also form a basis for a $P_n(Q)$ bimodule (for $p < n$).

6 Structure and Representation Theory of $P_n(Q)$

6.1 Filtration by ideals

The above picture is particularly useful for envisaging and constructing representations. The number of distinct connections running from $t = 0$ to $t = T$ is evidently non-increasing in any composition (it is a measure of the number of distinct bits of information which can be simultaneously propagated through the bond covering, which cannot exceed the number propagated across any fixed time slice). So for example, writing simply P_n for $P_n(Q)$, and defining idempotents

$$I_k = \prod_{i>k} \frac{A_{i.}}{\sqrt{Q}}$$

($Q \neq 0$) then I_0 allows *no* connections from $t = 0$ to $t = T$, so $P_n I_0 P_n$ is the invariant subspace of P_n where

$$\not\exists\ A, i, j\ \ s.t.\ \ i \sim^A j'.$$

We thus have

Proposition 8 *For $Q \neq 0$ the element I_0 is a primitive idempotent.*

Recall that since $A \oplus B$ a proper decomposition of $P_n I_0$ implies $I_0 A = I_0 B = 0$ (a contradiction) then

Corollary 8.1 *The left ideal $P_n I_0$ is indecomposable (and generically simple).*

Note that $dim(P_n I_0) = |S_n|$. Similarly

Proposition 9 *The element I_1 is primitive in the quotient algebra $P_n / P_n I_0 P_n$.*

so again $P_n I_1$ is indecomposable in this quotient.

Now I_2 is not primitive in $P_n / P_n I_1 P_n$ since, for example

$$I_2 1_{12} I_2 \sim 1_{12} I_2 \not\propto I_2.$$

On the other hand $\frac{(1+1_{12})}{2} I_2$ and $\frac{(1-1_{12})}{2} I_2$ are primitive idempotents.

Similarly I_3 is not primitive in $P_n / P_n I_2 P_n$, but, for example

$$\Sigma_{\pm} I_3 = \frac{(1 \pm 1_{12} \pm 1_{23} \pm 1_{13} + 1_{12} 1_{23} + 1_{13} 1_{23})}{3!} I_3$$

and two further combinations (with $\lambda = (2,1)$ symmetries) are.

From the definition of I_i we have $P_n I_{i-1} P_n \subset P_n I_i P_n$ and a nest of short exact sequences of ideals, $i = 1, 2, ..., n$

$$0 \to P_n I_{i-1} P_n \to P_n I_i P_n \to P_n I_i P_n / P_n I_{i-1} P_n \to 0$$

where finally $I_n = 1$.

Definition 17 *Let us define the algebra $P_n[i] = P_n I_i P_n / P_n I_{i-1} P_n$.*

This is the algebra of elements with not more than i distinct connections running, as it were, from $t = 0$ to $t = T$, quotiented by the invariant subspace of all elements with strictly less than i distinct connections from 0 to T.

Proposition 10 *In the quotient $P_n[i]$*

$$I_i \Sigma_n I_i = \Sigma_i \; I_i$$

(we take $\Sigma_0 = \Sigma_1 = 1$).

Proof: Any element of Σ_n not in the subgroup is killed by the quotient.

Recall that the $\mathbb{C}$ structure of the permutation group is known [21]. In particular there are standard constructions for primitive idempotents for each $\lambda \vdash i$. Then

Corollary 10.1 *For $\lambda \vdash i$ and Σ_λ an appropriate primitive idempotent of Σ_i, then $I_i \Sigma_\lambda$ is a primitive idempotent (mod $P_n I_{i-1} P_n$).*

Corollary 10.2 *The classification scheme in corollary 6.1 extends to include all non-semi simple $P_n(Q)$ except $P_n(0)$.*

Proof: By corollary (10.1) $I_i\Sigma_\lambda$ induces an indecomposable projective module with a simple invariant subspace distinct (because of the $P_n[i]$ quotient) for each λ. QED.

Remark: The case $P_n(0)$ is degenerate rather than exceptional in this respect, and can easily be dealt with.

Proposition 11 *Let Σ be any left Σ_i module. Then we can write the left $P_n[i]$ module*

$$P_n(Q)\,(I_i\Sigma) = D_{\underline{n}}(Q)\,(I_i\Sigma)$$

Proof: By proposition 7

$$P_nI_r\Sigma = \Sigma_n D_{\underline{n}} I_r\Sigma.$$

For each word $BA(I_r\Sigma)$ on the right there are three cases to consider for each letter in B, moving from right to left. Firstly, the letter permutes nodes isolated (in the connectivity sense) from Σ by the word AI_r: In this case its effect can be ignored, e.g.

$$1_{12}A_{1.}A_{2.} = A_{1.}A_{2.}.$$

Secondly, the letter permutes nodes neither of which is isolated by AI_r: Again the effect can be ignored, as

$$1_{12}I_r\Sigma = I_r 1_{12}\Sigma = I_r\Sigma.$$

Thirdly, the letter permutes an isolated and a non-isolated node. In this case there exists an alternative formulation of the word where that letter is replaced by letters not in Σ_n, for example

$$1_{12}A_{2.}A_{3.} = A_{1.}A_{12}A_{2.}A_{3.},$$

$$1_{13}A_{2.}A_{3.} = A_{2.}A_{23}A_{1.}A_{12}A_{2.}A_{3.}$$

(note that the alternative formulation is not usually unique). More generally, suppose that the letter is 1_{ij} with i isolated and j not, then as i is isolated we can always arrange it so that 1_{ij} appears here in the combination $1_{ij}A_{i.}$. But

$$1_{ij}A_{i.} = A_{j.}A_{ij}A_{i.}.$$

QED.

6.2 Explicit construction of irreducible representations:

Our procedure is to disect the regular representation of $P_n(Q)$ provided by S_m, using $I_i\Sigma_\lambda$ from corollary 10.1. That is, we form bases from $S_mI_i\Sigma_\lambda$. There are three stages:

1. The presence of I_i says: discard all but partitions of the form $(....((i+1)')((i+2)')...(n')...)$;

2. The quotient says: discard all but partitions in which the remaining primed elements $(1', 2', ..., i')$ each appear in a distinct part, and together with at least one unprimed element;

3. The Σ_λ says: form each basis state from a certain linear combination of elements of the subset of the remaining partitions which are related by simple permutation of the primed elements. Each such subset contains $i!$ elements (all possible arrangements of the primed elements). In each such subset, once we choose an arrangement to call the identity permutation, then we have a basis for the regular representation of Σ_i. The action of Σ_λ is to project from this onto a basis for the λ irreducible representation (i.e. altogether $\dim(\Sigma_i\Sigma_\lambda)$ linear combinations will survive - an invariant subspace of Σ_i - from each subset).

Let us first consider the fully symmetrized case for the left Σ_i module in proposition 11, call it Σ^s, in each sector i (i.e. $\Sigma^s = \Sigma_\lambda$ for $\lambda = (i) \vdash i$ so $P\Sigma^s = \Sigma^s$ for all $P \in \Sigma_i$). Then we get a basis for the left $P_n[i]$ module $P_n I_i \Sigma^s$ from a generalisation of the set S_m as follows. List the elements as partitions of $1, 2, ..., n$, ignoring $1', 2', ..., n'$ except in so far as to note which parts originally contained primed elements (we may mark them with a prime *outside* the bracket - $(M_i)'$). Now discard duplicate copies of partitions not distinguished by this property, and partitions in which other than i parts originally contained primed elements. We call the resultant set $S_n(i)$ (see also section 7.1). For example,

$$S_2(1) = \{((12)'), ((1)'(2)), ((1)(2)')\}.$$

We do not need to keep track of exactly *which* unprimed nodes were connected to *which* primed nodes here, since the symmetriser makes all these permutations equivalent. In other words the set $S_n(i)$ is the set of all possible ways of arranging the elements of S_n (c.f. $S_m = S_{2n}$) so that i parts are distinguished from the rest. An element of S_n with $p \geq i$ parts produces $p!/((p-i)!i!)$ elements of the basis $S_n(i)$ (and produces none if $p < i$). Note that

$$\sum_{i=0}^{n} |S_n(i)| = 2^n |S_n|. \tag{22}$$

The action of the generators on such a basis is just the usual product from equation 12 pulled through from the regular representation (remembering the $P_n[i]$ quotient, and that primed parts beget primed parts [3] e.g. $\sqrt{Q}A_{1.}((12)') = ((1)(2)')$). We will prove irreducibility of these representations in section 6.2.2.

Moving to the case where we take some other left Σ_i module in proposition 11, then our $S_n(i)$ basis must simply be (semi) direct producted with a basis for this new module (rule 3). Some permuting actions will act on the primes and hence on the Σ_i module rather than, or as well as, the partitions. There is usually an ambiguity in the choice of an identity permutation here, corresponding to a basis change in the eventual representation. We will resolve it, for the sake of definiteness, by labelling primes in a standard order (details of a standard order are given in section 7.1). If an action changes the order then this permutation acts on Σ. For example, for the $i = 2$ antisymmetriser $\Sigma_- = \Sigma_{(1^2)} \mapsto 1 - \sigma_{12}$ (the permutation action of σ_{12} is on the primes with respect to the standard order, not on the elements of M) and $S_2(2)$ we have (single element) basis

$$S_2(2)\Sigma_{(1^2)} = \{(((11')(22')) - ((12')(21')))\}$$

so

$$1_{12}(((11')(22')) - ((12')(21'))) = -(((11')(22')) - ((12')(21')))$$

gives the representation $\mathcal{R}_{(1^2)}(1_{12}) = -1$.

6.2.1 The case $n = 3$

We can well illustrate all of the above points with an extended example. Let us consider $n = 3$. The available partition shapes λ in S_6 are:

$$(6),(5,1),(4,2),(3^2),(4,1^2),(3,2,1),(2^3),(3,1^3),(2^2,1^2),(2,1^4),(1^6)$$

with corresponding multiplicities $\mathcal{D}_\lambda$:

$$1,6,15,10,15,60,15,20,45,15,1$$

giving total dimension $|S_6| = 203$.

On the other hand the dimensions of the bases described above are

$$5,10,6\ dim(\Sigma_2),1\ dim(\Sigma_3)$$

i.e., explicitly, the bases are

$$\{((123):\emptyset),((12)(3):\emptyset),((13)(2):\emptyset),((23)(1):\emptyset),((1)(2)(3):\emptyset)\},$$

$$\{(\emptyset:(123)),((12):(3)),((3):(12)),((13):(2)),((2):(13)),$$
$$((23):(1)),((1):(23)),((1)(2):(3)),((1)(3):(2)),((2)(3):(1))\},$$

$$\{(\emptyset:(12)(3)),(\emptyset:(23)(1)),(\emptyset:(2)(13)),((1):(2)(3)),((2):(1)(3)),((3):(1)(2))\}\times\Sigma_\pm$$
$$\{(\emptyset:(1)(2)(3))\}\times\Sigma_3$$

where all parts to the right of the colon are to be understood primed (c.f. [3]).

In full the $S_3(2)\Sigma_-$ basis may be written

$$\{(((121')(32'))-((122')(31'))),\ \ (((232')(11'))-((231')(12'))),$$
$$(((22')(131'))-((21')(132'))),\ \ (((1)(21')(32'))-((1)(22')(31'))),$$
$$(((2)(11')(32'))-((2)(12')(31'))),\ \ \ (((3)(11')(22'))-((3)(12')(21')))\ \}$$

so for example the representation of 1_{12} is

$$\mathcal{R}_{(1^3)}(1_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally, then, noting the multiplicities of inequivalent generically irreducible representations at level i we have

$$5^2+10^2+6^2.(1+1)+1^2.(1+2^2+1) = 203 \tag{23}$$

which coincides with the total dimesion, so we have, for example, the complete set of inequivalent irreducible representations for the semi-simple cases. Note that all the $i = 3$ representations reduce to (direct sums of) the same representation in $D_{\underline{n}}(Q)$, because none of the permutations can actually be realized in this subalgebra.

6.2.2 The 'algebra' $P_n^{\sim}(Q)$

Since we know the structure of the symmetric group (algebra) Σ_i (see, for example, [20, 21]) it behoves us to divide up our analysis by first considering the 'completely Σ-symmetrised algebra', $P_n^{\sim}(Q)$, which we define below (and in which the symmetric group effect is quotiented out). The rest will then follow from changing the left Σ_i module in propostion 11.

Definition 18 *We define an equivalence relation $\sim$ on S_m by $A \sim B$ iff they are the same up to a permutation of the connections made by the connectivities from $t = 0$ (unprimed elements) to $t = T$ (primed).*

That is to say, if $A \sim B$ then the connections amongst unprimed nodes are the same, the connections amongst primed nodes are the same, the number of instances of primed and unprimed nodes in the same part are the same, and the subset of nodes in such mixed parts is the same. So in case of two parts in a partition having both primed and unprimed nodes, the primed nodes in one can be swapped for the primed nodes in the other without changing the $\sim$ equivalence class. We will give an alternative definition later.

We write $P_n^{\sim} = P_n(Q)/\sim$ for the 'quotient' obtained by the linear extension to $P_n(Q)$.

In this case there is a Dirac bra-ket notation for $\sim$-cosets of S_m. Every coset may be written uniquely in the form $|a\rangle\langle b|$ where $a, b \in S_n(i)$ for some i (conversely every such pair defines a unique coset). There is then an inner product $\langle b|c\rangle$ in each $P_n[i]/\sim$, obtained from

$$|a\rangle\langle b||c\rangle\langle d| = \langle b|c\rangle \;\; |a\rangle\langle d|. \tag{24}$$

Note that the inner product is symmetric. In the sense of equation 24 the set $S_n(i)$ forms a basis for a representation of $P_n[i]/\sim$, and hence of $P_n(Q)/\sim$ and of $P_n(Q)$.

Proposition 12 *The $n + 1$ representations of $P_n(Q)/\sim$ with bases $S_n(i)$ ($i = 0, 1, 2, .., n$) and canonical action (up to the $P_n I_{i-1} P_n$ quotient) are each irreducible, except at the roots of a finite order polynomial in Q.*

Outline proof: For all $b \in S_n(i)$ the power of Q given by $\langle b|b\rangle$ is not exceeded by any $\langle b|c\rangle$; and there exists at least one b such that $\langle b|b\rangle$ is the unique maximum power of Q for any $\langle b|c\rangle$ (specifically any b of exactly n parts, for which $\langle b|b\rangle = Q^{n-i}$). It follows that the determinant of the Gram matrix [17, 18] is polynomial in Q with coefficient of the leading power unity. Therefore the inner product is non-degenerate and the Gram matrix is simple in case Q is an indeterminate. Taken with the outer product also implicit in equation 24 this ensures that the representation with basis $\{|a\rangle : a \in S_n(i)\}$ is surjective, since there exists another basis $\{|\alpha\rangle\}$ (say) for which $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$ so that $\{|\alpha\rangle\langle\beta| : \alpha, \beta \in S_n(i)\}$ is a complete set of elementary matrices. QED.

Corollary 12.1 *These representations are inequivalent.*

We will abuse the symbol for the set $S_n(i)$ to denote also the left $P_n(Q)$ module it spans.

Corollary 12.2 *Any representation of $P_n(Q)$ built from proposition 11 with Σ an irreducible Σ_i module is irreducible for Q indeterminate.*

Corollary 12.3 *$P_n(Q)$ is semi-simple for Q indeterminate and for all $Q \in \mathbb{C}$ except for the roots of a finite order polynomial in Q for any finite n.*

Proof: We have at least two proofs! Firstly, by noting that each of a complete set of indecomposable projective modules is in fact simple (the above proposition taken with corollary 1.1). Secondly, by counting and combinatorics: the irreducible representations account for the full dimension of the algebra. We will show this explicitly in the next section. Thirdly, by another counting argument - see section 5.

The Bratelli diagram for the restriction corresponding to $P_n^{\sim}(Q) \supset P_{n-1}^{\sim}(Q)$ on these irreducible representations is as follows, with top line $n = 0$, and leftmost column $i = 0$ (i.e. generated by I_0 for each n). We write only the dimension for each module, thus starting with $P_0 = \mathbb{C}$, then $P_1 = \mathbb{C}1 \oplus \mathbb{C}U_1$., we have

$$
\begin{array}{ccccccccccccc}
1 \\
\uparrow & \nwarrow \\
1 & & 1 \\
\uparrow & \nwarrow\!\!\!\nearrow & \uparrow\uparrow & \nwarrow \\
2 & & 3 & & 1 \\
\uparrow & \nwarrow\!\!\!\nearrow & \uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow & \nwarrow \\
5 & & 10 & & 6 & & 1 \\
\uparrow & \nwarrow\!\!\!\nearrow & \uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow & \nwarrow \\
15 & & 37 & & 31 & & 10 & & 1 \\
\uparrow & \nwarrow\!\!\!\nearrow & \uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow\uparrow & \nwarrow \\
52 & & 151 & & 160 & & 75 & & 15 & & 1 \\
\uparrow & \nwarrow\!\!\!\nearrow & \uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow\uparrow & \nwarrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow & \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow & \nwarrow \\
203 & & 674 & & 856 & & 520 & & 155 & & 21 & & 1
\end{array}
\tag{25}
$$

and so on. These restrictions are forced by proposition 5 - c.f. [3]. To see this note that the morphism of categories in the corollary to theorem 1 takes a layer of the above diagram to the layer below it (each node is mapped vertically down, since the idempotent e_n cuts at most one connection, e.g. $e_2P_2e_1e_2 \to P_1e_1$ and $P_2e_2 \otimes P_1e_1 \to P_2e_1e_2$). The 1 at the right hand side of the lower layer is missing in this map, of course, as this is the trivial representation of Σ_n. Consequently (i.e. as a knock on effect from the *previous* layer) the restriction information for the next two modules to the left - $S_n(n-2)$ and $S_n(n-1)$ - is incomplete. However, the only possibility is for the restrictions to include some copies of the trivial representation, and these may be filled in by dimension counting (we know the dimensions of all $S_n(i)$, as we will see shortly) or by noting that, with Id_k the $k \times k$ identity matrix denoting multiplicity k,

$$
{}_{n-1}\!\downarrow S_n(n-1) = Id_n \otimes \overbrace{S_{n-1}(n-1)}^{\text{trivial representation}} \oplus S_{n-1}(n-2)
$$

(the multiplicity n occurs since there are $n-1$ ways in which the n^{th} node can be in a primed part in $S_n(n-1)$, and one way in which it can be in an unprimed part on its own) and

$$\begin{aligned} &{}_{n-1}\downarrow S_n(n-2) = \\ &\qquad\qquad (Id_{n-1}\otimes S_{n-1}(n-1)) \oplus (Id_{n-1}\otimes S_{n-1}(n-2)) \oplus S_{n-1}(n-3). \end{aligned} \tag{26}$$

For example, omitting node 3 in $S_3(2)$ we get

$$\begin{aligned} &\{((1)'(23)'),((12)'(3)'),((13)'(2)'),((1)'(2)'(3)),((1)'(2)(3)'),((1)(2)'(3)')\} \\ &\quad \to \{((1)'(2)'),((12)'),((1)'(2)'),((1)'(2)'),((1)'(2)),((1)(2)')\} = S_2(1) + 3.S_2(2). \end{aligned}$$

Note again that in omitting the last node (n) in this mnemonic if we have a part of the form $(ij...mn)$ (i.e. unprimed) then this maps to $(ij...m)'$, since the action of generators here is as if the part is connected to *something*!

We may generate bases for the representations in a row of equation 25 from those in the preceeding row in such a way that the intertwiner between representations corresponding to equation 26 is lower unitriangular (c.f. [18, 19]). The rules for using basis states from $S_{n-1}(i)$ to construct basis states at level n are:

1.(down left, i.e. to $i-1$) take a primed bracket, put in element n and remove the prime (generates i new states from each state);

2.(down, i.e. to i) add (n) or insert n into any primed bracket ($i+1$ new states from each state);

3.(down right, i.e. to $i+1$) add $(n)'$.

It follows from our restriction rules that this construction preserves the restriction subblocks in the order of equation (26), but with some additional entries below the block diagonal. The first few bases are then as below (we have indented columns to indicate the separation into restriction subblocks):

$$\begin{array}{llll}
(\emptyset) & & & \\
 & & & \\
((1)) & ((1)') & & \\
 & & & \\
((1)(2)) & ((1)(2)') & ((1)'(2)') & \\
\quad((12)) & \quad((1)'(2)) & & \\
 & \qquad((12)') & & \\
 & & & \\
((1)(2)(3)) & ((1)(2)(3)') & ((1)(2)'(3)') & ((1)'(2)'(3)') \\
((12)(3)) & ((12)(3)') & ((1)'(2)(3)') & \\
\quad((1)(23)) & \quad((1)(2)'(3)) & ((12)'(3)') & \\
\quad((13)(3)) & \quad((1)'(2)(3)) & \qquad((1)'(2)'(3)) & \\
\quad((123)) & \quad((12)'(3)) & \quad((1)'(23)') & \\
 & ((1)(23)') & \qquad((13)'(2)') & \\
 & ((13)'(2)') & & \\
 & ((123)') & & \\
 & \qquad((1)'(23)) & & \\
 & ((13)(2)') & &
\end{array}.$$

The usefulness of this construction lies in computing determinants of Gram matrices (and hence ultimately the exceptional structure of $P_n(Q)$). For example the representation ${}_2{\downarrow}\, S_3(2)$ is given by

$$U_{1.} = \begin{pmatrix} \sqrt{Q} \\ 0 & 0 \\ \frac{1}{\sqrt{Q}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{Q}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; U_{12} = \begin{pmatrix} 0 & 0 & \sqrt{Q} \\ 0 & 0 & \sqrt{Q} \\ 0 & 0 & \sqrt{Q} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; U_{2.} = \begin{pmatrix} 0 \\ 0 & \sqrt{Q} \\ 0 & \frac{1}{\sqrt{Q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{Q}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

(all omitted entries zero) so the intertwiner takes the form

$$W = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ X & Y & Z & \alpha & 1 \\ Y & X & Z & \beta & \gamma & 1 \end{pmatrix}$$

where $X = \frac{Q-1}{Q(Q-2)}$, $Y = \frac{1}{Q(Q-2)}$, $X + Y + Z = 0$ and the other constants will be determined shortly. The Gram matrix Γ for the inner product at $n = 3$, and the composite matrix Γ' at $n = 2$ are

$$\Gamma = \begin{pmatrix} Q & 0 & 1 & 0 & 1 & 0 \\ 0 & Q & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} ; \quad \Gamma' = \begin{pmatrix} Q & 0 & 1 & 0 & 0 & 0 \\ 0 & Q & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & C \end{pmatrix} .$$

The constants A, B, C will be determined, they arise because the irreducible inner products contained here (on the diagonal) are only unique up to a scalar factor (a basis change in general). Now putting $\Gamma = W\Gamma' W^T$ we obtain $\alpha = \beta = 1/Q$, $\gamma = \frac{-1}{Q-3}$ and $A = Q$, $B = \frac{(Q-1)(Q-3)}{Q(Q-2)}$, $C = \frac{(Q-1)(Q-4)}{Q(Q-3)}$. Altogether $\det(\Gamma) = \det(\Gamma') = Q(Q-2)ABC = (Q-1)^2(Q-4)$. This determinant tells us how the irreducible representation collapses at special values of Q (c.f. [10, 18]). Even in this example it is notable that all polynomials factorize over the integers.

6.3 General n (general symmetry)

For $\lambda \vdash i$ let us allow Σ_λ now to symbolize the whole simple Σ_i module associated to the partition λ (c.f. [18, 21, 22, 23], say). Then for $P_n(Q)$ the generic simple modules may be realised as $S_n(i) \otimes_i \Sigma_\lambda$ where the product is as discussed in sections 6.2 and 6.2.1.

The restriction rule here is given by

Proposition 13 *For $\triangleright$ meaning "one box added to" ([21]) the restriction from P_n to P_{n-1} is*

$$
\begin{aligned}
{}_{n-1}\downarrow (S_n(i) \otimes \Sigma_\lambda) = &\left(\bigoplus_{\lambda' \triangleright \lambda} (S_{n-1}(i+1) \otimes \Sigma_{\lambda'}) \right) \\
&\oplus \left(\bigoplus_{\lambda' \triangleleft\triangleright \lambda} (S_{n-1}(i) \otimes \Sigma_{\lambda'}) \right) \\
&\oplus \bigoplus_{\lambda' \triangleleft \lambda} (S_{n-1}(i-1) \otimes \Sigma_{\lambda'}) \, . \qquad (27)
\end{aligned}
$$

For example, abbreviating modules on the right to their partitions and denoting multiplicity 3 by 3.()

$$\downarrow \left(S_n(3) \otimes \Sigma_{(2,1)}\right) \;\cong\; (2,1^2) \oplus (2^2) \oplus (3,1) \bigoplus (1^3) \oplus 3.(2,1) \oplus (3) \bigoplus (1^2) \oplus (2).$$

Proof: The middle term in equation (27) is present because there are i ways of having node n in a primed part (so we can ignore it except in as much as it makes that part distinguishable) and one way of having it in a part on its own - altogether equivalent to induction followed by restriction on the Σ_λ factor.

The first sum is present because if n is in an unprimed part then in the restriction this part behaves as a new primed part. This is because the part is still connected to n, so isolating it (from n) would change the state, as it would if it were a primed part. The symmetric group factor thus moves to Σ_{i+1}, and the Σ module here is induced from $\lambda \otimes \Box$ in the usual way [21].

Finally, the last term comes from elements in which n is in a primed part alone. Then discarding it affects the Σ module just like ordinary one box restriction. QED.

Note that this is consistent with the symmetrised quotient case - associated to each of the entries in column i of the Bratelli diagram drawn there we have *here* a representation for each partition λ of i, of dimension $|S_n(i)|.|\Sigma_\lambda|$.

As noted in corollary 12.3, it follows from theorem 1 that $\{S_n(i) \otimes_i \Sigma_\lambda \; : \; i = 0,1,...,n; \lambda \vdash i\}$ is a complete list of generic irreducibles. We can check this another way - since it is not obvious that the dimension counting generalising equation 23 works here, but it does! Let us write $d_n(i)$ for the dimension of the i^{th} representation in row n (the i^{th} column, counting the left hand column as column 0). Then the total dimension of $P_n(Q)$ is bounded below by

$$\sum_{i=0}^{n} (i)!(d_n(i))^2 = |S_m(0)|$$

($m = 2n$). The identity is readily proved - the number of ways of moving along arrows from $S_{2n}(0)$ to $S_n(i)$ in the Bratelli diagram above is exactly $i!$ times the number of ways of moving from position $S_n(i)$ to $S_0(0)$, and the latter number is $|S_n(i)| = d_n(i)$. On the other hand $|S_m| = |S_m(0)|$ so the bound is saturated. We thus have the complete structure for all semi-simple cases.

Let us go into this in a little more detail. Define operators $m_d, h, h^\dagger$ on $\mathbb{C}^\infty$ by their actions on the standard ordered basis $\{|i\rangle \; : \; i = 1,2,3,...\}$

$$m_d|i\rangle = i|i\rangle$$

$$h|i\rangle = |i+1\rangle$$
$$h^\dagger|i\rangle = |i-1\rangle \qquad\qquad (h^\dagger|1\rangle = 0)$$

and define $\mathcal{M} = m_d + h$ and $\mathcal{M}' = m_d + h + m_d h^\dagger$. For example, as an infinite matrix we have (with omitted entries zero)

$$\mathcal{M} = \left(\begin{array}{cccccccc} 1 & & & & & & & \\ 1 & 2 & & & & & & \\ & 1 & 3 & & & & & \\ & & 1 & 4 & & & & \\ & & & 1 & 5 & & & \\ & & & & 1 & 6 & & \\ & & & & & 1 & 7 & \\ & & & & & & ... & ... \\ & & & & & & & ... \end{array} \right).$$

Then with the usual $\langle i| = |i\rangle^\dagger$ (so $\langle i|j\rangle = \delta_{ij}$) we readily see that if

Definition 19 *$\mathcal{S}_m[i]$ is the subset of S_m of elements with i parts*

(*not* the same as $S_m(i)$, which is not a subset) then

$$|\mathcal{S}_m[i]| = \langle i|\mathcal{M}^{m-1}|1\rangle$$

(Stirling numbers of the second kind) so that altogether

$$|S_m| = \sum_i \langle i|\mathcal{M}^{m-1}|1\rangle$$

(these work for any m, not just $2n$). Note that as m grows $\frac{|S_{m+1}|}{|S_m|}$ is bounded by the largest eigenvalue of m_d (i.e. it is unbounded!). Furthermore

$$\begin{aligned} d_m(j) = |S_m(j)| &= \sum_i \frac{(i)!}{(i-j)!j!}\langle i|\mathcal{M}^{m-1}|1\rangle \\ &= \langle j+1|(\mathcal{M}')^{m-1}|1\rangle \end{aligned} \tag{28}$$

(this last identity is not so obvious!). In any case, puting $j = 0$ we get $|S_m| = |S_m(0)|$.

Note that for fixed j the ratio $\frac{d_m(j)}{d_{m-1}(j)}$ is unbounded at large m (c.f. the Potts model representation).

6.4 The structure of $D_{\underline{n}}(Q)$

In $P_n(Q)$ the i^{th} entry in each row of the Bratelli diagram 25 corresponds to many representations (one for each partition of i), The only difference for $D_{\underline{n}}(Q)$ is that the rightmost (n^{th}) entry corresponds not to one representation for each $\lambda \vdash i$, but to a single one dimensional representation. This arises from the impossibility of any transverse movement of n distinct connected lines on a graph with only n nodes in each lateral subgraph. All the permutation representations collapse to direct sums of a trivial representation of the identity.

7 On generalisations of $P_n(Q)$

We describe here some of the basic building blocks of a categorical version of $P_n(Q)$. This generalises the TM formalism to the general 'surgery' of partition vectors [3] on latticised manifolds, which will be developed further elsewhere. It is also useful *here* for a different perspective on several earlier definitions.

Definition 20 *('Internalisation') For $N \subseteq M$ and K the field of rational functions in Q define*

$$In_N : S_M \to KS_N \qquad\qquad (N \subset M)$$

by

$$In_N : A \mapsto Q^{f(A)} A'$$

where A' is obtained by deleting all elements of M not in N from A, and $f(A)$ is the number of empty brackets this formally leaves (empty brackets are in practice omitted from A').

For example, with $M = \{1,2\}, N = \{1\}$

$$((1)(2)) \to ((1)()) \to Q((1)).$$

Then with $N \subseteq M \cup P$ and $Ag(A,B) = \mathcal{Q}(A \cup B)$ we have a composition $\mathcal{P}_N$ defined by commutativity of the following diagram:

$$\begin{array}{ccc} S_M \times S_P & \xrightarrow{\mathcal{P}_N} & KS_N \\ {\scriptstyle Ag}\searrow & S_{M\cup P} & \nearrow{\scriptstyle In} \end{array} . \tag{29}$$

The product $\mathcal{P}$ of section 2 is the special case $|M| = 2n$, $|P| = 2n$, $|M \cap P| = n$, $N = M \cup P - M \cap P$ ($|N| = 2n$).

In general terms the physical interpretation of this composition is as follows. The sets S_M and S_P represent the boundary configuration space bases for two disjoint (Q-state Potts model-like) statisitical mechanical systems with spins on their boundaries labelled by the objects of M and P respectively. Aggregation identifies part of one boundary (possibly empty, in general) with part of the other, thus combining the two systems. Internalisation then removes this part from the boundary to the interior (in case $N = M \cap P$) or, more generally, removes some other part to the interior. This is the TM composition generalised to the partition vector [3] formalism.

The above construction is for the case in which each boundary subgraph of the statistical mechanical lattice is the complete graph for the boundary nodes. The main physical interest comes in restricting this, and also the interior of the lattice, to sparser (e.g. hypercubical) graphs - c.f. [10].

7.1 Outer and Inner products on $S_m(i)$ and S_m

Recall that $S_m(i)$ is the set of possible ways of attaching a distinguishing mark to each of any i of the parts of each element of S_m. For example

$$S_3(2) = \{((12)'(3)'), ((13)'(2)'), ((1)'(23)'), ((1)'(2)'(3)), ((1)'(2)(3)'), ((1)(2)'(3)')\}.$$

Definition 21 *For $N \subseteq M$ define a variation on internalisation*

$$In'_N : S_M \to \bigcup_i S_n(i)$$

where $In'_N(A)$ is obtained by first (as a formal intermediate step) replacing every element in A not in N by a dot, then discarding every part consisting purely of dots, then priming each part containing at least one dot.

For example, with $M = \{1,2,3,4\}, N = \{1,2\}$

$$((13)(2)(4)) \to ((1.)(2)(.)) \to ((1)'(2)).$$

Let us define a standard form for writing out a partition in S_m or $S_m(i)$, i.e. a standard order for the objects in a part and the parts in a partition. The first will be the usual natural order of the natural numbers, the second will be the order obtained by writing out the part containing 1 first, then the part containing the lowest number not contained in the first part, and so on (e.g.

$$((123)(49)(578)(6))$$

is in standard form).

Let us define a series of maps. Firstly

Definition 22 ('Expansion') *For P a permutation of $\{1,2,...,i\}$ define*

$$Ex_P : S_m(i) \to S_{m+i} \tag{30}$$

by

$$Ex_P : a \mapsto A$$

where A is the partition obtained from a by inserting element $m + P(k)$ into the k^{th} primed part (when written in the standard form).

For example, for the trivial permutation $P = 1$ we have

$$Ex_1((123)'(4)(56)') = ((1237)(4)(568)).$$

Note that

$$In'_M(Ex_P(a)) = a.$$

Now consider $S_{I \cap J}$ where $I \cap J = \emptyset$, then each $A \in S_{I \cap J}$ induces a relation $R(A) \subseteq I \times J$ via $(a,b) \in R(A)$ iff $a \sim^A b$.

Definition 23 ('Projection') *Define*

$$Pr : S_{I \cup J} \to K\Sigma_i$$

by

$$A \mapsto Pr(A)$$

where $Pr(A) = R(A)$ if $R(A)$ is an isomorphism and $Pr(A) = 0$ otherwise.

Note that in order to have a concrete realisation of $R(A) \in \Sigma_i$ we need to adopt an isomorphism to act as identity. In practice there is usually a natural choice (see later).

For example if $I = \{1,2\}, J = \{1',2'\}$ then the identity isomorphism might as well take $i \mapsto i'$ so

$$Pr(((12')(21'))) = (12), \qquad Pr(((1)(1')(22'))) = 0$$

where (12) means the permutation in cycle notation (not an element of S_m!).

We then define an inner product on $S_m(i)$

$$S_m(i) \times S_m(i) \to K$$
$$(a,b) \mapsto \langle a|b\rangle \tag{31}$$

by the composite map

$$S_m(i) \times S_m(i) \overset{Ex_1 \times Ex_1}{\longrightarrow} S_{M\cup I} \times S_{M\cup J} \overset{Ag}{\to} S_{M\cup I\cup J} \overset{In}{\to} KS_{I\cup J} \overset{1\times Pr}{\longrightarrow} K\Sigma_i \to K$$

where in the second cartesian product we want to distinguish the sets I and J, both necessarily of order i, such that $I \cap J = \emptyset$, but not distinguish the first and second occurence of the set M. It is notationally convenient to take $J = I'$ (i.e. $i' \in J$ iff $i \in I$). The Pr map is present to take account of the irreducible representation filtration quotient (see later).

For example

$$((12)(3)') \times ((1)'(23)) \mapsto ((12)(34)) \times ((14')(23)) \mapsto ((12344')) \mapsto ((44')) \mapsto 1.1$$

(in the second expression we have distinguished I and J by a prime on elements of J, as suggested above) and

$$((1)(2)(3)') \times ((1)'(2)(3)) \mapsto ((1)(2)(34)) \times ((14')(2)(3))$$
$$\mapsto ((14')(2)(34)) \mapsto ((4')()(4)) \;=\; Q\;((4')(4)) \mapsto Q.0 = 0.$$

We define an Outer product for each $P \in \Sigma_i$

$$Out_P : S_m(i) \times S_m(i) \to S_{M\cup M'}$$
$$(a,b) \mapsto |a\rangle_P\langle b| \tag{32}$$

where M and M' are disjoint of order m, by the composite map

$$S_m(i) \times S_m(i) \overset{Ex_1 \times Ex_P}{\longrightarrow} S_{M\cup I} \times S_{M'\cup I} \overset{Ag}{\to} S_{M\cup M'\cup I} \overset{In}{\to} S_{M\cup M'}.$$

Note that Out_P is injective. The ranges of Out_P for each P and i are manifestly disjoint. The union over all these disjoint ranges is S_{2m}. This is proved by the same counting argument as for the irreducible representations.

There is a generalisation of the outer product to $S_m(i) \times S_n(i)$ with $m \neq n$. This is more easily constructed in reverse, so....

Conversely, for any partition of set M into two disjoint subsets N, R (not necessarily of the same order) there is a map from

$$S_M \to \bigcup_i \bigcup_{P \in \Sigma_i} S_n(i) \times S_r(i) \times P$$

given by

$$A \mapsto (In'_N(A), In'_R(A), P)$$

where P is the isomorphism from primed parts of $In'_N(A)$ to primed parts of $In'_R(A)$ realised by A.

This is the inverse of the outer product.

We note the capacity for a diadic form for the product $\mathcal{P}_N$ defined in the previous subsection (but with a filtration quotient in operation)

$$|a\rangle_P \langle b|.|c\rangle_{P'} \langle d| = \langle b|c\rangle \; |a\rangle_{PR(A)P'} \langle d|$$

where $R(A)$ is as in the definition of the inner product.

Here are a couple of all singing all dancing examples:

$$A = ((12)(31'2'5')(44')(566')(3')) \mapsto |((12)(3)'(4)'(56)')\rangle_1 \langle((125)'(3)(4)'(6)')|$$

$$B = ((1)(2)(341')(54'6')(62'3'5')) \mapsto |((1)(2)(34)'(5)'(6)')\rangle_{(23)} \langle((1)'(235)'(46)')|$$

but

$$\langle((125)'(3)(4)'(6)')||((1)(2)(34)'(5)'(6)')\rangle \mapsto 1.(12)$$

so altogether we have (with permutation $1.(12).(23) = (132)$)

$$|((12)(3)'(4)'(56)')\rangle_{(132)} \langle((1)'(235)'(46)')| \mapsto ((12)(34'6')(41')(562'3'5')) = AB$$

as required.

Finally let us give an alternative definition of the equivalence $\sim$ on S_{2m}. We have that $A \sim B$ iff there exist permutations P, Q and elements $a, b \in S_m(i)$ for some i, such that

$$B = Out_P((a,b)) = |a\rangle_P \langle b| \qquad\qquad A = Out_Q((a,b)) = |a\rangle_Q \langle b|.$$

For example

$$((1233')(41'2')(4')) \sim ((1231'2')(43')(4'))$$

since the former is $Out_P((((123)'(4)'), ((12)'(3)'(4))))$ with $P = (12)$ while the latter is

$$Out_1((((123)'(4)'), ((12)'(3)'(4)))).$$

8 Discussion

We have completely determined the generic structure of the algebra which contains the Potts model and dichromatic polynomial representations of the Temperley-Lieb algebra for statistical mechanics in arbitrary dimensions. These results govern the subalgebras appropriate for models in fixed dimensions. In subsequent work with H. Saleur [10] we examine these subalgebras in detail, using the filtering system established here.

We will determine the generic irreducible content of the Potts model representations. In particular, we will show that since the irreducible representations constructed here grow with n faster than Q^n for any Q (c.f. equation (28)) then the existence of the Potts representations ensure that at least some of the positive integer Q algebras are exceptional in any dimension. This means that extra symmetry turns up to simplify the spectrum (i.e. the long distance properties) of these models, as it does in two dimensions. In two dimensions this effect is closely related to a 'rational' conformal symmetry of the critical field theory limit [8]. In other dimensions it requires further investigation.

We will examine the non-generic representation theory of the partition algebra.

We will also use the scheme developed here as the basis for an analysis of the infinite Full Temperley-Lieb algebras, and to find out which of the infinity of irreducible representations are relevant for physics, and why.

Note that the subset of S_M containing partitions with only even numbers of elements in each part generates a subalgebra of $P_n(Q)$. The subset containing partitions with exactly two elements in each part generates the Brauer algebra (compare figure 1 with [11]).

Acknowledgements

I would like to thank H Saleur, the City University, the UK SERC and the Nuffield Foundation for support. I would also like to thank B Westbury, H Saleur and T Stanley for useful conversations. I would particularly like to thank Hubert Saleur for getting me to Yale, where this paper was written.

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