spin-chain braid representations

Paul Martin and Eric C. Rowell

Hubert Saleur Celebration, September 21, 2021

Inspirations

Hubert Saleur

but also work with Faria Martins, Damiani, Wang, Deguchi, and other wonderful collaborators, who will forgive today's focus on Hubert.

Spin-chain braid representations

- A braid representation is a monoidal functor from the braid category B.
- A rank-N charge-conserving representation (or spin-chain representation) is a monoidal functor from some monoidal category to the category Match^N of rank-N charge-conserving matrices (see below for definition).
- In this work we construct all spin-chain braid representations up to isomorphism.

The problem

Classification of braid representations is a major open problem — except that it is impossibly wild.

For applications, conceding this impossibility is not an acceptable outcome. So we seek a framework for a paradigm change. Clues?:

Higher rep theory (Mazorchuk and many others) Kapranov-Voevodsky

Higher lattice gauge theory (Faria Martins and many others)

...

In the end, how universal is the construction, the 'new paradigm'?

...

Braids

Monoidal composition

$^{\otimes 2}$

$=$

$=$

R-matrix

The braid category is monoidally generated by the elementary braid $\sigma \in \mathsf{B}(2,2)$. Thus a functor

$$F : \mathsf{B} \to \mathsf{Match}^N$$

(or indeed to any target) is determined by the image $F(\sigma)$.

The image $F(\sigma)$ is a sparse matrix whose rows (and columns) may be indexed by ordered pairs (i,j) with $i,j \in \underline{N} = \{1,2,...,N\}$. The non-zero blocks are 1×1 and 2×2, naturally in correspondence with the vertices and edges of the complete graph respectively.

Target category Match^N

Fix a commutative ring k (we take $k = \mathbb{C}$). Mat is the monoidal category of matrices and aB-convention Kronecker product. note convention arbitrary so there is $\mathbb{Z}_2$ action (monoidal functor between the two conventions)

Label rows of object N by $\underline{N} = \{1, 2, ..., N\}$. note symbols suggest an arbitrary total order, so there is a Σ_N action

Mat^N is monoidal subcategory generated by object N (which is renamed as object 1 in Mat^N). $\mathrm{Mat}^N(m, n) = \mathrm{Mat}(N^m, N^n)$

Consider $R \in \mathrm{Mat}^N(m, n)$ with entries $\langle w|R|v\rangle \in k$ for $w \in \underline{N}^m$ and $v \in \underline{N}^n$

$R \in \mathrm{Mat}^N(m, m)$ charge-conserving if $\langle w|R|v\rangle \neq 0$ implies w a perm of v.

These matrices form a subcategory, Match^N.

R-matrix

The braid category is monoidally generated by the elementary braid $\sigma \in \mathsf{B}(2,2)$. Thus a functor

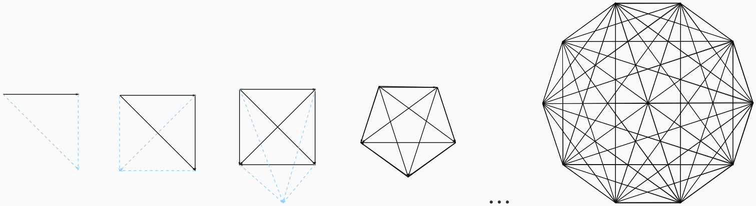
$$F : \mathsf{B} \to \mathsf{Match}^N$$

is determined by the image $F(\sigma) \in \mathsf{Match}^N(2,2)$.

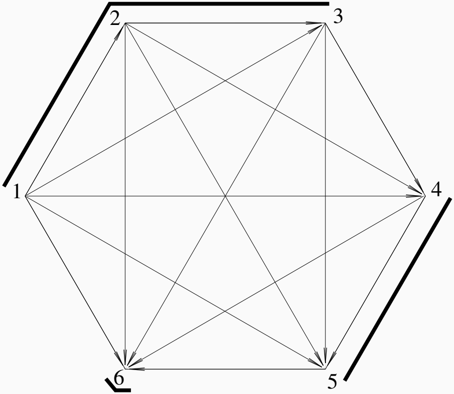
The image $F(\sigma)$ is a sparse matrix whose rows (and columns) may be indexed by ordered pairs (i,j) with $i,j \in \underline{N} = \{1,2,...,N\}$. The non-zero blocks are 1×1 and 2×2, naturally in correspondence with the vertices and edges respectively of the complete graph K_N.

Complete graph visualisation

Solutions can thus be visualised using K_N



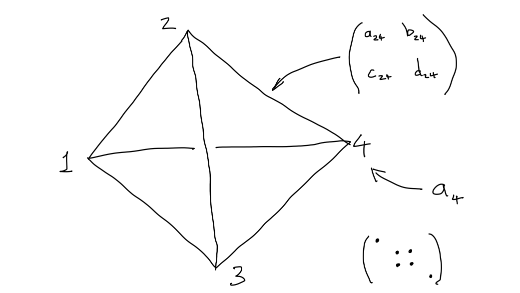
Here K_N denotes the directed graph with vertex set $\underline{N} = \{1, 2, ..., N\}$ and an edge (i, j) whenever $i < j$. Thus for example:



Here we have also indicated a partition of the vertices given by the integer partition $\lambda = (321)$. (for later use)

Gauge configuration

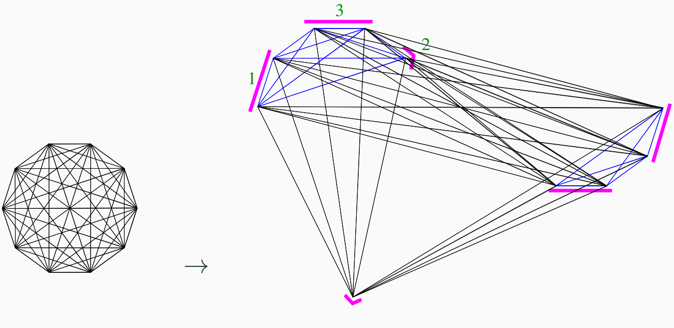
A configuration α on K_N is an assignment of a variable to each vertex and a 2×2 matrix of variables to each directed edge.



On classification theorems

- Aside. How to read the following Theorem: An example of a classification theorem is of course Young's classification of irreducible representations of the symmetric group over $\mathbb{C}$. Here one says that irreps may be classified up to isomorphism by the set of integer partitions (a set that is relatively well-understood). Analogously we will need to introduce notation for some further relatively-straightforward combinatorial structures. (Our Theorem also gives a *construction*. We will introduce notation for this too.)

Classification: first schematic



$$\mathfrak{S}_N = \{(p, q, \rho, s) \mid p < q \in \mathsf{P}(N);\ \rho \in Perm(q);\ s \in \mathsf{P}_2(q)\}$$

— rest of the talk = showing this set indexes all representations.

Classification: Notation

Next we construct two sets (for each N). One is the set $\mathfrak{S}_N$ to which we may apply an algorithm (given below) to construct all varieties of solutions. The other gives a transversal of this set under the Σ_N action (and frames the effect of the $\mathbb{Z}_2$ action), thus addressing the classification up to isomorphism.

Notation

We will need some notation.

Integer partitions and compositions

- We write Λ_N for the set of integer partitions of N. We may write $\lambda \vdash N$ for $\lambda \in \Lambda_N$.

- We write Γ_N for the set of compositions of N. For a composition we sometimes write $\lambda = \lambda_1\lambda_2 ... \lambda_L$ and sometimes write $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_L$.
If $\lambda = \lambda_1\lambda_2 ... \lambda_L$ is a composition then we write Γ_λ for the set

$$\Gamma_\lambda = \times_{i=1}^{L} \Gamma_{\lambda_i}$$

We may write $\mu \models \lambda$ for $\mu \in \Gamma_\lambda$.

Let $\lambda = \lambda_1\lambda_2 ... \lambda_L$ be a non-empty composition. We define

$$\gamma^l(\lambda) = \lambda_1\lambda_2 ... (\lambda_L + 1)$$

and $\gamma^r(\lambda) = \lambda_1\lambda_2 ... \lambda_L 1$. Observe that both define injective functions from Γ_N to Γ_{N+1}, and that $\gamma^l(\Gamma_N) \cap \gamma^r(\Gamma_N) = \emptyset$. Thus in particular

$$|\Gamma_N| = 2^{N-1}$$

and the orders of all Γ_λs follow.

Set partitions

- Given a set S we will write $\mathrm{P}(S)$ for the set of partitions; and $\mathrm{P}_2(S)$ for the set of partitions into at most 2 parts. If $N \in \mathbb{N}$ then $\mathrm{P}(N) = \mathrm{P}(\underline{N})$.
- "Children-first order": Given a partition p of $\{1, 2, ..., N\}$ (or any subset thereof), we will order the parts according to their lowest numbered element; and write p_i for the ith such part.
- A refinement of partition p is a further partition of the parts of p into possibly smaller parts. We will write $p < q$ if q is a refinement of p.
- Given a finite set S then $Perm(S)$ is the set of total orders of the elements, i.e. $Perm(S) = Hom^{iso}(S, \underline{|S|})$, the set of isomorphisms.

-should give an example in case S is q, a set partition,...

And now to reiterate:

$$\mathfrak{S}_N = \{(p, q, \rho, s) \mid p < q \in \mathrm{P}(N);\ \rho \in Perm(q);\ s \in \mathrm{P}_2(q)\}$$

We seek all

$$\mathrm{F} : \mathrm{B} \to \mathrm{Match}^N$$

i.e.

$$F : \sigma \mapsto \mathrm{Match}^N(2, 2)$$

Next we give

$$\mathrm{R} : \mathfrak{S}_N \to \mathrm{Match}^N(2, 2)$$

Recipe for solutions from $\mathfrak{S}_N$

The recipe for constructing a variety of solutions (or rather of elements of $\mathrm{Match}^N(2,2)$ that we shall prove later are all solutions) from $(p,q,\rho,s) \in \mathfrak{S}_N$ is as follows.

We call the parts of p 'nations' and order/number as above. Each edge that is between nation i and nation j is decorated with $/_{\mu_{ij}}$.

We call the parts of q 'counties', and each edge between vertices in the same county is decorated with 0.

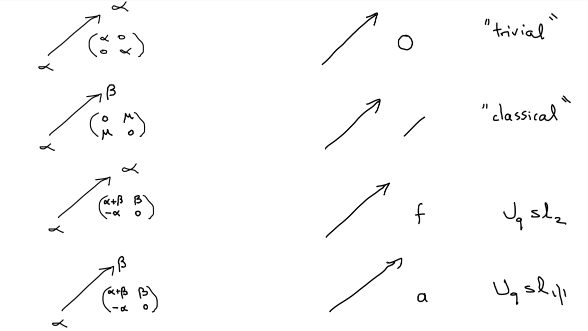
The remaining edges — between vertices in different counties in the same nation are decorated as follows. If the order on counties given by ρ places vertex v before w and also $v < w$ in the natural order then the vw edge is $+$; otherwise it is $-$.

Next for each vertex in nation i if it is in the first part in s decorate this vertex with α_i, else decorate with β_i.

Finally for each signed edge, if it is $+$ (resp. -) and the end vertices are both α_i then decorate with f_{β_i} (resp. $\underline{\mathrm{f}}_{\beta_i}$); or both β_i then decorate with f_{α_i} (resp. $\underline{\mathrm{f}}_{\alpha_i}$); else if end vertices are different then decorate with a (resp. $\underline{\mathrm{a}}$).

See below for the matrix implications of these decorations.

Edge symbol to matrix conversion



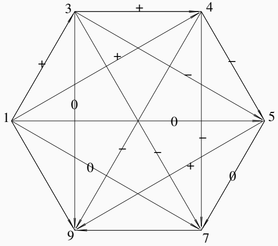
$$\mathrm{R} : \mathfrak{S}_N \to \mathrm{Match}^N(2,2)$$

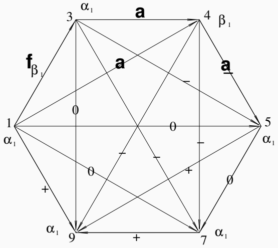
Examples for elements (p, q, ρ, s)

Consider a nation of size 6 in some complete graph. Say the first nation here:
$p = \{\{1, 3, 4, 5, 7, 9\}, \{2, 6, ...\}, \{8, ...\}, ..., \{...\}, ...\}$

A q refining this starts $q = \{\{1, 5, 7\}, \{3, 9\}, \{4\}, ...\}$ — writing out just the counties of the first nation.

For the order ρ of these counties let us consider the written order (which happens to be the nominal children-first order). Then we have the following signs in the first nation:





Thus for example the 35 edge is $-$ because 3 is higher in the nominal order but 5 comes first in the chosen county order.

Finally for s let us lump the first two counties together: $s = \{\{1, 5, 7, 3, 9\}, \{4\}\}$ (restricting here only to the first nation). Then the vertex variables are as shown on the right above - where some of the f/a labels are also shown.

Notation between integer and set partitions

- Given an integer partition, or indeed a composition, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ of N we have a partition of $\underline{N} = \{1, 2, ..., N\}$ into 'nations':

$$\mathsf{p}_\Lambda(\lambda) = \{\{1, 2, ..., \lambda_1\}, \{\lambda_1{+}1, \lambda_1{+}2, ..., \lambda_1{+}\lambda_2\},$$
$$\{\lambda_1{+}\lambda_2{+}1, \lambda_1{+}\lambda_2{+}2, ..., \lambda_1{+}\lambda_2{+}\lambda_3\}, ..., \{\left(\sum_{i=1}^{m-1} \lambda_i\right){+}1, ..., N\}\}$$

- Given a set partition p of a set S of order N we write $|p|$ for the integer partition of N given by the orders of the parts of p. (NB This does not require S to be ordered. If it is, then we may also extract a *composition* $||p||$.)

Notation - transversal

For the Σ_N-transversal we have:

$$\mathfrak{T}_N = \{(\lambda, \mu, s) \mid \lambda \vdash N, \mu \models \lambda, s \in \mathsf{P}_2(\mathsf{p}_\Lambda(\mu))\}$$

Here $\mu \models \lambda$ denotes that μ consists of a composition of each part of λ, with the compositions of order-tied parts written in lex order. For example $(1+1+1, 1+2, 1+2, 2+1, 1) \models (3, 3, 3, 3, 1) \vdash 13$.

The recipe for constructing an explicit element of $\mathfrak{S}_N$ from an element of the formal transversal $\mathfrak{T}_N$ (and hence a solution, via the above recipe) is as follows.
From λ we construct the set partition $\mathsf{p}_\Lambda(\lambda)$. From μ we construct the refinement $\mathsf{p}_\Lambda(\mu)$. The order on the counties is simply the natural one. (Example to follow.) From s ...

For the full transversal we look at the $\mathbb{Z}_2$-orbits of the Σ_N-transversal. The $\mathbb{Z}_2$ action breaks into "palindromes" and pairs.

Theorem

Theorem
For each $N \in \mathbb{N}$ the set $\mathfrak{S}_N$ indexes a transversal of the set of charge-conserving braid reps up to equivalence.

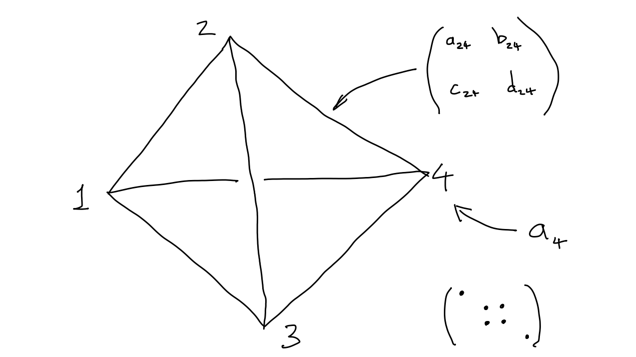
...We will conclude the construction once we have some machinery from the $N = 2, 3$ cases.

Outline of proof. We first give a Theorem formulating sufficient conditions for a solution. Then we solve these conditions in ranks 2 and 3. Then we will need a couple of Lemmas exploiting some magic in these solutions.

The proof

Recall

Conventions for $\text{Match}^N(2,2)$:



Theorem

Let $F : \mathsf{B} \to \mathsf{Mat}^N$ be a level-N charge-conserving monoidal functor, thus determined by $F(\sigma)$. The braid relations in the form

$$(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)$$

imply the following constraints on entries in the abcd form of $F(\sigma)$, together with the images of these constraints which are their orbits under the action of $\Sigma_N \times \mathbb{Z}_2$.

$$a_{12}(a_1^2 - a_{12}a_1 - b_{12}c_{12}) = 0, \qquad a_{12}(a_2^2 - a_{12}a_2 - b_{12}c_{12}) = 0, \tag{1.1}$$

$$a_{12}c_{12}d_{12} = 0 = a_{12}b_{12}d_{12}, \qquad a_{12}d_{12}(a_{12} - d_{12}) = 0, \tag{1.2}$$

$$c_{12}(d_{13}d_{23} - d_{12}d_{23} - a_{12}d_{13}) = 0, \qquad c_{12}(-a_{13}a_{23} + a_{12}a_{23} + d_{12}a_{13}) = 0 \tag{1.3}$$

$$-a_{13}d_{23}^2 + a_{13}^2 d_{23} - a_{12}b_{23}c_{23} + a_{12}b_{13}c_{13} = 0 \tag{1.4}$$

Cf. e.g. [Hietarinta].

Proof. Observe that the YBE can be verified in $\mathrm{Mat}^N(3,3)$ (since it can be formulated with three tensor factors; and the higher versions simply contain more copies of the same entries, by the tensor construction). One observes that this yields equations stretching across three indices, for example with $N=4$ the row of the matrix that must vanish $F(\varsigma_1\varsigma_2\varsigma_1 - \varsigma_2\varsigma_1\varsigma_2)$ with label $1\otimes 2\otimes 3$ (hereafter simply 123) is

$$
\begin{aligned}
&[0,0,0,0,0,0,a_{13}b_{12}c_{12}-a_{13}b_{23}c_{23}-a_{12}a_{23}+a_{12}a_{23},0,\\
&\qquad\qquad 0,a_{12}a_{13}b_{23}-a_{13}b_{23}d_{23}-a_{12}a_{23}b_{23},0,0,0,0,0,0,\\
&0,0,a_{13}b_{12}d_{12}-a_{13}a_{23}b_{12}+a_{12}a_{23}b_{12},0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]
\end{aligned}
$$

— column order is the usual lex order (so 111 112 113 114 121 122 123 124 then 131 132 ...). Observe that the non-zero entries are at matrix entries given by $123 \to 123$, $123 \to 132$, $123 \to 213$. The point we need here is (from charge conservation) that for a non-zero matrix entry the column index is at most a perm of the row index. Since the row index can contain at most three different colours (it contains three colours), the column index contains the same three. It follows that the level 3 case produces representatives of all possible orbits of constraint. □

Lemma

Let $F(\sigma)$ be a solution, with the submatrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{ij}$ *($ij = 12, 13, 23, ...$). Then changing these (independently) to* $\begin{pmatrix} a & xb \\ c/x & d \end{pmatrix}_{ij}$ *(any invertible x_{ij}) and hence in particular to* $\begin{pmatrix} a & cb \\ 1 & d \end{pmatrix}_{ij}$ *('lower-1 form') also gives a solution.*

In general writing $F'(\sigma)$ for the changed solution we say that F is X-related to F'. This is an equivalence relation, partitioning the set of all solutions into 'X-equivalence classes'. We write $F(\sigma) \stackrel{X}{=} F'(\sigma)$ for X-equivalent solutions.

(Note that F' is a solution not only in the same varietal branch, but with the same spectrum as F.)

Proof. Observe that the constraint equations above are satisfied by the substitutions given that they are satisfied by $F(\sigma)$, because the entries b_{ij}, c_{ij}, if appearing with non-zero coefficient, appear as $b_{ij}c_{ij}$. The equivalence relation property follows by construction. $\square$

Proposition

For $N = 2$ the following gives a complete classification of charge conserving functors F *from* B *up to* x *conjugation. We use the coefficient names given by* $\mathsf{F}(\sigma) = \begin{pmatrix} a_1 & & & \\ & a & b & \\ & c & d & \\ & & & a_2 \end{pmatrix}$. *We have:*

(0) the diagonal/'trivial' cases: $bc = 0$ *implies* $b = c = 0$ *and* $a_2 = a = d = a_1 \neq 0$.

(1) for $bc \neq 0$, *and hence* $ad = 0$:

(1.0) the cases $a = d = 0$: *eigenvalues* $a_2, a_1, \pm\sqrt{bc}$

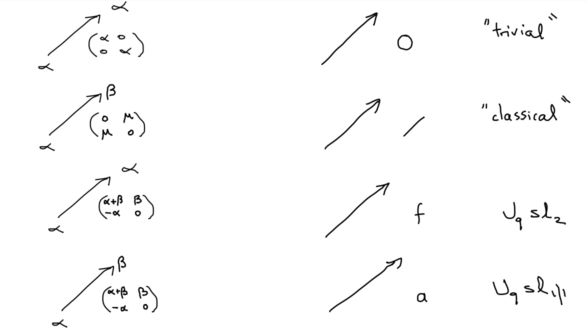
(1.1) the cases $a \neq 0$, $d = 0$: *here there are two subcases,* $a_2 \neq a_1$ *and* $a_2 = a_1$:

(1.1i) $a_2 \neq a_1$ *implies* $bc = -a_2 a_1$, $a = a_2 + a_1$ *(eigenvalues:* a_2, a_1*)*

(1.1ii) $a_2 = a_1$ *implies* bc *unconstrained (nonzero) (eigenvalues:* $a_2, -bc/a_2$*)*

(1.2) the case $d \neq 0$, $a = 0$ *(similar to (1.1) by symmetry).*

Summary of $N = 2$



proof. Invertibility requires that $a_2, a_1 \neq 0$ and $ad \neq bc$, so if $bc = 0$ then $ad \neq 0$. The braid relations require

$$abd = acd = 0, \quad (a-d)ad = 0. \tag{1.5}$$

$$a(a_1(a_1 - a) - bc) = 0, \quad a(a_2(a_2 - a) - bc) = 0, \tag{1.6}$$

$$d(a_1(a_1 - d) - bc) = 0, \quad d(a_2(a_2 - d) - bc) = 0.$$

(0) If $bc = 0$ we have $a_2 = a = d = a_1$ by invertibility, (1.5) and (1.6).

(1) If $bc \neq 0$ we have $ad = 0$ by (1.5).

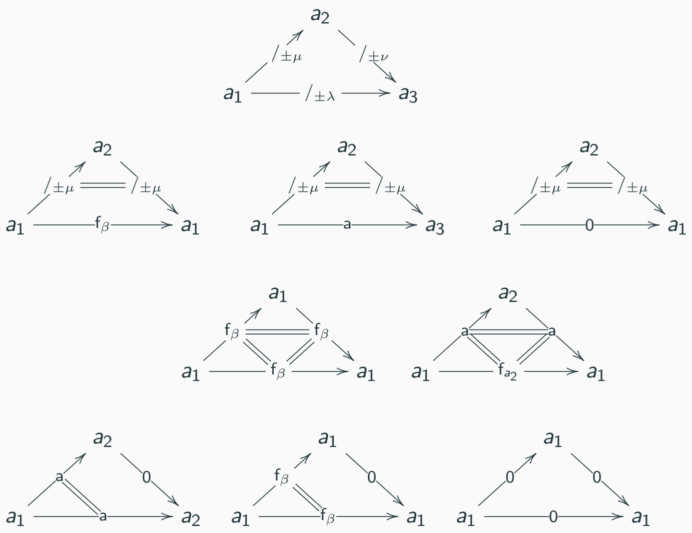
(1.0) If $a = d = 0$ the conditions are immediately satisfied.

(1.1) Taking $d = 0$ and $a \neq 0$, the remaining conditions are $bc = a_2(a_2 - a) = a_1(a_1 - a)$. Thus $a = a_2 - bc/a_2$ and the characteristic equation for the middle block is $(\lambda - a_2)(\lambda + bc/a_2) = 0$. If $a_2 \neq a_1$ then $bc = -a_2 a_1$ and $a = a_2 + a_1$. In either case, a_2 and $a - a_2$ are the eigenvalues (with multiplicities 3 and 1 in (1.1ii) and 2 and 2 in (1.1i)).

(1.2) similar.

□

Proposition. ('9-rule') For $N = 3$ the following triangles are allowed (showing one per S_3 orbit?):



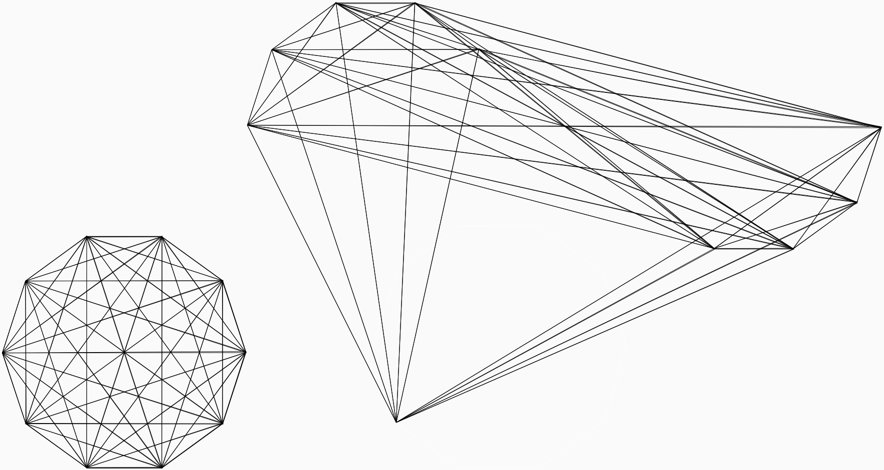
proof. From the $N = 2$ result there are 6^3 cases to try. Various miracles occur. $\square$

Outline proof of Theorem

A *configuration* of K_N is an assignment of a scalar to each vertex; and matrix to each edge, from the six types $/, 0, \mathsf{f}, \underline{\mathsf{f}}, \mathsf{a}, \underline{\mathsf{a}}$ as before. (Note that there are variables in all these components.)

An 'admissible' (or 'ground state') configuration is a configuration such that every triangle configuration is one of those nine forms given above. (Note that variables are constrained by these conditions, as well as types.)

Here we will enumerate groundstates (as varieties in the appropriate variables) in all ranks.



Lemma

(Rule-of-1 Lemma) An edge 2-colouring of complete graph K_N, with colour set $\{x, y\}$, is called 'parted' *if every triangle that has two y's has three.*

(I) The set of parted colourings is in bijection with the set of partitions of the vertices.

(II) The subset of colourings where no triangle has three x's is in bijection with the subset of partitions into two parts.

proof. (I) This amounts to a rearrangement of the definition of equivalence relation, where y on edge $\{v, v'\}$ means that $v \sim v'$.
(II) This gives the subset where no triple has three vertices in different parts. $\square$

Lemma
The / data of a configuration $F(\sigma)$ induces a partition $p_/(F)$ of the vertices. Specifically, exactly the edges between vertices in different parts are / edges. Furthermore, two / edges between the same two parts carry the same variable.

proof. Observe from the list of allowed triangle configurations that this data is parted with / in the role of x and everything else in the role of y. Now apply the Rule-of-1 Lemma.
The second part follows from the nature of triangles with two /s. □

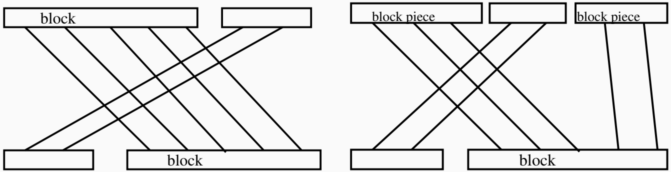
The remaining (non-/) edges connect within a part. We may refer to these /-parts as 'nations'. (A specific, albeit extrinsic, reason for this name choice will be given later.)

Lemma
For every level-N solution $F(\sigma)$, hence partition $p_/(F)$, there is an element in the $\Sigma_N \times \mathbb{Z}_2$ orbit such that

$$p_/(F') = \mathrm{p}_\Lambda(|p_/(F)|)$$

specifically given by parallel-transporting each part, so that labels and orientations are preserved in the blocks.

proof. This is a manifestation of the relabelling symmetry, just noting that if the permutation is relatively non-crossing within each part:



then orientations are preserved. □

Lemma
Consider a part of the partition induced by /. Edge 0's induce an equivalence and then arrows induce a 'weak total' order (i.e. a partial order that is a total order on equivalence classes of the equivalence relation).

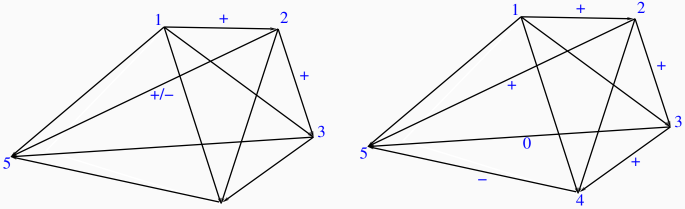
proof. Edge 0's induce an equivalence by comparing allowed triangle configurations with the Rule-of-1 Lemma. Configurations induce a well-defined order on the quotient since triangles are never cyclic-ordered and triangles with a 0 'collapse' consistently. □

It will be convenient to refer to f, a as $+$-oriented and $\underline{f}, \underline{a}$ as $-$-oriented.

6-rule: if an oriented chain of two edges is signed with the same sign in $F(\sigma)$ then the 'long' edge completing the triangle is signed with the same sign.
— To see this check the 9-rule list (**??**) and the orbit of $+++$ (**??**).
(This is called 6-rule since it reduces the number of possible edge $\pm$-colourings of a triangle from eight to six. It also reduces the number of ± 0-colourings.)

Lemma
Every $\Sigma_N \times \mathbb{Z}_2$ orbit of solutions contains an element with no $-$-orientation.
Furthermore, in such a non-$-$ configuration the elements of a 0-part are clustered with respect to the natural order in their nation, i.e. they are consecutive.

proof. First we note that the claim holds if it holds for 1-nation solutions, since there are perms that act non-trivially only on a single nation and the /-edges out of that nation, restricting to a complete set of perms on that nation. So now consider a single nation. We work by induction on N. The claim is true for $N < 4$ by inspection of our explicit solution sets. Suppose true at level-N and consider $N + 1$. WLOG by inductive assumption consider a configuration with all (signed) edges $+$ between vertices $1, 2, .., N$ (any non-signed edges are 0). Consider what configurations of edges to vertex $N+1$ are allowed here. Neglecting cases with 0s on the edges to $N+1$ for a moment we have, say, as on the left here:



By the 6-rule, if the i to $N+1$ edge is $+$ then, since the $i-1$ to i edge is $+$, the $i-1$ to $N+1$ edge is also $+$. Indeed this is also forced in the case where the $i-1$ to i edge is 0. Thus there are N possible configurations of form $++...+--..-$ (NB all these arrows point to $N+1$). The i-th of these configurations is taken to $+++...+$ by the perm $(i...N)$, so we are done in these cases. In case of 0 on a long edge i to $N+1$, the edges j ($j<i$) to i and j to

$N+1$ must be the same, so these edges satisfy the no-$-$ claim. Thus it remains only to address the edges after the last 0. Suppose this last 0 is i to $N+1$. The edges k to $N+1$ with $k > i$ will all be opposite to the corresponding i to k edge (which is either $+$ or 0), so either $-$ or 0. Now apply the perm $(i\ N{+}1)$. This has the effect of making the $N+1$ vertex the 'new' i vertex and incrementing i et seq. Thus all the edges that were k to $N+1$ are reversed, so all the $-$s become $+$, while the relative positions of all other vertices are preserved, so signs do not change — i.e. they remain non-$-$.

The second claim follows from the 9-rule. □

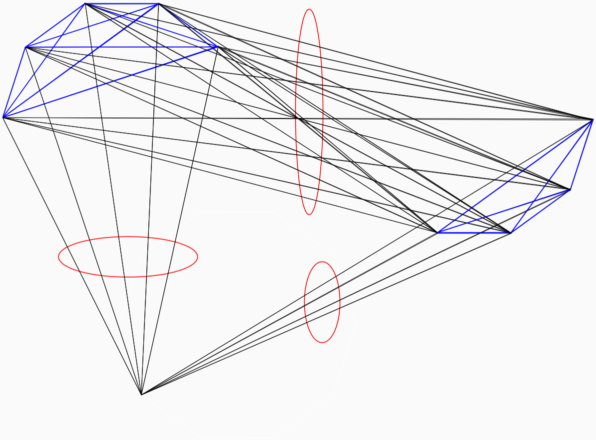
Lemma
Consider a part of the partition induced by /. The af data induces a partition on the parts of the part. It is a partition into (at most) two parts only because there is no aaa triangle.

proof. This is just a matter of unpacking the definitions and then using the Rule-of-1 Lemma for a third (!) time, this time using part (II). □

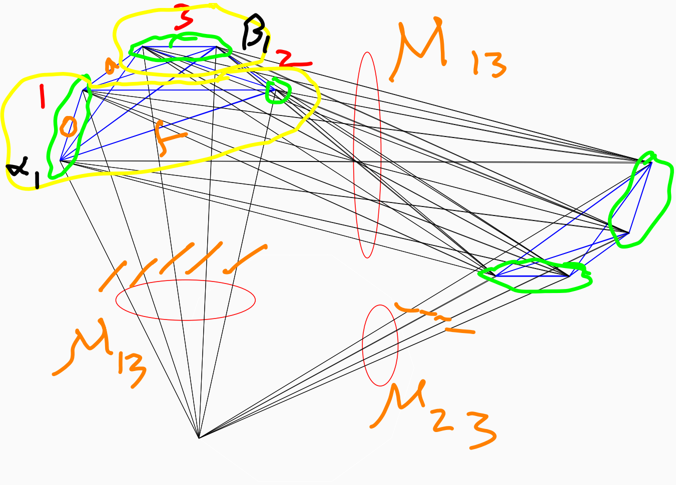
In short then, a configuration should be organised firstly as a partition according to / (we say a partition into 'nations'). Then according to 0 (into 'counties'). Then ...
... as in our pictures above taking elements of $\mathfrak{S}_N$ to elements of $\text{Match}^N(2, 2)$.

Done.

The schematic again



The schematic again



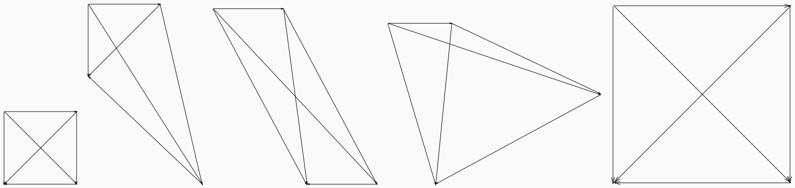
Worked example

Example: rank 4

Since we may work up to the action of the symmetric group permuting the vertices, the set partitions are effectively indexed by the corresponding integer partitions. So for example in rank 4 we have nation partitions $\lambda = 4$, 31, 22, 211, 1111 (in the obvious shorthand).

Case $\lambda = 4$ has no /s. The possible partitions according to 0s are $\mu = 4$, 31, 22, 211, 1111, which give weak orders 4, 3/1, 2/2, 2/1/1 and 1/2/1, 1/1/1/1 respectively (in the obvious shorthand; note for example that 1/3 is in the $\mathbb{Z}_2$-orbit of 3/1 so we do not include separately).
These correspond to edge labellings: 000000, 000+++, 0++++0, 0+++++ and ++0+++, ++++++, respectively (ordering edges as 12,13,23,14,24,34).
Then the f-level partitions are 4 (trivially); 31 and 4, corresponding to 000aaa and 000fff; 22 and 4, corresponding to 0aaaa0 and 0ffff0; 22 and 31 and 4, corresponding to 0aaaaf and 0aaffa and 0fffff; ...respectively.

‘Cubist’ schematics for nation partitions: rank 4



Here is an everything table for Case λ =4 then 31 then 22 then 211 then 1111:

$0-pt$	4	31	22	211		1111
weak	4	3/1	2/2	2/1/1	1/2/1	1/1/1/1
$0+$	000000	$000+++$	$0++++0$	$0+++++$	$++0+++$	$++++++$
$f-pt$	4	31 4	22 4	22 31 4	22 31 4	22 31 4
fa	000000	000*aaa* 000*fff*	0*aaaa*0 0*ffff*0	0*aaaaf* 0*aaffa* 0*fffff*		*faaaaf* *fffaaa* *ffffff*

$0-pt$	3|1	21|1	111|1
weak	3|1	2/1|1	1/1/1|1
$0+$	000///	$0++$///	$+++$///
$f-pt$	3|1	21|1 3|1	21|1 3|1
fa	000///	0*aa*/// 0*ff*///	*aaf*/// *fff*///

$0-pt$	2|2	11|2	11|11	2|1|1	11|1|1	1|1|1|1
weak	2|2	1/1|2	1/1|1/1	2|1|1	1/1|1|1	1|1|1|1
$0+$	0////0	+////0	+////+	0/////	+/////	//////
$f-pt$	2|2	11|2 2|2	11|11 2|11 2|2	2|1|1	11|1|1 2|1|1	1|1|1|1
fa	0////0	*a*////0 *f*////0	*a*////*a* *f*////*a* *f*////*f*	0/////	*a*///// *f*/////	//////

Signatures: how different are formally inequivalent solutions

Then signatures for all.

Viewed top-down, we now have some interesting questions to address about old and new paradigms for equivalence in representation theory.
Staying fairly close to traditional paradigms, we can consider 'signatures'.

So, what is a *signature* again? It is roughly the eigenvalue spectrum of the solution matrix for each solution type. Since the actual eigenvalues come from setting indeterminates from the formal solutions, what we consider here is the degeneracies (or generic degeneracies) rather than the values. So, ...

For $N = 3$:

$000 \to 9$ (i.e. degeneracy 9, all eigenvalues same)

$0aa \to 63$

$0ff \to 72$

$afa \to 54$

$fff \to 63$ (NB ... Jordan form)

$0// \to 4221$

$f// \to 32211$

$a// \to 22221$

$/// \to 1^9$

For $N = 4$:

$000000 \to 16$

$000aaa \to 12\ 4$

$000fff \to 13\ 3$

$0aaaa0 \to 12\ 4$ (NB ...)

$ffffff \to 10\ 6$

...

$////// \to 1^{16}$

Prospects

At this point a number of possibilities for developments, applications and generalisations arise.

Just a few examples: In this framework it is natural to consider extension to representations of loop braids, and thence to motion groupoids and TQFTs. Generalisation of the target category. Extensions to systems with tetrahedron constraints as well as vertex, edge and triangle. Treatment of classification-set combinatorics (related to combinatorics of social hierarchies). Higher(/lower!) representation theory. ...

Happy birthday Hubert!

A very small subset of references follow.

H. A. Dye.
Unitary solutions to the Yang-Baxter equation in dimension four.
Quantum Inf. Process., 2(1-2):117–151 (2003), 2002.

J.A. Green.
***Polynomial Representations of GLn*, volume 830 of *Lecture notes in mathematics*.**
Springer, 1980.

Jarmo Hietarinta.
All solutions to the constant quantum Yang-Baxter equation in two dimensions.
Phys. Lett. A, 165(3):245–251, 1992.

J E Humphreys.
***Reflection groups and Coxeter groups*.**
Cambridge, 1994.

Z. Kadar, P. Martin, E. Rowell, and Z. Wang.
Local representations of the loop braid group.
Glasgow Math. J., pages 1–20, 006 2016.

S Mac Lane.
***Categories for the working mathematician*.**
Springer, 1998.

P P Martin.
***Potts Models and related problems in statistical mechanics*.**
World Scientific, Singapore, 1991.